ADVANCES IN MATHEMATICS 35, 30–49 (1980)

Post's Problem without Admissibility

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INTRODUCTION

This paper is a contribution to β-recursion theory; i.e., recursion theory on arbitrary limit ordinals. The basic definitions, motivation, and results were set forth in [4]. In brief, the setting for β-recursion theory is S_β, the βth level of Jensen's S-hierarchy for L (see [1, p. 82]) and β-recursively enumerable (β-r.e.) sets are those subsets of S_β which are Σ_1-definable over $\langle S_\beta, \epsilon\rangle$. $A \subseteq S_\beta$ is β-recursive if both A and $S_\beta - A$ are β-r.e. and is β-finite if $A \in S_\beta$. As in α-recursion theory, there is a β-r.e. enumeration $\{W_e\}_{e \in S_\beta}$ of the β-r.e. sets and using this we define: A is weakly β-recursive in B ($A \leqslant_{w\beta} B$) if for some e

$$x \in A \leftrightarrow \exists K_1\, \exists K_2[\langle 0, x, K_1, K_2\rangle \in W_e \text{ and } K_1 \subseteq B \text{ and } K_2 \subseteq S_\beta - B],$$
$$x \in S_\beta - A \leftrightarrow \exists K_1\, \exists K_2[\langle 1, x, K_1, K_2\rangle \in W_e \text{ and } K_1 \subseteq B \text{ and } K_2 \subseteq S_\beta - B],$$

where K_1, K_2 vary over β-finite sets. A is β-recursive in B ($A \leqslant_\beta B$) if $\{Z \in S_\beta \mid Z \subseteq A\}$ and $\{Z \in S_\beta \mid Z \subseteq S_\beta - A\}$ are both weakly β-recursive in B. $\leqslant_\beta$ is transitive and the β-degree of $A = \{B \mid A \leqslant_\beta B, B \leqslant_\beta A\}$.

The program of β-recursion theory is to generalize theorems of ordinary recursion theory to arbitrary limit ordinals. This paper focuses on the β-degrees of β-r.e. sets and provides a partial solution to:

POST'S PROBLEM. Show that there are $\leqslant_{w\beta}$-incomparable β-r.e. sets.

Post's problem was solved in ordinary recursion theory by Friedberg [2] and Muchnik [10] independently, and generalized to α-recursion theory (i.e., recursion theory on admissible ordinals) by Sacks and Simpson [14]. (Purely "syntactic" techniques suffice to solve the following weaker formulation of Post's problem for every inadmissible β: Find a β-r.e. set of β-degree strictly between 0 and 0′, the largest β-degree of a β-r.e. set. See Theorem 3.4 of [4].)

Sacks and Simpson used ideas from the structure of L devised by Gödel to prove the Generalized Continuum Hypothesis in L [6]. We extend the application of techniques from the fine structure of L to recursion theory by making use of Jensen's $\diamondsuit$-principle in our solution to Post's problem. An effectivized version of Fodor's theorem is also developed and used in our proof. For background material on $\diamondsuit$ and stationary sets, see [1].

0001-8708/80/010030-20$05.00/0

In Section 1, Fodor's theorem, $\diamondsuit$, and other facts about stationary sets are reviewed. Section 2 uses these ideas to solve Post's problem in the special case when $\beta^* = \aleph_1^L$. Finally, in Section 3 the material of the first section is effectivized and applied to the case where β^* is only regular with respect to functions Σ_1 over S_β. Thus Post's problem is solved if β^* is a successor β-cardinal. Also, the generalization of the construction to Σ_n predicates is discussed in Section 4.

1. Review of Stationary Sets and the $\diamondsuit$-Principle

We begin by recalling some basic definitions and facts from combinatorial set theory.

Definition. Let $\kappa > \omega$ be a regular cardinal. $C \subseteq \kappa$ is closed if for all $\gamma < \kappa$, $C \cap \gamma$ unbounded in $\gamma \to \gamma \in C$. $S \subseteq \kappa$ is stationary if $S \cap C \neq \varnothing$ for every closed unbounded $C \subseteq \kappa$.

Proposition 1. *Suppose $\{C_\alpha\}_{\alpha<\gamma}$ is a collection of closed unbounded sets and $\gamma < \kappa$. Then $C = \bigcap_{\alpha<\gamma} C_\alpha$ is closed and unbounded.*

Proof. Let $\beta_{0,0}$ be arbitrary. Inductively, define

$$\beta_{n,0} = \text{some member of } C_0 \text{ greater than } \bigcup_{\alpha<\gamma} \beta_{n-1,\alpha},$$

$$\beta_{n,\delta} = \text{some member of } C_\delta \text{ greater than } \bigcup_{\alpha<\delta} \beta_{n,\alpha} \text{ if } \delta > 0.$$

Then $\beta = \bigcup_n \beta_{n,0} = \bigcup_n \beta_{n,\alpha}$ for all α. So $\beta \in \bigcap_\alpha C_\alpha$. Thus C is unbounded. But $C \cap \delta$ unbounded in δ

$$\begin{aligned} &\to C_\alpha \cap \delta \quad \text{unbounded in } \delta, \forall\alpha \\ &\to \delta \in C_\alpha \quad \text{for all } \alpha \\ &\to \delta \in C. \quad \blacksquare \end{aligned}$$

Corollary 2. *If $C \subseteq \kappa$ is closed and unbounded, then C is stationary.*

Proof. Just apply Proposition 1 when $\gamma = 2$. ∎

Fodor's Theorem. *Suppose $S \subseteq \kappa$ is stationary, $f\colon S \to \kappa$, and $f(\alpha) < \alpha$ for every $\alpha \in S$ (that is, f is regressive on S). Then for some γ, $\{\alpha \in S \mid f(\alpha) < \gamma\}$ is a stationary subset of S.*

Proof. Suppose not. Let $\gamma_0 < \kappa$ be arbitrary. Choose a closed unbounded C_0 so that $\alpha \in C_0 \cap S \to f(\alpha) \geqslant \gamma_0$. Let $\gamma_0 < \gamma_1 \in C_0$. Choose a closed unbounded $C_1 \subseteq C_0$ so that $\alpha \in C_1 \cap S \to f(\alpha) \geqslant \gamma_1$ (by Proposition 1). Choose

$\gamma_0 < \gamma_1 < \gamma_2 \in C_1$. Continuing in this way define $\gamma_0 < \gamma_1 < \cdots$, $C_0, C_1, \ldots$ so that for all $\beta < \kappa$

(a) $\gamma_\beta \in C_{\beta'}$ for all $\beta' < \beta$,

(b) β limit $\rightarrow \gamma_\beta = \bigcup_{\beta'<\beta} \gamma_{\beta'}$, $C_\beta = \bigcap_{\beta'<\beta} C_{\beta'}$,

(c) $\alpha \in C_\beta \cap S \rightarrow f(\alpha) \geqslant \gamma_\beta$.

This is possible as β limit $\rightarrow \gamma_\beta \cap C_{\beta'}$ unbounded in γ_β for $\beta' < \beta \rightarrow \gamma_\beta \in C_\beta$. Then $\gamma_\beta \in S$ for some limit β, as S is stationary and $\{\gamma_\beta \mid \beta \text{ limit}\}$ is closed, unbounded. But then $\gamma_\beta \in C_\beta$ and so $f(\gamma_\beta) \geqslant \gamma_\beta$ by (c), contradicting the hypothesis that f is regressive on S. ∎

Jensen's $\diamondsuit$-principle (see [7]) is a strong axiom true in Gödel's L which can be used to diagonalize over subsets of κ in only κ-many steps. Jensen used it to construct a Souslin Tree in L.

Let κ be a regular cardinal. $\diamondsuit_\kappa$ says: There is a sequence $\langle S_\alpha \mid \alpha < \kappa \rangle$ such that

(i) $S_\alpha \subseteq \alpha$ for each α,

(ii) If $X \subseteq \kappa$, then $\{\alpha \mid S_\alpha = X \cap \alpha\}$ is stationary.

PROPOSITION 3. (a) *$\diamondsuit_{\kappa^+}$ implies that $2^\kappa = \kappa^+$.*

(b) *$\diamondsuit_\kappa$ implies that there is a collection $\{X_\alpha \mid \alpha < \kappa\}$ of stationary subsets of κ such that $\alpha_1 \neq \alpha_2$ implies $X_{\alpha_1} \cap X_{\alpha_2} = \varnothing$.*

Proof. (a) Define $f: 2^\kappa \rightarrow \kappa^+$ by $f(Y) =$ least $\alpha > \kappa$ such that $S_\alpha = Y$. Then f is well defined and 1–1.

(b) For each $\alpha < \kappa$, let $X_\alpha = \{\beta > \alpha \mid S_\beta = \{\alpha\}\}$. Then each X_α is stationary and the X_α's are pairwise disjoint. ∎

THEOREM 4 (Jensen). *$\diamondsuit_\kappa$ is true in L. Moreover, there is a $\diamondsuit_\kappa$-sequence which is Σ_1-definable over $\langle L_\kappa, \epsilon \rangle$.*

Proof. We define S_α by induction on α. $S_0 = \varnothing$. $S_{\alpha+1} = \varnothing$ for all α. If S_α has been defined for all $\alpha < \lambda$, λ limit, we define S_λ as follows: Let $\langle X, C \rangle$ be the least (in the canonical well-ordering of L) pair so that

(i) $X, C \subseteq \lambda$,

(ii) C is closed, unbounded in λ,

(iii) $\alpha \in C \rightarrow S_\alpha \neq X \cap \alpha$.

Define $S_\lambda = X$ if such a pair exists, and $S_\lambda = \varnothing$ otherwise. Clearly, $\langle S_\alpha \mid \alpha < \kappa \rangle$ is Σ_1-definable over $\langle L_\kappa, \epsilon \rangle$.

We claim that $\langle S_\alpha \mid \alpha < \kappa\rangle$ satisfies $\diamondsuit_\kappa$. If not, let $\langle X, C\rangle$ be the least (in the canonical well-ordering of L) pair so that

(i) $X, C \subseteq \kappa$,

(ii) C is closed, unbounded in κ,

(iii) $\alpha \in C \to S_\alpha \neq X \cap \alpha$.

Now, $\langle X, C\rangle \in L_{\kappa^+}$. Let $M \prec L_{\kappa^+}$ so that

(a) M has (L-) cardinality $<\kappa$.

(b) $\kappa, \langle X, C\rangle \in M$, $\kappa \cap M$ is an ordinal.

By Gödel Condensation, there is a unique $\pi: \langle M, \epsilon\rangle \simeq \langle L_\alpha, \epsilon\rangle$, $\alpha < \kappa$. Then let $\pi(\kappa) = \lambda$. It is easily seen that $\pi \mid \lambda = \mathrm{id} \mid \lambda$, so $\pi(\langle X, C\rangle) = \langle X \cap \lambda, C \cap \lambda\rangle$. Moreover, $\pi(S_\alpha) = S_\alpha$ for $\alpha < \lambda$. Thus, since π is an isomorphism,

$\langle X \cap \lambda, C \cap \lambda\rangle$ is the least (in the canonical well-ordering of L)

pair so that

(i) $X \cap \lambda, C \cap \lambda \subseteq \lambda$,

(ii) $C \cap \lambda$ is closed, unbounded in λ,

(iii) $\alpha \in C \cap \lambda \to S_\alpha \neq X \cap \lambda \cap \alpha = X \cap \alpha$.

But then by definition, $S_\lambda = X \cap \lambda$. But since C is closed and $C \cap \lambda$ is unbounded in λ, $\lambda \in C$. This contradicts $\alpha \in C \to S_\alpha \neq X \cap \alpha$. ∎

Let E be a stationary subset of κ. We relativize $\diamondsuit_\kappa$ to E. $\diamondsuit_\kappa(E)$ says: There is a sequence $\langle S_\alpha \mid \alpha \in E\rangle$ so that

(i) $S_\alpha \subseteq \alpha$ for $\alpha \in E$,

(ii) If $X \subseteq \kappa$, then $\{\alpha \in E \mid S_\alpha = X \cap \alpha\}$ is a stationary subset of E.

THEOREM 5 (Jensen). *$\diamondsuit_\kappa(E)$ is true in L. Moreover, there is a $\diamondsuit_\kappa(E)$-sequence which is Σ_1-definable over $\langle L_\kappa, \epsilon, E\rangle$.*

Proof. Define S_α for $\alpha \in E$ by induction on α. $S_0 = \varnothing$. $S_{\alpha+1} = \varnothing$ for every α. If S_α has been defined for $\alpha < \lambda$, λ limit, let $\langle X, C\rangle$ be the least pair such that

(i) $X, C \subseteq \lambda$,

(ii) C is closed, unbounded in λ,

(iii) $\alpha \in C \cap E \to S_\alpha \neq X \cap \alpha$.

If such a pair exists, let $S_\lambda = X$; $S_\lambda = \varnothing$ otherwise. Clearly $\langle S_\alpha \mid \alpha \in E\rangle$ is Σ_1-definable over $\langle L_\kappa, \epsilon, E\rangle$.

Suppose $\langle S_\alpha \mid \alpha \in E\rangle$ does not satisfy $\diamondsuit_\kappa(E)$. Let $\langle X, C\rangle$ be the least pair so that

(i) $X, C \subseteq \kappa$,

(ii) C is closed, unbounded in κ,

(iii) $\alpha \in C \cap E \rightarrow S_\alpha \neq X \cap \alpha$.

We define a κ-sequence $M_0 \prec M_1 \prec \cdots \prec M_\alpha \prec \cdots$ of elementary submodels of L_{κ^+} as follows:

M_0 = an elementary submodel of L_{κ^+} such that $E, X, C, \kappa \in M_0$ and $M_0 \cap \kappa$ = an ordinal less than κ,

$M_{\alpha+1}$ = an elementary submodel of L_{κ^+} such that $M_\alpha \cup \{M_\alpha\} \subseteq M_{\alpha+1}$ and $M_{\alpha+1} \cap \kappa$ = an ordinal less than κ,

$M_\lambda = \bigcup_{\alpha<\lambda} M_\alpha$ for limit λ.

Let $\beta_\alpha = M_\alpha \cap \kappa$. Then $\{\beta_\alpha \mid \alpha < \kappa\}$ is a closed unbounded subset of κ, so there is a $\lambda = \beta_\alpha \in E$, since E is stationary. Let π: $M_\lambda \simeq L_\gamma$. Then $\pi(E) = E \cap \lambda$, $\pi(\langle X, C\rangle) = \langle X \cap \lambda, C \cap \lambda\rangle$ since $\pi(\kappa) = \beta_\alpha = \lambda$. So $\langle X \cap \lambda, C \cap \lambda\rangle$ is the least pair so that

(i) $X \cap \lambda, C \cap \lambda \subseteq \lambda$,

(ii) $C \cap \lambda$ is closed, unbounded in λ,

(iii) $\alpha \in C \cap \lambda \cap E \cap \lambda \rightarrow S_\alpha \neq X \cap \lambda \cap \alpha$.

But then $S_\lambda = X \cap \lambda$ by definition and $\lambda \in C$. As before this contradicts the choice of $\langle X, C\rangle$. ∎

2. Post's Problem When $\beta^* = \aleph_1^L$

The theory of stationary sets and $\diamondsuit$ can be useful in a priority construction. In this section we prove

THEOREM 6. *Suppose β is a limit ordinal and $\beta > \beta^* = \aleph_1^L$. Then there exist β-r.e. sets A, $B \subseteq \aleph_1^L$ such that $A \not\leqslant_{w\beta} B$, $B \not\leqslant_{w\beta} A$.*[1]

Proof. As in any construction to solve Post's problem, we wish to satisfy the requirements

$$R_e^A\colon \aleph_1^L - B \neq W_e^A, \qquad R_e^B\colon \aleph_1^L - A \neq W_e^B, \qquad e \in S_\beta .$$

[1] The argument that we present will also handle the case $\beta = \aleph_1^L$, or β^* = any successor L-cardinal.

Here, W_e^A = eth set β-r.e. in A. If $\{W_e\}_{e\in S_\beta}$ is a uniformly β-r.e. listing of the β-r.e. sets, then

$$W_e^A = \{x \mid \exists z_1\, \exists z_2[z_1, z_2 \in S_\beta, \langle x, z_1, z_2\rangle \in W_e \text{ and } z_1 \subseteq A, z_2 \subseteq S_\beta - A]\}.$$

In this construction, there will be added requirements to ensure that A, B are simple and weakly-t.r.e.

DEFINITIONS. $A \subseteq \beta^*$ is *simple* if A is β-r.e., $\beta^* - A$ is unbounded in β^*, and A intersects any β-r.e. B which is unbounded in β^*.

$A \subseteq \beta$ is *tamely-r.e.* (t.r.e.) if $\{z \in S_\beta \mid z \subseteq A\}$ is β-r.e. A is *weakly-t.r.e.* if $\{z \in S_\beta \mid z - A \text{ is bounded in } \beta^*\}$ is β-r.e.

The requirements for simplicity are

$$S_e^A\colon W_e \text{ unbounded in } \aleph_1^L \to A \cap W_e \neq \varnothing,$$

$$S_e^B\colon W_e \text{ unbounded in } \aleph_1^L \to B \cap W_e \neq \varnothing, \qquad e \in S_\beta.$$

Of course, we must also guarantee that $\beta^* - A$ is unbounded in β^*.

T.r.e.-ness is automatic when β is Σ_1-admissible. However, we cannot hope to build t.r.e. sets when $\Sigma_1 cf\beta = \omega$ (see [4, p. 36]). As a result, our attempts at R_e^A will not only keep ordinals out of A, but put ordinals into A. Weakly-t.r.e.-ness is essential in that it allows the positive part of these attempts to be countable. The requirements for weakly-t.r.e.-ness are

$$T_e^A\colon e - A \text{ bounded in } \aleph_1^L \to \exists \text{ stage } \sigma[e - A^\sigma \text{ is bounded in } \aleph_1^L],$$

$$T_e^B\colon e - B \text{ bounded in } \aleph_1^L \to \exists \text{ stage } \sigma[e - B^\sigma \text{ is bounded in } \aleph_1^L]$$

for $e \in S_\beta$. A^σ = the part of A enumerated by stage σ of the construction.

Remark about stages. Unless $\omega^\beta = \beta$ (for example, β primitive-recursively closed), one cannot naturally identify members of S_β with ordinals less than β. However, there is always a canonical β-recursive well-ordering $<_\beta$ of S_β with the property that $\mathrm{pr}_\beta(x) = \{z \mid z <_\beta x\}$ is a β-recursive function from S_β into S_β (see [1, p. 84]). Accordingly, we can identify stages σ with members of S_β, viewed as positions in the canonical well-ordering $<_\beta$.

Let $f\colon S_\beta \to \aleph_1^L$ be β-recursive and 1–1. f can be used to arrange the above requirements into a list of order type $\aleph_1^L$, assigning each requirement a "priority" less than $\aleph_1^L$. For example, we can assign:

$$f'(R_e^A) = f(\langle 0, e\rangle), \qquad f'(R_e^B) = f(\langle 1, e\rangle),$$

$$f'(S_e^A) = f(\langle 2, e\rangle), \qquad f'(S_e^B) = f(\langle 3, e\rangle),$$

$$f'(T_e^A) = f(\langle 4, e\rangle), \qquad f'(T_e^B) = f(\langle 5, e\rangle).$$

This assigns a single priority to each requirement. However, this is not good enough for our purposes. We would like to allow each requirement to have a *stationary set* of different priorities.

By Theorem 4, let $\langle S_\alpha \mid \alpha < \aleph_1^L \rangle$ be a β-finite $\diamondsuit_{\aleph_1^L}$-sequence. As in Proposition 3(b), $\{\alpha \mid S_\alpha = \{\gamma\}\}$ is stationary for each $\gamma < \aleph_1^L$. Define $g\colon \aleph_1^L \to \{\text{Requirements}\}$ by $g(\delta) = R$ if and only if $S_\delta = \{f'(R)\}$; $g(\delta)$ undefined if no such R exists. Then each requirement R has the stationary set of priorities $g^{-1}(\{R\})$. Note that $f' \circ g$ is partial β-recursive.

Thus each requirement R can be viewed as the whole stationary collection of auxiliary requirements R_δ, $g(\delta) = R$. The reason for arranging this is that unlike priority arguments in the admissible case, we will not be able to guarantee that every auxiliary requirement R_δ can eventually be satisfied. Instead, we can argue (rather easily) that a closed unbounded collection of auxiliary requirements R_δ will be satisfied and in fact will never be injured. Then since $g^{-1}(\{R\})$ is stationary, there will be an auxiliary requirement R_δ, $g(\delta) = R$ which will be satisfied and hence so will R.

We now describe how an attempt is made at an auxiliary requirement R_δ. All attempts will be of the form (K, H) where $K, H \in L_{\aleph_1^L}$; if (K, H) is an A-attempt then the members of K will be put into A by the attempt, and the attempt tries to preserve $A \cap H = \varnothing$. Similarly for B-attempts. In addition, if (K, H) is an attempt at R_δ, then $K \cup H$ will contain no ordinals $< \delta$. Finally (much in contrast to the admissible case):

(1) At most one A-attempt and at most one B-attempt will be made at each R_δ.

(2) If (K, H) is an attempt at R_δ at stage σ, then no attempt can be made after stage σ at any $R_{\delta'}$, $\delta < \delta' \leqslant \sup(K \cup H)$.

A consequence of (2) is that if $\delta < \delta'$, then no attempt at $R_{\delta'}$ can ever injure an attempt at R_δ. (The A-attempt (K_1, H_1) injures the A-attempt (K_2, H_2) if $K_1 \cap H_2 \neq \varnothing$. Similarly for B-attempts.)

Having laid the groundwork, we are now ready to examine the separate cases $g(\delta) = R_e^A, S_e^A, T_e^A$. The cases R_e^B, S_e^B, T_e^B are handled similarly.

If $g(\delta) = S_e^A$, then an A-attempt is made at R_δ at stage σ if $g(\delta)$ is defined by stage σ and

(a) No attempt has already been made at R_δ.

(b) No attempt (K, H) at some $R_{\delta'}$, $\delta' < \delta$ has been made such that $\sup(K \cup H) \geqslant \delta$;

(c) $A^\sigma \cap W_e^\sigma = \varnothing$, but $\exists\gamma\, (\gamma \in W_e^\sigma \wedge \gamma > \delta)$.

Then the A-attempt at R_δ at stage σ is $(\{\gamma_0\}, \varnothing)$, where $\gamma_0 = \mu\gamma(\gamma \in W_e^\sigma \wedge \gamma > \delta)$. No B-attempts are ever made at R_δ.

If $g(\delta) = T_e^A$, then an A-attempt is made at R_δ at stage σ if $g(\delta)$ is defined by stage σ and

(a) As before,

(b) as before,

(c) $e \not\subseteq A^\sigma \cup (\delta + 1)$.

Then the A-attempt at R_δ at stage σ is $(\varnothing, \{\gamma_0\})$ where $\gamma_0 = \mu\gamma(\gamma \in e - A^\sigma \wedge \gamma > \delta)$. No B-attempts are ever made at R_δ.

The case $g(\delta) = R_e^A$ requires an extended discussion. We would like to find an argument $x > \delta$ and a pair (z_1, z_2) of β-finite sets such that $A^\sigma \cap z_2 = \varnothing$ and $\langle x, z_1, z_2\rangle \in W_e^\sigma$. If we can make the A-attempt (z_1, z_2) and the B-attempt $(\{x\}, \varnothing)$, then (assuming $z_2 \cap A = \varnothing$ is preserved):

$$x \in B \qquad \text{and} \qquad x \in W_e^A.$$

Thus R_e^A will be satisfied. Requirements $\{S_e^A, T_e^A\}_{e \in S_\beta}$ guarantee that A will be both simple and weakly-t.r.e., so it suffices to look for (z_1, z_2) as above where $z_1 \subseteq A^\sigma \cup z_1'$ and z_1', z_2 are disjoint and countable. Then we would like to make the A-attempt (z_1', z_2).

Unfortunately, it may happen that $z_1' \cup z_2$ contains ordinals less than δ (and hence (z_1', z_2) cannot qualify as an attempt at R_δ). We can, however, make the A-attempt $(z_1' - \delta, z_2 - \delta)$. What is needed is some way to "guess" $A \cap \delta$.[2]

This guessing procedure is provided by $\diamondsuit_{\aleph_1^L}(E)$. Let $\langle S_\delta \mid \delta \in E\rangle$ be a β-finite $\diamondsuit_{\aleph_1^L}(E)$-sequence, where $E = \{\delta \mid g(\delta) = R_e^A\}$, provided by Theorem 5. We use S_δ as our guess at $A \cap \delta$. Since $\{\delta \in E \mid A \cap \delta = S_\delta\}$ is a stationary subset of E, this guess will be correct for stationary-many δ's.

We are now ready to describe how attempts are made at R_δ when $g(\delta) = R_e^A$. Attempts are made at stage σ if $g(\delta)$ is defined by stage σ and

(a) As before,

(b) as before,

(c) there is an $x > \delta$ and a pair (z_1, z_2) such that $\langle x, z_1, z_2\rangle \in W_e^\sigma$, $A^\sigma \cap z_2 = \varnothing$, $z_1 \cap \delta \subseteq S_\delta$, $z_2 \cap S_\delta = \varnothing$, $z_1 - A^\sigma$, z_2 are disjoint and countable.

Then the A-attempts $\langle z_1 - A^\sigma - \delta, z_2 - \delta\rangle$ and the B-attempt $\langle\{x\}, \varnothing\rangle$ are made at R_δ at stage σ.

Having described how attempts are made at the auxiliary requirements R_δ, one may describe the construction as follows: Let L: $S_\beta \to \aleph_1^L$ be an enumeration

[2] In the admissible case, one can simply use the guess $A^\sigma \cap \delta$. Then for σ large enough, $A^\sigma \cap \delta = A \cap \delta$.

of $\aleph_1^L$ (in the sense of [4, p. 19]) such that $L^{-1}(\{\delta\})$ is unbounded in $<_\beta$, for all $\delta < \aleph_1^L$. For example,

$$\begin{aligned} L(z) &= \delta, \qquad z = \langle x, \delta\rangle \qquad \text{for some } x, \\ &= 0, \qquad \text{otherwise.)} \end{aligned}$$

Then at stage σ, attempts are made at $R_{L(\sigma)}$ subject to the conditions described above. If the A-attempt (K, H) is made at stage σ, then members of K are put into A and $A^{\sigma+1} = A^\sigma \cup K$. Similarly for B.

We are now ready to verify that the requirements S_e^A, T_e^A, R_e^A will be satisfied (the argument for S_e^B, T_e^B, R_e^B is similar).

CLAIM 1. *For $\delta < \aleph_1^L$, let*

$$\begin{aligned} h(\delta) &= \max\{\sup(K \cup H) \mid (K, H) \textit{ is an attempt at } R_\delta\} \\ &= \delta \qquad \textit{if no attempts are made at } R_\delta. \end{aligned}$$

Then $h: \aleph_1^L \to \aleph_1^L$ and $C = \{\delta \mid h[\delta] \subseteq \delta\}$ is a closed unbounded set.

Proof. Since $\{(K, H) \mid (K, H)$ is an attempt at $R_\delta\}$ is a set of cardinality $\leqslant 2$, clearly $h: \aleph_1^L \to \aleph_1^L$. C is certainly closed: if $\gamma_0 < \gamma_1 < \cdots$ and $\gamma = \bigcup_n \gamma_n$, then

$$h[\gamma_n] \subseteq \gamma_n \,\forall n \Rightarrow h[\gamma] = \bigcup_n h[\gamma_n] \subseteq \bigcup_n \gamma_n = \gamma.$$

Now define $h'(\delta) = \mu\gamma[h(\gamma) \geqslant \delta]$. Clearly $h'(\delta) \leqslant \delta\ \forall\delta$. But also, $\{\delta \mid h'(\delta) < \gamma\}$ is bounded by $\sup h[\gamma] < \aleph_1^L$. So by Fodor's theorem, $\{\delta \mid h'(\delta) < \delta\}$ is not stationary, so certainly $\{\delta \mid h'(\delta) = \delta\}$ is unbounded. But this latter set is C. ▮

Let $A = \bigcup_{\sigma\in S_\beta} A^\sigma$, $B = \bigcup_{\sigma\in S_\beta} B^\sigma$.

CLAIM 2. *$\aleph_1^L - A$, $\aleph_1^L - B$ are unbounded in $\aleph_1^L$.*

Proof. We just consider $\aleph_1^L - A$. Let $S = \{\delta \mid g(\delta) = S_0^A\}$. Then S is stationary (by the definition of g) and hence $S' = S \cap C$ is unbounded.

SUBCLAIM. *If (K, H) is an A-attempt, then $K \cap S' = \varnothing$.*

Proof. Let $\delta' \in S'$. Let (K, H) be an attempt at R_δ. If $\delta > \delta'$, then $K \cap \delta = \varnothing \to \delta' \notin K$. If $\delta < \delta'$, then since $\delta' \in C$, $\sup(K \cup H) < \delta'$ and so again $\delta' \notin K$. If $\delta = \delta'$, then $\delta' \notin K$ by construction (see definition of attempts at R_δ when $g(\delta) = S_e^A$). ▮

But of course $A = \bigcup \{K \mid K$ is an A-attempt$\}$. Thus $A \cap S' = \varnothing$. ▮

CLAIM 3. *The requirements S_e^A, S_e^B are satisfied for all $e \in S_\beta$.*

Proof. We just consider S_e^A. Suppose W_e is unbounded in $\aleph_1^L$. Let $S = \{\delta \mid g(\delta) = S_e^A\}$. As S is stationary, let $\delta \in S \cap C$.

Let σ be a stage such that $L(\sigma) = \delta$, $g(\delta)$ is defined by stage σ and $W_e^\sigma - (\delta + 1) \neq \varnothing$. σ exists since W_e is unbounded in $\aleph_1^L$ and $L^{-1}(\{\delta\})$ is unbounded in $<_\beta$. If an A-attempt is made at R_δ at stage σ, then some member of W_e^σ is put into A and so S_e^A is satisfied. If not, then either

(a) An attempt has already been made at R_δ, or

(b) an attempt (K, H) has been made at some $R_{\delta'}$, $\delta' < \delta$ such that $\sup(K \cup H) \geqslant \delta$.

Condition (b) contradicts the assumption $\delta \in C$. Condition (a) implies $A^\sigma \cap W_e^\sigma \neq \varnothing$, so again S_e^A is satisfied. ▮

CLAIM 4. *The requirements T_e^A, T_e^B are satisfied for all $e \in S_\beta$.*

Proof. We just consider T_e^A. Suppose $e - A$ is bounded in $\aleph_1^L$ (where $e \subseteq \aleph_1^L$). Let $S = \{\delta \mid g(\delta) = T_e^A\}$. As S is stationary, $S \cap C$ is unbounded in $\aleph_1^L$.

Let $\delta \in S \cap C$, and let σ be a stage such that $g(\delta)$ is defined by stage σ and $L(\sigma) = \delta$. If $e \subseteq A^\sigma \cup (\delta + 1)$, then $e - A^\sigma$ is bounded and T_e^A is satisfied. Otherwise, as $\delta \in C$, either an A-attempt at R_δ has already been made or one will be made at stage σ. Let $(\varnothing, \{\gamma_0\})$ be an A-attempt at R_δ made at some stage $\sigma' \leqslant \sigma$.

SUBCLAIM. *If (K, H) is an A-attempt made after stage σ', then $\gamma_0 \notin K$.*

Proof. Let (K, H) be an A-attempt at $R_{\delta'}$. If $\delta' < \delta$, then $\sup(K \cup H) < \delta < \gamma_0$ since $\delta \in C$. If $\delta' > \delta$, then since $\sup(\varnothing \cup \{\gamma_0\}) = \gamma_0$, we must have $\gamma_0 < \delta'$ (otherwise no A-attempt could be made at $R_{\delta'}$). But then $K \cap \delta' = \varnothing \rightarrow \gamma_0 \notin K$. No A-attempts at R_δ can be made after stage σ' since one has already been made. ▮

But then $\gamma_0 \notin A = \bigcup \{K \mid (K, H)$ is an A-attempt$\}$. So we have shown that either

(a) $\exists \sigma[e - A^\sigma$ is bounded$]$, or

(b) $\forall \delta \in S \cap C[\exists \gamma_0 > \delta$ s.t. $\gamma_0 \in e - A]$.

Condition (b) contradicts the assumption that $e - A$ is bounded. ▮

CLAIM 5. *Each R_e^A, R_e^B is satisfied, $e \in S_\beta$.*

Proof. We just consider R_e^A. Let $E = \{\delta \mid g(\delta) = R_e^A\}$, a stationary subset of $\aleph_1^L$.

Let $\langle S_\delta \mid \delta \in E\rangle$ be the $\diamondsuit_E$-sequence chosen earlier (and used in our attempts at R_δ, $\delta \in E$). Then $E' = \{\delta \mid \delta \in E \text{ and } S_\delta = A \cap \delta\}$ is a stationary subset of E. Now pick any $\delta \in E' \cap C$.

Suppose $\aleph_1^L - B = W_e^A$. We work now toward a contradiction. By Claim 2, choose $x \in \aleph_1^L - B$ such that $x > \delta$. Then there is a pair (z_1, z_2) such that

(i) $z_1 \subseteq A$, $z_2 \subseteq \aleph_1^L - A$,

(ii) $\langle x, z_1, z_2\rangle \in W_e$.

By Claim 3, z_2 is bounded in $\aleph_1^L$ (i.e., is countable). By Claim 4, there is a stage σ such that $L(\sigma) = \delta$, $\langle x, z_1, z_2\rangle \in W_e^\sigma$ and $z_1 - A^\sigma$ is countable. Also, since $S_\delta = A \cap \delta$, we have $z_1 \cap \delta \subseteq S_\delta$ and $z_2 \cap S_\delta = \varnothing$. Thus since $\delta \in C$, either a pair of attempts is made at R_δ at stage σ, or a pair of attempts at R_δ was already made at some earlier stage.

SUBCLAIM. *Let the A-attempt $\langle z_1 - A^\sigma - \delta, z_2 - \delta\rangle$ and the B-attempt $\langle\{x\}, \varnothing\rangle$ be made at R_δ at stage σ. Then $z_2 \cap A = \varnothing$.*

Proof. Otherwise some A-attempt (K, H) made at some $R_{\delta'}$, at some later stage has $K \cap z_2 \neq \varnothing$. If $\delta' < \delta$, this can't happen since $\delta \in C \Rightarrow \sup(K \cup H) < \delta$ and $z_2 \cap \delta \subseteq (\delta - S_\delta) = \delta - A$. If $\delta' = \delta$, then no attempt can be made at $R_{\delta'}$ at any stage after stage σ. If $\delta' > \delta$, then $\delta' > \sup((z_1 - A^\sigma) \cup z_2)$ since otherwise no attempt can be made after stage σ at $R_{\delta'}$. But then $K \cap \delta' = \varnothing \Rightarrow K \cap z_2 = \varnothing$. ∎

The pair of attempts in the subclaim guarantees that $z_1 \subseteq A$, $z_2 \subseteq \aleph_1^L - A$, $\langle x, z_1, z_2\rangle \in W_e$ and $x \in B$. Thus $W_e^A \neq \aleph_1^L - B$. ∎

3 THE GENERAL CASE

In this section we treat the case: β^* is β-recursively regular. This means that every β-recursive function $f: \gamma \to \beta^*$, $\gamma < \beta^*$ has range bounded in β^*. By Theorem 3.13 of [4], we may assume that $\Sigma_1 cf\beta < \beta^*$; this assumption is actually not needed for our proof when β^* is a successor β-cardinal (or $\beta^* = \beta$ and there exists a largest β-cardinal).

PROPOSITION 7. *If β^* is a successor β-cardinal, then β^* is β-recursively regular.*

Proof. See [4, Proposition 1.18]. ∎

We begin by "effectivizing" Fodor's theorem and $\diamondsuit$ to S_β. The proof of Theorem 6 suggests that the effective analog of "closed unbounded subset of $\aleph_1^L$" should be "$\Pi_1^{S_\beta}$-definable closed unbounded subset of β^*" and "stationary" should be replaced by "intersects every $\prod_1^{S_\beta}$-definable closed unbounded subset of β^*." However, with only the assumption of β-recursive regularity of β^*, it may be that there exist $\prod_1^{S_\beta}$-definable unbounded subsets of β^* of order type ω; in this case, sets having the above property analogous to stationary must be final segments of β^*. So we discard the concept of "stationary" and instead choose to effectivize the following weaker versions of Fodor's theorem and $\diamondsuit$

WEAK FODOR'S THEOREM. *Suppose* $\gamma < \kappa$ *and* $f\colon \kappa \times \gamma \to \kappa$. *Then* $\{\delta < \kappa \mid f[\delta \times \gamma] \subseteq \delta\}$ *is a closed unbounded subset of* κ.

WEAK $\diamondsuit_\kappa$-PRINCIPLE. There exists a sequence $\langle S_\delta \mid \delta < \kappa\rangle$ such that

(i) $S_\delta \subseteq 2^\delta$, $\bar{S}_\delta < \kappa$,

(ii) If $X \subseteq \kappa$, then $\{\delta \mid X \cap \delta \in S_\delta\}$ is a closed unbounded subset of κ.

Proof of Weak Fodor. Let

$$\begin{aligned} g(\delta) &= \mu\delta' < \delta[f[\delta' \times \gamma] \nsubseteq \delta] \\ &= \delta, \qquad \text{otherwise.} \end{aligned}$$

Clearly $g(\delta) \leqslant \delta$ for all $\delta < \kappa$, and also $g^{-1}[\delta]$ is bounded for all δ since κ is regular and f is a function. Applying Fodor's theorem to g, we get that $\{\delta \mid g(\delta) = \delta\}$ is certainly unbounded. But $g(\delta) = \delta \leftrightarrow f[\delta \times \gamma] \subseteq \delta$, and since $\{\delta < \kappa \mid f[\delta \times \gamma] \subseteq \delta\}$ is certainly closed, we are done. ▮

The proof of Weak $\diamondsuit_\kappa$ will be deferred until the proof of its effective version.

We proceed to describe our effective versions of Weak Fodor and Weak $\diamondsuit$. In our application of these effectivized principles to some particular β-recursive f (in Weak Fodor) and β-r.e. X (in Weak $\diamondsuit$), it will be important to know that

$$\{\delta \mid f[\delta \times \gamma] \subseteq \delta\} \cap \{\delta \mid X \cap \delta \in S_\delta\}$$

is closed, unbounded. As these sets might have ordertype ω, we cannot simply argue that the intersection of two closed unbounded sets is closed unbounded (β^* is not regular *enough*). Instead our effective versions will specify the above sets exactly, given defining parameters for f as a β-recursive function and X as a β-r.e. set.

DEFINITION. Let β^* be β-recursively regular and $p \in S_\beta$. Then $\delta < \beta^*$ is *p-stable* if $h_1[(\delta \cup \{p\}) \times \omega] \cap \beta^* \subseteq \delta$, where h_1 is a parameter-free Σ_1 Skolem function for S_β. (For a summary of the basic facts concerning Skolem functions, see [4, Chap. 1].)

Since $h_1[(\delta \cup \{p\}) \times \omega]$ is a Σ_1-elementary substructure of S_β, we see that $\delta < \beta^*$ is p-stable if and only if there is an $H \prec_{\Sigma_1} S_\beta$ such that $H \cap \beta^* = \delta$. Let $C_p = \{\delta < \beta^* \mid \delta \text{ is } p\text{-stable}\}$.

LEMMA 8. *For each $p \in S_\beta$, C_p is a closed unbounded subset of β^*.*

Proof. *Case* 1. β^* is a successor β-cardinal. It is enough to show that C_p is unbounded. Let $\gamma_0 =$ the largest β-cardinal less than β^*. Let $\gamma > \gamma_0$, $\gamma < \beta^*$ be arbitrary. Let $H = h_1[(\gamma \cup \{p\}) \times \omega]$. As h_1 is Σ_1, $H \cap \beta^*$ must be bounded in β^*.

CLAIM. *$H \cap \beta^* =$ an ordinal.*

Proof. This is because $\delta \in H \cap \beta^* \to \exists f \in H[f: \delta \to^{1-1} \gamma_0]$. Then $f^{-1}[\gamma_0] \subseteq H$ since $\gamma_0 \subseteq H$. ▌

Let $\delta = H \cap \beta^*$. Then $\delta \geqslant \gamma$, $\delta \in C_p$.

Case 2. β^* is a limit β-cardinal. In this case, we use the assumption $\Sigma_1 cf\beta < \beta^*$. Let $\gamma_0 = \Sigma_1 cf\beta < \beta^*$ and let $f: \gamma_0 \to \beta$ be β-recursive, unbounded, and order preserving. As h_1 is Σ_1, let $h_1{}^\sigma$ be the part of graph(h_1) enumerated by stage σ (in some fixed β-recursive enumeration of graph(h_1)). Let $\delta_0 < \beta^*$ be arbitrary. Define, inductively,

$$\delta_1 = \sup h_1^{f(0)}[(\delta_0 \cup \{p\}) \times \omega] \cap \beta^*,$$

$$\delta_{\gamma+1} = \sup h_1^{f(\gamma)}[(\delta_\gamma \cup \{p\}) \times \omega] \cap \beta^*,$$

$$\delta_\lambda = \sup\{\delta_\gamma \mid \gamma < \lambda\} \qquad \text{for limit } \lambda$$

for $\gamma, \lambda < \gamma_0$. Then $\delta_{\gamma_0} = \sup\{\delta_\gamma \mid \gamma < \gamma_0\} < \beta^*$ and $\beta^* \cap h_1[(\delta_{\gamma_0} \cup \{p\}) \times \omega] \subseteq \delta_{\gamma_0}$. Then $\delta_0 \leqslant \delta_{\gamma_0} \in C_p$.

Case 3. $\beta^* = \beta$. Then $C_p = \{\gamma < \beta \mid p \in S_\gamma \text{ and } \gamma \text{ is } \beta\text{-stable}\}$ which is unbounded in β. ▌

EFFECTIVIZED FODOR. *Suppose $\gamma < \beta^*$, $A \subseteq \beta^* \times \gamma$ and the partial function $f: A \to \beta^*$ is Σ_1 over S_β with parameter p. Then $\delta > \gamma$, $\delta \in C_p$ implies $f[\delta \times \gamma] \subseteq \delta$.*

EFFECTIVIZED $\diamondsuit_{\beta^*}$. *There exists a β^*-recursive sequence $\langle S_\delta \mid \delta < \beta^* \rangle$ such that*

(i) $S_\delta \subseteq 2^\delta$, $\quad$ *β-card.* $S_\delta < \beta^*$,

(ii) *if $X \subseteq \beta^*$ is Σ_1 over S_β with parameter p, then $\delta \in C_{\langle p, \beta^* \rangle}$ implies $X \cap \delta \in S_\delta$.*

Proof of Effectivized Fodor. Let $\delta > \gamma$, $\delta \in C_p$ and $H = h_1[(\delta \cup \{p\}) \times \omega]$. Then $\delta \times \gamma \subseteq H$ and so $f[\delta \times \gamma] \subseteq H$ since $p \in H$. But $H \cap \beta^* = \delta$.

Proof of Effectivized $\diamondsuit_{\beta^*}$. *Case* 1. There is a largest β-cardinal less than β^*. Then let this β-cardinal be γ_0. If $\delta \leqslant \gamma_0$, set $S_\delta = \varnothing$. Otherwise, if $\gamma_0 < \delta < \beta^*$, let $S_\delta = 2^\delta \cap J_{\bar\delta}$, where $\bar\delta = \mu\gamma \geqslant \delta[J_\gamma \models$ "δ is not a cardinal"].

If $\delta \in C_{\langle p, \beta^* \rangle}$, $X \subseteq \beta^*$ is Σ_1 over S_β with parameter p, then let $H = h_1[(\delta \cup \{p, \beta^*\}) \times \omega]$. Thus $\gamma_0 \in H$ and $\delta = H \cap \beta^*$. So $\gamma_0 < \delta$.

Let $c: \langle H, \epsilon \rangle \cong \langle J_\gamma, \epsilon \rangle$ be the transitive collapse. Now $J_\gamma \models$ "δ is a cardinal" since $\delta = c(\beta^*)$. But X is Σ_1-definable with parameter p over S_β, and hence $X \cap H$ is Σ_1-definable with parameter p over H. Since $X \cap H = X \cap \delta$ and $c \mid \delta = \mathrm{id} \mid \delta$, $X \cap \delta$ is definable over J_γ. But then $X \cap \delta \in J_{\gamma+1} \subseteq J_{\bar\delta}$. So $X \cap \delta \in S_\delta$.

Case 2. β^* is the limit of smaller β-cardinals. Then if $\delta < \beta^*$, let $S_\delta = 2^\delta \cap J_{\beta^*}$. Since X β-r.e., $\delta < \beta^*$ implies $X \cap \delta \in J_{\beta^*}$, we are done. ∎

We are now ready to prove:

THEOREM 9. *If* β^* *is* β*-recursively regular, then there exist* β-r.e. $A, B \subseteq \beta^*$ such that $A \not\leqslant_{w\beta} B$, $B \not\leqslant_{w\beta} A$.

Proof. As before, we have the requirements

$$R_e^A: \beta^* - B \neq W_e^A, \qquad R_e^B: \beta^* - A \neq W_e^B,$$

$$S_e^A: W_e \text{ unbounded in } \beta^* \rightarrow A \cap W_e \neq \varnothing,$$

$$S_e^B: W_e \text{ unbounded in } \beta^* \rightarrow B \cap W_e \neq \varnothing,$$

$$T_e^A: (e - A) \text{ bounded in } \beta^* \rightarrow \exists\sigma[e - A^\sigma \text{ bounded in } \beta^*],$$

$$T_e^B: (e - B) \text{ bounded in } \beta^* \rightarrow \exists\sigma[e - B^\sigma \text{ bounded in } \beta^*]$$

for $e \in S_\beta$. Previously, we considered auxiliary requirements $\{R_\delta\}_{\delta<\beta^*}$ so that each R_δ constituted an attempt at one of the above requirements, each of them being attempted by stationary-many R_δ's. In this construction, each $\delta < \beta^*$ will be responsible for a whole (size $<\beta^*$) collection of requirements, each requirement being attempted by a final segment of δ's.

More precisely, fix a 1–1 β-recursive $f: S_\beta \rightarrow \beta^*$ and let p' be the parameter which defines f. ($\langle p', \beta^* \rangle$ will be the parameter for the construction if $\beta^* < \beta$.) Then δ is responsible for R_e^A, R_e^B, S_e^A, S_e^B, T_e^A, T_e^B where $f(e) < \delta$. As before, "guesses" will have to be made at $A \cap \delta$, $B \cap \delta$ for the sake of requirements R_e^A, R_e^B; as $\diamondsuit_{\beta^*}$ provides us with a collection of such guesses, the requirements R_e^A, R_e^B will in fact be treated by δ as collections of requirements, one for each guess given by $\diamondsuit_{\beta^*}$.

The auxiliary requirements. Fix a β^*-recursive $\diamondsuit_{\beta^*}$ sequence $\langle S_\delta \mid \delta < \beta^* \rangle$. Then the collection $\mathscr{R}_\delta$ of auxiliary requirements at level δ consists of:

(a) $\{S_\gamma^A \mid \gamma < \delta\} \cup \{S_\gamma^B \mid \gamma < \delta\}$,

(b) $\{T_\gamma^A \mid \gamma < \delta\} \cup \{T_\gamma^B \mid \gamma < \delta\}$,

(c) $\{(R_\gamma^A, g) \mid \gamma < \delta, g \in S_\delta\} \cup \{(R_\gamma^B, g) \mid \gamma < \delta, g \in S_\delta\}$.

S_γ^A (T_γ^A) is intended to represent S_e^A (T_e^A) where $f(e) = \gamma$. Similarly for S_γ^B, T_γ^B. (R_γ^A, g) represents R_e^A with the guess g for $A \cap \delta$, where $f(e) = \gamma$. Similarly for (R_γ^B, g). Note that there may be $\gamma < \delta$ which are not in range f, in which case auxiliary requirements with subscript γ will never be active.

Now for each $\delta < \beta^*$, $\mathscr{R}_\delta$ has β-cardinality less than β^*, so let $\{R_\xi^\delta\}_{\xi<\kappa<\beta^*}$ be a fixed well-ordering of $\mathscr{R}_\delta$.

As before, each δ will make certain A-attempts and B-attempts of the form (K, H) where K, H are bounded in β^* and $K \cap \delta = \varnothing = H \cap \delta$. δ will make at most one A-attempt and one B-attempt at each R_ξ^δ. Also, if δ makes the attempt (K, H) at stage σ, then no δ' such that $\delta < \delta' \leqslant \sup(K \cup H)$ can ever make an attempt at any later stage.

Two A-attempts (K_1, H_1) and (K_2, H_2) are compatible if $K_1 \cap H_2 = \varnothing = K_2 \cap H_1$. Similarly for B-attempts. At each stage σ, some $\delta < \beta^*$ will be examined, and then A- and B-attempts will be made successively at R_1^δ, R_2^δ, R_3^δ,...; an attempt at R_ξ^δ at this stage must be compatible with attempts at $R_{\xi'}^\delta$, $\xi' < \xi$ at this stage and all attempts at members of $\mathscr{R}_\delta$ at earlier stages. Thus no two attempts at members of $\mathscr{R}_\delta$ will ever conflict, and each member of $\mathscr{R}_\delta$ has an opportunity to act at each stage where δ is being examined.

We now describe exactly how attempts are made and how the construction proceeds. Let $L\colon S_\beta \to \beta^*$ be a fixed enumeration of β^* such that for each $\delta < \beta^*$, $L^{-1}(\{\delta\})$ is unbounded in $<_\beta$ and L is Σ_1-definable over S_β with parameter β^*. (For example,

$$L(x) = \delta \quad \text{if} \quad x = \langle \delta, y \rangle, \quad \delta < \beta^*$$
$$= 0, \quad \text{otherwise}$$

is an example of such an L.)

Stage σ. We consider $L(\sigma) = \delta$. If some attempt (K, H) has been made at some $\delta' < \delta$ where $\sup(K \cup H) \geqslant \delta$, go to the next stage. Otherwise, begin by making the A-attempt and B-attempt $(\varnothing, \{\delta\})$. (This is to ensure $\beta^* - A$, $\beta^* - B$ unbounded in β^*.) Then successively make A- and B-attempts at R_1^δ, R_2^δ,... as follows: Assume by induction that we have finished with $R_\delta^{\xi'}$, $\xi' < \xi$. We describe how to act on R_ξ^δ. If an attempt has already been made at R_ξ^δ at some earlier stage σ', $L(\sigma') = \delta$, then go on to $R_{\xi+1}^\delta$. Now assume R_ξ^δ is of one of the forms S_γ^A, T_γ^A, (R_γ^A, g). The cases S_γ^B, T_γ^B, (R_γ^B, g) are treated similarly. If γ has not yet appeared in Range f (in some fixed enumeration of the β-r.e. set range f), go on to $R_{\xi+1}^\delta$. Otherwise, let $f(e) = \gamma$.

$R_\xi^\delta = S_\gamma^A$. If there is an A-attempt $(\{\gamma'\}, \varnothing)$ compatible with earlier A-attempts at members of $\mathscr{R}_\delta$ such that $\gamma' > \delta$ and $\gamma' \in W_e^\sigma$, then make the A-attempt $(\{\gamma_0\}, \varnothing)$ where γ_0 is the least such γ'. Otherwise go to $R_{\xi+1}^\delta$. Make no B-attempts.

$R_\xi^\delta = T_\gamma^A$. If there is an A-attempt $(\varnothing, \{\gamma'\})$ compatible with earlier A-attempts at members of $\mathscr{R}_\delta$ such that $\gamma' > \delta$ and $\gamma' \in e - A^\sigma$, then make the A-attempt $(\varnothing, \{\gamma_0\})$ where γ_0 is the least such γ'. Otherwise go to $R_{\xi+1}^\delta$. Make no B-attempts.

$R_\xi^\delta = (R_\gamma^A, g)$. Suppose that there are $x > \delta$ and a pair (z_1, z_2) such that $\langle x, z_1, z_2\rangle \in W_e^\sigma$, $z_1 \cap \delta \subseteq g$, $z_2 \cap g = \varnothing$, $z_2 \cap A^\sigma = \varnothing$ and $z_1 - A^\sigma$, z_2 are disjoint and bounded in β^*. Then if such an $x > \delta$ and (z_1, z_2) can be found so that the A-attempt $\langle z_1 - A^\sigma - \delta, z_2 - \delta\rangle$ and the B-attempt $\langle\{x\}, \varnothing\rangle$ are compatible with earlier attempts at members of $\mathscr{R}_\delta$, make the least such pair of attempts. Otherwise go to $R_{\xi+1}^\delta$.

This ends the description of the construction. Note that the only parameters needed in the construction are p' and β^*. Let p'' be a parameter which defines a β-recursive $g: \Sigma_1 cf\beta \to \beta$, range g unbounded. Let $p = \langle p', \beta, p''\rangle$.

CLAIM 1. *Suppose* $\delta \in C_p$. *Let* (K, H) *be an attempt at some member of* $\mathscr{R}_{\delta'}$, $\delta' < \delta$. *Then* $\sup(K \cup H) < \delta$.

Proof. Case 1. There is a largest β-cardinal less than β^*, γ. Then each $\mathscr{R}_{\delta'}$ is well-ordered in length γ. Define the partial Σ_1 function f by $f(\delta', \gamma') = \max\{\sup(K \cup H) \mid (K, H)$ is an attempt at $R_{\gamma'}^{\delta'}\}$. Then by Effectivized Fodor, $\delta \in C_p$, $\delta > \gamma \to f[\delta \times \gamma] \subseteq \delta$. But it is easily seen that $\delta \in C_p \to \delta > \gamma$ and so we are done.

Case 2. Otherwise. In this case we use the assumption $\Sigma_1 cf\beta < \beta^*$. Let $\gamma = \Sigma_1 cf\beta$ and define f by

$$f(\delta', \gamma') = \sup\{\sup(K \cup H) \mid (K, H) \text{ is an attempt at some member of } \mathscr{R}_{\delta'} \text{ made by stage } g(\gamma')\}.$$

Then by Effectivized Fodor, $\delta \in C_p \to f[\delta \times \gamma] \subseteq \delta$. So we are done. ∎

CLAIM 2. *If* $\delta \in C_p$, *then* $A \cap \delta$, $B \cap \delta \in S_\delta$.

Proof. This is immediate from Effectivized $\diamondsuit_{\beta^*}$. ∎

Thus we see that any A-attempt (K, H) made at a member of $\mathscr{R}_\delta$, $\delta \in C_p$ is ***permanent***; i.e., $K \subseteq A$ and $A \cap H = \varnothing$. (Similarly, for B-attempts.) Moreover, by Claim 2, a correct "guess" was made at $A \cap \delta$ and at $B \cap \delta$. These facts make it easy to check that the desired requirements have been met.

CLAIM 3. *$\beta^* - A$, $\beta^* - B$ are unbounded in β^*.*

Proof. The first attempts made by $\delta \in C_p$ are the A-attempts and B-attempts $(\varnothing, \{\delta\})$. These attempts are permanent so $\delta \notin A \cup B$. C_p is unbounded by Lemma 8. ▮

CLAIM 4. *The requirements S_e^A, S_e^B are satisfied for all $e \in S_\beta$.*

Proof. Let $f(e) = \gamma$. We just consider S_e^A. Choose $\delta \in C_p$, $\delta > \gamma$. If W_e is unbounded in β^*, there must be a stage σ such that

(i) $L(\sigma) = \delta$,

(ii) $f(e)$ is defined by stage σ,

(iii) there is an $x \in W_e^\sigma$, $x > \delta'$, where δ' = least member of C_p greater than δ.

Then by Claim 1, the A-attempt $(\{x\}, \varnothing)$ must be compatible with earlier A-attempts at members of $\mathscr{R}_\delta$. So either an attempt must be made at S_γ^A at stage σ or one must have already been made. But then $A \cap W_e \neq \varnothing$. ▮

CLAIM 5. *The requirements T_e^A, T_e^B are satisfied for all $e \in S_\beta$.*

Proof. Let $f(e) = \gamma$. We just consider T_e^A. Suppose that for no stage σ, do we have $e - A^\sigma$ bounded. We show that $e - A$ is unbounded. Let $\gamma' > \gamma$ be arbitrary and $\delta \in C_p$, $\delta > \gamma'$. Let σ be any stage such that $L(\sigma) = \delta$ and $f(e)$ is defined by stage σ. Since $e - A^\sigma$ is unbounded, there must be an A-attempt $(\varnothing, \{\gamma_0\})$, $\gamma_0 \in e - A^\sigma$, which is compatible with all earlier A-attempts at members of $\mathscr{R}_\delta$. Then either such an A-attempt was made at stage σ or an earlier attempt was made at T_γ^A. But this attempt is permanent since $\delta \in C_p$. So we have shown that $\exists x > \gamma'[x \in e - A]$. Since γ' was arbitrary, $e - A$ is unbounded. ▮

CLAIM 6. *Each R_e^A, R_e^B is satisfied, $e \in S_\beta$.*

Proof. We just consider R_e^A. Let $f(e) = \gamma$ and $\gamma < \delta \in C_p$. Let δ' = least member of C_p greater than δ. Since $\beta^* - B$ is unbounded choose $x \in \beta^* - B$, $x > \delta'$. Suppose $\beta^* - B = W_e^A$. Then there must be a pair (z_1, z_2) such that $(x, z_1, z_2) \in W_e$, $z_1 \subseteq A$, $z_2 \subseteq \beta^* - A$. Since A is simple (Claim 4) and weakly-t.r.e. (Claim 5), there is a stage σ such that

(i) $L(\sigma) = \delta, f(e)$ is defined by stage σ,

(ii) $z_1 - A^\sigma$, z_2 are bounded in β^*.

Since $\delta \in C_p$, there is a $g \in S_\delta$, $g = A \cap \delta$. Then of course $z_1 \cap \delta \subseteq g$, $z_2 \cap g = \varnothing$. Then the A-attempt $(z_1 - A^\sigma - \delta, z_2 - \delta)$ and the B-attempt $(\{x\}, \varnothing)$ must be compatible with all earlier attempts since all of these attempts are permanent. But then attempts at (R_γ^A, g) must have been made at stage σ or some earlier stage. Since this attempt must be permanent, it guarantees $B \neq W_e^A$. ∎

4. Generalizing to Σ_n, $n > 1$

Jensen's master codes can be used to extend the above results to Σ_n sets when the Σ_n-projectum of β, $\rho_n{}^\beta$, is regular with respect to the functions Σ_n over S_β.

Recall [7]

Theorem (Jensen). *For each $n > 0$ there is a subset $A_n{}^\beta$ of $\rho_n{}^\beta$ which is Σ_n over S_β such that*

$$\Sigma_{n+m}^{S_\beta} \cap 2^{\rho_n{}^\beta} = \Sigma_m^{\langle S_{\rho_n{}^\beta}, A_n{}^\beta\rangle}$$

for all $m > 0$.

Thus, as far as subsets of $\rho_n{}^\beta$ are concerned, Σ_{n+m}-definability over S_β reduces to Σ_m-definability over the amenable structure $\langle S_{\rho_n{}^\beta}, A_n{}^\beta\rangle$.

Now for $A, B \subseteq S_\beta$, define $A \leqslant_{w\beta}^n B$ if there are Σ_n predicates W_1, W_2 such that

$$x \in A \leftrightarrow \exists z_1 \exists z_2[\langle x, z_1, z_2\rangle \in W_1 \wedge z_1 \subseteq B \wedge z_1 \subseteq S_\beta - B],$$
$$x \notin A \leftrightarrow \exists z_1 \exists z_2[\langle x, z_1, z_2\rangle \in W_2 \wedge z_1 \subseteq B \wedge z_2 \subseteq S_\beta - B].$$

Then $\leqslant_{w\beta} = \leqslant_{w\beta}^1$.

Theorem 10. *Suppose $\rho_n{}^\beta$ is regular with respect to functions Σ_n over S_β. Then there exist sets $A, B \subseteq \rho_n{}^\beta$ which are Σ_n over S_β and such that $A \not\leqslant_{w\beta}^n B$, $B \not\leqslant_{w\beta}^n A$.*

Definition. If $\mathfrak{A} = \langle S_\beta, \epsilon, A\rangle$ is an amenable structure, then if $B, C \subseteq S_\beta$ we define: $B \leqslant_{\mathfrak{A}} C$ if there are predicates W_1, W_2 Σ_1 over $\mathfrak{A}$ such that

$$x \in B \leftrightarrow \exists z_1 \exists z_2[\langle x, z_1, z_2\rangle \in W_1 \wedge z_1 \subseteq C \wedge z_2 \subseteq S_\beta - C],$$
$$x \notin B \leftrightarrow \exists z_1 \exists z_2[\langle x, z_1, z_2\rangle \in W_2 \wedge z_1 \subseteq C \wedge z_2 \subseteq S_\beta - C].$$

Proof of Theorem 10. Let $\mathfrak{A} = \langle S_{\rho_{n-1}^\beta}, \epsilon, A_{n-1}^\beta\rangle$ (we can assume $n \geqslant 2$). A straightforward relativization of Theorem 9 yields: There are $A, B \subseteq \rho_1{}^{\mathfrak{A}} = \rho_n{}^\beta$ which are Σ_1 over $\mathfrak{A}$ (hence Σ_n over S_β) such that $A \not\leqslant_{\mathfrak{A}} B$, $B \not\leqslant_{\mathfrak{A}} A$. If $\rho_n{}^\beta < \rho_{n-1}^\beta$, then every β-finite subset of $\rho_n{}^\beta$ belongs to $S_{\rho_{n-1}^\beta}$, so $\leqslant_{\mathfrak{A}} = \leqslant_{w\beta}^n$ for subsets of $\rho_n{}^\beta$. Thus $A \not\leqslant_{w\beta}^n B$, $B \not\leqslant_{w\beta}^n A$. Otherwise, $\mathfrak{A}$ is an admissible structure (since then $\rho^{\beta}_{n-1} = \rho_n{}^{\beta}$ which is Σ^{β}_n-regular by assumption). Then do the usual construction [16] of incomparable Σ_1 sets A, B over the admissible structure $\mathfrak{A}$, but in addition guaranteeing

$$R \in \Sigma_1, \qquad R \text{ unbounded in } \rho^{\beta}_{n-1} \to A \cap R \neq \varnothing, \quad B \cap R \neq \varnothing.$$

Then all β-finite subsets of $S_{\rho^{\beta}_{n-1}} - A$, $S_{\rho^{\beta}_{n-1}} - B$ belong to $S_{\rho^{\beta}_{n-1}}$. Using this, it can be seen that $A \not\leqslant^n_{w\beta} B$, $B \not\leqslant^n_{w\beta} A$. ∎

Last, it is quite pertinent to ask for a stronger incomparability in Theorem 10: If $A, B \subseteq S_\beta$, say that A is Δ_n in B if A is Δ_n-definable over $\langle S_\beta[B], \epsilon, B\rangle$. Then do there exist Σ_n sets A, B such that neither is Δ_n in the other? Shore [15] handles the case where β is Σ_n-admissible. See [17] for progress on the case where $\rho_n{}^{\beta}$ is regular with respect to functions Σ_n over S_β.

5. Recent Developments

We have now produced ordinals for which Post's problem has a negative solution: Let $\beta = \aleph^L_{\omega_1} + \omega$. If $\alpha = \aleph^L_{\omega_1}$ then there are no α-degrees strictly between $0'$ and $0''$. These results will appear in a series of papers entitled "Some Negative Solutions to Post's Problem," the first of which is listed as [18].

However, the complete situation regarding Post's problem is not fully understood. We end with some questions.

(1) For which β can Post's problem be solved positively? See [18] for a conjecture on this.

(2) For which β are there incomparable β-r.e. degrees?

(3) For which β are the β-r.e. degrees dense?

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