Descriptive Set Theory, Sommersemester 2003

1.Vorlesung

My last course dealt with the techniques of Pure Set Theory:

Constructible Universe L
Set-forcing over L
Class-forcing over L
Inner models K for large cardinals
Set-forcing and Class-forcing over K

In the present course I consider applications of these techniques. There are two kinds of applications:

Consistency results
Con(ZFC + a large cardinal) $\rightarrow$ Con(something interesting).
Some examples:
Con(GCH)
Con(Suslin's Hypothesis)
Con(Inaccessible) $\rightarrow$ Con(All projective sets of reals are measurable)
Con(Hypermeasurable) $\rightarrow$ Con(Failure of the singular cardinal hypothesis)
Con(Woodin) $\rightarrow$ Con(The nonstationary ideal on ω_1 is saturated)

Theorems
ZFC + a large cardinal $\rightarrow$ something interesting.
Some examples in descriptive set theory:
Σ^1_1 sets of reals are measurable
Measurable cardinal $\rightarrow$ Σ^1_2 sets of reals are measurable
Infinitely many Woodin cardinals $\rightarrow$ All sets of reals in $L(\mathcal{R})$ are measurable

There is a good reason why the latter examples are taken from descriptive set theory: Large cardinals appear to give a complete understanding of the behaviour of sets of reals in $L(\mathcal{R})$. But if we go beyond that, we are faced with CH, which remains undecidable even with the addition of large cardinal hypotheses.

The aim of this course is to study applications of the second type, within descriptive set theory. After establishing as much as possible in ZFC alone, we shall introduce large cardinals in order to complete the picture. Here is an outline:

ZFC Results
The Borel and Projective Hierarchies
Borel= Δ^1_1
The Suslin Property and Regularity for Analytic Sets
Determinacy and the Wadge Property for Borel Sets
Π^1_1 Uniformisation

The Constructible Universe
Failure of Δ^1_2 Regularity
Failure of Π^1_1 Determinacy and Wadge
Projective Uniformisation

Forcing Extensions of L
Solovay's Model: Projective Regularity
Failure of Π^1_2 Uniformisation

Sharps
Regularity for Σ^1_2 Sets
Π^1_1 Determinacy and Wadge
Π^1_2 Uniformisation

Woodin Cardinals
Projective Regularity
Projective Uniformisation
Projective Determinacy and Wadge
Projective Equivalence Relations
Projective Basis Theorems

Part 1: ZFC Results

A *Polish space* is a topological space that is homeomorphic to a complete, separable metric space.

Lemma 1. Let X be a Polish space. Then there is a continuous function from Baire Space $\mathcal{N} = \{f \mid f \text{ is a function from } N \text{ to } N\}$ onto X.

Proof. By induction on the length of $s \in \text{Seq}$ = the set of finite sequences of elements of N, we define C_s such that $C_\emptyset = X$ and for nonempty s:

i. C_s is a closed ball of diameter $\leq 1/n$, where $n = \text{length } s$.
ii. $C_s \subseteq \bigcup_{k=0}^{\infty} C_{s*k}$ (all $s \in \text{Seq}$).
iii. $s \subseteq t \rightarrow \text{center}(C_t) \in C_s$.

For each $a \in \mathcal{N}$ let $f(a)$ be the unique point in $\bigcap\{C_s \mid s \subseteq a\}$. Then f is continuous and has range X. $\square$

Borel Sets

Let X be a Polish space. $A \subseteq X$ is *Borel* iff it belongs to the smallest σ-algebra of subsets of X containing all closed sets. The Borel hierarchy is defined as follows: For each $\alpha < \omega_1$ define $\mathbf{\Sigma}^0_\alpha$, $\mathbf{\Pi}^0_\alpha$ as follows:

$\mathbf{\Sigma}^0_1$ = Open Sets
$\mathbf{\Pi}^0_1$ = Closed Sets
$\mathbf{\Sigma}^0_\alpha$ = All sets $A = \bigcup_{n=0}^{\infty} A_n$, where each A_n belongs to $\mathbf{\Pi}^0_\beta$ for some $\beta < \alpha$
$\mathbf{\Pi}^0_\alpha$ = Complements of sets in $\mathbf{\Sigma}^0_\alpha$

It is clear that the above sets are all Borel. As every open set is the countable union of closed sets, we have $\mathbf{\Sigma}^0_1 \subseteq \mathbf{\Sigma}^0_2$ and then by induction:

$\alpha < \beta \rightarrow \mathbf{\Sigma}^0_\alpha \cup \mathbf{\Pi}^0_\alpha \subseteq \mathbf{\Sigma}^0_\beta \cap \mathbf{\Pi}^0_\beta$.

Thus $\bigcup_\alpha \mathbf{\Sigma}^0_\alpha = \bigcup_\alpha \mathbf{\Pi}^0_\alpha$ is the collection of Borel sets. For each α, $\mathbf{\Sigma}^0_\alpha$, $\mathbf{\Pi}^0_\alpha$ are closed under finite unions, finite intersections and inverse images by continuous functions.

We show that in the case where X is the Baire space, the Borel hierarchy is strict: $\mathbf{\Sigma}^0_\alpha \nsubseteq \mathbf{\Pi}^0_\alpha$ and therefore $\mathbf{\Sigma}^0_\alpha \neq \mathbf{\Sigma}^0_{\alpha+1}$ for each α. The proof generalises to arbitrary uncountable Polish spaces.

Lemma 2. (Universal $\mathbf{\Sigma}^0_\alpha$ Sets) For each $\alpha \geq 1$ there exists a set $U \subseteq \mathcal{N}^2$ such that U is $\mathbf{\Sigma}^0_\alpha$ and for every $\mathbf{\Sigma}^0_\alpha$ set $A \subseteq \mathcal{N}$ there is $a \in \mathcal{N}$ such that $A = \{x \mid (x, a) \in U\}$.

Proof. By induction on α. For $\alpha = 1$ let $G_1, G_2, \ldots$ be a list of all basic open sets and $G_0 = \emptyset$. We define $U = \{(x, y) \mid x \in G_{y(n)} \text{ for some } n\}$. U

is open and if G is an arbitrary open set, then $G = \{x \mid (x, a) \in U\}$ where $G = \bigcup_n G_{a(n)}$.

Suppose now that U_β is a universal $\mathbf{\Sigma}^0_\beta$ set for each $\beta < \alpha$ and we shall construct a universal $\mathbf{\Sigma}^0_\alpha$ set U. Choose $\alpha_0 \leq \alpha_1 \leq \cdots$ less than α either with supremum α or maximum the ordinal predecessor to α. Choose a continuous mapping of $\mathcal{N}$ onto $\mathcal{N}^\omega$ and for each $a \in \mathcal{N}$ let $(a)_n$ be the n-th coordinate of the image of a. We define $U = \{(x, y) \mid (x, (y)_n) \notin U_{\alpha_n}$ for some $n\}$. Then U is $\mathbf{\Sigma}^0_\alpha$. If A is $\mathbf{\Sigma}^0_\alpha$ then A is the union of sets A_n where A_n is $\mathbf{\Pi}^0_{\alpha_n}$. For each n let a_n be such that $A_n = \{x \mid (x, a_n) \notin U_{\alpha_n}\}$ and let a be such that $(a)_n = a_n$ for each n; then $A = \{x \mid (x, a) \in U\}$. □

Corollary 3. For each $\alpha \geq 1$ there is a set $A \subseteq \mathcal{N}$ that is $\mathbf{\Sigma}^0_\alpha$ but not $\mathbf{\Pi}^0_\alpha$.

Proof. Let U be a universal $\mathbf{\Sigma}^0_\alpha$ set and consider $A = \{x \mid (x, x) \in U\}$. □

2.Vorlesung

Analytic Sets

The continuous image of a Borel set need not be Borel. Let X be a Polish space.

Definition. $A \subseteq X$ is *analytic* iff there is a continuous function $f : \mathcal{N} \to X$ with range A. The *projection* of a set $S \subseteq X \times Y$ is the set $P = \{x \mid (x, y) \in S$ for some $y \in Y\}$.

Lemma 4. The following are equivalent:
(a) A is the continuous image of a Borel set in some Polish space.
(b) A is analytic.
(c) A is the projection of a closed set in $X \times \mathcal{N}$.
(d) A is the projection of a Borel set in $X \times Y$ for some Polish space Y.

Proof. We will show that every Borel set is analytic. Then (a)→(b) follows. (b)→(c) holds since if A is the range of $f : \mathcal{N} \to X$ then A is the projection of the closed set $\{(f(x), x) \mid x \in \mathcal{N}\} \subseteq X \times \mathcal{N}$. (c)→(d)→(a) is trivial.

Note that every closed set in a Polish space forms a Polish space and therefore is analytic by Lemma 1. So it suffices to show that each Borel subset of a Polish space X is the projection of a closed set in $X \times \mathcal{N}$. We show that the family P of all projections of closed sets in $X \times \mathcal{N}$ is closed under countable unions and intersections. As P clearly contains all closed sets and each open set is the countable union of closed sets, it follows that P contains all Borel sets, as desired.

Let A_n be the projection of the closed set $F_n \subseteq X \times \mathcal{N}$ for each n. We shall show that $\bigcup_n A_n$, $\bigcap_n A_n$ are projections of closed sets. As before, choose a continuous mapping of $\mathcal{N}$ onto $\mathcal{N}^\omega$ and for each $a \in \mathcal{N}$ let $(a)_n$ be the n-th coordinate of the image of a.

$x \in \bigcup_n A_n \leftrightarrow$
$\exists n\ \exists a\ (x, a) \in F_n \leftrightarrow$
$\exists a\ \exists b\ (x, a) \in F_{b(0)} \leftrightarrow$
$\exists c\ (x, (c)_0) \in F_{(c)_1(0)}$.

$x \in \bigcap_n A_n \leftrightarrow$
$\forall n\ \exists a\ (x, a) \in F_n \leftrightarrow$
$\exists c\ \forall n\ (x, (c)_n) \in F_n \leftrightarrow$
$\exists c\ (x, c) \in \bigcap_n \{(x, c) \mid (x, (c)_n) \in F_n\}$.

Hence $\bigcup_n A_N$ is the projection of the closed set $\{(x, c) \mid (x, (c)_0) \in F_{(c)_1(0)}\}$ and $\bigcap_n A_N$ is the projection of an intersection of closed sets. □

Lemma 5. The collection of analytic sets is closed under countable unions and intersections, as well as continuous images and preimages.

Proof. Closure under countable unions and intersections was established in the proof of the previous lemma. Closure under continuous images is clear by definition. Suppose that $x \in A \leftrightarrow f(x) \in B$, where f is continuous and B is analytic. Write $y \in B \leftrightarrow \exists z\ C(y, z)$, where C is Borel. Then $x \in A \leftrightarrow \exists z\ C(f(x), z)$, so A is the projection of the Borel set $\{(x, z) \mid C(f(x), z)\}$ and therefore is analytic. □

The Projective Hierarchy

The collection of analytic sets is not closed under complementation. For $n \geq 1$ we define the $\mathbf{\Sigma}^1_n$, $\mathbf{\Pi}^1_n$ and $\mathbf{\Delta}^1_n$ subsets of a Polish space X as follows:

$\mathbf{\Sigma}^1_1$ = Analytic
$\mathbf{\Pi}^1_1$ = Coanalytic = Complements of Analytic sets
$\mathbf{\Sigma}^1_{n+1}$ = Projections of $\mathbf{\Pi}^1_n$ subsets of $X \times \mathcal{N}$

$\mathbf{\Pi}^1_{n+1}$ = Complements of $\mathbf{\Sigma}^1_{n+1}$ sets
$\mathbf{\Delta}^1_n = \mathbf{\Sigma}^1_n \cap \mathbf{\Pi}^1_n$.

A set is *projective* iff it is $\mathbf{\Sigma}^1_n$ or $\mathbf{\Pi}^1_n$ for some n. It is obvious that $\mathbf{\Delta}^1_n \subseteq \mathbf{\Sigma}^1_n \subseteq \mathbf{\Delta}^1_{n+1}$, $\mathbf{\Delta}^1_n \subseteq \mathbf{\Pi}^1_n \subseteq \mathbf{\Delta}^1_{n+1}$; we shall show that $\mathbf{\Sigma}^1_n \neq \mathbf{\Pi}^1_n$ and therefore these inclusions are proper.

Lemma 6. (Universal $\mathbf{\Sigma}^1_n$ Sets) For each $n \geq 1$ there is a set $U \subseteq \mathcal{N}^2$ such that U is $\mathbf{\Sigma}^1_n$ and for for every $\mathbf{\Sigma}^1_n$ $A \subseteq \mathcal{N}$ there is some $v \in \mathcal{N}$ such that $A = \{x \mid (x, v) \in U\}$.

Proof. Let h be a homeomorphism of $\mathcal{N}^2$ with $\mathcal{N}$. For notational convenience, define $\mathbf{\Sigma}^1_0$ to be $\mathbf{\Sigma}^0_1$. We prove the Lemma by induction on $n \geq 0$. By Lemma 2 there does exists a universal $\mathbf{\Sigma}^1_0$ set. Inductively, assume that V is a universal $\mathbf{\Sigma}^1_{n-1}$ set and define $U = \{(x, y) \mid (h(x, a), y) \notin V$ for some $a \in \mathcal{N}\}$. Then U is $\mathbf{\Sigma}^1_n$. If $A \subseteq \mathcal{N}$ is $\mathbf{\Sigma}^1_n$ then there is a $\mathbf{\Pi}^1_{n-1}$ set B such that $A = \{x \mid (x, a) \in B$ for some $a \in \mathcal{N}\}$. The set $C = \mathcal{N} - h[B]$ is $\mathbf{\Sigma}^1_{n-1}$ and since V is universal there exists a v such that $C = \{u \mid (u, v) \in V\}$; then

$x \in A \leftrightarrow$
$(x, a) \in B$ for some $a \leftrightarrow$
$h(x, a) \notin C$ for some $a \leftrightarrow$
$(h(x, a), v) \notin V$ for some $a \leftrightarrow$
$(x, v) \in U$.

So U is a universal $\mathbf{\Sigma}^1_n$ set. □

Corollary 7. For each $n \geq 1$ there is a $\mathbf{\Sigma}^1_n$ set which is not $\mathbf{\Pi}^1_n$.

Every Borel set is $\mathbf{\Delta}^1_1$. This follows from the fact that the latter is closed under countable unions and intersections, and contains all open and closed sets. Conversely:

Theorem 8 (Suslin's Theorem). Every $\mathbf{\Delta}^1_1$ set is Borel.

Proof. We say that a set D *separates* two disjoint sets A, B iff A is contained in D and B is disjoint from D. We shall show that any two disjoint analytic sets can be separated by a Borel set, which clearly implies the Theorem.

First note that if $A = \bigcup_n A_n$, $B = \bigcup B_n$ and the pair A_m, B_n can be separated by a Borel set $D_{m,n}$ for each m, n, then the pair A, B can be separated by a Borel set, namely by $D = \bigcup_m \bigcap_n D_{m,n}$.

Now let A, B be analytic and choose continuous functions f, g such that $A = f[\mathcal{N}], B = g[\mathcal{N}]$. For each $s \in \mathrm{Seq}$ let $A_s = f[\mathcal{N}_s], B_s = g[\mathcal{N}_s]$, where $\mathcal{N}_s$ is the basic open set $\{f \mid s \subseteq f\}$ in Baire space. For each $a, b \in \mathcal{N}$, $\{f(a)\} = \bigcap_n A_{a\restriction n}$ and $\{g(b)\} = \bigcap_n B_{b\restriction n}$. It follows that for any $a, b \in \mathcal{N}$, there exists n such that $A_{a\restriction n}$ and $B_{b\restriction n}$ can be separated by an open set, as if U_a, U_b are disjoint open sets separating $f(a)$ from $g(b)$, the sets $A_{a\restriction n}$, $B_{b\restriction n}$ will be contained in U_a, U_b, respectively, for large enough n.

Now suppose that A, B cannot be separated by a Borel set. Then for some m_0, n_0, $A_{\langle m_0\rangle}, B_{\langle n_0\rangle}$ cannot be separated by a Borel set. Then for some m_1, n_1, the sets $A_{\langle m_0,m_1\rangle}, B_{\langle n_0,n_1\rangle}$ cannot be separated by a Borel set, etc. Let $a = \langle m_0, m_1, \ldots\rangle$ and $b = \langle n_0, n_1, \ldots\rangle$. Then $A_{a\restriction n}, B_{b\restriction n}$ cannot be separated by a Borel set for any n, in contradiction to the previous paragraph. $\Box$

For any set S let $\mathrm{Seq}(S)$ denote the collection of finite sequences of elements of S. A *tree* T on S is a subset of $\mathrm{Seq}(S)$ closed under initial segments. A *branch* through T is a function $a : \omega \to S$ such that $a \restriction n \in T$ for all n. T is *well-founded* iff T has no branch.

Let κ be an infinite cardinal. A *tree on* $\omega \times \kappa$ is a set T of pairs $(s, h) \in \mathrm{Seq}(\omega) \times \mathrm{Seq}(\kappa)$ such that length (s) = length (h) and for each $n \leq$ length (s), $(s \restriction n, h \restriction n) \in T$. For each $x \in \mathcal{N}$, $T(x) = \{h \mid (x \restriction n, h) \in T$, where $n =$ length $(h)\}$. Then $T(x)$ is a tree on κ for each $x \in \mathcal{N}$. The *projection* of T is defined by

$$p[T] = \{x \in \mathcal{N} \mid T(x) \text{ has a branch}\}.$$

A set of reals A is *κ-Suslin* iff it is the projection of a tree on $\omega \times \kappa$.

Proposition 9. Analytic subsets of $\mathcal{N}$ are ω-Suslin.

Proof. If A is analytic, then A is the projection of a closed set C on $\mathcal{N} \times \mathcal{N}$. Now consider the following tree on $\omega \times \omega$:

$$T = \{(a \restriction n, b \restriction n) \mid (a, b) \in C\}.$$

A branch through T is essentially a pair $(a, b) \in C$, and conversely, every pair $(a, b) \in T$ is a branch through T. As A is the projection of C, it is also the projection of the tree T. □

3.Vorlesung

A set C is *perfect* iff it is nonempty, closed and has no isolated points. ZFC^- is the theory obtained from ZFC by restricting the Replacement Axiom to Σ_{100} formulas.

Theorem 10. Suppose that A is the projection of a tree T on $\omega \times \kappa$ which belongs to the transitive ZFC^- model M. If A has an element not belonging to M then A contains a perfect subset.

Corollary 11. An uncountable analytic set has a perfect subset.

Proof of Corollary 11. If A is analytic then choose a tree T on $\omega \times \omega$ such that $A = p[T]$. Choose a countable ZFC$^-$ model M such that $T \in M$. By the Theorem, A is either a subset of M, and therefore countable, or contains a perfect subset. $\square$

Proof of Theorem 10: Define $T_0 = T$ and inductively:

$T_{\alpha+1} = \{(s,h) \in T_\alpha \mid$ There exist $(s_0,h_0),(s_1,h_1) \in T_\alpha$ extending (s,h) such that s_0, s_1 are incompatible$\}$.

For limit λ, $T_\lambda = \bigcap\{T_\alpha \mid \alpha < \lambda\}$. As T belongs to M and M is a model of ZFC$^-$, the sequence $\langle T_\beta \mid \beta \in \mathrm{Ord}(M)\rangle$ is definable in M and for some least ordinal $\alpha \in M$, $T_\alpha = T_{\alpha+1}$.

Let x belong to A and choose f so that (x,f) is a branch through T. If (x,f) is not a branch through T_α then there is some least ordinal β such that (x,f) is a branch through T_β but not through $T_{\beta+1}$. So there is some $(s,h) \subseteq (x,f)$ in $T_\beta - T_{\beta+1}$ and therefore x is the union of $\{s' \mid (s',h')$ extends (s,h) and belongs to $T_\beta\}$. It follows that x belongs to M, since T_β does.

As A has an element not belonging to M, it must be that T_α has a branch and therefore is nonempty. If (s,h) belongs to T_α then we can choose $(s_0,h_0),(s_1,h_1)$ extending (s,h) in T_α with s_0, s_1 incompatible. Then we can choose extensions $(s_{00},h_{00}),(s_{01},h_{01})$ of (s_0,h_0) in T_α such that s_{00}, s_{01} are incompatible, and similarly for (s_1,h_1). Continuing in this way we can build a subtree of T_α whose projection is a perfect subset of A. $\square$

A *null set* is a set of reals of Lebesgue measure 0. A *meager set* is the countable union of nowhere dense sets. A set of reals is *measurable* iff it differs by a null set from a Borel set (equivalently, from a countable union of closed sets or from a countable intersection of open sets). It has the *Baire property* iff it differs by a meager set from a Borel set (equivalently, from an open set).

Theorem 12. Suppose that A is the projection of a tree T on $\omega \times \kappa$ which belongs to the transitive ZFC$^-$ model M. Suppose that M has only countably many reals. Then A is measurable and has the property of Baire.

Corollary 13. Analytic sets are measurable and have the Baire property.

Proof of Corollary 13. If A is analytic then A is the projection of a tree T on $\omega \times \omega$. There is a countable ZFC$^-$ model that contains T. So by the Theorem, A is measurable and has the Baire property. $\square$

Before proving Theorem 12, we must introduce *Borel codes* and *absoluteness.* Let $I_1, I_2, \ldots$ be a recursive enumeration of the basic open sets of $\mathcal{N}$. Let c belong to $\mathcal{N}$. We define $u(c) \in \mathcal{N}$ by $u(c)(n) = c(n+1)$ for all n. Let Γ be a recursive bijection from $N \times N$ onto N. For each $i \in N$ we define $v_i(c)$ by $v_i(c)(n) = c(\Gamma(i,n)+1)$ for all n. For each positive countable ordinal α we define coding sets Σ_α, Π_α as follows:

$c \in \Sigma_1$ iff $c(0) > 1$
$c \in \Pi_\alpha$ iff either $c \in \Sigma_\beta \cup \Pi_\beta$ for some $\beta < \alpha$ or $c(0) = 0$ and $u(c) \in \Sigma_\alpha$
For $\alpha > 1$: $c \in \Sigma_\alpha$ iff either $c \in \Sigma_\beta \cup \Pi_\beta$ for some $\beta < \alpha$ or $c(0) = 1$ and $v_i(c) \in \bigcup_{\beta<\alpha}(\Sigma_\beta \cup \Pi_\beta)$ for all i.

If $c \in \Sigma_\alpha$ we call c a $\boldsymbol{\Sigma}^0_\alpha$*-code*, similarly for $\boldsymbol{\Pi}^0_\alpha$*-codes*. The union of all Σ_α is the set BC of *Borel codes.* The Borel code c codes the Borel set A_c defined as follows:

If $c \in \Sigma_1$ then $A_c = \bigcup\{I_n \mid c(n) = 1\}$
If $c \in \Pi_\alpha$ and $c(0) = 0$ then $A_c = \sim A_{u(c)}$
If $c \in \Sigma_\alpha$ and $c(0) = 1$ then $A_c = \bigcup_i A_{v_i(c)}$.

It is clear that for every $\alpha > 0$, if $c \in \Sigma_\alpha$ then $A_c \in \boldsymbol{\Sigma}^0_\alpha$, similarly for Π_α. Conversely, every $\boldsymbol{\Sigma}^0_\alpha$, $\boldsymbol{\Pi}^0_\alpha$ set B is coded by some $c \in \Sigma_\alpha$, Π^0_α, respectively. Thus $\{A_c \mid c \in \mathrm{BC}\}$ is the collection of all Borel sets.

We introduce the hierarchy of Σ^1_n *and* Π^1_n *formulas.* A Σ^1_1 *formula with parameter* $p \in \mathcal{N}$ is a formula of the form

$$\varphi(y_1, \ldots, y_n) \leftrightarrow \exists z \ \psi(y_1, \ldots, y_n, z, p),$$

where ψ is *arithmetical*, i.e., a formula in the language of second-order arithmetic with only number quantifiers. The subsets of $\mathcal{N}^n$ which are definable by Σ^1_1 formulas with parameters are exactly the analytic subsets of $\mathcal{N}^n$. A Π^1_1 *formula with parameter* p is the negation of a Σ^1_1 formula with parameter p. Inductively: A Σ^1_{k+1} *formula with parameter* p is a formula of the form $\varphi(y_1, \ldots, y_n) \leftrightarrow \exists z \ \psi(y_1, \ldots, y_n, z)$, where ψ is a Π^1_k formula with parameter

p and a Π^1_{k+1} *formula with parameter* p is the negation of such a formula. A subset of $\mathcal{N}^n$ is $\Sigma^1_n(p)$, $\Pi^1_n(p)$ iff it is definable by a Σ^1_n, Π^1_n formula with parameter p and is $\Delta^1_n(p)$ iff it is both $\Sigma^1_n(p)$ and $\Pi^1_n(p)$. When p is recursive, we write Σ^1_n, Π^1_n, Δ^1_n.

Lemma 14. (a) The set BC of all Borel codes is Π^1_1.
(b) There is a Δ^1_1 relation R such that for Borel codes c, $R(a,c)$ iff $a \in A_c$.
(c) The following properties of Borel codes are Π^1_1:
$A_c \subseteq A_d$
$A_c = A_d$
$A_c = \emptyset$
$A_c = A_d \cup A_e$
$A_c = \sim A_d$
$A_c = A_d \cap A_e$
$A_c = A_d \triangle A_e$
$A_c = \bigcup_n A_{c_n}$.

Proof. (a) Define the relation E by:

xEy iff either ($y(0) = 0$ and $x = u(y)$) or ($y(0) = 1$ and $x = v_i(y)$ for some i).

Then y is a Borel code iff there is no infinite sequence $y = z_0, z_1, \ldots$ with $z_{n+1} E z_n$ for each n. As the relation E is arithmetical, it follows that BC is Π^1_1.

(b) For any $c \in \mathcal{N}$ there is a smallest countable $T = T_c \subseteq \mathcal{N}$ with the property:

$(*)_c$ $c \in T$ and whenever $y \in T$, zEy then $z \in T$.

And if c is a Borel code, $a \in \mathcal{N}$ then there is a unique function $h = h_{a,c}$ on T_c such that for all $y \in T_c$:

$(**)_a$ If $y(0) > 1$ then $h(y) = 1$ iff $a \in I_n$ for some n such that $y(n) = 1$
If $y(0) = 0$ then $h(y) = 1$ iff $h(u(y)) = 0$
If $y(0) = 1$ then $h(y) = 1$ iff $h(v_i(y)) = 1$ for some i.

For $y \in T_c$ and h as above we have $h(y) = 1$ iff $a \in A_y$. Thus for a Borel code c:

$a \in A_c$ iff
For all countable T satisfying $(*)_c$ and all h defined on T satisfying $(**)_a$, $h(c) = 1$ iff
There is a countable T satisfying $(*)_c$ and an h defined on T satisfying $(**)_a$ such that $h(c) = 1$.

As $(*)_c$ is Σ^1_1 and $(**)_a$ is arithmetical, this gives the desired result.

(c) This follows easily from (b). $\Box$

Lemma 15. (Mostowski Absoluteness) Suppose that M is a transitive model of ZFC$^-$.
(a) If $\varphi(y_1, \ldots, y_n)$ is a Σ^1_1 formula with parameter in M then for all $y_1, \ldots, y_n$ in M:

$$M \vDash \varphi(y_1, \ldots, y_n) \text{ iff } \varphi(y_1, \ldots, y_n) \text{ is true.}$$

(b) If $\varphi(y_1, \ldots, y_n)$ is a Σ^1_2 formula with parameter in M then for all $y_1, \ldots, y_n$ in M:

$$M \vDash \varphi(y_1, \ldots, y_n) \text{ implies } \varphi(y_1, \ldots, y_n) \text{ is true.}$$

Proof. (a) Using a recursive homeomorphism between $\mathcal{N}^n$ and $\mathcal{N}$ we can assume that $n = 1$. In both M and the universe we have that $\varphi(y)$ holds iff $T(y)$ has a branch, where T is a tree on $\omega \times \omega$. If $T(y)$ has a branch in M then of course it also has one in the universe. If $T(y)$ has no branch in M then $T(y)$ is well-founded in M and therefore there exists an order-preserving function in M from $T(y)$ into the ordinals of M. It follows that there is such a function in the universe and therefore $T(y)$ has no branch in the universe.

(b) Write $\varphi(y_1, \ldots, y_n) = \exists z\ \psi(y_1, \ldots, y_n, z)$, where ψ is Π^1_1. If M satisfies $\varphi(y_1, \ldots, y_n)$ then choose $y \in M$ such that $\psi(y_1, \ldots, y_n, z)$ holds in M. It follows from (a) that the latter also holds in the universe, and therefore so does $\varphi(y_1, \ldots, y_n)$. $\Box$

For a transitive model M of ZFC$^-$, let BC^M denote the set of Borel codes, as interpreted in M, and for $c \in \mathrm{BC}^M$, let A^M_c denote A_c as interpreted in M.

Corollary 16. Suppose that M is a transitive model of ZFC$^-$. Then:
(a) $\mathrm{BC}^M = \mathrm{BC} \cap M$.

(b) If c belongs to BC^M then $A_c^M = A_c \cap M$.
(c) The following properties of Borel codes in M hold iff they hold in M:
$A_c \subseteq A_d$
$A_c = A_d$
$A_c = \emptyset$
$A_c = A_d \cup A_e$
$A_c = \sim A_d$
$A_c = A_d \cap A_e$
$A_c = A_d \triangle A_e$
$A_c = \bigcup_n A_{c_n}$.

4.Vorlesung

Lemma 17. The following sets are both Σ_2^1 and Π_2^1 definable:
(a) $\{c \mid c$ is a Borel code and A_c is null$\}$.
(b) $\{c \mid c$ is a Borel code and A_c is meager$\}$.

Proof. For a Borel code c:

A_c is null iff
For each n there exists a Σ_1 code d such that $A_c \subseteq A_d$ and A_d has measure less than $1/n$ iff
For all Π_1 codes e, if $A_e \subseteq A_c$ then A_e has measure 0.

As the properties "d is a Σ_1 code and A_d has measure less than $1/n$" and "e is a Π_1 code and A_e has measure 0" are arithmetical, the above provides both Σ_2^1 and Π_n^1 definitions for $\{c \mid c$ is a Borel code and A_c is null$\}$.

Also:

A_c is meager iff
There exist Π_1 codes c_n, $n \in N$ such that $A_c \subseteq \bigcup_n A_{c_n}$ and each A_{c_n} is nowhere dense iff
For all Σ_1 codes d, if A_d is nonempty then $A_c \triangle A_d$ is not meager.

As the property "c is a Π_1 code and A_c is nowhere dense" is arithmetical, the second line above gives a Σ_2^1 definition of $\{c \mid c$ is a Borel code and A_c is meager$\}$, and using this, the third line above gives a Π_2^1 definition. $\square$

Corollary 18. Suppose that c is a Borel code and c belongs to the transitive ZFC^- model M. Then A_c is null iff $M \vDash A_c$ is null, and A_c is meager iff $M \vDash A_c$ is meager.

Proof. Use Lemma 17 and part (b) of Lemma 15. □

We consider $\mathcal{B}_m$ and $\mathcal{B}_c$, the quotients of the σ-algebra of Borel sets by the ideals $\mathcal{I}_m$ of null sets and $\mathcal{I}_c$ of meager sets. For each $B \in \mathcal{B}$, let $[B]_m$, $[B]_c$ denote the equivalence class of B in $\mathcal{B}_m$, $\mathcal{B}_c$, respectively. We view $\mathcal{B}_m$, $\mathcal{B}_c$ as forcing notions by discarding $[\emptyset]_m$, $[\emptyset]_c$ and using the natural order of inclusion modulo $\mathcal{I}_m$, $\mathcal{I}_c$, respectively.

Lemma 19. (a) If G is $\mathcal{B}_m$-generic then there is a unique real x_G such that for all non-null $B \in \mathcal{B}$:

$$[B]_m \in G \leftrightarrow x_G \in B^*,$$

where B^* denotes $A_c^{V[G]}$ for any Borel code c for B (this definition is independent of the choice of c). And a real x (in an outer model of V) is of this form iff $x \notin B^*$ for each null $B \in \mathcal{B}$. Such reals are called *random* reals.
(b) If G is $\mathcal{B}_c$-generic then there is a unique real x_G such that for all non-meager $B \in \mathcal{B}$:

$$[B]_c \in G \leftrightarrow x_G \in B^*,$$

where B^* denotes $A_c^{V[G]}$ for any Borel code c for B. And a real x (in an outer model of V) is of this form iff $x \notin B^*$ for each meager $B \in \mathcal{B}$. Such reals are called *Cohen* reals.

Proof. (a) For convenience we work not in Baire space but in the (real) real numbers $\mathcal{R}$. Define $x_G = \sup\{r \mid r \text{ is rational and } [(r,\infty)] \in G\}$. We show that x_G belongs to A_c^* iff $[A_c] \in G$, by induction on $c \in \mathrm{BC}$. If c is a Σ_1 code for a rational interval (p,q) then we have:

$x_G \in A_c^*$ iff
$p < x_G < q$ iff
$p < \sup\{r \in \mathcal{Q} \mid [(r,\infty)] \in G\} < q$ iff
$[(p,\infty)] \in G$ and $[(q,\infty)] \notin G$ iff
$[(p,q)] \in G$
$[A_c] \in G$.

If c is a Σ_1 code for the union of rational intervals $A_c = \bigcup_n I_{k_n}$ then:

$x \in A_c^*$ iff $x \in \bigcup_n I_{k_n}^*$ iff
$[I_{k_n}] \in G$ for some n iff
$[\bigcup_n I_{k_n}] \in G$ iff
$[A_c] \in G$.

Inductively, if α is countable and c is a Σ_α code, then the result holds by induction by the same argument as above. If c is a Π_α code, we may assume that $c(0) = 0$ and therefore $u(c)$ is a Σ_α code, $A_{u(c)} = \mathcal{R} - A_c$ and we have:

$x \in A_c^*$ iff
$x \notin A_{u(c)}^*$ iff
$[A_{u(c)}] \notin G$ iff
$[A_c] \in G$.

This proves the first part of (a), as the uniqueness of x_G is clear.

Suppose that $x = x_G$ is random. If A_c is null then $[A_c] \notin G$ and therefore by the first part of (a), $x \notin A_c^*$. Conversely, suppose that $x \notin A_c^*$ whenever A_c is null. Note that if $[A_c] = [A_d]$ then $A_c \triangle A_d$ is null, $A_c^* \triangle A_d^*$ is null and thus $x \in A_c^*$ iff $x \in A_d^*$. Now let $G = \{[A_c] \mid x \in A_c^*\}$. It is easy to check that G is a filter on $\mathcal{B}_m$. We claim that G is $\mathcal{B}_m$-generic: Since $\mathcal{B}_m$ satisfies the countable chain condition, it suffices to show that if $\{[A_{c_n}] \mid n \in N\}$ is a maximal antichain in $\mathcal{B}_m$ then x belongs to $A_{c_n}^*$ for some n. But the maximality of this antichain implies that x belongs to $(\bigcup_n A_{c_n})^*$ and the latter equals $\bigcup_n A_{c_n}^*$.

(b) This is proved exactly as part (a), using the fact that $\mathcal{B}_c$ also satisfies the countable chain condition. □

We can now prove Theorem 12. As M has only countably many reals, it follows that the set of reals which are not random over M is null. Thus to show that A is measurable, it suffices to show that $\{x \in A \mid x$ is random over $M\}$ is Borel. Suppose that x is random over M and let $x = x_G$, where G is $\mathcal{B}_m$-generic over M. Then $M[x]$ is a model of ZFC$^-$ and we have:

$x \in A$ iff
$x \in p[T]$ iff
$M[x] \models x \in p[T]$ iff

For some $[B]_m \in G$, $[B]_m \Vdash x_G \in p[T]$ iff
For some $[B]_m$, $x \in B^*$ and $[B]_m \Vdash x_G \in p[T]$ iff
$x \in \bigcup\{B^* \mid [B]_m \Vdash x_G \in p[T]\}$,

and the latter is a Borel property of x. The same proof shows that A has the property of Baire. $\square$

We next consider the Ramsey property. For an infinite set $A \subseteq \omega$ we let $[A]^\omega$ denote the set of all infinite subsets of A. Is $S \subseteq [\omega]^\omega$ then we say that an infinite $H \subseteq \omega$ is *homogeneous for* S iff either $[H]^\omega \subseteq S$ or $[H]^\omega \cap S = \emptyset$. We say that $S \subseteq [\omega]^\omega$ is *Ramsey* iff there is an infinite homogeneous set H for S.

Theorem 20. Suppose that A is the projection of the tree T on $\omega \times \kappa$, where T belongs to the transitive ZFC$^-$ model M and M has only countably many sets of reals. Then A is Ramsey.

The proof of this result makes use of *Mathias forcing*. A condition is a pair (s, A) where s is a finite subset of ω and A is an infinite subset of ω such that $\max s < \min A$. A condition (s, A) extends a condition (t, B) iff

1. t is an initial segment of s.
2. $A \subseteq B$.
3. $s - t \subseteq B$.

If G is Mathias generic then G is determined by the real

$$x_G = \bigcup\{s \mid (s, A) \in G \text{ for some } A\},$$

since $G = \{(s, A) \mid s \subseteq x_G \subseteq s \cup A\}$. The real x_G is called a *Mathias real*. We shall prove:

Lemma 21. Let φ be a sentence of the forcing language and (s, A) a condition. Then there exists an infinite $B \subseteq A$ such that (s, B) decides φ (i.e., forces either φ or $\sim \varphi$).

Lemma 22. A real x is Mathias over the transitive ZFC$^-$ model M iff x is infinite and for each maximal almost disjoint family $\mathcal{A} \in M$ of subsets of ω, there is an $A \in \mathcal{A}$ such that x is almost contained in A.

Given these Lemmas we prove Theorem 20 as follows: Suppose that M is a transitive ZFC$^-$ model with only countably many sets of reals and that A is the projection of the tree $T \in M$. For any real x, if $M[x]$ satisfies ZFC$^-$, then:

x belongs to A iff
$T(x)$ has a branch iff
$M[x] \models T(x)$ has a branch.

Now let φ be the sentence "$T(x_G)$ has a branch". By Lemma 21 there is a condition of the form $(\emptyset, A)$ which decides φ; assume that $(\emptyset, A) \Vdash \varphi$. As M has only countably many sets of reals, there exists a Mathias generic G over M which contains the condition $(\emptyset, A)$. Thus $M[x_G] \models \varphi$. By Lemma 22 every infinite $y \subseteq x$ is a Mathias real over M, and also the generic determined by y contains the condition $(\emptyset, A)$. Thus $T(y)$ has a branch for each infinite $y \subseteq x_G$ and therefore x_G is homogeneous for A.

5.Vorlesung

Lemma 21. Let φ be a sentence of the forcing language and (s, A) a condition. Then there exists an infinite $B \subseteq A$ such that (s, B) decides φ (i.e., forces either φ or $\sim \varphi$).

Lemma 22. A real x is Mathias over the transitive ZFC$^-$ model M iff x is infinite and for each maximal almost disjoint family $\mathcal{A} \in M$ of subsets of ω, there is an $A \in \mathcal{A}$ such that x is almost contained in A.

Proof of Lemma 21. For any $A \subseteq \omega$ and $k \in \omega$ we let $A(> k)$ denote $\{n \in A \mid n > k\}$. If s is a finite subset of ω then $A(> s)$ denotes $A(> \max s)$. We first construct an infinite $B \subseteq A$ such that

$(*)$ If (t, C) extends (s, B) and decides φ then so does $(t, B(> t))$.

By induction we define $b_k = \min B_k$ for $k \in \omega$: Set $B_0 = A$. To define B_{k+1} let $\{t_1, \ldots, t_l\}$ be a list of all subsets of $\{b_1, \ldots, b_k\}$. Construct $B_k = B^1_{k+1} \supseteq B^2_{k+1} \supseteq \cdots \supseteq B^l_{k+1} = B_{k+1}$ as follows: Given B^j_{k+1}, if there exists $C \subseteq B^j_{k+1}$ such that $(s \cup t_{j+1}, C)$ decides φ then set $B^{j+1}_{k+1} = C$; otherwise, $B^{j+1}_{k+1} = B^j_{k+1}$. Then $B = \{b_k \mid k \in \omega\}$ is as desired.

So we can suppose that A satisfies $(*)$. Now define $b_k = \min B_k$ by induction on k as follows: Set $B_0 = A$. Given B_k construct B_{k+1} such that for each $t \subseteq \{b_1, \ldots, b_k\}$ exactly one of the following holds:

$(s \cup t \cup \{n\}, B_{k+1}(> n)) \Vdash \varphi$ for each $n \in B_{k+1}$
$(s \cup t \cup \{n\}, B_{k+1}(> n)) \Vdash \sim \varphi$ for each $n \in B_{k+1}$
$(s \cup t \cup \{n\}, B_{k+1}(> n))$ does not decide φ for each $n \in B_{k+1}$.

We claim that (s, B) decides φ: Let (t, C) be an extension of (s, B) deciding φ. Assume that Length(t) is minimal. If Length(t) = Length(s) then we are done since it follows from $(*)$ for A that (s, B) decides φ. Otherwise let $m = \max t$ and write $t = s \cup t' \cup \{m\}$ where $t' \subseteq \{b_1, \ldots, b_k\}$. By construction $(s \cup t' \cup \{m\}, B_{k+1}(> m))$ either forces φ or $\sim \varphi$; assume the former. Then the same holds for each $m \in B_{k+1}$. It follows that $(s \cup t', C)$ decides φ, contradicting the minimality of Length(t). So (s, B) is the desired extension of (s, A) which decides φ. $\square$

Proof of Lemma 22. If x is Mathias over M and $\mathcal{A}$ is a maximal almost disjoint family in M then $D = \{(s, B) \mid B$ is almost contained in an element of $\mathcal{A}\}$ is dense, and hence there exists $(s, B) \in G_x \cap D$; as x is almost contained in B and B is almost contained in an element of $\mathcal{A}$, it follows that x is almost contained in an element of $\mathcal{A}$.

Conversely, suppose that x is infinite and each maximal almost disjoint family of M has an element which almost contains x. Let $D \in M$ be dense open for Mathias forcing; we must show that D has an element (s, A) such that $s \subseteq x \subseteq s \cup A$. We say that an infinite A *captures* (s, D) iff

$(**)$ For all infinite $B \subseteq A(> s)$ there exists a finite initial segment t of B such that $(s \cup t, A(> (s \cup t)))$ belongs to D.

Main Claim. For any infinite A, there is an infinite $A^* \subseteq A$ such that A^* captures (s, D) for each s with $\max s \in A^*$.

Proof of Main Claim. It suffices to show that for each infinite A and s there is an infinite $A^* \subseteq A$ such that A^* captures (s, D). For then, we can inductively define $k_n = \min A_n$ by setting $A_0 = A$ and choosing an infinite $A_{n+1} \subseteq A_n$ which captures (s, D) for all s with $\max s \in \{k_0, \ldots, k_n\}$; then $A^* = \{k_0, k_1, \ldots\}$ is as desired.

So suppose that A and s are given, with $\max s \in A$. We may assume that for all finite $t \subseteq A(> s)$ if $(s \cup t, B)$ belongs to D for some $B \subseteq A(> (s \cup t))$ then in fact $(s \cup t, A(> (s \cup t)))$ belongs to D. Now let S be the set of all B such that either B is not contained in $A(> s)$ or $(s \cup t, A(> (s \cup t)))$ belongs to D for some finite initial segment t of B. Then S is an open set (in the space of infinite subsets of ω). We shall show that open sets are *completely Ramsey* (see below), which implies that there exists an infinite $B \subseteq A(> s)$ all of whose infinite subsets either belong to S or to the complement of S. By our assumption about A and the density of D, it must be the case that all infinite subsets of B belong to S. It follows that B captures (s, D), as desired.

Finally, we establish the *complete Ramseyness* of open sets: S is *completely Ramsey* iff for any condition (s, A) there exists $A^* \subseteq A$ such that either $[s, A^*] \subseteq S$ or $[s, A^*] \cap S = \emptyset$, where $[s, A^*] = \{B \mid s \subseteq B \subseteq s \cup A^*\}$. It suffices to show that open sets are Ramsey, as given (s, A) we can consider $S^* = \{B \mid f^*(B) \in S\}$, where $f^*(B) = s \cup f[B]$ and f is the increasing enumeration of A; the Ramseyness of the open set S^* implies the complete Ramseyness of S with respect to (s, A). We say that A *accepts* s iff $[s, A] \subseteq S$ and *rejects* s iff no $B \subseteq A$ accepts s.
(i) There is an $A = \{k_0, k_1, \ldots\}$ which either accepts or rejects each of its finite subsets, obtained by inductively defining $k_n = \min A_n$ where $A_0 = \omega$ and $A_{n+1} \subseteq A_n$ accepts or rejects each subset of $\{k_0, \ldots, k_n\}$.
(ii) In fact there is an $A = \{k_0, k_1, \ldots\}$ that either accepts $\emptyset$ or rejects each of its finite subsets, obtained by by choosing A_0 as in (i) to reject $\emptyset$ (without loss of generality) and assuming that A_0 rejects each subset of $\{k_0, \ldots, k_{n-1}\}$, choosing k_n as follows: For every subset s of $\{k_0, \ldots, k_{n-1}\}$ there are only finitely many $z \in A_0$ such that A_0 accepts $s \cup \{z\}$, as otherwise there is an infinite $Z \subseteq A_0$ such that A_0 accepts $s \cup \{z\}$ for each $z \in Z$ and therefore A_0 accepts s, in contradiction to the assumption that A_0 rejects s. Thus we can choose $k_n \in A_0 - \{k_0, \ldots, k_{n-1}\}$ such that A_0 rejects each subset of $\{k_0, \ldots, k_n\}$. A_0 rejects each finite subset of $A = \{k_0, k_1, \ldots\}$ and therefore so does A itself.
(iii) Now if the set A constructed in (ii) accepts $\emptyset$ we have established the Ramseyness of S. Otherwise A rejects each of its finite subsets. We claim that no infinite subset of A belongs to S. Otherwise there is an infinite $B \subseteq A$ which belongs to S and since S is open there is a finite initial segment s of B such that $[s, B(> s)]$ is contained in S; this contradicts the fact that A rejects s. $\Box$ (Main Claim).

Now apply the hypothesis on x to obtain A almost containing x which captures (s, D) for all s with $\max s \in A$. Choose a nonempty finite initial segment s of x such that $\max s \in A$ and $x \subseteq s \cup A(> s)$. Consider the tree T of $t \subseteq A(> s)$ such that $(s \cup t, A(> (s \cup t)))$ does not belong to D, ordered by end-extension. Then T is well-founded, as A captures (s, D). By absoluteness, T is well-founded in V and therefore $x(> s)$ is not a branch through T. But then for some initial segment t of $x(> s)$, the condition $(s \cup t, A(> (s \cup t)))$ belongs to D and satisfies $s \cup t \subseteq x \subseteq s \cup t \cup A(> (s \cup t))$, as desired. □

6.Vorlesung

Borel Determinacy

A *pruned tree* T on a set A is a nonempty set of finite sequences of elements of A, closed under initial segments, such that each element of T has proper extension in T. We let $[T]$ denote the collection of infinite branches through T. For any $X \subseteq [T]$, we define the game $G(T, X)$ as follows: Players I and II alternately choose $a_0, a_1, \ldots$ in A so that for each n the sequence $(a_0, \ldots, a_n)$ belongs to T. Player I wins the game iff the infinite sequence $(a_0, a_1, \ldots)$ belongs to X. A *strategy* for I assigns an extension of length $n+1$ in T to each element of T of even length n; similarly for II with "even" replaced by "odd". A strategy σ for I is a *winning* strategy iff I always wins the game using σ, no matter how II plays; we similarly define a *winning* strategy for II. The game $G(T, X)$ is *determined* iff either I or II has a winning strategy.

Theorem 23. If $X \subseteq [T]$ is either closed or open, then $G(T, X)$ is determined.

Proof. Suppose that X is closed and that II has no winning strategy. Consider the strategy for I in which he plays in such a way as to guarantee that II still has no winning strategy afterwards. Inductively this is possible, as otherwise II would have had a winning strategy before I plays his next move. We claim that this is in fact a winning strategy for I: Otherwise there is a play of the game $(a_0, a_1, \ldots)$ where I follows the described strategy but I loses. Since X is closed this means that for some n, $(a_0, a_1, \ldots, a_{2n})$ has no extension in X (i.e., I has already lost at a finite stage of the game). But this contradicts the definition of I's strategy!

Similarly, if X is open and I has no winning strategy, then II uses the strategy which always guarantees that I still has no strategy. This must win for II, as otherwise II loses at a finite stage, contradicting the definition of his strategy. □

Theorem 24. (Martin) If $X \subseteq [T]$ is Borel then $G(T, X)$ is determined.

Proof. Let T be a nonempty pruned tree on a set A. A *covering* of T is a triple $(\tilde{T}, \pi, \varphi)$ where

i. $\tilde{T}$ is a nonempty pruned tree on some set $\tilde{A}$.
ii. $\pi : \tilde{T} \to T$ is monotone (i.e., $s \subseteq t \to \pi(s) \subseteq \pi(t)$) with Length$(\pi(s)) =$ Length(s). Thus π gives rise to a continuous function $\pi : [\tilde{T}] \to [T]$.
iii. φ maps strategies for I (II, respectively) in $\tilde{T}$ to strategies for I (II, respectively) in T in such a way that $\varphi(\tilde{\sigma})$ restricted to positions of length $\leq n$ depends only on $\tilde{\sigma}$ restricted to positions of length $\leq n$ for each n.
iv. If $\tilde{\sigma}$ is a strategy for I (II, respectively) in $\tilde{T}$ and $x \in [T]$ is played according to $\varphi(\tilde{\sigma})$ then there is $\tilde{x} \in [\tilde{t}]$ played according to $\tilde{\sigma}$ such that $\pi(\tilde{x}) = x$.

It follows that if $X \subseteq [T]$ and $\tilde{\sigma}$ is a winning strategy for I (II, respectively) in $G(\tilde{T}, \tilde{X})$, where $\tilde{X} = \pi^{-1}[X]$, then $\varphi(\tilde{\sigma})$ is a winning strategy for I (II, respectively) in $G(T, X)$. A *k-covering* of T is a covering $(\tilde{T}, \pi, \varphi)$ of T such that $T \restriction 2k = \tilde{T} \restriction 2k$ and π is the identity on $\tilde{T} \restriction 2k$. For $X \subseteq [T]$ we say that $(\tilde{T}, \pi, \varphi)$ *unravels* X iff $\pi^{-1}[X] = \tilde{X}$ is a clopen subset of $[\tilde{T}]$.

Thus Theorem 24 follows from:

Main Lemma 25. If T is a nonempty pruned tree and $X \subseteq [T]$ is Borel then for each k there is a k-covering of T which unravels X.

We prove the Main Lemma using:

Lemma 26. The Main Lemma holds for closed $X \subseteq [T]$.

Lemma 27. Fix k. Let $(T_{i+1}, \pi_{i+1}, \varphi_{i+1})$ be a $(k+i)$-covering of T_i for each i. Then there is a pruned tree T_∞ and $\pi_{\infty,i}$, $\varphi_{\infty,i}$ such that for each i $(T_\infty, \pi_{\infty,i}, \varphi_{\infty,i})$ is a $(k+i)$-covering of T_i and $\pi_{\infty,i} = \pi_{i+1} \circ \pi_{\infty,i+1}$, $\varphi_{\infty,i} = \varphi_{i+1} \circ \varphi_{\infty,i+1}$.

Proof of Main Lemma. We show by induction on $\xi > 0$ that for all T, all k and all $\mathbf{\Sigma}^0_\xi$ subsets X of T, there is a k-covering of T that unravels X. Notice that if a k-covering unravels X it also unravels $\sim X$, so by Lemma 26 we are done if $\xi = 1$. Assume $\xi > 1$ and that the desired property holds for all $\eta < \xi$. So for each T, each $\eta < \xi$, each $\mathbf{\Pi}^0_\eta$ subset Y of T, and each k there is a k-covering that unravels $\sim Y$, and therefore also Y. Let X be $\mathbf{\Sigma}^0_\xi$ and $k \in N$. Then $X = \bigcup_i X_i$ with $X_i \in \mathbf{\Pi}^0_{\xi_i}$, $\xi_i < \xi$. Let (T_1, π_1, φ_1) be a k-covering of $T_0 = T$ that unravels X_0. Then $\pi_1^{-1}[X_i]$ is also $\mathbf{\Pi}^0_{\xi_i}$ on T_1 for each i. By recursion define $(T_{i+1}, \pi_{i+1}, \varphi_{i+1})$ to be a $(k+i)$-covering of T_i that unravels $\pi_i^{-1} \circ \pi_{i-1}^{-1} \circ \cdots \circ \pi_1^{-1}[X_i]$. Let $(T_\infty, \pi_{\infty,i}, \varphi_{\infty,i})$ be as in Lemma 27. Then $(T_\infty, \pi_{\infty,0}, \varphi_{\infty,0})$ unravels every X_i. Thus $\pi_{\infty,0}^{-1}[X] = \bigcup_i \pi_{\infty,0}^{-1}[X_i]$ is open in $[T_\infty]$. Finally, let $(\tilde{T}, \pi, \varphi)$ be a k-covering of T_∞ that unravels $\pi_{\infty,0}^{-1}[X]$, by Lemma 26. Then $(\tilde{T}, \pi_{\infty,0} \circ \pi, \varphi_{\infty,0} \circ \varphi)$ is a k-covering of T that unravels X.

7.Vorlesung

Proof of Lemma 27. Note that for any finite sequence s if $\text{Length}(s) \le 2(k+i)$ then whether $s \in T_i$ or not is independent of i. So we define:

$s \in T_\infty$ iff
$s \in T_i$ for any i with $\text{Length}(s) \le 2(k+i)$.

It is clear that T_∞ is a pruned tree and that T_∞, T_i have the same first $2(k+i)$ levels.

We define $\pi_{\infty,i}$: If $\text{Length}(s) \le 2(k+i)$ then $\pi_{\infty,i}(s) = s$. If $2(k+i) < \text{Length}(s) \le 2(k+j)$ we put $\pi_{\infty,i}(s) = \pi_{i+1} \circ \pi_{i+2} \circ \cdots \circ \pi_j(s)$ (this is independent of j).

We similarly define $\varphi_{\infty,i}$: If σ_∞ is a strategy for T_∞ let $\varphi_{\infty,i}(\sigma_\infty) \restriction 2(k+i) = \sigma_\infty \restriction 2(k+i)$ and for $j > i$, $\varphi_{\infty,i}(\sigma_\infty) \restriction 2(k+j) = \varphi_{i+1} \circ \varphi_{i+2} \circ \cdots \circ \varphi_j(\sigma_\infty \restriction 2(k+j))$.

It remains to verify condition (iv) of the definition of covering. Suppose that σ_∞ is a strategy for T_∞ and let $x_i \in [\varphi_{\infty,i}(\sigma_\infty)]$ (i.e., x_i is a play according to the strategy $\varphi_{\infty,i}(\sigma_\infty)$). Let $x_{i+1} \in [\varphi_{\infty,i+1}(\sigma_\infty)]$, $x_{i+2} \in [\varphi_{\infty,i+2}(\sigma_\infty)], \ldots$ come from (iv) for the coverings $(T_{i+1}, \pi_{i+1}, \varphi_{i+1})$, $(T_{i+2}, \pi_{i+2}, \varphi_{i+2}), \ldots$ applied to the strategies $\varphi_{j+1}(\varphi_{\infty,j+1}(\sigma_\infty)) = \varphi_{\infty,j}(\sigma_\infty)$ for $j \ge i$, so that $\pi_{j+1}(x_{j+1}) = x_j$ for any $j \ge i$. Since π_{j+1} is the identity on sequences of length $\le 2(k+j)$, it follows that $(x_i, x_{i+1}, x_{i+2}, \ldots)$ converges to a sequence

x_∞ defined by $x_\infty \restriction 2(k+j) = x_j \restriction 2(k+j)$ for $j \geq i$. Now σ_∞ and $\varphi_{\infty,j}(\sigma_\infty)$ agree on sequences of length $\leq 2(k+j)$ so as x_j follows the strategy $\varphi_{\infty,j}(\sigma_\infty)$ for $j \geq i$ we have that x_∞ follows the strategy σ_∞. Finally it is clear that $\pi_{\infty,i}(x_\infty) = x_i$. □

Proof of Lemma 26. We first introduce *quasistrategies.* If T is a nonempty pruned tree then a *quasistrategy for I in T* is a nonempty pruned subtree $\Sigma \subseteq T$ such that if $(a_0, \ldots, a_{2j}) \in \Sigma$ and $(a_0, \ldots, a_{2j}, a_{2j+1}) \in T$ then $(a_0, \ldots, a_{2j}, a_{2j+1}) \in \Sigma$. Similarly we define *quasistrategies for II in T.* If $X \subseteq [T]$ is given we say that a quasistrategy Σ for I is *winning in $G(T, X)$* iff $[\Sigma] \subseteq X$ (similarly for II). Thus a winning strategy for I in $G(T, X)$ can be identified with a winning quasistrategy for I with the additional property that for each $(a_0, \ldots, a_{2j-1}) \in \Sigma$ there is a *unique* a_{2j} such that $(a_0, \ldots, a_{2j}) \in \Sigma$. If X is closed, then there is a *canonical quasistrategy for I in $G(T, X)$*, defined by $\Sigma = \{p \in T \mid p \text{ is not losing for } I\}$. If I has a winning quasistrategy then his canonical quasistrategy is winning.

Fix k, T and X as in the Lemma and let T_X be the subtree of T whose branches are the elements of X. For a tree S, S_u denotes $\{v \mid u * v \in S\}$ and for $Y \subseteq [S]$, Y_u denotes $\{x \mid u * x \in Y\}$. The desired k-covering $(\tilde{T}, \pi, \varphi)$ is described via the following auxiliary game:

(i) Players start with moves $x_0, x_1, \ldots, x_{2k-2}, x_{2k-1}$ such that $(x_0, \ldots, x_{2k-1}) \in T$.
(ii) In his next move, I plays (x_{2k}, Σ_I) where $(x_0, \ldots, x_{2k}) \in T$ and Σ_I is a quasistrategy for I in $T_{(x_0,\ldots,x_{2k})}$ (with the convention that II starts first in games on $T_{(x_0,\ldots,x_{2k})}$).
(iii) II now has two options:
Option 1: II plays (x_{2k+1}, u) where $(x_0, \ldots, x_{2k+1}) \in T$ and u is a sequence of even length such that $u \in T_{(x_0,\ldots,x_{2k+1})}$ and $u \in (\Sigma_I)_{(x_{2k+1})} - (T_X)_{(x_0,\ldots,x_{2k+1})}$. From then on I and II play $x_{2k+2}, x_{2k+3}, \ldots$ so that $(x_0, \ldots, x_j) \in T$ for all j and $u \subseteq (x_{2k+2}, x_{2k+3}, \ldots)$.
Option 2: II plays (x_{2k+1}, Σ_{II}) where $(x_0, \ldots, x_{2k+1}) \in T$ and Σ_{II} is a quasistrategy for II in $(\Sigma_I)_{(x_{2k+1})}$ with $\Sigma_{II} \subseteq (T_X)_{(x_0,\ldots,x_{2k+1})}$. From then on I and II play $x_{2k+2}, x_{2k+3}, \ldots$ so that $(x_{2k+2}, x_{2k+3}, \ldots, x_l) \in \Sigma_{II}$ for all $l \geq 2k+2$.

The map π is given by $\pi(x_0, \ldots, x_{2k-1}, (x_{2k}, *), (x_{2k+1}, *), x_{2k+2}, \ldots, x_l) = (x_0, \ldots, x_l)$. As $\tilde{x} \in \pi^{-1}(X)$ iff $\tilde{x}(2k+1)$ is of the form (x_{2k+1}, Σ_{II}), it follows that $\pi^{-1}(X)$ is clopen.

Finally we must define φ so that given a strategy $\tilde{\sigma}$ on $\tilde{T}$, the strategy $\sigma = \varphi(\tilde{\sigma})$ on T has the property that for any $x \in [\sigma]$ there is $\tilde{x} \in [\tilde{\sigma}]$ with $\pi(\tilde{x}) = x$. We now describe the strategy σ. There are two cases.

Case 1. $\tilde{\sigma}$ is a strategy for I in $\tilde{T}$.

For the first $2k$ moves, σ is just $\tilde{\sigma}$. Then $\tilde{\sigma}$ provides I with (x_{2k}, Σ_I); σ has I play x_{2k}. Then II plays x_{2k+1}. There are two subcases.

Subcase 1. I has a winning strategy in $G((\Sigma_I)_{(x_{2k+1})}, [(\Sigma_I)_{(x_{2k+1})}] - X_{(x_0,\ldots,x_{2k+1})})$.

Then σ requires I to play this winning strategy. After finitely many moves a shortest position u of even length is reached for which $u \notin (T_X)_{(x_0,\ldots,x_{2k+1})}$, say $u = (x_{2k+2}, \ldots, x_{2l-1})$. Then $(x_0, \ldots, x_{2k-1}, (x_{2k}, \Sigma_I), (x_{2k+1}, u), x_{2k+2}, \ldots, x_{2l-1})$ is a legal position in $\tilde{T}$, and σ requires I from then on to play following $\tilde{\sigma}$.

Subcase 2. II has a winning strategy in $G((\Sigma_I)_{(x_{2k+1})}, [(\Sigma_I)_{(x_{2k+1})}] - X_{(x_0,\ldots,x_{2k+1})})$.

Let Σ_{II} be II's canonical quasistrategy in this game. From then on, I plays following $\tilde{\sigma}$, assuming that in the game on $\tilde{T}$, II played (x_{2k+1}, Σ_{II}). I can do this as long as II collaborates and plays so that $(x_{2k+2}, \ldots, x_{2l-1}) \in (\Sigma_{II})_{(x_0,\ldots,x_{2k+1})}$, since then we have legal positions in $\tilde{T}$. But if for some l with $2l - 1 > 2k + 2$, II plays so that $(x_{2k+2}, \ldots, x_{2l-1}) \notin (\Sigma_{II})_{(x_0,\ldots,x_{2k+1})}$, then by definition of Σ_{II}, it follows that I has a winning strategy in $G((\Sigma_I)_{(x_{2k+1},\ldots,x_{2l-1})}, [(\Sigma_I)_{(x_{2k+1},\ldots,x_{2l-1})}] - X_{(x_0,\ldots,x_{2k+1},\ldots,x_{2l-1})})$. Then I continues as in Subcase 1.

Case 2. $\tilde{\sigma}$ is a strategy for II in $\tilde{T}$.

Again for the first $2k$ moves σ is just $\tilde{\sigma}$. Next I plays x_{2k}. Put $U = \{(x_{2k+1}) * u \in T_{(x_0,\ldots,x_{2k})} \mid u$ has even length and there is a quasistrategy Σ_I for I in $T_{(x_0,\ldots,x_{2k})}$ such that $\tilde{\sigma}$ requires II to play (x_{2k+1}, u) when I plays $(x_{2k}, \Sigma_I)\}$. Then

$$\mathcal{U} = \{x \in [T_{(x_0,\ldots,x_{2k})}] \mid x \supseteq (x_{2k+1}) * u \text{ for some } (x_{2k+1}) * u \in U\}$$

is an open set in $[T_{(x_0,\ldots,x_{2k})}]$.

Consider now the game on $T_{(x_0,\ldots,x_{2k})}$ where II plays first, the players produce $x_{2k+1}, x_{2k+2}, \ldots$ and II wins iff $(x_{2k+1}, x_{2k+2}, \ldots)$ belongs to $\mathcal{U}$.

Subcase 1. II has a winning strategy in this game.

Then σ follows this winning strategy until a position $(x_{2k+1}, \ldots, x_{2l-1}) = u$ is reached which belongs to U. Let Σ_I witness $u \in U$. II then follows $\tilde{\sigma}$ after the position $(x_0, \ldots, x_{2k-1}, (x_{2k}, \Sigma_I), (x_{2k+1}, u), x_{2k+2}, \ldots, x_{2l-1})$.

Subcase 2. I has a winning strategy in this game.

Let Σ_I be the canonical quasistrategy for I. Then if I plays (x_{2k}, Σ_I) in the game on $\tilde{T}$, $\tilde{\sigma}$ cannot ask II to play something of the form (x_{2k+1}, u), because then $(x_{2k+1}) * u$ belongs to U and by the rules of $\tilde{T}$, $(x_{2k+1}) * u$ belongs to Σ_I, contradicting the fact that no sequence in Σ_I can belong to U.

So if I plays (x_{2k}, Σ_I) in the game on $\tilde{T}$, $\tilde{\sigma}$ asks II to play (x_{2k+1}, Σ_{II}). So σ has II play x_{2k+1} and then follow $\tilde{\sigma}$ as long as I collaborates so that $(x_{2k+2}, \ldots, x_{2l})$ belongs to Σ_{II}. If for some $l \geq k+1$, I plays x_{2l} with $(x_{2k+2}, \ldots, x_{2l}) \notin \Sigma_{II}$, then since Σ_{II} is a quasistrategy for II in $(\Sigma_I)_{(x_{2k+1})}$ (and therefore I's moves are untrestricted as long as they are in Σ_I) it follows that $(x_{2k+2}, \ldots, x_{2l}) \notin (\Sigma_I)_{(x_{2k+1})}$ and we are back in Subcase 1 again.

This completes the proof of Lemma 26. $\Box$

8.Vorlesung

The Wadge Property

Suppose that A, B are subsets of Baire space. We say that A is *Wadge reducible* to B, written $A \leq_w B$, iff there is a continuous function f such that $x \in A$ iff $f(x) \in B$.

Theorem 28. If A, B are Borel then either $A \leq_w B$ or $B \leq_w \sim A$.

Proof. Consider the *Wadge game* $G_w(A, B)$ where players I and II alternately choose natural numbers $x(0), y(0), x(1), y(1), \ldots$ and II wins iff $(x \in A \leftrightarrow y \in B)$. This is a Borel game and therefore determined. If σ is a winning strategy for II then σ induces a continuous function σ^* from Baire space to Baire space such that $x \in A \leftrightarrow \sigma^*(x) \in B$. If σ is a winning strategy for I then σ induces a continuous function σ^* from Baire space to Baire space such that $y \in B \leftrightarrow \sigma^*(y) \in \sim A$. $\Box$

The *Wadge degree* of A is its equivalence class $[A]_w$ under the equivalence relation $A \equiv_w B$ iff ($A \leq_w B$ and $B \leq_w A$). Theorem 28 says that the ordering of Wadge degrees is almost a linear ordering, in the sense that the only incomparable pairs of Wadge degrees are of the form $[A]_w$, $[\sim A]_w$. It is also possible that $[A]_w = [\sim A]_w$; an example is $A = \{x \mid x(0) \text{ is even}\}$.

Theorem 29. The ordering of Wadge degrees is well-founded.

Proof. If not then there is a sequence $A_0, A_1, \ldots$ with $A_n \not\leq_w A_{n+1}$ and $A_n \not\leq \sim A_{n+1}$ for each n. Let σ_n^0 be a winning strategy for I in $G_w(A_n, A_{n+1})$ and σ_n^1 a winning strategy for I in $G_w(A_n, \sim A_{n+1})$.

Fix $x \in 2^N$. We define plays of games $G_0^x, G_1^x, \ldots$ as follows: In game G_n^x, I applies the strategy $\sigma_n^{x(n)}$, and the k-th move of II is the k-th move of I in G_{n+1}^x. Let $y_n(x)$ be the play of I in game G_n^x. Then

$$(*)\ y_n(x) \notin A_n \leftrightarrow y_{n+1}(x) \in A_{n+1}^{x(n)},$$

where $A_{n+1}^0 = A_{n+1}$, $A_{n+1}^1 = \sim A_{n+1}$. Consider now $X = \{x \in 2^N \mid y_0(x) \in A_0\}$.

Claim. If $x, \bar{x}$ differ at exactly one argument, we have $x \in X$ iff $\bar{x} \notin X$.

Proof of Claim. Suppose that x and $\bar{x}$ differ exactly at argument k. Then $y_{k+1}(x) = y_{k+1}(\bar{x})$, since these depend only on $x(n)$ for $n > k$. So by $(*)$, $y_k(x) \notin A_k \leftrightarrow y_{k+1}(x) \in A_{k+1}^{x(k)} \leftrightarrow y_{k+1}(\bar{x}) \in A_{k+1}^{x(k)} \leftrightarrow y_{k+1}(\bar{x}) \notin A_{k+1}^{\bar{x}(k)} \leftrightarrow y_k(\bar{x}) \in A_k$. Since x, $\bar{x}$ agree at arguments less than k, it follows again by $(*)$ that $y_0(x) \notin A_0 \leftrightarrow y_0(\bar{x}) \in A_0$; i.e., $x \notin X$ iff $\bar{x} \in X$. □ (Claim)

X is Borel and therefore has the Baire property. Therefore either X or $\sim X$ is comeager inside some basic open set $N_s = \{x \mid s \subseteq x\}$, $s \in 2^{<\omega}$. But the homeomorphism that switches the value of x at $n = \text{Length}(s)$ sends $X \cap N_s$ to $\sim X \cap N_s$ and therefore gives two disjoint comeager subsets of N_s, a contradiction. □

Uniformisation of Π_1^1, Σ_2^1 Relations

Let $P \subseteq \mathcal{N} \times \mathcal{N}$ and let F be a function with domain and range contained in $\mathcal{N}$. We say that F *uniformises* P iff the domain of F equals $\{x \mid \exists y\ (x, y) \in P\}$ and for x in the domain of F, $(x, F(x)) \in P$.

Theorem 30. Every Π^1_1 relation can be uniformised by a function F whose graph $\{(x,y) \mid y = F(x)\}$ is Π^1_1.

Proof. First we show how to pick a special element from a given Π^1_1 set P. Let U be a recursive tree on $\omega \times \omega$ such that

$$x \in P \text{ iff } U(x) \text{ is well-founded.}$$

Let $\langle u_n \mid n \in N\rangle$ be a recursive enumeration of Seq $= \omega^{<\omega}$ where $u_0 = \emptyset$ and length $u_n \leq n$ for each n. Define the tree T on $\omega \times \omega_1$ by:

$(s,h) \in T \leftrightarrow \forall m, n < \text{Length}(s)[\text{ If } u_m \supseteq u_n \text{ and } (s \restriction \text{Length}(u_m), s \restriction u_m) \in U \text{ then } h(m) < h(n)]$.

Then $x \in P \leftrightarrow T(x)$ has a branch. Also, if x belongs to P then $T(x)$ has a *least* branch, i.e., a branch g_x with the property that $g_x(n) \leq f(n)$ whenever f is a branch through $T(x)$: g_x is defined by $g_x(n) = \text{Rank}(u_n)$ in $T(x)$ if $u_n \in U(x)$; 0 otherwise.

Now define $P_0 = P$, $P_{2n+1} = \{x \in P_{2n} \mid g_x(n) \text{ is least}\}$, $P_{2n+2} = \{x \in P_{2n+1} \mid x(n) \text{ is least}\}$. Then the intersection of the P_n's has a unique element a.

Claim. $\{a\}$ is Π^1_1.

Proof of Claim. We have:

$x \in P_1 \leftrightarrow$
$x \in P \wedge \forall y(\text{Rank}(u_0) \text{ in } U(x) \leq \text{Rank}(u_0) \text{ in } U(y))$
$x \in P_2 \leftrightarrow$
$x \in P_1 \wedge \forall y((\text{Rank}(u_0) \text{ in } U(x) < \text{Rank}(u_0) \text{ in } U(y)) \vee (\text{Rank}(u_0) \text{ in } U(x) = \text{Rank}(u_0) \text{ in } U(y) \wedge x(0) \leq y(0)))$
$x \in P_3 \leftrightarrow$
$x \in P_2 \wedge \forall y((\text{Rank}(u_0) \text{ in } U(x) < \text{Rank}(u_0) \text{ in } U(y)) \vee (\text{Rank}(u_0) \text{ in } U(x) = \text{Rank}(u_0) \text{ in } U(y) \wedge x(0) < y(0)) \vee (\text{Rank}(u_0) \text{ in } U(x) = \text{Rank}(u_0) \text{ in } U(y) \wedge x(0) = y(0) \wedge \text{Rank}(u_1) \text{ in } U(y) \leq \text{Rank}(u_1) \text{ in } U(x))$
Etc.

The above are Π^1_1 definitions, using the equivalences:

Rank(u) in $U(x) \leq$ Rank(u) in $U(y)$ iff
There exists an order-preserving function from $(U(x))_u$ into $(U(y))_u$

Rank(u) in $U(x) <$ Rank(u) in $U(y)$ iff
There exists an order-preserving function from $(U(x))_u$ into $(U(y))_{u^*}$ for some u^* properly extending u.

Now we prove the Uniformisation Theorem. Let P be a Π^1_1 subset of $\mathcal{N} \times \mathcal{N}$. Let U be a tree in $\omega \times \omega \times \omega$ such that

$$(x, y) \in P \text{ iff } U(x, y) \text{ is well-founded.}$$

For each $x \in \mathcal{N}$ let P^x be the Π^1_1 set $\{y \mid (x, y) \in P\}$ and define $P^x = P^x_0 \supseteq P^x_1 \supseteq \cdots$ exactly as we defined P_n, $n \in N$, with P, $U(y)$ replaced by P^x, $U(x, y)$. Let Q^x the intersection of the P^x_n's. Then Q^x has a single element, which we denote by $F(x)$. The Π^1_1 expression we gave above applies to Q^x, and shows that the graph of F, $\{(x, y) \mid y \in Q^x\}$, is Π^1_1, as desired. □

Corollary 31. Every Σ^1_2 binary relation can be uniformised by a Σ^1_2 function.

Proof. Suppose that $P \subseteq \mathcal{N} \times \mathcal{N}$ is Σ^1_2; then there is a Π^1_1 relation $Q \subseteq \mathcal{N}^3$ such that P is the projection of Q; i.e., $P = \{(x, y) \mid \exists z((x, y, z) \in Q)\}$. Apply Π^1_1 uniformisation to get a Π^1_1 function $F : \mathcal{N} \to \mathcal{N}^2$ with domain $\{x \mid \exists y, z((x, y, z) \in Q)\}$ such that for x in this set, $F(x) = (F(x)_0, F(x)_1)$ where $(x, F(x)_0, F(x)_1) \in Q$. Then F^* uniformises P, where $F^*(x) = F(x)_0$. And the graph of F^* is Σ^1_2 since

$$y = F^*(x) \text{ iff } \exists z \ (y, z) = F(x).$$

Thus F^* is the desired Σ^1_2 uniformising function. □.

9.Vorlesung

Part 2: The Constructible Universe

We showed that the regularity properties measurability, Baire property, perfect set property and Ramsey property hold for $\mathbf{\Sigma}^1_1$ sets, provably in ZFC. However this does not extend to $\mathbf{\Delta}^1_2$ sets:

Theorem 32. In L, there is a wellordering $<$ of the reals of length ω_1 such that the relation $\{(x, y) \mid y$ codes the set of $<$-predecessors of $x\}$ is Δ^1_2. (Such a wellordering is called a *good* Δ^1_2 *wellordering*.)

Proof. $<$ is just the usual canonical wellordering of L, restricted to the reals. We have:

y codes the set of $<$-predecessors of x iff
$\exists z$(z codes an $L_\alpha \models \mathrm{ZFC}^-$ containing x, and y codes the set of $<_{L_\alpha}$-predecessors of x in L_α) iff
$\forall z$(If z codes an $L_\alpha \models \mathrm{ZFC}^-$ containing x then y codes the set of $<_{L_\alpha}$-predecessors of x in L_α)

Thus we have a good Δ^1_2 wellordering. □

Corollary 33. In L, there are Δ^1_2 sets which are not measurable and do not have the Baire, perfect set or Ramsey properties.

Proof. List the reals using a good Δ^1_2 wellordering as $\langle x_\alpha \mid \alpha < \omega_1 \rangle$ and build X by inductively deciding $x_\alpha \in X$, diagonalising over reals which might witness one of the four regularity properties. □

We proved determinacy for Borel sets in ZFC. This does not extend to Π^1_1 sets: A *Turing set* of reals is a set of reals X with the property: $x \in X$, $x =_T y \to y \in X$, where $=_T$ denotes Turing equivalence. A *cone* is a Turing set of the form $\{y \mid x \leq_T y\}$ for some x.

Lemma 34. (Martin's Lemma) Suppose that X is a determined Turing set. Then either X or $\sim X$ contains a cone.

Proof. If I has a winning strategy σ in the game for X then for any possible play y for player II, $\sigma(y)$ belongs to X, where $\sigma(y)$ is the result of the game where II plays y and I follows his strategy σ. Thus if $\sigma \leq_T y$, it follows that $\sigma(y) =_T y$ belongs to X. If II has the winning strategy τ, then we get $\tau(x) =_T x \in \sim X$ for all $x \geq_T \tau$. □

Corollary 35. In L, Π^1_1 determinacy fails.

Proof. It suffices to find a Turing set X such that neither X nor $\sim X$ contains a cone. Consider $X = \{x \mid x \text{ is the first-order theory of some } L_\alpha \models \mathrm{ZFC}^-\}$. Clearly $\sim X$ cannot contain a cone as every real is recursive in some element of X. And if x belongs to X, then x'(= the Turing jump of x) does not, so $\sim X$ does not contain a cone. □

We showed that for Borel sets A and B either $A \leq_W B$ or $B \leq_W \sim A$. The same property for Π^1_1 would imply that any two non-Borel Π^1_1 sets have the same Wadge degree. The latter fails in L:

Theorem 36. In L, not all non-Borel Π^1_1 sets have the same Wadge degree.

Proof. Let $\langle \alpha_i \mid 0 \leq i < \omega_1 \rangle$ be the increasing enumeration of the countable ordinals α such that L_α is admissible. Let WO be the set of x such that x codes an ordinal. WO is a complete Π^1_1 set, and therefore each Π^1_1 set is Wadge reducible to WO. Let X be the set of x such that x codes an ordinal between α_i and α_{i+1} for an *even* i. X is Π^1_1 and not Borel. Now suppose that WO were Wadge reducible to X and let a be a real coding a continuous function f such that $x \in \mathrm{WO} \leftrightarrow f(x) \in X$. Choose an odd i such that a belongs to L_{α_i} and α_i is countable in $L_{\alpha_{i+1}} = M$. Then $x \in \mathrm{WO} \cap M \leftrightarrow f(x) \in X \cap M$, and $X \cap M$ is $\Delta_1(M)$. So $\mathrm{WO} \cap M$ is $\Delta_1(M)$, which is impossible since M is the least admissible set containing some real. $\square$

The Uniformisation Property does extend in L beyond Σ^1_2:

Theorem 37. In L: Each Σ^1_n binary relation can be uniformised by a Σ^1_n function, for each $n \geq 2$.

Proof. Let $<$ be a good Δ^1_2 wellordering of the reals. If $R(x,y) \leftrightarrow \exists z\, S(x,y,z)$ with S Π^1_{n-1}, $n \geq 2$, then define:

$$S^*(x,y,z) \leftrightarrow (S(x,y,z) \wedge \forall (y',z') < (y,z)(\sim S(x,y',z'))).$$

Then S^* is Σ^1_n, as $<$ is a good Δ^1_2 wellordering and $n \geq 2$. So R is uniformised by $R^*(x,y) \leftrightarrow \exists z\, S^*(x,y,z)$. $\square$

10.Vorlesung

Part 3: Forcing Extensions of L

Is it possible to have regularity properties beyond Σ^1_1 in a forcing extension of L? Solovay constructed, using an L-inaccessible, a generic extension of L in which *all* projective sets are regular.

Theorem 38. (Solovay) Assume that there is an L-inaccessible. Then there is a set-generic extension of L in which all projective sets are measurable and have the Baire, perfect set and Ramsey properties.

Proof. Let κ be inaccessible in L and let P be the Lévy collapse to make κ equal to ω_1: A condition in P is a finite function $p : F \to \kappa$, where $F \subseteq \omega \times \kappa$ and $p(n, \alpha) < \alpha$ for each $(n, \alpha) \in \mathrm{Dom}\, p$. P makes each $\alpha < \kappa$ countable and has the κ-cc; it follows that in a P-generic extension $L[G]$, κ is ω_1. Also P is homogeneous: If p, q are elements of P then there is an automorphism of P which sends p to a condition compatible with q.

We show that in $L[G]$, each projective set of reals X is measurable. Suppose that $x \in X \leftrightarrow L[G] \vDash \varphi(x, a)$ where φ is a formula of second-order arithmetic and $a \in L[G]$ is a real parameter. We may choose $\alpha < \kappa$ such that a belongs to $L[G_\alpha]$, where $G_\alpha = G \cap L_\alpha$ is $P_\alpha = P \cap L_\alpha$-generic over L. Then $L[G]$ is a P-generic extension of $L[G_\alpha]$, since $P \approx P_\alpha \times P$.

Almost every real (in the sense of Lebesgue measure) is random over $L[G_\alpha]$ and therefore it suffices to show that X agrees with some Borel set on the reals which are random over $L[G_\alpha]$. Suppose that x is such a real; then $L[G] = L[G_\alpha][G_x][H_x]$, where $P \approx P_\alpha * Q * R$, G_x is the generic corresponding to x for random real forcing Q over $L[G_\alpha]$ and H_x is generic over $L[G_\alpha][G_x]$ for some forcing R. It is not difficult to show that R is equivalent to P, and so we can assume $R = P$. Thus by the homogeneity of P:

$x \in X$ iff
$L[G_\alpha][G_x][H_x] \vDash \varphi(x, a)$ iff
$L[G_\alpha][G_x] \vDash (P \Vdash \varphi(x, a))$ iff
For some $B \in G_x$, $B \Vdash_Q (P \Vdash \varphi(\dot{x}, a))$ iff
x belongs to the union of the $B^{L[G]}$, $B \in Q$ such that $B \Vdash_Q (P \Vdash \varphi(\dot{x}, a))$.

As $L[G_\alpha]$ has only countably many reals, it follows that Q is countable and therefore this last condition on x is a Borel condition. So X is measurable.

The same argument works for the Baire property, using the Cohen algebra in place of the random algebra.

If X is an uncountable projective set of reals in $L[G]$ then again we can write $x \in X$ iff $L[G_\alpha][G_x] \vDash (P \Vdash \psi(x, a))$, where a belongs to $L[G_\alpha]$, $\alpha < \kappa$ and G_x is generic over $L[G_\alpha]$ for some forcing Q_x. As X is uncountable, we can choose $x \in X$ to not belong to $L[G_\alpha]$ and for this x can assume that $Q_x \Vdash (\dot{x} \notin L[G_\alpha]$ and $(P \Vdash \psi(\dot{x}, a)))$. As the power set of Q_x in $L[G_\alpha]$ is countable in $L[G]$, we can build a perfect set of y's with corresponding Q_x-generics G_y such that $L[G_\alpha][G_y] \vDash (P \Vdash \psi(\dot{y}, a))$. Thus y belongs to X for each such y, showing that X has a perfect subset.

For the Ramsey property, we again write $x \in X$ iff $L[G_\alpha][G_x] \vDash (P \Vdash$

$\psi(x,a))$. There is a Mathias condition $(\emptyset, A)$ which forces over $L[G_\alpha]$ that $L[G_\alpha][G_{\dot{x}}] \models (P \Vdash \psi(\dot{x},a))$ where $\dot{x}$ denotes the Mathias generic real (or the same with ψ replaced by $\sim \psi$; assume the former). Then if x is Mathias generic over $L[G_\alpha]$, $x \subseteq A$, it follows that $L[G_\alpha][G_y] \models (P \Vdash \psi(y,a))$ for all infinite $y \subseteq x$, as any such y is also Mathias generic. Thus $y \in X$ for such y and x is homogeneous for X. $\Box$

Shelah showed that the use of an L-inaccessible as above is necessary to obtain the measurability of Σ^1_3 sets. (It is however not necessary for the measurability of Δ^1_2 sets.) He also showed that an L-inaccessible is not needed to obtain the Baire property for all projective sets. It is not difficult to show that an L-inaccessible is necessary to obtain the perfect set property for Π^1_1 sets. The situation with regard to the Ramsey property is still open.

11.Vorlesung

We showed that both Π^1_1 determinacy and the Wadge property for Π^1_1 sets fail in L. These properties also fail in all set-generic extensions of L, by similar arguments. Are these properties in fact refutable in ZFC? We will see that they are implied by the existence of large cardinals, which makes this rather unlikely.

We showed that projective relations can be projectively uniformised in L. This can fail in a set-generic extension of L:

Theorem 39. There is a forcing extension of L in which some Π^1_2 binary relation has no projective uniformisation.

Proof. Add ω_1 Cohen reals to L. In the extension, we have: For each real x, there is a real y which is Cohen over $L[x]$. Consider now the relation $R(x,y) \leftrightarrow y$ is Cohen over $L[x]$. This is a total Π^1_2 relation. Suppose that f were a projective function with parameter a which uniformised R. Then our model can be written as $L[b,G]$ where a belongs to $L[b]$ and G is generic for adding ω_1 Cohen reals over $L[b]$. But by homogeneity, $f(b)$ must belong to $L[b]$, which contradicts the fact that $f(b)$ is Cohen over $L[b]$. $\Box$

Part 4: Sharps

Assume that every real has a #. We will show that the regularity, determinacy, Wadge and uniformisation properties that we established in ZFC can be extended to one more level of the projective hierarchy.

Theorem 40. Suppose that ω_1 is inaccessible to reals, i.e., $L[a]$ contains only countably many reals for each real a. Then every $\mathbf{\Sigma}^1_2$ set is measurable and has the Baire, perfect set and Ramsey properties.

Proof. By Theorems 10, 12 and 20, it suffices to show that each $\mathbf{\Sigma}^1_2$ set A is the projection of a tree T on $\omega \times \kappa$ for some κ, where T belongs to a transitive ZFC$^-$ model containing only countably many reals. Let U be a tree on $\omega \times \omega \times \omega$ such that

$$x \in A \text{ iff } \exists y\ U(x, y) \text{ is well-founded.}$$

Let $\langle u_n \mid n \in N\rangle$ be a recursive enumeration of Seq such that $u_0 = \emptyset$ and $\text{Length}(u_n) \leq n$ for each n. Define the tree T on $\omega \times \omega \times \omega_1$ by:

$(u, v, h) \in T \leftrightarrow \forall m, n < \text{Length}(u) = \text{Length}(v)$ [If $u_m \supseteq u_n$ and $(u \restriction \text{Length}(u_m), v \restriction \text{Length}(u_m), u_m) \in U$ then $h(m) < h(n)$].

Then $x \in A \leftrightarrow T(x)$ has a branch. T can be identified with a tree on $\omega \times \omega_1$ and belongs to $L[a]$, where a is a real coding the tree U. It follows that A is the projection of a tree which belongs to a transitive ZFC$^-$ model all of whose reals are constructible from a. By hypothesis this model contains only countably many reals. □

Theorem 41. Suppose that $x^\#$ exists for every real x. Then every $\mathbf{\Sigma}^1_1$ set is determined.

Proof. Let A be $\mathbf{\Sigma}^1_1$; we want to show that the game G_A (played on ω) is determined. Let T be a tree on Seq^2 suich that

$$x \in A \leftrightarrow T(x) \text{ has a branch.}$$

Let $<^*$ be the Kleene-Brouwer ordering on Seq: $s <^* t$ iff either s properly extends t or $s(n) < t(n)$ where n is least so that $s(n) \neq t(n)$. For $s \in Seq$ let $T_s = \{t \mid (u, t) \in T$ for some $u \subseteq s\}$. Also let $t_0, t_1, \ldots$ be an enumeration of Seq and if s has length $2n$, let K_s be the set $\{t_0, \ldots, t_{n-1}\} \cap T_s$, $k_s =$ Card (K_s).

We define an auxiliary game G^* as follows: Player I plays natural numbers $a_0, a_1, \ldots$ and player II plays pairs $(b_0, h_0), (b_1, h_1), \ldots$ where the b_i are natural numbers and for each n, h_n is an order-preserving function from $(K_{(a_0, b_0, \ldots, a_n, b_n)}, <^*)$ into ω_1 and $i < j \rightarrow h_i \subseteq h_j$. If Player II can follow these rules for infinitely many moves then he wins; otherwise player I wins. Clearly G^* is determined, as it is a closed game.

If II has a winning strategy in G^* then of course he also has a winning strategy in G_A.

Let σ^* be the canonical winning strategy for I in G^*. We shall construct a winning strategy σ for I in G_A. Suppose that $s^* = (a_0, (b_0, h_0), \ldots, a_n, (b_n, h_n))$ is a play in G^*, and let E be the range of h_n. Then the h_i's are uniquely determined by E, as h_n is the unique order-preserving function from the ordering $(K_{(a_0,b_0,\ldots,a_n,b_n)}, <^*)$ into ω_1, and the h_i are the appropriate restrictions of h_n. So s^* is uniquely determined by $s = (a_0, b_0, \ldots, a_n, b_n)$ and E. For each such s let F_s be the function $F_s(E) = \sigma^*(s^*)$, where s^* is uniquely determined as above by E. Then as σ^* is constructible from a real and this real has a $\#$, it follows that there is an uncountable $H \subseteq \omega_1$ which is homogeneous for F_s for every s, i.e., for all s, $F_s(E)$ is the same for all $E \subseteq H$ of cardinality k_s. Define the strategy σ by: $\sigma(s) = F_s(E)$, for any $E \subseteq H$ of cardinality k_s.

We claim that this is a winning strategy for I. If not, let x be a result of playing this strategy with $x \notin A$. Then there is an order-preserving embedding h of $(T(x), <^*)$ into H, as H is uncountable and $(T(x), <^*)$ is a countable well-ordering. But consider now the play of G^* where II plays restrictions of h to the appropriate K_s's. This is a play of G^* according to the strategy σ^*, in contradiction to the fact that σ^* is a winning strategy for I in G^*. □

Martin showed that the existence of $\#$'s is also sufficient for the determinacy Boolean combinations of $\mathbf{\Pi}^1_1$ sets. Thus from this assumption the Wadge property holds for $\mathbf{\Pi}^1_1$ sets.

12.Vorlesung

Theorem 42. Assume that every real has a $\#$ and $A(x,y)$ is $\mathbf{\Pi}^1_2$. Then A can be uniformised by a $\mathbf{\Pi}^1_3$ function.

Proof. First we show how to choose a canonical element from a nonempty $\mathbf{\Pi}^1_2$ set A. It will be convenient to work with the space 2^ω instead of Baire space. Write $x \in A \leftrightarrow \forall y\, T(x,y)$ has a branch, where T is a tree on $2 \times 2 \times \omega$. Then $x \in A \leftrightarrow U(x)$ is well-founded, where a node in U is of the form (s,t,f) with s and t finite $0,1$-sequences of the same length, f an order-preserving function from $(T(s,t), <^*)$ into ω_1 and $<^*$ the Kleene-Brouwer order. Let L^* denote $\bigcup\{L[x] \mid x \text{ a real}\}$.

Now suppose that s, t are finite $0,1$-sequences of the same length and $F, G \in L^*$ are functions from $U(s,t)$ into Ord. We write $F \leq^* G$ iff for some CUB $C \subseteq \omega_1$ in L^*, $F(f) \leq G(f)$ for all $f \in U(s,t)$ with Range $(f) \subseteq C$. For any F and G, either $F \leq^* G$ or $G \leq^* F$, since we have assumed #'s and F, G are constructible from reals. Therefore $\leq^*$ gives a wellordering if we identify F with G when $F =^* G$.

For $x \in A$ let F^x be the canonical ranking function on $U(x)$. Then F^x belongs to L^* since it is constructible from T and x. Fix n and let $t_1, t_2, \ldots, t_{2^n}$ list the $0,1$-sequences of length n in lexicographical order. Then set $\alpha_n^x = \langle \beta_1, \ldots, \beta_{2^n} \rangle$, where β_i is the rank of $F^x \restriction U(x \restriction n, t_i)$ in $\leq^*$.

We can now define a canonical element of A. Choose x_1 to minimize $\alpha_1^x, x(0)$ (in the lexicographical ordering of finite sequences of ordinals) for $x \in A$, and set $n_0 = x_1(0)$. Then choose x_2 to minimize $\alpha_2^x, x(1)$ for $x \in A$ which minimize $\alpha_1^x, x(0)$, and set $n_1 = x_2(1)$. Continue in this way, producing a real $x^* = \langle n_0, n_1, \ldots \rangle$.

We claim that x^* belongs to A. Indeed, for each n choose $F^n(t)$ with domain $U(x^* \restriction n, t)$, t of length n, whose ranks realize $\alpha_n^{x_n}$. Then for some CUB C, the $F^n(t)$ restricted to elements of $U(x^* \restriction n, t)$ with range in C cohere. It follows that $U(x^*)$ is well-founded, and therefore x^* belongs to A.

We show that $\{x^*\}$ is Π_3^1 in a real coding T. Indeed:

$y = x^*$ iff $y \in A$ and
$\forall z, k$ (If z belongs to A, $z \restriction k = y \restriction k$ and $\alpha_l^z = \alpha_l^y$ for $l < k$ then α_k^z is greater than α_k^y or $z(k) \geq y(k)$).

As α_l^y and α_l^z can be compared in $L[(y,z)^\#]$, this gives a Π_3^1 definition. And as in the proof of Π_1^1 uniformisation, the above relativises uniformly to give that a binary $\mathbf{\Pi}_2^1$ relation can be $\mathbf{\Pi}_3^1$ uniformised. $\square$

Corollary 43. Assume #'s for reals. Then every binary $\mathbf{\Sigma}_3^1$ relation can be $\mathbf{\Sigma}_4^1$ uniformised.

We cannot expect to extend these results to higher projective levels only under the assumption of #'s. There is a smallest inner model closed under #, and this model has a Δ_3^1 wellordering of its reals. Thus in this model regularity must fail for Δ_3^1 sets. And in this model there is a Π_2^1 set A such that both A and the complement of A are cofinal in the Turing degrees, showing that Π_2^1 determinacy must fail. (In fact Δ_2^1 determinacy fails.) The

Wadge property also fails for Π^1_2 sets. And in the forcing extension obtained by adding ω_1 Cohen reals, there is a Π^1_3 binary relation which cannot be projectively uniformised.

To extend descriptive set theory, one must a large cardinal hypothesis which violate the existence of a Δ^1_3 wellordering of the reals. This hypothesis is: There is a Woodin cardinal κ such that $V_\kappa^\#$ exists. With this hypothesis, one obtains Σ^1_3 regularity, Π^1_2 determinacy and the uniformisation of Π^1_3 binary relations by Π^1_3 functions. More Woodin cardinals carry the theory to the higher levels of the projective hierarchy. Thus with infinitely many Woodin cardinals, one has a very satisfying theory of the projective sets.