Internal and External Consistency, Wintersemester 2005

1.-6.Vorlesungen

There are two standard ways to establish consistency in set theory. One is to prove consistency using *inner models*, in the way that Gödel proved the consistency of GCH using the inner model L. The other is to prove consistency using *outer models*, in the way that Cohen proved the consistency of the negation of CH by enlarging L to a forcing extension $L[G]$.

But we can demand more from the outer model method, and we illustrate this by examining Easton's strengthening of Cohen's result:

Theorem 1 *(Easton's Theorem) There is a forcing extension $L[G]$ of L in which GCH fails at every regular cardinal.*

Assume that the universe V of all sets is rich in the sense that it contains inner models with large cardinals. Then what is the relationship between Easton's model $L[G]$ and V? In particular, are these models *compatible*, in the sense that they are inner models of a common third model? If not, then the failure of GCH at every regular cardinal is consistent only in a weak sense, as it can only hold in universes which are incompatible with the universe of all sets. Ideally, we would like $L[G]$ to not only be compatible with V, but to be an inner model of V.

We say that a statement is *internally consistent (relative to large cardinals)* iff it holds in some inner model (under the assumption that there are inner models with large cardinals). By specifying what large cardinals are required, we obtain a new type of consistency result. Let Con (ZFC + φ) stand for "ZFC + φ is consistent" and Icon(ZFC + φ) stand for "there is an inner model of ZFC + φ". A typical consistency result takes the form

$$\text{Con (ZFC + LC)} \rightarrow \text{Con (ZFC} + \varphi)$$

where LC denotes some large cardinal axiom. An *internal* consistency result takes the form

$$\text{Icon(ZFC + LC)} \rightarrow \text{Icon(ZFC} + \varphi).$$

Thus a statement φ is internally consistent relative to large cardinals iff Icon(ZFC + φ) follows from Icon(ZFC + LC) for some large cardinal axiom LC.

A statement can be consistent without being internally consistent relative to large cardinals. . An example is the statement that there are no transitive models of ZFC, which fails in any inner model (assuming there are inner models with inaccessible cardinals). Another example is:

For each infinite regular cardinal κ there is a nonconstructible subset of κ whose proper initial segments are constructible.

This can be forced over L, but does not hold in any inner model, assuming the existence of $0^{\#}$.

If the consistency of a statement without parameters is shown using set forcing, then it is usually easy to prove its internal consistency relative to large cardinals. (Some examples are mentioned below.) But this is not the case for statements that contain uncountable parameters or for statements whose consistency is shown through the use of class forcing. In these latter cases, questions of internal consistency and of internal consistency strength can be quite interesting, as we shall see.

Large Cardinals and L-like Universes

The second part of this course addresses the following question: Can we simultaneously have the advantages of both the axiom of constructibility and the existence of large cardinals? Unfortunately even rather modest large cardinal hypotheses, such as the existence of a measurable cardinal, refute $V = L$. We can however hope for the following compromise:

V is an "L-like" model containing large cardinals.

In this article we explore the possibilities for this assertion, for various notions of "L-like" and for various types of large cardinals.

There are two approaches to this problem. The first approach is via the

Inner model program. Show that any universe with large cardinals has an L-like inner model with large cardinals.

The inner model program, through use of fine structure theory and the theory of iterated ultrapowers, has succeeded in producing very L-like inner models containing many Woodin cardinals.

An alternative approach is given by the

Outer model program. Show that any universe with large cardinals has an L-like outer model with large cardinals.

We will show that L-like outer models with extremely large cardinals can be obtained using the method of iterated forcing.

Part One. Internal Consistency

We turn now to a detailed study of internal consistency, beginning with Easton's theorem.

Let Reg denote the class of infinite regular cardinals and Card the class of all infinite cardinals. An *Easton function* is a class function $F : \text{Reg} \to \text{Card}$ such that:

For all $\kappa \leq \lambda$ in Reg: $F(\kappa) \leq F(\lambda)$.
For all $\kappa \in \text{Reg}$: $\text{cof}\ (F(\kappa)) > \kappa$.

Easton showed that if F is an Easton function in L, then there is a cofinality-preserving class forcing extension $L[G]$ of L in which $2^\kappa = F(\kappa)$ for all regular κ. We say that the model $L[G]$ *realises* the Easton function F.

Which Easton functions in L can be realised in an inner model? The following results were obtained jointly with Pavel Ondrejović.

Theorem 2 *Suppose that $0^\#$ exists and F is an Easton function in L which is L-definable using parameters which are countable in V. Then there exists an inner model with the same cofinalities as L in which $2^\kappa = F(\kappa)$ for each infinite regular κ.*

Corollary 3 *The statement*

$$2^\kappa = \kappa^{++} \text{ for all infinite regular } \kappa$$

is internally consistent relative to $0^\#$.

Theorem 4 *Assume that $0^\#$ exists, κ is a regular uncountable cardinal and $\alpha < \kappa^+$. Then there is an inner model with the same cofinalities as L in which GCH holds below κ and $2^\kappa > \alpha$.*

Corollary 5 *Assume that $0^\#$ exists. Then there is an inner model with the same cofinalities as L in which the GCH holds below $\aleph_1^V$ but fails at $\aleph_1^V$.*

Proof of Corollary 3. We consider the following reverse Easton iteration (defined in L): P_0 is trivial. P_λ, λ limit, is the direct limit of the P_i, $i < \lambda$, if λ is regular, and is the inverse limit of the P_i, $i < \lambda$, if λ is singular. $P_{\alpha+1} = P_\alpha * Q_\alpha$ for every α, where Q_α is trivial unless α is a limit cardinal, in which case:

$Q_\alpha = \prod_{n\in\omega} \text{Add}\ (\alpha^{+n}, \alpha^{+(n+2)})$, if α is regular;
$Q_\alpha = \prod_{0<n\in\omega} \text{Add}\ (\alpha^{+n}, \alpha^{+(n+2)})$, if α is singular.

The forcing Add (β, γ) (for regular β) adds γ subsets of β: Conditions are functions $p : d \to 2$, d a subset of $\beta \times \gamma$ of size less than β, ordered by extension.

For any regular α, P factors as $P(\leq \alpha) * P(> \alpha)$ where $P(\leq \alpha)$ is α^+-cc and $P(\leq \alpha)$ forces that $P(> \alpha)$ is α^+-closed. It follows that cofinalities are preserved by P. Also P forces that $2^\alpha = \alpha^{++}$ for every regular α.

We show that if $0^\#$ exists, then there is a P-generic over L. By induction on $i \in I =$ the Silver indiscernibles we construct a $P(\leq i)$-generic $G(\leq i)$. To facilitate limit stages of the construction, we maintain the following property:

$(*)$ $i < j \to G(\leq i)$ embeds into $G(\leq j)$ in the sense that $\pi_{ij}[G(\leq i)] \subseteq G(\leq j)$.

In $(*)$, π_{ij} is the "shift map" defined as follows. Let $\langle i_\alpha \mid \alpha \in \text{Ord}\rangle$ be the increasing enumeration of I. Then $\pi_{i_\alpha i_\beta}$ fixes indiscernibles less than i_α and sends $i_{\alpha+\gamma}$ to $i_{\beta+\gamma}$. This definition of π_{ij} on indiscernibles lifts uniquely to an elementary embedding $\pi_{ij} : L \to L$.

Lemma 6 *Suppose that j is a limit indiscernible and $G(\leq i)$ is defined for $i \in I \cap j$ so as to satisfy $(*)$. Define:*
$G(\leq j) = \bigcup_{i\in I\cap j} \pi_{ij}[G(\leq i)]$.
Then $G(\leq j)$ is $P(\leq j)$-generic over L.

Proof. If $D \in L$ is open dense on $P(\leq j)$ then write D as $t(\vec{i}, j, \vec{\infty})$ where t is a Skolem term for L and $\vec{i} < j < \vec{\infty}$ are indiscernibles. Let k be an indiscernible such that $\vec{i} < k < j$ and set $\bar{D} = t(\vec{i}, k, \vec{\infty})$. Then $\bar{D}$ is open dense on $P(\leq k)$ and is therefore met by some condition p in $G(\leq k)$. But then $\pi_{kj}(p)$ belongs to $G(\leq j) \cap D$. □

Now suppose that $G(\leq i)$ is defined and we wish to define $G(\leq i^*)$, where i^* is the least indiscernible greater than i, obeying property $(*)$. First we make a preliminary choice for $G(\leq i^*)$. Note that $P(\leq i^*)$ factors as $P(\leq i) * P(i, i^*]$ where $P(\leq i)$ forces that $P(i, i^*]$ is i^+-closed.

Lemma 7 *Let $\mathcal{D}$ be the collection of open dense subsets of $P(i, i^*]$ which belong to $L[G(\leq i)]$. Then $\mathcal{D}$ is the union of sets $\mathcal{D}_n$, $n \in \omega$, which belong to $L[G(\leq i)]$ and have cardinality at most i in that model.*

Proof. Every $D \in \mathcal{D}$ is named by a term in $L_{(i^*)^{+L}} \subseteq L_{i^{**}}$. So it suffices to show that $L_{i^{**}}$ can be written as the countable union of sets which belong to L and have size at most i in L. Any element of $L_{i^{**}}$ is of the form $t(\alpha, i, i^*, \vec{\infty})$ where t is a Skolem term for L, α is an ordinal less than i and $\vec{\infty}$ is a finite initial segment of $I - (i^* + 1)$. For a fixed t, the collection of $t(\alpha, i, i^*, \vec{\infty})$, $\alpha < i$, is a constructible set of L-cardinality at most i. As there are only countably many t's, we are done. □

By Lemma 7 we can build a $P(i, i^*]$-generic $G'(i, i^*]$ over $L[G(\leq i)]$ in ω steps, using the i^+-closure of $P(i, i^*]$ to meet all open dense sets in $\mathcal{D}_n$ at step n. This yields a generic $G'(\leq i^*) = G(\leq i) * G'(i, i^*]$ for $P(\leq i^*)$.

However we must also ensure property $(*)$. Let π denote π_{i,i^*}. As π is the identity on L_i we do have $G(< i) = \pi[G(< i)] \subseteq G(< i^*)$; however we must modify $G'(i^*)$. Recall that Q_{i^*} is the forcing $\prod_{n \in \omega} \text{Add}\ ((i^*)^{+n}, (i^*)^{+(n+2)})$. Therefore $G'(i^*)$ can be written as $\prod_{n \in \omega} G'(i^*)_n$, where $G'(i^*)_n$ is generic for $\text{Add}\ ((i^*)^{+n}, (i^*)^{+(n+2)})$. We show how to modify $G'(i^*)_0$ so as to guarantee $(*)$; the modification of the entire $G'(i^*)$ is similar.

Each condition p' in $G'(i^*)_0$ is a function from a subset of $i^* \times (i^*)^{++}$ into 2. Its modification p has the same domain as p' and is defined by:

$p(\alpha, \beta) = p'(\alpha, \beta)$ if $\alpha \geq i$ or $\beta \notin \text{Range}\ (\pi)$
$p(\alpha, \beta) = G(i)_0(\alpha, \bar{\beta})$ if $\alpha < i$ and $\pi(\bar{\beta}) = \beta$.

Lemma 8 *(1) If p' is a condition in $G'(i^*)_0$ then so is its above modification p.*
(2) Let $G(i^*)_0$ consist of all modifications p of conditions p' in $G'(i^*)_0$. Then $G(i^*)_0$ is Add $(i^*, (i^*)^{++})$-generic over $L[G(< i^*)]$.

Proof. (1) We first show that if X is a constructible set of L-cardinality i^* and $\bar{X} = \pi^{-1}(X)$ then $\pi \restriction \bar{X}$ is constructible.

Suppose that X is an element of Range (π). Then Range (π) also contains a bijection f between X and i^*, and $X \cap \text{Range } \pi = f^{-1}[i^* \cap \text{Range } (\pi)] = f^{-1}[i]$. Now $\pi^{-1}(f) = \bar{f}$ is a bijection between $\pi^{-1}(X) = \bar{X}$ and i, and for $x \in \bar{X}$ we have: $\bar{f}(x) = \pi(\bar{f}(x)) = \pi(\bar{f})(\pi(x)) = f(\pi(x))$, so $\pi(x) = f^{-1} \circ \bar{f}(x)$. Thus $\pi \restriction \bar{X}$ is the composition of two constructible functions.

If X is not an element of Range (π), then write X as $t(\alpha, i, i^*, \vec{j})$ where $\alpha < i$ and $\vec{j}$ are indiscernibles greater than i^*. Let Y be the union of all $t(\alpha, \beta, i^*, \vec{j})$ of L-cardinality i^*, where $\alpha < \beta < i^*$. Then Y contains X as a subset, has L-cardinality i^* and is an element of Range $(\pi) = \text{Hull}(I - \{i\})$. By the above argument, $\pi \restriction \bar{Y}$ is constructible, where $\bar{Y} = \pi^{-1}[Y \cap \text{Range } (\pi)]$. As $X \cap \text{Range } (\pi) = X \cap (Y \cap \text{Range } (\pi))$ it follows that $X \cap \text{Range } (\pi)$ is constructible and therefore $\pi \restriction \bar{X}$ is too, where $\bar{X} = \pi^{-1}[X \cap \text{Range } (\pi)]$.

We now show that the modification p of any p' in $G'(i^*)_0$ is a condition. Note that Dom (p') is an element of $L[G(< i^*)]$ of size less than i^* and therefore is a subset of a constructible set of size i^*. It follows from the above that Dom $(p') \cap$ Range (π) and $\pi^{-1} \restriction (\text{Dom } (p') \cap \text{Range } (\pi))$ belong to $L[G(< i^*)]$. It follows that p, which is obtained by modifying p' on Dom $(p') \cap$ Range (π) using π^{-1}, also belongs to $L[G(< i^*)]$, as desired.

(2) We claim that if $D \in L[G(< i^*)]$ is open dense on Add $(i^*, (i^*)^{++})$ then there is a condition $p' \in G'(i^*)_0$ which *strongly meets* D, i.e., any modification of p' on a set of size $\leq i^{++}$ meets D. This implies the genericity of $G(i^*)_0$, as the above modification p of p' takes place on a set of size i^{++}.

It suffices to show that the set of q which strongly meet D is dense. Given q_0, extend q_0 to q_1 meeting D. Then temporarily modify q_1 (in i^{++} places) and extend this modification to meet D; unmodify this extension to obtain an extension q_2 of q_1. Continue this process for i^{+++} steps, ensuring at the end that any modification of the final condition has been considered at some stage of the construction. Then the final condition strongly meets D. □ (Lemma 8)

We now verify (∗) as follows. The embedding $\pi = \pi_{i,i^*}$ from L to L can be extended to an embedding π^* from $L[G(<i)]$ to $L[G(<i^*)]$ (sending $G(<i)$ to $G(<i^*)$) as $G(<i) \subseteq G(<i^*)$. By choice of $G_0(i^*)$, π^* can be further extended to an embedding from $L[G(<i)][G_0(i)]$ to $L[G(<i^*)][G_0(i^*)]$ (sending $G_0(i)$ to $G_0(i^*)$). We can similarly modify each of the $G'_n(i^*)$, and therefore π^* can in fact be extended to an embedding π^{**} from $L[G(<i)][G(i)]$ to $L[G(<i^*)][G(i^*)]$ (sending $G(i)$ to $G(i^*)$). It follows that π^{**} sends $G(\leq i)$ to $G(\leq i^*)$ as these are $G(<i) * G(i)$, $G(<i^*) * G(i^*)$, respectively. So $\pi[G(\leq i)] \subseteq G(\leq i^*)$, as stated in (∗). □

Proof of Theorem 4. First obtain a generic for the reverse Easton iteration P, defined just as in the previous proof, but with Q_α nontrivial at exactly the regular cardinals, where it is taken to be simply Add (α, α). Let G be P-generic over L and set $g = G(\kappa)$, a generic for Add (κ, κ) over the ground model $L[G(<\kappa)]$.

Now let $\kappa \leq \alpha < (\kappa^+)^V$. We will show that a generic for Add (κ, α) over $L[G(<\kappa)]$ can be obtained by "stretching" g. For this purpose we need a special type of bijection between α and κ:

Definition. A bijection $f : \alpha \to \kappa$ is *good* iff $f \restriction X$ is constructible whenever $X \subseteq \alpha$ is constructible and has L-cardinality κ.

Lemma 9 *For any $\alpha < (\kappa^+)^V$ there exists a good bijection $f : \alpha \to \kappa$.*

Proof. We prove by induction on $i \in I \cap [\kappa, (\kappa^+)^V)$ that there is a good bijection $f_i : i \to \kappa$. Set $f_\kappa =$ the identity. If the result holds for i then prove it for i^*, the I-successor to i, as follows: For each $n \in \omega$ set $H_n = i^* \cap \mathrm{Hull}(i + 1 \cup \{i^*, i^{**}, \ldots, i^{*n}\})$, where Hull denotes Skolem hull in L and i^{*n} is the n-th indiscernible greater than i. Then each H_n has L-cardinality i and any subset of i^* of L-cardinality i is contained in some H_n. Let $X_0 = H_0$ and $X_{n+1} = H_{n+1} - H_n$. Using the inductively defined f_i, we can create a bijection f^* between i^* and $\kappa \times \omega$ with the property that for each n, $f^* \restriction X$ is constructible for any constructible $X \subseteq X_n$ of L-cardinality κ. As any constructible $X \subseteq i^*$ of L-cardinality κ is contained in finitely many X_n's it follows that f^* is a good bijection between i^* and $\kappa \times \omega$. Obtain f_{i^*} by composing f^* with a constructible bijection between $\kappa \times \omega$ and κ.

Suppose that i is a limit indiscernible and let γ be its cofinality (in V). Let $\langle i_\alpha \mid \alpha < \gamma \rangle$ be increasing, continuous and cofinal in i, with $i_0 = 0$ and each i_α a multiple of κ. A constructible subset of i of L-cardinality κ intersects only finitely many of the intervals $[i_\alpha, i_{\alpha+1})$. Using the f_{i_α} we can therefore create a good bijection between i and $\kappa \times \gamma$. Then f_i is obtained by composing this good bijection with a constructible bijection between $\kappa \times \gamma$ and κ. $\square$ (Lemma 9)

It follows that for any $\alpha < (\kappa^+)^V$ there is a bijection $f : \alpha \to \kappa$ which is $L[G(<\kappa)]$-*good*, i.e., good with L replaced by $L[G(<\kappa)]$. This is because any set in $L[G(<\kappa)]$ of $L[G(<\kappa)]$-cardinality κ is a subset of a constructible set of L-cardinality κ.

Fix an $L[G(<\kappa)]$-good bijection $f : \alpha \to \kappa$. For a condition $p \in \text{Add }(\kappa, \alpha)$ define $f(p)$ as follows: (β, γ) is in the domain of $f(p)$ iff $(\beta, f^{-1}(\gamma))$ is in the domain of p, in which case $f(p)(\beta, \gamma) = p(\beta, f^{-1}(\gamma))$. Note that $f(p)$ is a condition in Add (κ, κ) as f is $L[G(<\kappa)]$-good. (In fact, we only need goodness here for sets in $L[G(<\kappa)]$ of $L[G(<\kappa)]$-cardinality strictly less than κ.)

Claim. Let h be the set of $p \in \text{Add }(\kappa, \alpha)$ such that $f(p)$ belongs to g. Then h is Add (κ, α)-generic over $L[G(<\kappa)]$.

Proof. Clearly h is a compatible, upward-closed set of conditions. We must show that if $A \subseteq \text{Add }(\kappa, \alpha)$ is a maximal antichain in $L[G(<\kappa)]$ then $f(p)$ belongs to g for some $p \in A$. It suffices to show that $B = \{f(p) \mid p \in A\}$ is a maximal antichain in Add (κ, κ), as by the $L[G(<\kappa)]$-goodness of f, B does belong to $L[G(<\kappa)]$.

Let $D(A)$ be the union of the domains of the conditions in A and $D(B)$ the union of the domains of the conditions in B. Then $\text{id} \times f$ maps $D(A)$ onto $D(B)$ and by the $L[G(<\kappa)]$-goodness of f, $\text{id} \times f \restriction D(A)$ belongs to $L[G(<\kappa)]$. It follows that any condition $q \in \text{Add }(\kappa, \kappa)$ with domain included in $D(B)$ is of the form $f(p)$ for some condition $p \in \text{Add }(\kappa, \alpha)$.

Now let q be an arbitrary condition in Add (κ, κ). We must show that q is compatible with some element of B. It suffices to show that $q_0 = q \restriction D$

is compatible with some element of B, as incompatibility between q and a condition in B can only occur on D. Now by the above, q_0 is of the form $f(p_0)$ for some $p_0 \in \text{Add}\ (\kappa, \alpha)$. As A is a maximal antichain in Add (κ, α), p_0 is compatible with some $r \in A$. Then $f(p_0) = q_0$ is compatible with $f(r) \in B$, as desired. □

Conjecture. Assume that $0^\#$ exists. Then an L-definable Easton function F can be realised in an inner model M having the same cofinalities as L iff it satisfies: $F(\kappa) < (\kappa^{++})^V$ for all $\kappa \in \text{Reg}^L$.

Singular Jonsson cardinals

A *Jonsson cardinal* is a cardinal κ with the property that every structure for a countable language of size κ has a proper substructure of size κ.

κ is *Ramsey* iff $\kappa \to (\kappa)^{<\omega}$, i.e., whenever F is a function from $[\kappa]^{<\omega}$ into 2, there is $H \subseteq \kappa$ of size κ such that F is constant on $[H]^n$ for each n. In the large cardinal hierarchy, Ramsey cardinals lie strictly between $0^\#$ and a measurable cardinal.

Theorem 10 *The following are equiconsistent:*
a. There is a Jonsson cardinal.
b. There is a Ramsey cardinal.

The equiconsistency is in fact an *internal* equiconsistency: If κ is Jonsson then κ is Ramsey in an inner model (the Dodd-Jensen core model); conversely, if κ is Ramsey then κ is Jonsson in an inner model (in fact, in V itself).

Theorem 11 *The following are equiconsistent:*
a. There is a singular Jonsson cardinal.
b. There is a measurable cardinal.

The direction Con $a \to$ Con b is internal: Mitchell showed that if there is a singular Jonsson cardinal then there is an inner model with a measurable cardinal. The usual proof of Con $b \to$ Con a is however via forcing, and therefore *not* internal. Here is the argument:

Theorem 12 *(Prikry) Suppose that κ is measurable. Then there is a generic extension in which κ is a singular Jonsson cardinal.*

Proof. We use Prikry forcing. Fix a normal, κ-complete nonprincipal ultrafilter U on κ. Conditions are pairs (s, A) where s is a finite subset of κ and A is an element of U whose min is greater than $\max(s)$. When strengthening (s, A), s is end-extended and A is shrunk to a subset. Prikry shows that for any sentence φ of the forcing language and condition (s, A), φ is decided by an extension of (s, A) of the form (s, B). It follows using the κ-additivity of U that bounded subsets of κ are not added. Also the forcing is κ^+-cc, so cardinals greater than κ are preserved. If G is generic then the union of the first components of conditions in G is an unbounded subset of κ of ordertype ω.

It remains to show that κ is Jonsson in the Prikry extension. For this we use the following combinatorial form of the Jonsson property:

Proposition 13 *κ is Jonsson iff for every $F : [\kappa]^{<\omega} \to \kappa$ there is $H \subseteq \kappa$ of size κ such that $F \restriction [H]^{<\omega} : [H]^{<\omega} \to \kappa$ is not onto.*

This is easily proved: If κ is Jonsson then a proper substructure of (κ, F) of size κ provides the required H; conversely, if $\mathcal{A}$ is a structure on κ then define F so that its range on any nonempty set is the universe of a substructure of $\mathcal{A}$.

Now suppose that $(s, A) \Vdash F : [\kappa]^{<\omega} \to \kappa$; we claim that there is a set B in U and a countable subset x of ω_1 such that $B \subseteq A$ and (s, B) forces that whenever t belongs to $[B]^{<\omega}$ and $F(t)$ is countable, then $F(t)$ belongs to x.

To obtain B and x, consider the function $G : [A]^{\omega} \times [A]^{\omega} \to \kappa$ defined by: $G(t, u) = \alpha$ iff α is countable and $(s \cup t, C) \Vdash F(u) = \alpha$ for some $C \subseteq A$; if there is no such α then set $G(t, u) = 0$. Now we use:

Lemma 14 *If U is a normal, κ-complete ultrafilter on κ and $G : [\kappa]^{<\omega} \to \lambda$, $\lambda < \kappa$, then there is a set $B \in U$ such that G is constant on $[B]^n$ for each n.*

Using this Lemma, there is a $B \in U$, $B \subseteq A$, such that G takes only countably-many values on $[B]^{<\omega} \times [B]^{<\omega}$. It follows that (s, B) forces F to take only countably-many countable values on $[B]^{<\omega}$, as desired. □

But is the equiconsistency in Theorem 11 internal? By Theorem 12, it suffices to show that the existence of a measurable cardinal implies the existence of an *inner* model which is a Prikry extension.

Theorem 15 *Suppose that there is a measurable cardinal. Then there is an inner model of the form $V^*[G^*]$ where G^* is Prikry generic over V^*.*

Proof. Let U be a normal, κ-additive nonprincipal ultrafilter on κ. The *ultrapower* of (V, U) is formed by taking all equivalence classes $[f]_U$ of $f : \kappa \to V$ in V under the equivalence relation $f =_U g$ iff $\{\alpha \mid f(\alpha) = g(\alpha)\} \in U$, with the relation $\in_U$ defined by $[f]_U \in_U [g]_U$ iff $\{\alpha \mid f(\alpha) \in g(\alpha)\} \in U$. This ultrapower is well-founded as U is countably complete and therefore is isomorphic to (V_1, U_1), where V_1 is a transitive inner model and in V_1, U_1 is a normal, κ_1-additive nonprincipal ultrafilter on some κ_1. We obtain an elementary embedding π_U from V into V_1 by sending x to $[c_x]_U$, where c_x is the constant function on κ with value x. Every element of V_1 is of the form $\pi_U(f)(\kappa)$ where f is a function on κ, as the latter equals $[f]_U$.

We can iterate this ω times, forming successive well-founded ultrapowers (V_n, U_n), $n \in \omega$, with corresponding measurable cardinals κ_n. As each V_n canonically embeds into V_{n+1}, we can form the direct limit of the V_n's, V_ω.

Lemma 16 *(1) V_ω is well-founded and therefore isomorphic to a transitive inner model V^*.*
(2) The image of (κ, U) under the canonical embedding of (V, U) into V^* is (κ^*, U^*), where κ^* is the supremum of the κ_n's and U^* is a normal, κ^*-additive nonprincipal ultrafilter on κ^* in V^*.
(3) The sequence $\{\kappa_n \mid n \in \omega\}$ is a Prikry sequence over V^* for the measure U^*.

This will finish the proof, as then $V^*[G^*]$ is an inner model which is a Prikry extension of V^*, where G^* is the Prikry generic corresponding to $\{\kappa_n \mid n \in \omega\}$.

Proof of Lemma 16. (1) First note that if $\pi : (\bar{V}, \bar{U}) \to (V, U)$ is elementary then there is an elementary embedding $\pi^* : \mathrm{Ult}(\bar{V}, \bar{U}) \to \mathrm{Ult}(V, U)$ such that $\pi^* \pi_{\bar{U}} = \pi_U \pi$ (where π_U, $\pi_{\bar{U}}$ are the canonical embeddings of (V, U), $(\bar{V}, \bar{U})$ into their ultrapowers). π^* is defined by: $\pi^*(\pi_{\bar{U}}(f)(\bar{\kappa})) = \pi_U(\pi(f))(\kappa)$, where $\bar{U}$ is an ultrafilter on $\bar{\kappa}$. It follows that the entire ω-iteration $(\bar{V}, \bar{U}) \to (\bar{V}_1, \bar{U}_1) \to \cdots$ of $(\bar{V}, \bar{U})$ embeds into the ω-iteration $(V, U) \to (V_1, U_1) \to \cdots$ of (V, U) in the following sense: Suppose that π_{ij} ($\bar{\pi}_{ij}$) is the canonical embedding of (V_i, U_i) into (V_j, U_j) (of $(\bar{V}_i, \bar{U}_i)$ into $(\bar{V}_j, \bar{U}_j)$). Then there are embeddings $\tau_n : (\bar{V}_n, \bar{U}_n) \to (V_n, U_n)$ with $\pi_{ij} \tau_i = \tau_j \bar{\pi}_{ij}$. It follows that there is an embedding τ^* from $(\bar{V}_\omega, \bar{U}_\omega)$, the direct limit of the $(\bar{V}_n, \bar{U}_n)$'s, into (V_ω, U_ω).

Now note that if V_ω is ill-founded then $(\bar{V}, \bar{U})$ can be chosen to be countable and so that the associated τ^* has an infinite descending sequence in its range. So it suffices to show: If $\pi : (\bar{V}, \bar{U}) \to (V, U)$ is elementary with $\bar{V}$ countable, then the ω-iteration of $(\bar{V}, \bar{U})$ is well-founded.

Lemma 17 *Suppose that $\pi : (\bar{V}, \bar{U}) \to (V, U)$ and $\bar{V}$ is countable. Then there is $\sigma : Ult(\bar{V}, \bar{U}) \to (V, U)$ such that $\sigma\pi_{\bar{U}} = \pi$.*

Given the Lemma, we can successively embed the n-th iterate $(\bar{V}_n, \bar{U}_n)$ into (V, U), and therefore the ω-th iterate $(\bar{V}_\omega, \bar{U}_\omega)$ as well, proving its well-foundedness.

Proof of Lemma 17. As the intersection of countably many elements of U is nonempty, we can choose an ordinal α which belongs to $\pi(A)$ for every $A \in \bar{U}$. Now define $\sigma(\pi_{\bar{U}}(f)(\bar{\kappa})) = \pi(f)(\alpha)$. We must check that this is well-defined:

$\pi_{\bar{U}}(f)(\bar{\kappa}) = \pi_{\bar{U}}(g)(\bar{\kappa}) \to$
$[f]_{\bar{U}} = [g]_{\bar{U}} \to$
$\{\bar{\alpha} \mid f(\bar{\alpha}) = g(\bar{\alpha})\} \in \bar{U} \to$
$\alpha \in \pi(\{\bar{\alpha} \mid f(\bar{\alpha}) = g(\bar{\alpha})\}) \to$
$\alpha \in \{\beta \mid \pi(f)(\beta) = \pi(g)(\beta)\} \to$
$\pi(f)(\alpha) = \pi(g)(\alpha)$,

as desired. □ (Lemma 17)

(2) The canonical embedding $\pi_{n,n+1}$ from V_n to V_{n+1} is the identity on κ_n and sends κ_n to κ_{n+1}; it follows that the image of $\kappa = \kappa_0$ in the ω-th iterate is the supremum of the κ_n's.

(3) First note that the set C of κ_n's has the following property:

$(*)$ For $A \subseteq \kappa^*$ in V^*: $A \in U^*$ iff A contains all but finitely many elements of C.

For, if $A \subseteq \kappa^*$ and A belongs to U^* then for large enough n we can write A as $\pi^*_n(A_n)$, where $A_n \subseteq \kappa_n$ belongs to U_n and π^*_n is the canonical embedding of V_n into V^*. Then for such n, κ_n belongs to $\pi_{n,n+1}(A_n)$ and therefore to A. If

A does not belong to U^* then its complement does and therefore A contains only finitely many elements of C.

Lastly we show:

Lemma 18 *Suppose that U is a normal, κ-complete nonprincipal ultrafilter on κ in V and C is a subset of κ of ordertype ω such that for $A \subseteq \kappa$ in V: $A \in U$ iff A contains all but finitely many elements of C. Then C is a Prikry sequence for U over V.*

Proof. Let P denote Prikry forcing for the ultrafilter U over V. We show that $G = \{(s, A) \in P \mid s \text{ is a finite initial segment of } C \text{ and } C \setminus s \subseteq A\}$ is P-generic over V.

Let $D \in V$ be open dense on P. For each $s \in [\kappa]^{<\omega}$ let $F_s : [\kappa]^{<\omega} \to 2$ be defined by $F(t) = 1$ iff $\max(s) < \min(t)$ and for some X, $(s \cup t, X) \in D$. Choose $A_s \in U$ such that F_s is constant on $[A_s]^n$ for each n. If there is an X such that $(s, X) \in D$ then choose such an $X = X_s$ and define $B_s = A_s \cap X_s$; otherwise set $B_s = A_s$.

Let A be the diagonal intersection of the B_s, i.e., the set of $\alpha < \kappa$ such that α belongs to B_s whenever $\max(s)$ is less than α. (It is easy to show that A belongs to U, as each B_s does.) Note that for all s, if $(s, X) \in D$ for some X, then $(s, B_s) \in D$ and therefore $(s, A \setminus (\max(s) + 1)) \in D$.

By hypothesis there is a finite initial segment s of C such that A contains $C \setminus s$. As D is dense we can choose an extension $(s \cup t, X)$ of (s, A) which belongs to D (with $\max(s) < \min(t)$). As $A \setminus (\max(s) + 1)$ is homogeneous for F_s, it follows that for any $u \subseteq C \setminus (\max(s) + 1)$ of the same size as t, $(s \cup u, Y)$ belongs to D for some Y and therefore $(s \cup u, A \setminus (\max(u) + 1))$ belongs to D. By choosing u to be an initial segment of $C \setminus s$, the condition $(s \cup u, A \setminus (\max(u) + 1))$ is in $D \cap G$, as desired. □

7.Vorlesung

The singular cardinal hypothesis

The *Singular cardinal hypothesis (SCH)* is the statement: For every singular cardinal κ, if $2^{\text{cof } \kappa} < \kappa$ then $\kappa^{\text{cof } \kappa} = \kappa^+$.

Note that $\kappa^{\mathrm{cof}\,\kappa}$ is always at least κ^+, as by König's theorem, if $\langle \kappa_i \mid i < \mathrm{cof}\,\kappa \rangle$ is a cofinal increasing sequence of cardinals less than κ, then $\kappa = \sum_{i<\mathrm{cof}\,\kappa} \kappa_i < \prod_{i<\mathrm{cof}\,\kappa} \kappa = \kappa^{\mathrm{cof}\,\kappa}$. Thus the SCH follows from the GCH. And the SCH implies that the GCH must hold at singular strong limit cardinals, as for such λ, $2^{\mathrm{cof}\,\lambda} < \lambda$ and therefore $2^\lambda = \lambda^{\mathrm{cof}\,\lambda} = \lambda^+$ by the SCH.

Under SCH, cardinal exponentiation is completely determined by the behaviour of the continuum function $\kappa \mapsto 2^\kappa$ for regular κ:

Theorem 19 *Assume SCH.*
(a) If κ is a singular cardinal then 2^κ is $2^{<\kappa}$ if the continuum function is eventually constant below κ, and is $(2^{<\kappa})^+$ otherwise.
(b) If κ, λ are any infinite cardinals, then:
(b1) If $\kappa \leq 2^\lambda$ then $\kappa^\lambda = 2^\lambda$.
(b2) If $2^\lambda < \kappa$ and $\lambda <$ cof κ then $\kappa^\lambda = \kappa$.
(b3) If $2^\lambda < \kappa$ and cof $\kappa \leq \lambda$ then $\kappa^\lambda = \kappa^+$.

Proof. (a) For any limit cardinal κ, $2^\kappa = (2^{<\kappa})^{\mathrm{cof}\,\kappa}$. If κ is singular and the continuum function is eventually constant below κ then choose $\mu < \kappa$ such that cof $\kappa < \mu$ and $2^\mu = 2^{<\kappa}$; then we have $2^\kappa = (2^{<\kappa})^{\mathrm{cof}\,\kappa} = (2^\mu)^{\mathrm{cof}\,\kappa} = 2^\mu = 2^{<\kappa}$. If the continuum function is not eventually constant below κ then $\lambda = 2^{<\kappa}$ has cofinality cof κ and $2^{\mathrm{cof}\,\kappa} < \lambda$; by the SCH, $2^\kappa = (2^{<\kappa})^{\mathrm{cof}\,\kappa} = \lambda^{\mathrm{cof}\,\lambda} = \lambda^+ = (2^{<\kappa})^+$.
(b) Fix λ; we prove this by induction on κ. (b1) holds as $\kappa^\lambda \leq (2^\lambda)^\lambda = 2^\lambda \leq \kappa^\lambda$. Assume that $2^\lambda < \kappa$. If κ is a successor cardinal ν^+ then by induction ν^λ is either 2^λ, ν or ν^+; in any case it is at most κ. So as $\lambda < \kappa$, $\kappa^\lambda = (\nu^+)^\lambda = \nu^+ \cdot \nu^\lambda = \kappa$. If κ is a limit cardinal, then by induction $\nu^\lambda < \kappa$ for all $\nu < \kappa$. So if $\lambda <$ cof κ we have $\kappa^\lambda = \kappa$. If $\lambda \geq$ cof κ then we have $\kappa^\lambda \leq \prod_{i<\mathrm{cof}\,\kappa} \kappa_i^\lambda \leq \prod_{i<\mathrm{cof}\,\kappa} \kappa = \kappa^{\mathrm{cof}\,\kappa} \leq \kappa^\lambda$; so $\kappa^\lambda = \kappa^{\mathrm{cof}\,\kappa}$ and as $2^{\mathrm{cof}\,\kappa} \leq 2^\lambda < \kappa$, the latter is κ^+ by the SCH. □

The analog of Cohen's result for the SCH is:

Theorem 20 *(Gitik) Suppose that K is an inner model satisfying GCH which contains a* totally measurable *cardinal κ (i.e., a cardinal κ of Mitchell order κ^{++}). Then there is a generic extension $K[G]$ of K in which κ is a singular strong limit cardinal of cofinality ω and GCH fails at κ.*

By work of Mitchell, a totally measurable cardinal is necessary. Now consider the following weak analogue of Easton's result for the singular cardinal hypothesis:

(Global Gitik) GCH fails on a proper class of singular strong limit cardinals.

The proof of the previous theorem shows:

Theorem 21 *Suppose that K is an inner model satisfying GCH which contains a proper class of totally measurable cardinals. Then there is a generic extension $K[G]$ of K in which Global Gitik holds.*

Is Global Gitik internally consistent relative to large cardinals? In analogy to Easton's theorem, we might expect to show that the generic extension $K[G]$ of Theorem 21 can be obtained as an inner model. This is however not true for the natural choice of K, using covering arguments. The following is joint with Tomáš Futáš.

Theorem 22 *Suppose that there is a # for a proper class of totally measurable cardinals and let K be the "natural" inner model with a class of totally measurable cardinals. (K is obtained by taking the least iterable mouse m with a measurable limit of totally measurable cardinals and iterating its top measure to infinity.) Then there is no inner model of the form $K[G]$, where G is generic over K, in which Global Gitik holds.*

On the other hand, it is possible to choose K differently, so as to witness the internal consistency relative to large cardinals of Global Gitik:

Theorem 23 *Suppose that there is an inner model containing a measurable limit κ of totally measurable cardinals, where κ is countable in V. Then there is an inner model in which Global Gitik holds.*

This is proved as follows: Using the proof of Theorem 21, force over the given inner model to obtain a failure of the GCH on a set of singular strong limit cardinals cofinal in κ. This forcing preserves the measurability of κ. Using the countability of κ, the generic exists in V. Now iterate κ to infinity; the resulting model is a model of Global Gitik.

What is the internal consistency strength of Global Gitik, i.e., what large cardinal hypothesis must hold to guarantee an inner model of Global Gitik?

Theorem 23 provides an upper bound, but the optimal upper bound is not yet known.

8.Vorlesung

Part Two. External Consistency: Large Cardinals and L-like Universes

We now turn to a detailed discussion of the outer model program. First we have to say what we mean by "large cardinals".

A cardinal κ is *inaccessible* iff it is uncountable, regular and larger than the power set of any smaller cardinal. It is *measurable* iff there is a κ-complete, nonprincipal ultrafilter on κ.

Measurability is equivalent to a property expressed in terms of *embeddings*, and stronger large cardinal properties are also expressed in this way. As usual, V denotes the universe of all sets. Let M be an inner model, i.e., a transitive proper class that satisfies the axioms of ZFC. A class function $j : V \to M$ is an *embedding* iff it preserves the truth of formulas with parameters in the language of set theory and is not the identity. If j is an embedding then there is a least ordinal κ such that $j(\kappa) \neq \kappa$, called the *critical point* of j, which is a measurable cardinal.

For an ordinal α, $j : V \to M$ is *α-strong* iff V_α is contained in M. A cardinal κ is α-strong iff there is an α-strong embedding with critical point κ. *Strong* means α-strong for all α.

Kunen showed that no embedding is strong. However a cardinal can be strong, as embeddings witnessing its α-strength can vary with α. Stronger properties are obtained by requiring $j : V \to M$ to have strength depending on the image under j of its critical point. For example, κ is *superstrong* iff there is a nontrivial elementary embedding $j : V \to M$ with critical point κ which is $j(\kappa)$-strong. An important weakening of superstrength is the property that for each $f : \kappa \to \kappa$ there is a $\bar{\kappa} < \kappa$ closed under f and a nontrivial elementary embedding $j : V \to M$ with critical point $\bar{\kappa}$ which is $j(f)(\bar{\kappa})$-strong; such κ are known as *Woodin cardinals.* The consistency strength of the existence of a Woodin cardinal is strictly between that of a strong cardinal and a superstrong cardinal.

We can demand more than superstrength. A cardinal κ is *hyperstrong* iff it is the critical point of an embedding $j : V \longrightarrow M$ which is $j(\kappa)+1$-strong. For a finite $n > 0$, *n-superstrength* is obtained by requiring j to be $j^n(\kappa)$-strong, where $j^1 = j$, $j^{k+1} = j \circ j^k$. Finally, κ is *ω-superstrong* iff it is the critical point of an embedding $j : V \longrightarrow M$ which is n-superstrong for all n. Kunen's result shows that no embedding j with critical point κ is $j^\omega(\kappa) + 1$-strong, where $j^\omega(\kappa)$ is the supremum of the $j^n(\kappa)$ for finite n.

Large cardinals and L-like universes

Regarding the *inner model program*: If κ is inaccessible, then κ is also inaccessible in L, the most L-like model of all. This is not the case for measurability, however if κ is measurable then κ is measurable in an inner model $L[U]$, where U is an ultrafilter on κ, which has a definable wellordering and in which GCH, $\diamondsuit$, $\square$ hold and gap 1 morasses exist. For a strong cardinal κ there is a similarly L-like inner model $L[E]$ in which κ is strong, where E now is not a single ultrafilter, but rather a sequence of generalised ultrafilters, called *extenders*. More recent work yields similar results for Woodin cardinals, and even for Woodin limits of Woodin cardinals.

However, progress beyond that has been impeded by the so-called *iterability problem.*

The outer model program: Can we obtain L-like *outer* models with large cardinals? For inaccessibles one has the following result of Jensen:

Theorem 24 *(L-coding) There is a generic extension $V[G]$ of V such that*

a. ZFC holds in $V[G]$.
b. $V[G] = L[R]$ for some real R.
c. Every inaccessible cardinal of V remains inaccessible in $V[G]$.

There are similar $L[U]$ and $L[E]$ coding theorems, providing outer models of the form $L[U][R]$ and $L[E][R]$, R a real, which are just as L-like as $L[U]$ and $L[E]$, preserving measurability and strength, respectively.

However the approach via coding is limited in its use. Obtaining L-like outer models via coding depends on the existence of L-like inner models, such as $L[U]$ or $L[E]$, which, as we have observed, are not known to exist beyond

Woodin limits of Woodin cardinals. And there are problems with the coding method itself which arise already just past a strong cardinal.

A more promising approach is to use iterated forcing. To illustrate this, consider the problem of making the GCH true in an outer model. Begin with an arbitrary model V of ZFC. Using forcing, we can add a function from $\aleph_1$ onto $2^{\aleph_0}$ without adding reals, thereby making CH true. By forcing again, we add a function from (the possibly new) $\aleph_2$ onto (the possibly new) $2^{\aleph_1}$ without adding subsets of $\aleph_1$, thereby obtaining $2^{\aleph_1} = \aleph_2$. Continue this indefinitely (via a reverse Easton iteration) and the result is a model of the GCH.

Do we preserve large cardinal properties if we make GCH true in this way? The answer is Yes.

9.Vorlesung

Theorem 25 *(Large cardinals and the GCH) If κ is superstrong then there is an outer model in which κ is still superstrong and the GCH holds. The same holds for hyperstrong, n-superstrong for finite n and ω-superstrong.*

Proof. First we describe in more detail the above iteration to make GCH true. By induction on α we define the iteration P_α of length α: P_0 is the trivial forcing. For limit λ, P_λ is the inverse limit of the P_α, $\alpha < \lambda$, if λ is singular and is the direct limit of the P_α, $\alpha < \lambda$, if λ is regular. For successor $\alpha + 1$, $P_{\alpha+1} = P_\alpha * Q_\alpha$, where Q_α is the forcing that collapses $2^{\aleph_\alpha}$ to $\aleph_{\alpha+1}$ using conditions of size at most $\aleph_\alpha$. For any cardinal κ of the form $\beth_{\alpha+1}$, the entire iteration P can be factored as $P_\kappa * P^\kappa$, where P_κ has a dense subset of size κ and P^κ is κ^+-closed. In particular, strongly inaccessible cardinals remain strongly inaccessible after forcing with P.

Now suppose that κ is superstrong, witnessed by the embedding $j : V \to M$, and that G is P-generic. Let P^* denote M's version of P. To show that κ is superstrong in $V[G]$, it suffices to find a P^*-generic G^* such that $V_{j(\kappa)}^{V[G]} \subseteq M[G^*]$ and G^* contains $j[G]$, the pointwise image of G under j, as a subclass: Given such a G^*, extend the embedding j to an embedding $j^* : V[G] \to M[G^*]$ by sending σ^G to $j(\sigma)^{G^*}$, for an arbitrary P-name σ. This is well-defined and elementary by the truth lemma, as G^* contains $j[G]$. And j^* witnesses the superstrength of κ in $V[G]$ as $V_{j(\kappa)}^{V[G]} \subseteq M[G^*]$.

Now P^*_α is the same as P_α for $\alpha < j(\kappa)$, as j is a superstrong embedding. The first difference between P^* and P is at $j(\kappa)$: $P^*_{j(\kappa)}$ is the direct limit of the P_α, $\alpha < j(\kappa)$, as $j(\kappa)$ is inacessible in M; but $j(\kappa)$ is not necessarily regular in V and therefore it is possible that $P_{j(\kappa)}$ is the *inverse* limit of the P_α, $\alpha < j(\kappa)$. So we cannot simply choose $G^*_{j(\kappa)}$ to be $G_{j(\kappa)}$, as the latter is generic for the wrong forcing.

But this problem is easily fixed: As $j(\kappa)$ is in fact Mahlo in M, it follows that $P^*_{j(\kappa)}$ has the $j(\kappa)$-cc in M: If $\Delta \in M$ is a maximal antichain in $P^*_{j(\kappa)}$ then by Mahloness $\Delta_0 = \Delta \cap P^*_\alpha$ is a maximal antichain in P^*_α for some regular $\alpha < j(\kappa)$; but then Δ_0 is a maximal antichain in the entire $P^*_{j(\kappa)}$ as by Easton support, any condition in $P^*_{j(\kappa)}$ is the join of a condition in P^*_α with a condition with no support below α, and therefore is compatible with an element of Δ_0. So any $G^*_{j(\kappa)}$ contained in $P^*_{j(\kappa)}$ whose intersection with each P_α, $\alpha < j(\kappa)$, is P_α-generic must also be $P^*_{j(\kappa)}$-generic. It follows that we can take $G^*_{j(\kappa)}$ to simply be the intersection of $G_{j(\kappa)}$ with $P^*_{j(\kappa)}$. Notice that $G^* \cap V_{j(\kappa)}$ equals $G \cap V_{j(\kappa)}$ and trivially contains the pointwise image of G_κ under j as j is the identity below κ.

Finally we must define a generic $G^{*,\, j(\kappa)}$ for the "upper part" $P^{*,\, j(\kappa)}$ of the P^* iteration, which starts at $j(\kappa)$ and is defined in the ground model $M[G^*_{j(\kappa)}]$. In addition, $G^{*,\, j(\kappa)}$ must contain the pointwise image of G^κ under j^*, where j^* is the lifting of j to $V[G_\kappa]$ and G^κ is generic for P^κ, an iteration starting at κ defined over the ground model $V[G_\kappa]$.

In fact this latter requirement completely determines $G^{*,\, j(\kappa)}$:

Lemma 26 *$j^*[G^\kappa]$ generates a $P^{*,\, j(\kappa)}$-generic over $M[G^*_{j(\kappa)}]$, i.e., each predense subclass of $P^{*,\, j(\kappa)}$ which is definable over $M[G^*_{j(\kappa)}]$ has an element which is extended by a condition in $j^*[G^\kappa]$.*

Proof. We only consider predense *subsets* of $P^{*,\, j(\kappa)}$ in $M[G^*_{j(\kappa)}]$; a similar argument works for predense subclasses.

We can assume that $j : V \to M$ is given as an extender ultrapower embedding. This means that each element of M is of the form $j(f)(a)$, where a belongs to $V^M_{j(\kappa)} = V_{j(\kappa)}$ and f is a function (in V) with domain V_κ. To see this, it suffices to show that the class $H = \{j(f)(a) \mid a \in V_{j(\kappa)},\ f$ a function with domain $V_\kappa\}$ is an elementary submodel of M, for then we can replace j by πj, where π is the transitive collapse of H. Now for any $f_1,\dots,f_n$ with domain V_κ and any formula $\varphi(x_1,\dots,x_n,y)$ let f be a function with domain V_κ such that for any $b_1,\dots,b_n$ in V_κ, if $\varphi(f(b_1),\dots,f(b_n),y)$ holds for some y then y can be taken to be $f(\langle b_1,\dots,b_n\rangle)$. Then for any $a_1,\dots,a_n$ in $V^M_{j(\kappa)}=V_{j(\kappa)}$, if $\varphi(j(f_1)(a_1),\dots,j(f_n)(a_n),y)$ holds for some y in M then y can be taken to be $j(f)(\langle a_1,\dots,a_n\rangle)$. It follows that H is elementary in M.

Now let D be a predense subset of $P^{*,\ j(\kappa)}$ in $M[G^*_{j(\kappa)}]$. D is of the form $\sigma^{G^*_{j(\kappa)}}$ where the name σ can be written as $j(f)(a)$ with f and a as above. Now using the κ^+-closure of P^κ, choose a condition p in G^κ which extends an element of $f(\bar a)$ whenever $\bar a$ belongs to V_κ and $f(\bar a)^{G_\kappa}$ is predense on $P^{*,\ \kappa}$. Then $j^*(p)$ belongs to $j^*[G^\kappa]$ and extends an element of $j(f)(a)^{G^*_{j(\kappa)}}=\sigma^{G^*_{j(\kappa)}}=D$, as desired. □ (Lemma 26)

This completes the construction of G^* and therefore the proof that P preserves superstrong cardinals.

10.-12.Vorlesungen

Now suppose that κ is hyperstrong. We need to find a P^*-generic G^* such that $V^{V[G]}_{j(\kappa)+1}\subseteq M[G^*]$ and G^* contains $j[G]$ as a subclass. Note that $V^{V[G]}_{j(\kappa)+1}$ equals $V_{j(\kappa)+1}[G_{j(\kappa)}]$ so for the former condition it suffices to have $G^*_{j(\kappa)}=G_{j(\kappa)}$.

The forcings $P_{j(\kappa)+1}=P_{j(\kappa)}*Q_{j(\kappa)}$ and $P^*_{j(\kappa)+1}$ agree as $j(\kappa)$ is regular in V and M contains $V_{j(\kappa)+1}$. We take $G^*_{j(\kappa)}$ to be $G_{j(\kappa)}$. Also, $j^*[g_\kappa]$, where j^* is the lifting of j to $V[G_\kappa]$ and g_κ is the $Q^{G_\kappa}_\kappa$-generic chosen by G at stage κ of the iteration, is a set of conditions in $Q^{G_{j(\kappa)}}_{j(\kappa)}$ which belongs to $M[G_{j(\kappa)}]$ and has size 2^κ there; therefore $j^*[g_\kappa]$ has a lower bound in $Q^{G_{j(\kappa)}}_{j(\kappa)}$. By choosing our generic G so that $g_{j(\kappa)}$ includes this lower bound (or by modifying G to a P-generic G' in $V[G]$ so that $g'_{j(\kappa)}$ will contain this lower bound), we can succeed in lifting j to $V[G_{\kappa+1}]$. We may assume that $j:V\to M$ is *given by a hyperextender*; this means that each element of M is of the form $j(f)(a)$ where f is a function in V with domain $V_{\kappa+1}$ and a is an element of $V_{j(\kappa)+1}$. Then we can use the argument from the superstrong case to generate the entire generic G^* containing $j[G]$.

The case of n-superstrongs raises a new difficulty. We first treat the case $n = 2$. As in the superstrong case, P and P^* may take different limits at $j^2(\kappa)$, as the latter may be singular in V. As in that case, we can obtain a $P^*_{j^2(\kappa)}$-generic by intersecting $G_{j^2(\kappa)}$ with $P^*_{j^2(\kappa)}$. However we must also ensure that $G^*_{j^2(\kappa)}$ contain $j[G_{j(\kappa)}]$ as a subset. Write $P_{j(\kappa)}$ as $P_\kappa * P^\kappa_{j(\kappa)}$; it suffices to arrange that $G^*_{j^2(\kappa)}$ contain $j^*[G^\kappa_{j(\kappa)}]$ as a subset, where j^* is the lifting of j to $V[G_\kappa]$ and $G^\kappa_{j(\kappa)}$ is $P^\kappa_{j(\kappa)}$-generic over $V[G_\kappa]$.

We argue as follows. If $j[j(\kappa)]$ is bounded in $j^2(\kappa)$ then the set of conditions $j^*[G^\kappa_{j(\kappa)}]$ has a lower bound in $P^*_{j^2(\kappa)} \subseteq P_{j^2(\kappa)}$. Otherwise $j^2(\kappa)$ is singular, so $P_{j^2(\kappa)}$ is an inverse limit and again the set of conditions $j^*[G^\kappa_{j(\kappa)}]$ has a lower bound in $P_{j^2(\kappa)}$. We therefore assume that our generic G has been chosen so that $G_{j^2(\kappa)}$ contains the greatest lower bound of $j^*[G^\kappa_{j(\kappa)}]$. Then we can take $G^*_{j^2(\kappa)}$ to be the intersection of $G_{j^2(\kappa)}$ with $P^*_{j^2(\kappa)}$ and thereby obtain $j[G_{j(\kappa)}] \subseteq G^*_{j^2(\kappa)}$. This allows us to lift j to $V[G_{j(\kappa)}]$. Then we can use the argument from the superstrong case to generate the entire generic G^* containing $j[G]$.

For the case $n > 2$ the argument is similar; we must choose $G_{j^n(\kappa)}$ to contain the greatest lower bound of $j^*[G^{j^{n-2}(\kappa)}_{j^{n-1}(\kappa)}]$, where j^* is the lifting of j to the model $V[G_{j^{n-2}(\kappa)}]$.

Finally we consider ω-superstrength. Again we must choose G^* to be P^*-generic over M and to contain the pointwise image of G under j. Let $j^\omega(\kappa)$ denote the supremum of the $j^n(\kappa)$, $n \in \omega$. As before it suffices to find $G^*_{j^\omega(\kappa)}$ which is $P^*_{j^\omega(\kappa)}$-generic and contains $j[G_{j^\omega(\kappa)}]$ as a subset. Note that $j[G_\kappa] = G_\kappa$ is trivially contained in $G_{j^\omega(\kappa)}$ and $j^*[G^\kappa_{j^\omega(\kappa)}]$ has a lower bound in $P^\kappa_{j^\omega(\kappa)}$ (as defined in $V[G_\kappa]$); by choosing $G_{j^\omega(\kappa)}$ to contain this lower bound we can take $G^*_{j^\omega(\kappa)}$ to be $G_{j^\omega(\kappa)} \cap P^*_{j^\omega(\kappa)}$ and thereby obtain $j[G_{j^\omega(\kappa)}] \subseteq G^*_{j^\omega(\kappa)}$. And again we can use the argument from the superstrong case to generate the entire generic G^* containing $j[G]$. So it only remains to show:

Lemma 27 *$G_{j^\omega(\kappa)} \cap P^*_{j^\omega(\kappa)}$ is $P^*_{j^\omega(\kappa)}$-generic over M.*

Proof. Suppose that $D \in M$ is dense on $P^*_{j^\omega(\kappa)}$ and write D as $j(f)(a)$ where f has domain $V_{j^\omega(\kappa)}$ and a belongs to $V_{j^{n+1}(\kappa)}$ for some n. (We may assume that every element of M is of this form.) Choose p in $G_{j^\omega(\kappa)}$ such that p *reduces*

$f(\bar{a})$ below $j^n(\kappa)$ whenever $\bar{a}$ belongs to $V_{j^n(\kappa)}$ and $f(\bar{a})$ is open dense on $P_{j^\omega(\kappa)}$, in the sense that if q extends p then q can be further extended into $f(\bar{a})$ without changing q at or above $j^n(\kappa)$. Such a p exists using the $j^n(\kappa)^+$-closure of $P^{j^n(\kappa)}_{j^\omega(\kappa)}$ in $V[G_{j^n(\kappa)}]$. Then $j(p)$ belongs to $j[G_{j^\omega(\kappa)}]$ and reduces D below $j^{n+1}(\kappa)$. As $G_{j^{n+1}(\kappa)}$ is $P_{j^{n+1}(\kappa)}$-generic and P, P^* agree below $j^{n+1}(\kappa)$, it follows that $G_{j^\omega(\kappa)} \cap P^*_{j^\omega(\kappa)}$ intersects D, as desired. $\square$ (Lemma 27)

This completes the proof of the Theorem. $\square$

Jensen's (global) $\square$ principle asserts the existence of a sequence $\langle C_\alpha \mid \alpha$ singular$\rangle$ such that C_α has ordertype less than α for each α and $C_{\bar{\alpha}} = C_\alpha \cap \bar{\alpha}$ whenever $\bar{\alpha} \in \text{Lim } C_\alpha$. The following strengthens a result of Doug Burke:

Theorem 28 *(Superstrong cardinals and $\square$) If κ is superstrong then there is an outer model in which κ is still superstrong and $\square$ holds.*

Proof. We may assume the GCH. Consider now the reverse Easton iteration P where at the regular stage α, Q_α is a P_α-name for the forcing which adds a $\square$-sequence on the singular limit ordinals less than α. A condition in Q_α is a sequence $\langle C_\beta \mid \beta \leq \gamma,\, \beta \text{ singular}\rangle$, $\gamma < \alpha$, such that C_β has ordertype less than β for each β and $C_{\bar{\beta}} = C_\beta \cap \bar{\beta}$ whenever $\bar{\beta}$ belongs to Lim C_β.

Using the fact that P_α forces $\square$-sequences of any regular length less than α, it is easy to verify by induction that any condition in Q_α can be extended to have arbitrarily large length less than α. Also Q_α, and indeed the entire iteration from stage α on, is α-distributive.

Let P^* denote M's version of P. We want to construct G^* to be P^*-generic over M, to agree with G strictly below $j(\kappa)$ and to contain $j[G]$ as a subclass. As in earlier arguments, P and P^* agree strictly below $j(\kappa)$ but not necessarily at $j(\kappa)$, which is regular in M but may be singular in V; as before we take $G^*_{j(\kappa)}$ to be $G_{j(\kappa)} \cap P^*_{j(\kappa)}$. Our new task is to define a $Q^*_{j(\kappa)}$-generic g over $M[G^*_{j(\kappa)}]$.

Lemma 29 *Assume GCH and let $j : V \to M$ witness the superstrength of κ with $j(\kappa)$ minimal. Then $j(\kappa)$ has cofinality κ^+.*

Proof. Let $\langle f_i \mid i < \kappa^+\rangle$ be a list of all functions from κ to κ. Then the sequence $\langle j(f_i) \mid i < \kappa\rangle$ belongs to M, as it equals $j(\langle f_i \mid i < \kappa^+\rangle) \restriction \kappa$. For any ordinal $\alpha < \kappa^+$ we can use a bijection between α and κ and similarly conclude that $\langle j(f_i) \mid i < \alpha \rangle$ belongs to M.

Now for each $\alpha < \kappa^+$ let κ_α be least so that κ_α is closed under each $j(f_i)$, $i < \alpha$. Then κ_α is less than $j(\kappa)$, as $j(\kappa)$ is regular in M. Let κ^* be the supremum of the κ_α's. It suffices to show that there is a superstrong embedding j^* with critical point κ such that $j^*(\kappa) = \kappa^*$; then by the minimality of $j(\kappa)$, we must have $j(\kappa) = \kappa^*$ and therefore $j(\kappa)$ has cofinality κ^+.

To obtain j^* define $H = \{j(f)(a) \mid f : V_\kappa \to V,\ a \in V_{\kappa^*}\}$. Then H is an elementary submodel of M and $H \cap j(\kappa) = \kappa^*$. Let $\pi : H \simeq M^*$; then $j^* = \pi j : V \to M^*$ witnesses the superstrength of κ and $j^*(\kappa) = \pi(j(\kappa)) = \kappa^*$, as desired. $\Box$

We can assume that j is given by an ultrapower, and therefore that j is continuous at κ^+. It follows that $(j(\kappa)^+)^M$ has cofinality κ^+. Therefore we can write the collection of κ^+-many open dense subsets of $Q^*_{j(\kappa)}$ as the union of κ^* subcollections, each of size less than $j(\kappa)$. Now we can build g in κ^+ steps, using $j(\kappa)$-distributivity to meet fewer than $j(\kappa)$ open dense sets at each step (and defining the $\Box$-sequence coherently at limit stages). We must also ensure that g extend g_κ; but this is easy to arrange as the latter is a condition in the forcing $Q^*_{j(\kappa)}$.

Finally the rest of G^* can be generated from $j[G]$ as before. $\Box$

The proof of the previous theorem does not work for hyperstrong κ, and there is a good reason for this. κ is *subcompact* iff for any $B \subseteq H_{\kappa^+}$ there are $\mu < \kappa$, $A \subseteq H_{\mu^+}$ and an elementary embedding $j : (H_{\mu^+}, A) \to (H_{\kappa^+}, B)$ with critical point μ. (Note that by elementarity, j must send μ to κ.)

Proposition 30 *(a) If κ is hyperstrong then κ is subcompact. (b) (Jensen) If there is a subcompact cardinal then $\Box$ (even when restricted to ordinals between κ and κ^+) fails.*

Proof. (a) Suppose that $j : V \to M$ witnesses hyperstrength. Then for all subsets B of $j(\kappa)^+$ in the range of j, j gives an elementary embedding of (H_{κ^+}, A) into $(H_{j(\kappa)^+}, B)$, where $j(A) = B$; moreover this embedding belongs to M as j is hyperstrong and $j \restriction H_{\kappa^+}$ belongs to $H_{j(\kappa)^+}$. As the range of j is an elementary submodel of M, it follows that there is an elementary embedding

of some (H_{μ^+}, A) into $(H_{j(\kappa)^+}, B)$ (sending μ to $j(\kappa)$) which belongs to the range of j. So $j(\kappa)$ is subcompact in Range j and therefore by elementarity subcompact in M. As j is elementary, κ is subcompact in V.
(b) Suppose that κ is subcompact and $\vec{C} = \langle C_\alpha \mid \kappa < \alpha < \kappa^+,\ \alpha \text{ singular}\rangle$ has the properties of a $\Box$-sequence. By thinning out the C_α's we can ensure that each has ordertype at most κ. Let j be an embedding from $(H_{\mu^+}, \vec{\bar{C}})$ to $(H_{\kappa^+}, \vec{C})$, sending μ to κ. Let α be the supremum of the ordinals in the range of j. Then α has cofinality μ^+. The ordinals in the range of j form a $< \mu$-closed and therefore ω-closed unbounded subset of α. And Lim C_α is a closed unbounded subset of α. Therefore the intersection D of these two sets is unbounded in α. By the coherence property of $\vec{C}$, the ordertype of C_β for sufficiently large β in D is at least μ. But as the ordertype of C_α is at most κ (in fact less than κ), the ordertype of C_β for all β in D is strictly less than κ. Thus there are β in $D \subseteq$ Range j with C_β of ordertype not in Range j, contradicting the elementarity of j. $\Box$

13.-14.Vorlesungen

Another important property of L is the existence of a definable wellordering of the universe.

Theorem 31 *(Large cardinals and definable wellorderings) If κ is superstrong then there is an outer model in which κ is still superstrong and there is a definable wellordering of the universe. The same holds for hyperstrong, n-superstrong for finite n and ω-superstrong.*

Proof. We may assume the GCH. Let κ have one of the large cardinal properties mentioned in the theorem, as witnessed by the embedding $j : V \to M$. Choose λ to be a cardinal greater than $j^\omega(\kappa)$. By the method of L-coding, we can enlarge V without adding subsets of λ to a universe of the form $L[A]$, A a subset of λ^+. By an earlier argument, the embedding j lifts to $L[A]$ and therefore κ retains its large cardinal properties.

Now we introduce a definable wellordering. Perform a reverse Easton iteration of length λ^+, indexed by successor cardinals greater than λ^+, where at the i-th successor cardinal, an i^+-Cohen set is added iff i belongs to A. The result is that i belongs to A iff not every subset of the successor of the i-th successor cardinal is constructible from a subset of the i-th successor cardinal. Now the result of this iteration is a model of the form $L[B]$ where B is a subset of $\lambda^{(\lambda^+)}$, the "λ^+-th cardinal greater than λ". Repeat this to code B using the next interval of successor cardinals. Continuing this indefinitely yields a model with a wellordering definable from the parameter λ.

To eliminate the parameter λ, use a pairing function $f : \mathrm{Ord} \times \mathrm{Ord} \to \mathrm{Ord}$ on the ordinals and arrange that the universe is of the form $L[C]$ where C is a class of ordinals and for any i, i is in C iff some subset of the successor to the $f(i,j)$-th successor cardinal is not constructible from a subset of the $f(i,j)$-th successor cardinal, for all sufficiently large j. $\Box$

Jensen's (global) $\Box$ principle asserts the existence of a sequence $\langle C_\alpha \mid \alpha$ singular$\rangle$ such that C_α has ordertype less than α for each α and $C_{\bar\alpha} = C_\alpha \cap \bar\alpha$ whenever $\bar\alpha \in \mathrm{Lim}\ C_\alpha$. The following strengthens a result of Doug Burke.

Theorem 32 *(Superstrong cardinals and $\Box$) If κ is superstrong then there is an outer model in which κ is still superstrong and $\Box$ holds.*

Proof. We may assume the GCH. Consider now the reverse Easton iteration P where at the regular stage α, Q_α is a P_α-name for the forcing which adds a $\Box$-sequence on the singular limit ordinals less than α. A condition in Q_α is a sequence $\langle C_\beta \mid \beta \le \gamma,\ \beta \text{ singular}\rangle$, $\gamma < \alpha$, such that C_β has ordertype less than β for each β and $C_{\bar\beta} = C_\beta \cap \bar\beta$ whenever $\bar\beta$ belongs to $\mathrm{Lim}\ C_\beta$.

Using the fact that P_α forces $\Box$-sequences of any regular length less than α, it is easy to verify by induction that any condition in Q_α can be extended to have arbitrarily large length less than α. Also Q_α, and indeed the entire iteration from stage α on, is α-distributive.

Let P^* denote M's version of P. We want to construct G^* to be P^*-generic over M, to agree with G strictly below $j(\kappa)$ and to contain $j[G]$ as a subclass. As in earlier arguments, P and P^* agree strictly below $j(\kappa)$ but not necessarily at $j(\kappa)$, which is regular in M but may be singular in V; as before we take $G^*_{j(\kappa)}$ to be $G_{j(\kappa)} \cap P^*_{j(\kappa)}$. Our new task is to define a $Q^*_{j(\kappa)}$-generic g over $M[G^*_{j(\kappa)}]$.

Lemma 33 *Assume GCH and let $j : V \to M$ witness the superstrength of κ with $j(\kappa)$ minimal. Then $j(\kappa)$ has cofinality κ^+.*

Proof. Let $\langle f_i \mid i < \kappa^+\rangle$ be a list of all functions from κ to κ. Then the sequence $\langle j(f_i) \mid i < \kappa\rangle$ belongs to M, as it equals $j(\langle f_i \mid i < \kappa^+\rangle) \restriction \kappa$. For any ordinal $\alpha < \kappa^+$ we can use a bijection between α and κ and similarly conclude that $\langle j(f_i) \mid i < \alpha\rangle$ belongs to M.

Now for each $\alpha < \kappa^+$ let κ_α be least so that κ_α is closed under each $j(f_i)$, $i < \alpha$. Then κ_α is less than $j(\kappa)$, as $j(\kappa)$ is regular in M. Let κ^* be the supremum of the κ_α's. It suffices to show that there is a superstrong embedding j^* with critical point κ such that $j^*(\kappa) = \kappa^*$; then by the minimality of $j(\kappa)$, we must have $j(\kappa) = \kappa^*$ and therefore $j(\kappa)$ has cofinality κ^+.

To obtain j^* define $H = \{j(f)(a) \mid f : V_\kappa \to V,\ a \in V_{\kappa^*}\}$. Then H is an elementary submodel of M and $H \cap j(\kappa) = \kappa^*$. Let $\pi : H \simeq M^*$; then $j^* = \pi j : V \to M^*$ witnesses the superstrength of κ and $j^*(\kappa) = \pi(j(\kappa)) = \kappa^*$, as desired. $\Box$

We can assume that j is given by an ultrapower, and therefore that j is continuous at κ^+. It follows that $(j(\kappa)^+)^M$ has cofinality κ^+ and $H(j(\kappa)^+)^M$ is closed under κ-sequences. Therefore we can write the collection of $(j(\kappa)^+)^M$-many open dense subsets of $Q^*_{j(\kappa)}$ as the union of κ^+ subcollections, each of which belongs to $M[G^*_{j(\kappa)}]$ and has size less than $j(\kappa)$ there. Now we can build g in κ^+ steps, using $j(\kappa)$-distributivity to meet fewer than $j(\kappa)$ open dense sets at each step (and defining the $\Box$-sequence coherently at limit stages). We must also ensure that g extend g_κ; but this is easy to arrange as the latter is a condition in the forcing $Q^*_{j(\kappa)}$.

Finally the rest of G^* can be generated from $j[G]$ as before. $\Box$

The proof of the previous theorem does not work for hyperstrong κ, and there is a good reason for this. κ is *subcompact* iff for any $B \subseteq H_{\kappa^+}$ there are $\mu < \kappa$, $A \subseteq H_{\mu^+}$ and an elementary embedding $j : (H_{\mu^+}, A) \to (H_{\kappa^+}, B)$ with critical point μ. (Note that by elementarity, j must send μ to κ.)

Proposition 34 *(a) If κ is hyperstrong then κ is subcompact. (b) (Jensen) If there is a subcompact cardinal then $\Box$ (even when restricted to ordinals between κ and κ^+) fails.*

Proof. (a) Suppose that $j : V \to M$ witnesses hyperstrength. Then for all subsets B of $j(\kappa)^+$ in the range of j, j gives an elementary embedding of

(H_{κ^+}, A) into $(H_{j(\kappa)^+}, B)$, where $j(A) = B$; moreover this embedding belongs to M as j is hyperstrong and $j \restriction H_{\kappa^+}$ belongs to $H_{j(\kappa)^+}$. As the range of j is an elementary submodel of M, it follows that there is an elementary embedding of some (H_{μ^+}, A) into $(H_{j(\kappa)^+}, B)$ (sending μ to $j(\kappa)$) which belongs to the range of j. So $j(\kappa)$ is subcompact in Range j and therefore by elementarity subcompact in M. As j is elementary, κ is subcompact in V.
(b) Suppose that κ is subcompact and $\vec{C} = \langle C_\alpha \mid \kappa < \alpha < \kappa^+,\ \alpha \text{ singular}\rangle$ has the properties of a $\square$-sequence. By thinning out the C_α's we can ensure that each has ordertype at most κ. Let j be an embedding from $(H_{\mu^+}, \vec{\bar{C}})$ to $(H_{\kappa^+}, \vec{C})$, sending μ to κ. Let α be the supremum of the ordinals in the range of j. Then α has cofinality μ^+. The ordinals in the range of j form a $< \mu$-closed and therefore ω-closed unbounded subset of α. And Lim C_α is a closed unbounded subset of α. Therefore the intersection D of these two sets is unbounded in α. By the coherence property of $\vec{C}$, the ordertype of C_β for sufficiently large β in D is at least μ. But as the ordertype of C_α is at most κ (in fact less than κ), the ordertype of C_β for all β in D is strictly less than κ. Thus there are β in $D \subseteq$ Range j with C_β of ordertype not in Range j, contradicting the elementarity of j. $\square$

For uncountable, regular κ, $\diamondsuit_\kappa$ says that there exists $\langle D_\alpha \mid \alpha < \kappa\rangle$ such that D_α is a subset of α for each α and for every subset D of κ, $\{\alpha < \kappa \mid D_\alpha = D \cap \alpha\}$ is stationary in κ. $\diamondsuit$ asserts that $\diamondsuit_\kappa$ holds for every uncountable, regular κ.

Theorem 35 *(Large cardinals and $\diamondsuit$) If κ is superstrong then there is an outer model in which κ is still superstrong and $\diamondsuit$ holds. The same holds for hyperstrong, n-superstrong for finite n and ω-superstrong.*

Proof. We use a reverse Easton iteration P where at each regular stage α, Q_α is α-distributive (in fact, in the present context the entire iteration starting with α is α-closed). A condition in Q_α is a sequence $\langle D_\beta \mid \beta < \gamma\rangle$, $\gamma < \alpha$, such that D_β is a subset of β for each $\beta < \gamma$. It is easy to show that a Q_α-generic yields a $\diamondsuit_\alpha$-sequence, using the α-closure of Q_α.

The proof in the superstrong case is just as in Theorem 32, where we take $G^*_{j(\kappa)}$ to be the intersection of $G_{j(\kappa)}$ with $P^*_{j(\kappa)}$ and then build a $Q^*_{j(\kappa)}$-generic containing the condition g_κ. For hyperstrong κ (witnessed by $j : V \to M$), we need only observe that $j[G_{\kappa^+}]$ has a lower bound in the forcing $P^*_{j(\kappa)^+}$

and choose $G^*_{j(\kappa)^+} = G_{j(\kappa)^+}$ to contain this lower bound. (This is where the argument with $\Box$ breaks down.)

For n-superstrongs, $1 < n$ finite, we can take $G^*_{j^n(\kappa)}$ to be the intersection of $G_{j^n(\kappa)}$ with $P^*_{j^n(\kappa)}$ (requiring the latter to contain the greatest lower bound of $j[G_{j^{n-1}(\kappa)}]$), but face the problem of defining a $Q^*_{j^n(\kappa)}$-generic containing the image of $g_{j^{n-1}(\kappa)}$ under (the lifting to $V[G_{j^{n-1}(\kappa)}]$ of) j. We use the following.

Lemma 36 *Suppose that n is greater than 1 and $j : V \to M$ witnesses the n-superstrength of κ, with $j^n(\kappa)$ chosen minimally. Then j is continuous at $j^{n-1}(\kappa)$ (i.e., the range of j is cofinal in $j^n(\kappa)$).*

Proof. Let κ^* be the supremum of the range of j intersect $j^n(\kappa)$. It suffices to show that there is an n-superstrong embedding j^* with critical point κ such that $(j^*)^n(\kappa) = \kappa^*$.

Let H consist of all elements of M of the form $j(f)(a)$, where $f : V_\kappa \to V$ and a belongs to V_{κ^*}. Then H is an elementary submodel of M: If $M \vDash \varphi(y, j(f_1)(a_1), \ldots, j(f_n)(a_n))$ for some y in M, where $f_i : V_\kappa \to V$, $a_i \in V_{\kappa^*}$ for each i, then choose $g : V_\kappa \to V$ so that $g(\langle x_1, \ldots, x_n\rangle) = y$ is a solution to $\varphi(y, f_1(x_1), \ldots, f_n(a_n))$ in V (if there is such a solution y in V). Then by elementarity, $M \vDash \varphi(y, j(f_1)(a_1), \ldots, j(f_n)(a_n))$ where y equals $j(g)(\langle a_1, \ldots, a_n\rangle)$. The latter is an element of H.

Note that $H \cap j^n(\kappa) = \kappa^*$: If $j(f)(a)$ is less than $j^n(\kappa)$, where $f : V_\kappa \to V$ and $a \in V_{\kappa^*}$, then $j(f)(a)$ is less than the supremum of $j(f)[V_\alpha] \cap j^n(\kappa)$ where $\alpha \in \text{Range }(j) \cap \kappa^*$ is large enough so that V_α contains a; as the latter supremum belongs to Range (j), it follows that $j(f)(a)$ is less than κ^*.

Now let $\pi : H \simeq M^*$ be the transitive collapse of H and define $j^* = \pi j$. Then $j^* : V \to M^*$ is elementary and has critical point κ. As π sends $j^n(\kappa)$ to κ^* and is the identity on κ^*, it follows that $(j^*)^m(\kappa) = j^m(\kappa)$ for $m < n$ and $(j^*)^n(\kappa) = \kappa^*$. And j^* is n-superstrong as $V_{\kappa^*} = V^M_{\kappa^*} = V^{M^*}_{\kappa^*}$. $\Box$

We may assume that every element of M is of the form $j(f)(a)$ where $f : V_{j^{n-1}(\kappa)} \to V$ and a belongs to $V_{j^n(\kappa)}$. Now we claim that the image of $g_{j^{n-1}(\kappa)}$ under (the lifting to $V[G_{j^{n-1}(\kappa)}]$ of) j generates a generic for $Q^*_{j^n(\kappa)}$, in the sense that every dense subset of $Q^*_{j^n(\kappa)}$ which belongs to $M[G^*_{j^n(\kappa)}]$ is met by a condition in $j[g_{j^{n-1}(\kappa)}]$. For, if D is such a dense set, then D has a name of the form $j(f)(a)$ where a belongs to $V_{j(\alpha)}$ for some $\alpha < j^{n-1}(\kappa)$. By the $j^{n-1}(\kappa)$-closure of $Q_{j^{n-1}(\kappa)}$, there is a condition $\bar{p} \in g_{j^{n-1}(\kappa)}$ which meets all dense sets with names of the form $f(\bar{a})$, $\bar{a} \in V_\alpha$; then $j(\bar{p}) = p$ meets D.

Finally, ω-superstrength is handled just as in the case of GCH. □

The technique of the previous proof can also be used to force a weakened form of □, preserving very large cardinals. □ *holds at small cofinalities* iff the □ principle holds when restricted to singular ordinals of cofinality at most the least superstrong cardinal.

Theorem 37 *If κ is hyperstrong then thre is an outer model in which κ is still hyperstrong and □ holds at small cofinalities.*

Proof. Perform a reverse Easton iteration where at each regular stage α, Q_α adds a □-sequence on the singular ordinals less than κ which have cofinality at most the least superstrong cardinal. If $j : V \to M$ witnesses that κ is hyperstrong, then we take $G^*_{j(\kappa)^+}$ to be $G_{j(\kappa^+}$ and observe that $j[g_{\kappa^+}]$ does have a greatest lower bound in $Q_{j(\kappa)^+}$, because its supremum is an ordinal of cofinality κ^+, greater than the least superstrong cardinal of M. By choosing $g^*_{j(\kappa)^+} = g_{j(\kappa)^+}$ to contain this greatest lower bound, we can lift j to $V[G_{\kappa^++1}]$, and then to all of $V[G]$. If $j : V \to M$ witnesses the 2-superstrength of κ then similarly we get a greatest lower bound for $j[G_{j(\kappa)}]$ in $P_{j^2(\kappa)}]$ as for each regular $\alpha \in (\kappa, j(\kappa))$, the supremum of $j[\alpha]$ is an ordinal of cofinality greater than κ, which is superstrong (and more) in M. Then we use the argument of the preceding proof to lift j to all of $V[G]$. A similar argument handles ω-superstrength. □

Theorem 38 *(Large cardinals and Gap* 1 *morasses) If κ is superstrong then there is an outer model in which κ is still superstrong and gap* 1 *morasses exist at each regular cardinal. The same holds for hyperstrong, n-superstrong for finite n and ω-superstrong.*

Proof. For the definition of a gap 1 morass we refer the reader to Devlin's book. Assume GCH and let κ be superstrong. We apply the reverse Easton iteration P where at each regular stage α, Q_α adds a gap 1 morass at α. A condition in Q_α is a size $< \alpha$ initial segment of a morass up to some top level, together with a map of an initial segment of this top level into α^+ which obeys the requirements of a morass map. To extend a condition, we end-extend the morass up to its top level and require that the map from the given initial segment of its top level into α^+ factor as the composition of a map into the top level of the stronger condition followed by the map given by the stronger condition into α^+. The forcing Q_α is α-closed and, using a Δ-system argument, is α^+-cc.

To obtain the desired G^*, we must build a $Q^*_{j(\kappa)}$-generic which extends the image under (the lifting to $V[G_\kappa]$ of) j of the Q_κ-generic g_κ. As in the case of $\Box$ we use minimisation of $j(\kappa)$ to ensure that it has cofinality κ^+ and then build a $Q^*_{j(\kappa)}$-generic in κ^+ steps. Note that any condition in $j[g_\kappa]$ is extended by one which has top level κ and maps an initial segment of the top level into $j(\kappa)^+$ using j. Now given fewer than $j(\kappa)$ maximal antichains in $M[G^*_{j(\kappa)}]$, we can choose $\alpha < j(\kappa)^+$ of cofinality $j(\kappa)$ in M so that these maximal antichains are maximal when restricted to conditions which are "below α" in the sense that they map an initial segment of their top level into α. Moreover, there is a condition which serves as a lower bound to all conditions in $j[g_\kappa]$ which are below α in this sense. Therefore we can choose a condition below α meeting all of the given maximal antichains compatibly with the conditions in $j[g_\kappa]$ which are below α, and therefore compatibly with all conditions in $j[g_\kappa]$. Repeating this in κ^+ steps for increasingly large $\alpha < j(\kappa)^+$ of M-cofinality $j(\kappa)$ (taking unions at limit stages) yields the desired $Q^*_{j(\kappa)}$-generic. The remainder of the generic G^* can be generated as before.

Now suppose that κ is hyperstrong. We must define a suitable $Q^*_{j(\kappa)^+}$-generic. We may assume that j is given by a hyperextender and therefore j is cofinal from κ^{++} into $j(\kappa)^{++}$ of M. Let S consist of those morass points at the top level (i.e., level κ^+) of g_{κ^+} which have cofinality κ^+. For each σ in S let $g_{\kappa^+} \upharpoonright \sigma$ denote the set of conditions in g_{κ^+} which are below σ. Then $j[g_{\kappa^+} \upharpoonright \sigma]$ has a greatest lower bound p_σ in $Q^*_{j(\kappa)^+}$.

The collection of maximal antichains of $Q^*_{j(\kappa)^+}$ which belong to $M[G^*_{j(\kappa)^+}]$ can be written as a union $\bigcup_{i<j(\kappa)^+} X_i$ where for each i and each σ in S, $X_i \upharpoonright j(\sigma)$ (the subset of X_i consisting of those maximal antichains all of whose elements are below $j(\sigma)$) is a set of size at most $j(\kappa)$ in M. By induction on $\sigma \in S$ choose a condition q_σ extending p_σ and all q_τ, $\tau \in S \cap \sigma$, which meets all antichains in $X_0 \upharpoonright j(\sigma)$. By hyperstrength, the sequence of $q_\tau \upharpoonright j(\sigma)$, $\tau \in S$, has a greatest lower bound p^1_σ for each $\sigma \in S$. Now repeat this construction

for $X_1, X_2, \ldots$ for $j(\kappa)^+$ steps, resulting in a set of conditions which generates a generic $g_{j(\kappa)^+}$ for $Q^*_{j(\kappa)^+}$. As before, the remainder of the generic G^* can be generated as before.

The cases of n-superstrength, $2 \leq n$ finite, are handled as in the proof of Theorem 35. ω-superstrength is handled as in the case of GCH. $\square$

Questions. 1. The above proofs show that one can force the GCH and $\square$ preserving the superstrength of *all* superstrong cardinals and GCH preserving the hyperstrength of *all* hyperstrong cardinals. Is it possible to force GCH preserving the 2-superstrength of all 2-superstrong cardinals?
2. It is possible to force a definable wellordering of the universe over a model of GCH preserving the superstrength of all superstrong cardinals, at the cost of some cardinal collapsing. Is it possible to do this without cardinal collapsing? Is it possible to preserve the superstrength of all superstrong cardinals while forcing not only the universe but also each $H(\kappa)$, $\kappa > \omega_1$, to have a definable wellordering?
3. Is it consistent with a superstrong cardinal to have a gap 2 morass at every regular cardinal?
4. To what extent are the condensation and hyperfine structural properties of L consistent with large cardinals?