Topics in Set Theory, Wintersemester 2007

1.Vorlesung

The singular cardinal hypothesis, SCH

Easton: The continuum function $C(\kappa) = 2^\kappa$ on *regular* cardinals is subject only to the following conditions:

Monotonicity: $\kappa_0 < \kappa_1 \to C(\kappa_0) \leq C(\kappa_1)$.

Cofinality jump: cof $C(\kappa) > \kappa$.

I.e., if GCH holds and C is any class function on the regular cardinals obeying the above conditions, there is a cofinality-preserving forcing extension where $2^\kappa = C(\kappa)$ for all regular κ.

Generalised continuum problem: What are the possible behaviours of the continuum function on *arbitrary* cardinals?

Key subproblem: Can the GCH fail at a singular strong limit cardinal?

Initially it was thought that a positive answer to this type of question would be obtained using a forcing method similar to Easton's. However Silver showed that the situation is not so simple.

Theorem 1 *Suppose that GCH holds below a singular cardinal λ of uncountable cofinality. Then the GCH holds at λ as well. If λ is a singular cardinal of uncountable cofinality and $2^\alpha \leq \alpha^{++}$ for $\alpha < \lambda$, then $2^\lambda \leq \lambda^{++}$. (And more.)*

This says nothing about singular cardinals of countable cofinality, and nothing about the possibility of getting the GCH to fail at a singular strong limit cardinal. In the wake of Silver's result, Jensen proved the following important result:

Theorem 2 *Suppose that $0^\#$ does not exist. Then every uncountable set of ordinals is a subset of a constructible set of ordinals of the same cardinality.*

Corollary 3 *In forcing extensions of L, the GCH holds at singular strong limit cardinals.*

Proof. Suppose that V is a forcing extension of L. Then $0^\#$ does not exist. By Jensen's theorem, every uncountable set of ordinals is a subset of a constructible set of the same size. Now suppose that λ is a singular strong limit cardinal and let α be the maximum of cof λ and ω_1. There are at most λ^+ constructible subsets of λ, as GCH holds in L. It follows from Jensen's theorem that there are at most $\lambda^+ \cdot 2^\alpha = \lambda^+$ subsets of λ of size α. But as λ is a strong limit cardinal, the number of subsets of λ is the same as the number of size α subsets of λ, and therefore 2^λ is λ^+. □

Thus if it is possible to violate the GCH at a singular strong limit cardinal, one must use large cardinals. Silver and Prikry achieved such a violation from a supercompact cardinal.

Theorem 4 *Suppose that GCH holds and κ is κ^{++}-supercompact. Then in a forcing extension, κ is a measurable cardinal where the GCH fails.*

Theorem 5 *Suppose that κ is measurable. Then in a cardinal-preserving forcing extension with no new bounded subsets of κ, κ is a strong limit cardinal of cofinality ω.*

Corollary 6 *Con(ZFC + there is a κ which is κ^{++}-supercompact) implies Con(ZFC + the GCH fails at a singular, strong limit cardinal).*

This work was the beginning of the study of the singular cardinal problem, a study which has involved some of the deepest work in large cardinal forcing.

2.Vorlesung

Basic Prikry forcing

Let κ be measurable and U a normal ultrafilter on κ.

Definition 7 *Let P be the set of pairs (p, A) such that:*

(1) p is a finite subset of κ.
(2) A is an element of U.
(3) $\min A > \max p$.

(p, A) is an *extension* of (q, B) ($(p, A) \leq (q, B)$) iff p end-extends q, A is a subset of B and $p \setminus q$ is contained in B. (p, A) is a *direct* or *Prikry extension* of (q, B) ($(p, A) \leq^* (q, B)$) iff $p = q$ and A is a subset of B.

The next lemma is easily verified.

Lemma 8 *(a) If G is P-generic then $\bigcup\{p \mid (p, A) \in G$ for some $A\}$ is an ω-sequence cofinal in κ.*
(b) P is κ^+-cc.
(c) The direct extension relation $\leq^*$ is κ-closed.

Lemma 9 (The Prikry property) *If σ is a sentence of the forcing language then every condition (q, B) has a direct extension (q, A) which decides σ (i.e., either forces σ or $\sim \sigma$).*

Proof. Define $h : [B]^{<\omega} \to 2$ as follows:

$h(s) = 1$ iff $(q \cup s, C) \Vdash \sigma$ for some C
$h(s) = 0$ otherwise.

As U is a normal ultrafilter, there is $A \in U$ which is homogeneous for h, i.e., for each $n \in \omega$ and s_1, $s_2 \in [A]^n$, $h(s_1) = h(s_2)$. We claim that (q, A) decides σ. Otherwise there would be extensions $(q \cup s_1, B_1)$, $(q \cup s_2, B_2)$ of (q, A) which force σ, $\sim \sigma$, respectively. We can assume that s_1 and s_2 have the same size n. Thus both s_1 and s_2 belong to $[A]^n$. But then $h(s_1) = 0$, $h(s_2) = 1$, contradicting the homogeneity of A. □

Corollary 10 *P does not add bounded subsets of κ.*

Proof. Suppose $(p, A) \Vdash \dot{a}$ is a subset of λ, where λ is less than κ. Set $(p, A_0) = (p, A)$ and by Lemma 9 choose a direct extension (p, A_1) of (p, A_0) which decides "$0 \in \dot{a}$". Then choose a direct extension (p, A_2) of (p, A_1) which decides "$1 \in \dot{a}$", etc. After λ steps one arrives at a direct extension (p, A_λ) of (p, A) which decides which ordinals less than λ belong to $\dot{a}$, and therefore forces $\dot{a}$ to belong to the ground model. □

Corollary 11 *If G is P-generic then κ has cofinality ω in $V[G]$ and V, $V[G]$ have the same cardinals and bounded subsets of κ. In particular, if GCH fails at κ in V, then in $V[G]$, κ is a singular strong limit cardinal where the GCH fails.*

Suppose that G is P-generic and let C be $\bigcup\{p \mid (p, A)$ belongs to G for some $A\}$. Then C is called a *Prikry sequence for U (over V)*. Note that the entire generic G can be recovered from C:

$$G = \{(p, A) \mid p \text{ is an initial segment of } C \text{ and } C \setminus p \text{ is a subset of } A\}.$$

The above holds as the set on the right is a compatible set of conditions containing G. Thus $V[G] = V[C]$. Also, C generates U in the sense that A belongs to U iff A belongs to V and C is almost contained in A. ("Almost" means "with finitely many exceptions".)

Theorem 12 *Suppose that M is an inner model containing the normal ultrafilter U on κ and C is an ordertype ω subset of κ which is almost contained in each element of U. Then C is a Prikry sequence for U (over M).*

Proof. We need to show that the set

$$G(C) = \{(p, A) \mid p \text{ is an initial segment of } C \text{ and } C \setminus p \text{ is contained in } A\}$$

is P-generic over M. It suffices to check that $G(C)$ intersects all dense subsets of P in M. First we show:

Lemma 13 *Suppose that (q, B) is a condition and D is open dense. Then there is a direct extension (q, A) of (q, B) and $m \in \omega$ such that for all $n \geq m$ and $s \in [A]^n$, the condition $(q \cup s, A(> \max s))$ belongs to D.*

Proof. Define $h : [B]^{<\omega} \to 2$ as follows:

$h(s) = 1$ iff $(q \cup s, C) \in D$ for some C
$h(s) = 0$ otherwise.

Let $A' \in U$, $A' \subseteq B$, be homogeneous for h. As (q, A') has an extension in D, there exists an m such that $h(s) = 1$ for all $s \in [A']^n$, all $n \geq m$. For each $s \in [A']^n$, $n \geq m$, choose $A_s \in U$, $A_s \subseteq A'$, so that $(q \cup s, A_s)$ belongs to D. Now we take A to be the "diagonal intersection" $\triangle\{A_s \mid s \in [A']^n, n \geq m\}$ of these A_s, where

$\triangle\{A_s \mid s \in [A']^n, n \geq m\} = \{\alpha < \kappa \mid \alpha \in A_s$ for all $n \geq m$ and for all $s \in [A']^n$ with $\max s < \alpha\}$.

Then A is in U as it contains the usual diagonal intersection of sets in U. The condition (q, A) is as desired, since for each $n \geq m$ and $s \in [A]^n$, we have $A(> \max s) \subseteq A_s$ and so $(q \cup s, A(> \max s))$ belongs to D. $\square$ (Lemma 13)

Now we show that $G(C)$ intersects all open dense D in M. For each finite $q \subseteq \kappa$, use the previous lemma to choose $m(q) \in \omega$ and $A(q) \in U$ so that $\min A(q) > \max q$ and for $n \geq m(q)$, s in $[A(q)]^n$, the condition $(q \cup s, A(q)(> \max s))$ belongs to D. Let A be the diagonal intersection $\triangle\{A(q) \mid q \in [\kappa]^{<\omega}\}$. As A belongs to U, there is $\tau < \kappa$ such that $C \setminus \tau$ is contained in $A \setminus \tau$. Consider the condition $(C \cap \tau, A \setminus \tau)$. Then for every $n \geq m(C \cap \tau)$ and every $s \in [C \setminus \tau]^n$ we have

$((C \cap \tau) \cup s, A(> \max s))$ belongs to D,

since s is contained in $A \setminus \tau$ and therefore in $A(C \cap \tau)$, and $A(> \max s))$ is contained in $A(C \cap \tau)(> \max s)$. Choose $s \in [C \setminus \tau]^n$ for some $n \geq m(C \cap \tau)$. Then $(C \cap \tau) \cup s$ is contained in C and $C(> \max s)$ is contained in $A(> \max s)$. So $((C \cap \tau) \cup s, A(> \max s))$ belongs to $G(C) \cap D$. $\square$

3.Vorlesung

Tree Prikry forcing

We eliminate the assumption of normality for the ultrafilter U in basic Prikry forcing. Assume only that U is a κ-complete non-principal ultrafilter on the measurable cardinal κ.

Definition 14 *A set T is called a U-*tree with trunk *t iff*

(1) T consists of finite increasing sequences below κ.
(2) $(T, \preceq)$ is a tree, where $\preceq$ is the initial segment relation.
(3) For every $\eta \in T$, $\eta \preceq t$ or $t \preceq \eta$.
(4) For every $\eta \in T$, if $t \preceq \eta$ then $\{\alpha < \kappa \mid \eta * \alpha \in T\}$ belongs to U.

For each $n \in \omega$ we let $\mathrm{Lev}_n(T)$ denote the set of nodes in T of length n.

The conditions in *Tree Prikry forcing* are the pairs (t, T) where T is a U-tree and t is the trunk of T. Extension is defined by: $(t, T) \leq (s, S)$ iff

$T \subseteq S$. Note that this implies $s \preceq t \in S$. If in addition $s = t$, then we say that (t, T) is a *direct or Prikry* extension of (s, S), written $(t, T) \preceq^* (s, S)$.

The following is an immediate consequence of the κ-completeness of the ultrafilter U.

Lemma 15 *Suppose that T_α, $\alpha < \lambda$, are U-trees with the same trunk t and λ is less than κ. Then the intersection of the T_α's is also a U-tree with trunk t.*

It now follows, as with basic Prikry forcing, that if P denotes Tree Prikry forcing, then for P-generic G, $\bigcup\{t \mid (t, T) \in G$ for some $T\}$ is an ω-sequence cofinal in κ, P is κ^+-cc and the direct extension relation $\preceq^*$ is κ-closed. We next prove the Prikry property.

Lemma 16 (The Prikry Property) *If (t, T) is a condition and σ is a sentence of the forcing language, then there is a direct extension (s, S) of (t, T) which decides σ.*

Proof. Let us say that a finite increasing sequence s is *indecisive* iff there is no U-tree S with trunk s such that (s, S) decides σ. If the lemma fails, then the node t is indecisive: For, if (t, S) decides σ then so does $(t, T \cap S)$, a direct extension of (t, T).

Now note that if s is indecisive, it must be the case that $s * \alpha$ is indecisive for a set of α in U: Otherwise we can choose $T(s * \alpha)$ for U-measure one α such that $(s * \alpha, T(s * \alpha))$ decides σ in the same way for all such α, and then form a U-tree S with trunk s by glueing together these $T(s * \alpha)$; the resulting condition (s, S) would then decide σ.

It follows that we can inductively form a U-tree S with trunk t consisting entirely of indecisive nodes. But this is impossible, as the condition (t, S) has some extension (u, R) which decides σ, demonstrating that the node u of S is not indecisive. $\square$

As an easy corollary we have:

Corollary 17 *Tree Prikry forcing at κ adds no bounded subsets of κ, preserves cardinals and gives κ cofinality ω.*

Prikry forcing at cofinality ω

Suppose that κ is the supremum of an increasing sequence of measurable cardinals κ_n, $n \in \omega$, where κ_n carries the nonprincipal κ_n-complete ultrafilter U_n. We describe a cofinality-preserving forcing P for adding an element of $\prod_n \kappa_n$ which eventually dominates each element of this product in the ground model.

Definition 18 *A condition in P is a sequence $p = \langle p_n \mid n \in \omega \rangle$ where:*

(1) For each n, p_n is either an element of U_n or an ordinal less than κ_n.
(2) There is an $l(p) < \omega$ such that for $n < l(p)$, p_n is an ordinal less than κ_n and for $n \geq l(p)$, p_n is an element of U_n.

We say that p *extends* q, written $p \leq q$, iff for each n, one of the following holds:

(a) $p(n) = q(n)$ is an ordinal less than κ_n.
(b) $p(n) \in q(n)$ where $q(n)$ is an element of U_n.
(c) $p(n) \subseteq q(n)$ both belong to U_n.

Note that $p \leq q$ implies that $l(p)$ is at least $l(q)$. We say that p is a *direct or Prikry* extension of q, written $p \leq^* q$, iff $p \leq q$ and $l(p) = l(q)$.

For each n and $p \in P$ we write $p(< n)$ for p restricted to n and $p(\geq n)$ for p restricted to $[n, \omega)$. For each n, P naturally factors as $P(< n) \times P(\geq n)$, where $P(< n)$ consists of all $p(< n)$, $p \in P$, and $P(\geq n)$ consists of all $p(\geq n)$, $p \in P$. Note that the direct extension relation on $P(\geq n)$ is κ_n-closed.

We now prove the important Prikry property.

Lemma 19 (The Prikry Property) *If $p = \langle p_n \mid n \in \omega \rangle$ is a condition and σ is a sentence of the forcing language, then p has a direct extension q which decides σ.*

Proof. Suppose that s is a finite sequence from $\prod_{m<n(s)} \kappa_m$ for some $n(s) \in \omega$. We say that s is *indecisive* iff there is no p deciding σ with $p(< n(s)) = s$ and $l(p) = n(s)$. If the lemma fails, then $p(< l(p))$ is indecisive.

Suppose that s is indecisive. Then for $U_{n(s)}$-measure one α, $s * \alpha$ is indecisive: Otherwise, we could choose $p(s * \alpha)$, with $p(s * \alpha)(< n(s) + 1) = s * \alpha$

and $l(p(s*\alpha)) = n(s)+1$, for $U_{n(s)}$-measure one α which decide σ in the same way, and then using the $\kappa^+_{n(s)}$-closure of U_n, $n > n(s)$, glue the $p(s*\alpha)$ together to a single p with $p(< n(s)) = s$, $l(p) = n(s)$, which would decide σ.

Now again using the closure properties of the ultrafilters U_n, we can build a condition $q \leq^* p$ so that $r(< l(r))$ is indecisive for each $r \leq q$. But this is impossible, as q has some extension r which decides σ, contradicting the indecisiveness of $r(< l(r))$. □

As an easy corollary we obtain:

Corollary 20 *Let P denote the above forcing. Then P adds no new bounded subsets of $\kappa = \bigcup_{n\in\omega} \kappa_n$, is κ^+-cc, preserves cofinalities and adds an element of $\prod_{n\omega} \kappa_n$ which eventually dominates each ground model element of that product.*

The last statement of the above corollary holds as if p_0 is an element of $\prod_{n\omega} \kappa_n$, it is dense for $p \in P$ to have the property that for $n \geq l(p)$, $\min p(n)$ is greater than $p_0(n)$.

Supercompact Prikry forcing

For $\kappa \leq \lambda$, κ regular, $P_\kappa\lambda$ denotes the set of size $< \kappa$ subsets of λ. An ultrafilter U on $P_\kappa\lambda$ is *fine* iff it contains the set $\{x \in P_\kappa\lambda \mid \alpha \in x\}$ for each $\alpha < \lambda$. A function $f : A \to \lambda$, $A \subseteq P_\kappa\lambda$, is *regressive* iff $f(a) \in a$ for each $a \in A$. An ultrafilter U on $P_\kappa\lambda$ is *normal* iff it is fine, κ-complete and any function which is *regressive* on a set in U is constant on a set in U.

Definition 21 *κ is* λ-strongly compact *iff there is a fine, κ-complete ultrafilter on $P_\kappa\lambda$. And κ is* λ-supercompact *iff there is a normal ultrafilter on $P_\kappa\lambda$.*

Prikry forcing with a normal ultrafilter on $P_\kappa\lambda$ is analogous to basic Prikry forcing, with κ replaced by $P_\kappa\lambda$ and the standard ordering on κ replaced with the following ordering on $P_\kappa\lambda$:

Definition 22 *For a, b in $P_\kappa\lambda$ we say that a is* strongly included *in b, written $a < b$, iff a is a subset of b and the ordertype of a is less than the ordertype of $b \cap \kappa$.*

Lemma 23 *Suppose that U is a normal ultrafilter on $P_\kappa\lambda$.*
(a) If F is a function defined on a set in U such that $F(a) < a$ for each a in the domain of F, then F is constant on a set in U.
(b) Suppose that A_a belong to U for each $a \in P_\kappa\lambda$. Then the diagonal intersection *$\triangle_a A_a = \{b \mid b \in A_a$ for each $a < b\}$ belongs to U.*

Proof. (a) Note that function $a \mapsto$ ordertype $(F(a))$ is regressive on a set in U and therefore constant with some value $\bar{\kappa} < \kappa$ on a set in U. Also, for each $\alpha < \bar{\kappa}$, the function $a \mapsto \alpha$-th element of $F(a)$ is regressive and therefore constant on a set in U. As there are fewer than κ possible α's, it follows that F is constant on a set in U.
(b) If not then there is a function G defined on a set in U such that $G(a) < a$ and a does not belong to $A_{G(a)}$ for each a. But then by (a), G is constant on a set in U, which contradicts the fact that each $A_{G(a)}$ belongs to U. □

Definition 24 *For $A \subseteq P_\kappa\lambda$, $[A]^{[n]}$ denotes the set of n-element subsets of A which are totally ordered by $<$, and $[A]^{[<\omega]}$ denotes the union of the $[A]^{[n]}$, $n \in \omega$.*

The following is a generalisation of the fact that measurable cardinals are "measure-one Ramsey". The proof is as in the measurable cardinal case, using Lemma 23 (b).

5.Vorlesung

Lemma 25 *Suppose that U is a normal ultrafilter on $P_\kappa\lambda$. If $F : [A]^{[<\omega]} \to 2$ and A belongs to U then there is $B \subseteq A$ in U such that $F \restriction [B]^{[n]}$ is constant for each $n \in \omega$.*

In our analysis of the effect of supercompact Prikry forcing on the cardinals, we will need the following important result of Solovay.

Theorem 26 *Suppose that κ is λ-strongly compact, $\kappa \leq \lambda$ regular. Then $\lambda^{<\kappa} = \lambda$.*

We need three lemmas.

Lemma 27 *Suppose that κ is λ-strongly compact, $\kappa \leq \lambda$ regular. Then there is an elementary embedding $j : V \to M$ with critical point κ such that every subset of M of size λ is covered by an element of M of M-cardinality less than $j(\kappa)$. In particular, $j(\kappa)$ is greater than λ and j is discontinuous at all regular cardinals in $[\kappa, \lambda]$.*

Proof. Suppose that U is a κ-complete fine ultrafilter on $P_\kappa\lambda$ and let $j : V \to M$ be the ultrapower via U. If $[f_\alpha]_U$, $\alpha < \lambda$, are elements of M then define $F(x) = \{f_\alpha(x) \mid \alpha \in x\}$. Then $[f_\alpha]_U$ belongs to $[F]_U$ for each α by the fineness of U and $[F]_U$ has M-cardinality less than $j(\kappa)$. Thus every subset of M of size at most λ is covered by an element of M of M-cardinality less than $j(\kappa)$. It follows that $\sup j[\mu]$ is singular in M for each regular $\mu \in (\kappa, \lambda]$ as $\sup j[\mu]$ has cofinality $\mu \leq \lambda$ in V and therefore M-cofinality less than $j(\kappa) < \sup j[\mu]$. As $j(\mu)$ is regular in M for regular μ, it follows that j is discontinuous at all regulars in $(\kappa, \lambda]$. The above covering property also implies that $j(\kappa)$ is greater than λ and the κ-completeness of U implies that κ is therefore the critical point of j. □ (Lemma 27)

Lemma 28 *Suppose that κ is λ-strongly compact, $\kappa \leq \lambda$ regular. Then if $\langle S_i \mid i < \gamma \rangle$, $\gamma < \kappa$, is a sequence of stationary subsets of $\lambda \cap$ Cof$(< \kappa)$, there exists $\bar\lambda < \lambda$ of cofinality strictly between $\aleph_0$ and κ such that $S_i \cap \bar\lambda$ is stationary for each $i < \gamma$.*

Proof. Let $j : V \to M$ be as in the previous Lemma. By elementarity, it suffices to show that in M there is $\delta < j(\lambda)$ of M-cofinality strictly between $\aleph_0$ and $j(\kappa)$ such that $j(S_i) \cap \delta$ is stationary in M for each $i < \gamma$. Let δ be $\sup j[\lambda] < \lambda$. Then δ has cofinality λ in V and therefore cofinality strictly between $\aleph_0$ and $j(\kappa)$ in M.

Note that $j[\lambda]$ is a $< \kappa$-closed unbounded subset of δ. Now suppose that C is closed unbounded in δ. Then $C' = C \cap j[\lambda]$ is $< \kappa$-closed unbounded in δ. Let D be $j^{-1}[C']$. Then D is $< \kappa$-closed unbounded in δ and therefore $S_i \cap D$ is nonempty. It follows that $j(S_i) \cap C'$ is nonempty and therefore we have shown that each $j(S_i)$, $i < \gamma$, is stationary in δ (in V). □ (Lemma 28)

Lemma 29 *If $\bar\lambda < \lambda$ are regular then there is exist pairwise disjoint $\langle S_i \mid i < \lambda \rangle$ such that each S_i is a stationary subset of $\lambda \cap$ Cof$\bar\lambda$, the set of ordinals less than λ of cofinality $\bar\lambda$.*

Proof. For each $\alpha \in$ Cof$\bar\lambda$ let $f_\alpha : \bar\lambda \to \alpha$ be cofinal and continuous. Now for each $\delta < \lambda$ and $\alpha \in (\delta, \lambda)$ choose $i_{\delta,\alpha}$ such that $f_\alpha(i_{\delta,\alpha})$ is greater than δ. Then there is a stationary set $S_\delta \subseteq \lambda \cap$ Cof$\bar\lambda$ such that $i_{\delta,\alpha} = i_\delta$ is constant for α in S_δ and also $f_\alpha(i_\delta) = \lambda_\delta$ is constant for $\alpha \in S_\delta$. Now choose λ-many S_δ's with the same i_δ and distinct λ_δ; these stationary subsets of $\lambda \cap$ Cof$\bar\lambda$ are pairwise disjoint. □ (Lemma 29)

Now we prove Theorem 26. Let $\langle S_i \mid i < \lambda \rangle$ be pairwise disjoint stationary subsets of $\lambda \cap$ Cofω. We may assume that λ is greater than κ, as $\kappa^{<\kappa} = \kappa$ follows from the strong inaccessibility of κ. For each x in $[\lambda]^{<\kappa}$ choose $\lambda_x < \lambda$ of cofinality strictly between $\aleph_0$ and κ such that $S_i \cap \lambda_x$ is stationary for each $i \in x$. Let C_x be a closed unbounded subset of λ_x of ordertype cof λ_x. Then $S_i \cap C_x$ is stationary and therefore nonempty for each $i \in x$. Thus if x, y are distinct elements of $[\lambda]^{<\kappa}$ and $\lambda_x = \lambda_y$ then $C_x = C_y$ and $\{S_i \cap C_x \mid i \in x\} \neq \{S_i \cap C_y \mid i \in y\}$. Now there are at most λ possibilities for λ_x and for each λ_x, there are at most $[2^{\text{cof }(\lambda_x)}]^{<\kappa} \leq \kappa$ possibilities for $\{S_i \cap C_x \mid i \in x\}$. It follows that $[\lambda]^{<\kappa}$ is λ, as desired. □

We can extend Theorem 26 to the case of singular λ using the following.

Lemma 30 *If κ is λ-strongly compact then κ is also $\lambda^{<\kappa}$-strongly compact.*

Proof. Let U be a fine, κ-complete ultrafilter on $P_\kappa\lambda$. For x in $P_\kappa\lambda$ let $x^* \in P_\kappa P_\kappa\lambda$ be $\{y \in P_\kappa\lambda \mid y < x\}$. Then define U^* contained in the power set of $P_\kappa P_\kappa\lambda$ by: $A^* \in U^*$ iff $\{x \mid x^* \in A^*\}$ belongs to U. As U is a κ-complete ultrafilter on $P_\kappa\lambda$ it follows that U^* is a κ-complete ultrafilter on $P_\kappa P_\kappa\lambda$. If y belongs to $P_\kappa\lambda$, then the set of $x \in P_\kappa\lambda$ such that $y < x$ belongs to U by the fineness and κ-completeness of U. It follows that $\{a \in P_\kappa P_\kappa\lambda \mid y \in a\}$ belongs to U^*, so U^* is fine. □ (Lemma 30)

Corollary 31 *Suppose that κ is λ-strongly compact, $\kappa \leq \lambda$. If λ has cofinality at least κ then $\lambda^{<\kappa} = \lambda$ and otherwise $\lambda^{<\kappa} = \lambda^+$.*

Proof. If λ is regular then this follows from Theorem 26. If λ is singular of cofinality at least κ then $\lambda^{<\kappa}$ is the supremum of $\bar{\lambda}^{<\kappa}$, $\bar{\lambda} < \lambda$, which by Theorem 26 is λ. If λ has cofinality less than κ then by Lemma 30, κ is $\lambda^{<\kappa}$-strongly compact and therefore λ^+-strongly compact. Thus again by Theorem 26, $\lambda^{<\kappa} \leq (\lambda^+)^{<\kappa} = \lambda^+$. As $\lambda^{<\kappa}$ is greater than λ in this case, it follows that $\lambda^{<\kappa}$ equals λ^+. □

6.Vorlesung

We now define supercompact Prikry forcing.

Definition. P consists of all pairs $((a_1, \ldots, a_n), A)$ such that:

(1) $a_1 < a_2 < \cdots < a_n$ belong to $P_\kappa\lambda$.
(2) $A \in U$.
(3) $a_n < a$ for each $a \in A$.

$((a_1, \ldots, a_n), A)$ *extends* $((b_1, \ldots, b_m), B)$, written $((a_1, \ldots, a_n), A) \leq ((b_1, \ldots, b_m), B)$, iff:

(a) $n \geq m$.
(b) For $k \leq m$, $a_k = b_k$.
(c) $A \subseteq B$.
(d) a_k belongs to B for each k in $[m+1, n]$.

$((a_1, \ldots, a_n), A)$ *directly extends* $((b_1, \ldots, b_m), B)$, written $((a_1, \ldots, a_n), A) \leq^* ((b_1, \ldots, b_m), B)$, iff $((a_1, \ldots, a_n), A)$ extends $((b_1, \ldots, b_m), B)$ and $n = m$. The relation $\leq^*$ is κ-closed.

Lemma 32 *If σ is a sentence of the forcing language then every condition in supercompact Prikry forcing has a direct extension which decides σ.*

Proof. The proof is exactly as in the measurable cardinal case, now using Lemma 25. Suppose that $((a_1, \ldots, a_n), A)$ is a condition and define $h : [A]^{[<\omega]} \to 2$ as follows:

$h(b_1, \ldots, b_m) = 1$ iff $((a_1, \ldots, a_n, b_1, \ldots, b_m), C)$ forces σ for some C
$h(b_1, \ldots, b_m) = 0$, otherwise.

By Lemma 25, there is $B \subseteq A$ which is homogeneous for h, i.e., for each $n \in \omega$, h is constant on $[B]^{[n]}$. We claim that $((a_1, \ldots, a_n), B)$ decides σ. Otherwise there would be extensions $((a_1, \ldots, a_n, b_1, \ldots b_m), B_1)$ and $((a_1, \ldots, a_n, c_1, \ldots c_l), B_2)$ of $((a_1, \ldots, a_n), B)$ which force σ, $\sim \sigma$, respectively. We can assume that l equals m. Thus both $(b_1, \ldots, b_m)$ and $(c_1, \ldots, c_m)$ belong to $[B]^{[m]}$. But then $h(b_1, \ldots, b_m) = 1$ and $h(c_1, \ldots, c_m) = 0$, contradicting the homogeneity of B.
□

It follows that P does not add bounded subsets of κ and therefore preserves cardinals up to κ.

Let G be P-generic and let $C = (a_1, a_2, \ldots)$ be the limit of the $(a_1, \ldots, a_n)$ such that $((a_1, \ldots, a_n), A) \in G$ for some A. An easy density argument shows that if $\delta \leq \lambda$, then $\delta = \bigcup_n (a_n \cap \delta)$. Therefore, if $\delta \leq \lambda$ had cofinality at least κ in V, it will have cofinality ω in $V[G]$. It follows that κ^+ in $V[G]$ is at least λ^+ of V.

Now as λ-supercompact Prikry forcing P is (Card $P_\kappa\lambda)^+$-cc and card$P_\kappa\lambda = \lambda^{<\kappa}$, it follows from Corollary 31 that P is λ^+-cc when λ has cofinality at least κ and λ^{++}-cc when λ has cofinality less than κ. Thus in the former case, cofinalities greater than λ are preserved and κ^+ of $V[G]$ equals λ^+ of V. In the latter case, cofinalities greater than λ^+ are preserved; the remaining question is what happens to λ^+ itself.

Lemma 33 *Suppose that λ has cofinality less than κ in V. Then P changes the cofinality of λ^+ to ω.*

Proof. Fix in V an increasing sequence $\langle \lambda_i \mid i < \text{cof}\,\lambda \rangle$ of regular cardinals cofinal in λ, $\lambda_0 \geq \kappa$. As $\lambda^{<\kappa} = \lambda^+$ the cardinality of $\prod_{i<cof\lambda} \lambda_i$ is λ^+. Now inductively build a sequence $\langle f_\alpha \mid \alpha < \lambda^+ \rangle$ of elements of $\prod_{i<cof\lambda} \lambda_i$ with the following properties, where $<^*$ denotes $<$ on a final segment of cof λ:

(1) $\alpha < \beta \rightarrow f_\alpha <^* f_\beta$.
(2) $g \in \prod_{i<cof\lambda} \lambda_i \rightarrow g <^* f_\alpha$ for some $\alpha < \lambda^+$.

Recall the sequence $C = (a_0, a_1, \ldots)$ derived from G. For each n consider $g_n \in \prod_{i<cof\lambda} \lambda_i$ defined by $g_n(i) = \sup(a_n \cap \lambda_i)$. Then as the range of each f_α is contained in a_n for sufficiently large n, it follows that the g_n's are cofinal modulo a final segment of cof λ in $\prod_{i<cof\lambda} \lambda_i$ and therefore λ^+ has cofinality ω in $V[G]$. $\Box$ (Lemma 33)

This completes the analysis of supercompact Prikry forcing.

7.Vorlesung

Strongly compact Prikry forcing

The construction here is entirely analogous to that of Tree Prikry forcing with a (possibly) non-normal measure on κ.

Let U be a fine κ-complete ultrafilter on $P_\kappa\lambda$ which may fail to be normal.

Definition 34 *A set T is called a U-*tree with trunk *t iff*

(1) T consists of finite sequences $(x_1, \ldots, x_n)$ from $P_\kappa\lambda$ which are increasing in the Magidor relation $<$.
(2) $(T, \preceq)$ is a tree, where $\preceq$ is the initial segment relation.
(3) For every $\eta \in T$, $\eta \preceq t$ or $t \preceq \eta$.
(4) For every $\eta \in T$, if $t \preceq \eta$ then $\{x \mid \eta * x \in T\}$ belongs to U.

The conditions in *Tree Prikry forcing* are the pairs (t, T) where T is a U-tree and t is the trunk of T. Extension is defined by: $(t, T) \leq (s, S)$ iff $T \subseteq S$. Note that this implies $s \preceq t \in S$. If in addition $s = t$, then we say that (t, T) is a *direct or Prikry* extension of (s, S), written $(t, T) \leq^* (s, S)$.

The following is an immediate consequence of the κ-completeness of the ultrafilter U.

Lemma 35 *Suppose that T_α, $\alpha < \lambda$, are U-trees with the same trunk t and λ is less than κ. Then the intersection of the T_α's is also a U-tree with trunk t.*

If P denotes the above Tree Prikry forcing, then for P-generic G, the limit of the $t = (x_1, \ldots, x_n)$ such that (t, T) belongs to G for some T is an ω-sequence in $P_\kappa\lambda$ whose union is all of λ. It follows that each cardinal in the interval $[\kappa, \lambda]$ of cofinality at least κ is forced by P to have cofinality ω. P is $(\lambda^{<\kappa})^+$-cc and therefore all cofinalities greater than $\lambda^{<\kappa}$ are preserved. The direct extension relation $\leq^*$ is κ-closed. We also have:

Lemma 36 (The Prikry Property) *If (t, T) is a condition and σ is a sentence of the forcing language, then there is a direct extension (s, S) of (t, T) which decides σ.*

The proof of this lemma is just as in the case of a Tree Prikry forcing with a measure on κ. It follows that P does not add bounded subsets of κ and therefore cofinalities less than κ are preserved. Now recall the following:

(a) If λ has cofinality at least κ then $\lambda^{<\kappa} = \lambda$.
(b) If λ has cofinality less than κ then $\lambda^{<\kappa} = \lambda^+$ and P adds an ω-sequence cofinal in λ^+.

It follows that cofinalities greater than λ are preserved when λ has cofinality at least κ (and therefore κ^+ becomes λ^+ in the generic extension), and cofinalities greater than λ^+ are preserved when λ has cofinality less than κ (and therefore κ^+ becomes λ^{++} in the generic extension). This completes the analysis of Tree Prikry forcing for a strong compact.

Extender-based Prikry forcing at cofinality ω

We have seen how to add a new ω-sequence to an ω-limit κ of measurables $\langle \kappa_n \mid n \in \omega \rangle$ without adding new bounded subsets of κ. Now we wish to add many to obtain a violation of the singular cardinal hypothesis.

Assume GCH and let λ be regular and greater than $\kappa = \sup_{n \in \omega} \kappa_n$. We wish to add at least λ-many sequences through the product of the κ_n's without adding bounded subsets of κ.

We suppose that each κ_n is $H(\lambda^+)$-strong; this means that there is an elementary embedding $j_n : V \to M_n$ with critical point κ_n such that $H(\lambda^+)$ belongs to M_n and $j_n(\kappa_n)$ is greater than λ. We may assume that j_n is an *ultrapower embedding* which is equivalent to saying that every element of M_n is of the form $j_n(f)(\alpha)$ for some $f : \kappa_n \to \kappa_n$ and $\alpha < \lambda^+$. This implies that M_n is closed under κ_n-sequences. For each $\alpha < \lambda$ we consider the κ_n-complete ultrafilter $U_{n\alpha}$ defined by

$X \in U_{n\alpha}$ iff $X \subseteq \kappa_n$ and $\alpha \in j_n(X)$.

For $\alpha \leq \beta < \lambda$ we define the following ordering (which depends on the choice of j_n):

$\alpha \leq_n \beta$ iff $\alpha \leq \beta$ and for some $f : \kappa_n \to \kappa_n$, $j_n(f)(\beta) = \alpha$.

Remark. This implies that $U_{n\alpha}$ is below $U_{n\beta}$ in the *Rudin-Keisler ordering* of ultrafilters on κ_n. The Rudin-Keisler ordering of ultrafilters on a cardinal κ is defined by: $U_0 \leq_{RK} U_1$ iff for some $f : \kappa \to \kappa$, $A \in U_0$ iff $f^{-1}[A] \in U_1$. If f witnesses $\alpha \leq_n \beta$ then f also witnesses $U_{n\alpha} \leq_{RK} U_{n\beta}$. But the converse does not hold in general.

Lemma 37 *The partial ordering $\leq_n$ is κ_n-directed and in fact each $x \in [\lambda]^{<\kappa_n}$ has λ-many upper bounds in $\leq_n$.*

Proof. Using GCH, let $\langle a_\alpha \mid \alpha < \kappa_n\rangle$ be an enumeration of $[\kappa_n]^{<\kappa_n}$ such that for each $x \in [\kappa_n]^{<\kappa_n}$, the set of α such that $x = a_\alpha$ is a cofinal subset of $(\sup x)^+$. Now note that as j_n is the identity below κ_n, $\langle a_\alpha \mid \alpha < \kappa_n\rangle$ is the restriction to κ_n of $j_n(\langle a_\alpha \mid \alpha < \kappa_n\rangle)$; let $\langle a_\alpha \mid \alpha < \lambda\rangle$ denote the restriction of $j_n(\langle a_\alpha \mid \alpha < \kappa_n\rangle)$ to λ. Then for each $x \in [\lambda]^{<\lambda} \cap M_n$, the set of α such that $x = a_\alpha$ is a cofinal subset of $(\sup x)^+$.

Now suppose that x belongs to $[\lambda]^{<\kappa_n}$; we find $\alpha < \lambda$ such that $\beta <_n \alpha$ for each $\beta \in x$. Enumerate x in increasing order as $\langle \beta_i \mid i < \gamma\rangle$, where γ is less than κ_n and choose $\alpha < \lambda$ so that a_α equals x. If $\beta = \beta_i$ belongs to x, then $\beta = j_n(f)(\alpha)$, where f is defined by: $f(\bar{\alpha}) =$ the i-th element (in increasing order) of $a_{\bar{\alpha}}$. So $\beta <_n \alpha$. As there are λ-many α such that x equals a_α, we are done. □

Fix $\pi_{\alpha\beta}$ witnessing $\beta \leq_n \alpha$, setting $\pi_{\alpha\alpha}$ to be the identity.

Lemma 38 *Suppose that $\gamma < \beta \leq \alpha$ with $\gamma \leq_n \alpha$ and $\beta \leq_n \alpha$. Then $\{\nu < \kappa_n \mid \pi_{\alpha\beta}(\nu) > \pi_{\alpha\gamma}(\nu)\}$ belongs to $U_{n\alpha}$.*

Proof. Let X denote $\{\nu < \kappa_n \mid \pi_{\alpha\beta}(\nu) > \pi_{\alpha\gamma}(\nu)\}$. We wish to show that α belongs to $j_n(X)$. But $j_n(X)$ equals $\{\nu < j_n(\kappa_n) \mid j_n(\pi_{\alpha\beta})(\nu) > j_n(\pi_{\alpha\gamma})(\nu)\}$, so we must show that $j_n(\pi_{\alpha\beta})(\alpha) = \beta > j_n(\pi_{\alpha\gamma})(\alpha) = \gamma$, which follows from our hypothesis. □

Lemma 39 *Suppose that x belongs to $[\lambda]^{<\kappa_n}$ and $\beta \leq_n \alpha$ for each $\beta \in x$. Then there is $A \in U_{n\alpha}$ such that $\pi_{\alpha\beta_0}$ agrees with $\pi_{\beta_1\beta_0}\pi_{\alpha\beta_1}$ on A whenever $\beta_0 \leq_n \beta_1$ belong to x.*

Proof. We must show that $\{\nu \mid \pi_{\alpha\beta_0}(\nu) = \pi_{\beta_1\beta_0}\pi_{\alpha\beta_1}(\nu)\}$ belongs to $U_{n\alpha}$. By the definition of $U_{n\alpha}$ this means that $j_n(\pi_{\alpha\beta_0})(\alpha) = j_n(\pi_{\beta_1\beta_0})j_n(\pi_{\alpha\beta_1})(\alpha)$, which by the choice of the π's just says $\beta_0 = j_n(\pi_{\beta_1\beta_0})(\beta_1) = \beta_0$, so we are done. □

8.Vorlesung

We are now ready to define extender-based Prikry forcing at cofinality ω. We first define forcings Q_n for each n, and then put them together to form the desired forcing P. Each Q_n is the union of Q_{n0} and Q_{n1}, which we define next.

Definition 40 *Q_{n1} consists of all functions f from a subset of λ of size at most κ into κ_n, ordered by: $f \le g$ iff f extends g as a function.*

Definition 41 *Q_{n0} consists of triples (a, A, f) such that:*

(1) f belongs to Q_{n1}.
(2a) a is a subset of λ of size less than κ_n with a maximum .
(2b) a is disjoint from Dom f.
(2c) $\alpha \le_n \max a$ for each $\alpha \in a$.
(3) A belongs to the ultrafilter $U_{n,\max a}$.
(4) Whenever $\alpha \ge_n \beta$ belong to a then $\pi_{\max a,\beta}(\mu) = \pi_{\alpha\beta}\pi_{\max a,\alpha}(\mu)$ for all μ in A.
(5) Whenever $\alpha > \beta$ belong to a then $\pi_{\max a,\alpha}(\mu) > \pi_{\max a,\beta}(\mu)$ for μ in A.

Extension is defined by: $(a, A, f) \le (b, B, g)$ iff f extends g, a contains b and $A \subseteq \pi^{-1}_{\max a,\max b}[B]$.

Remark. (4) above implies that whenever $\alpha \ge_n \beta \ge_n \gamma$ belong to a then $\pi_{\alpha\gamma}(\mu) = \pi_{\beta\gamma}\pi_{\alpha\beta}(\mu)$ for all μ in $\pi_{\max a,\alpha}[A]$, as if $\mu = \pi_{\max a,\alpha}(\nu)$ then by (4), the left side is $\pi_{\alpha\gamma}(\mu) = \pi_{\alpha\gamma}\pi_{\max a,\alpha}(\nu) = \pi_{\max a,\gamma}(\nu)$ and also the right side is $\pi_{\beta\gamma}\pi_{\alpha\beta}(\mu) = \pi_{\beta\gamma}\pi_{\alpha\beta}\pi_{\max a,\alpha}(\nu) = \pi_{\beta\gamma}\pi_{\max a,\beta}(\nu) = \pi_{\max a,\gamma}(\nu)$.

Let Q_n be the union of Q_{n0} and Q_{n1}. The *direct extension* relation $\le^*$ on Q_n is simply the union of the extension relations on Q_{n0} and Q_{n1}. The *extension* relation $\le$ on Q_n is defined by: $p \le q$ iff p is a direct extension of q or $p \in Q_{n1}$, $q = (a, A, f) \in Q_{n0}$ where:

(a) p extends f.
(b) Dom p contains a.
(c) $p(\max a) \in A$.
(d) For β in a, $p(\beta) = \pi_{\max a,\beta}(p(\max a))$.

At last we define the desired forcing P.

Definition 42 *P consists of $p = \langle p_n \mid n \in \omega\rangle$ such that for each n, p_n belongs to Q_n and for some finite $l(p)$, p_n belongs to Q_{n1} for n less than $l(p)$ and for n at least $l(p)$, $p_n = (a_n, A_n, f_n)$ belongs to Q_{n0} with $a_n \subseteq a_{n+1}$.*

$p \le q$ iff for each n, $p_n \le q_n$ in Q_n. And $p \le^* q$ (p is a direct *extension of q) iff for each n, p_n is a direct extension of q_n.*

9.Vorlesung

Lemma 43 *Suppose that* $\gamma < \beta \leq \alpha$ *with* $\gamma \leq_n \alpha$ *and* $\beta \leq_n \alpha$*. Then* $\{\nu < \kappa_n \mid \pi_{\alpha\beta}(\nu) > \pi_{\alpha\gamma}(\nu)\}$ *belongs to* $U_{n\alpha}$*.*

Lemma 44 *Suppose that* x *belongs to* $[\lambda]^{<\kappa_n}$ *and* $\beta \leq_n \alpha$ *for each* $\beta \in x$*. Then there is* $A \in U_{n\alpha}$ *such that* $\pi_{\alpha\beta_0}$ *agrees with* $\pi_{\beta_1\beta_0}\pi_{\alpha\beta_1}$ *on* A *whenever* $\beta_0 \leq_n \beta_1$ *belong to* x*.*

Lemma 45 P *is* κ^{++}*-cc.*

Proof. Let $p(\alpha)$, $\alpha < \kappa^{++}$, be elements of P and write $p(\alpha)$ as $\langle p(\alpha)_n \mid n \in \omega\rangle$, where for $n \geq l(p(\alpha))$, $p(\alpha)_n = (a(\alpha)_n, A(\alpha)_n, f(\alpha)_n)$. There is a stationary $S \subseteq \kappa^{++}$ such that for α, β in S we have:

(a) $l(p(\alpha)) = l(p(\beta)) = l$.
(b) For n less than l, the collection of Dom $(p(\alpha)_n)$, $\alpha \in S$, forms a Δ system on whose root $p(\alpha)_n$ and $p(\beta)_n$ agree.
(c) For n at least l, the collection of $a(\alpha)_n \cup$ Dom $(f(\alpha)_n)$, $\alpha \in S$, forms a Δ system on whose root $f(\alpha)_n$ and $f(\beta)_n$ agree. Moreover $a(\alpha)_n$ and $a(\beta)_n$ have the same intersection with this root and therefore $a(\alpha)_n$ is disjoint from Dom $(f(\beta)_n)$.

Now we claim that if α, β belong to S then $p(\alpha)$ and $p(\beta)$ are compatible. We construct q below both of these conditions as follows. For n less than l let q_n be $p(\alpha)_n \cup p(\beta)_n$, which by (b) above is a well-defined function. Now suppose n is at least l; we define $q_n = (b_n, B_n, g_n)$. We take g_n to be $f(\alpha)_n \cup f(\beta)_n$. To define b_n, choose ρ above all elements of $a(\alpha)_n \cup a(\beta)_n$ in the ordering $\leq_n$ and greater than all elements of Dom (g_n), then set $b_n = a(\alpha)_n \cup a(\beta)_n \cup \{\rho\}$. Finally, to define B_n, let α^*, β^* be $\max a(\alpha)_n$, $\max a(\beta)_n$, respectively, and let B'_n be the intersection of $\pi^{-1}_{\rho\alpha^*}[A(\alpha)_n] \cap \pi^{-1}_{\rho\beta^*}[A(\beta)_n]$. Now using Lemmas 43 and 44, choose $B_n \in U_{n\rho}$ to be a subset of B'_n such that:

i. Whenever $\alpha \geq_n \beta$ belong to b_n then $\pi_{\rho\beta}(\mu) = \pi_{\alpha\beta}\pi_{\rho\alpha}(\mu)$ for all μ in B_n.
ii. Whenever $\alpha > \beta$ belong to b_n then $\pi_{\rho\alpha}(\mu) > \pi_{\rho\beta}(\mu)$ for μ in B_n.

Then $q(n) = (b_n, B_n, g_n)$ belongs to Q_{n0} for each n, and q is a condition extending both $p(\alpha)$ and $p(\beta)$, as desired. □

10.-11.Vorlesungen

We wish to prove the following two lemmas.

Lemma 46 *(The Prikry Property) For any sentence σ, each condition in P has a direct extension that decides σ.*

Lemma 47 *P preserves κ^+.*

Both of these lemmas will follow rather easily, given a certain fact about "minimal extensions" of conditions, which we now describe. Recall the following notation: If $p = \langle p_n \mid n \in \omega \rangle$ is a condition and $n \geq l(p)$, we write p_n as $(a_n(p), A_n(p), f_n(p))$. Now suppose that $q \leq p$ belong to P. Define the condition $q \downarrow p = r$ as follows: For n not in the interval $[l(p), l(q))$, $r_n = p_n$. For n in the interval $[l(p), l(q))$, r_n is the union of $f_n(p)$ and $q_n \upharpoonright a_n(p)$. We say that q is a *minimal extension of* p iff $q = q \downarrow p$.

Note that minimal extensions can be alternatively described as follows. Suppose that m is at least $l(p)$ and choose $\vec{\nu} = \langle \nu_{l(p)}, \ldots, \nu_{m-1} \rangle$ in $\prod_{l(p) \leq k < m} A_k(p)$. Define the condition $q = p * \vec{\nu}$ as follows: $q_n = p_n$ for n not in $[l(p), m)$ and for n in $[l(p), m)$,

$$q_n = f_n(p) \cup \{\langle \beta, \pi_{\max a_n(p), \beta}(\nu_n) \rangle \mid \beta \in a_n(p)\}.$$

Then $p * \vec{\nu}$ is a minimal extension of p and every minimal extension of p is of this form, as $q \downarrow p$ is just the condition $p * \langle \nu_{l(p)}, \ldots, \nu_{l(q)-1} \rangle$ where $\nu_n = q_n(\max a_n(p))$.

The main fact we need is the following.

Sublemma 48 *Suppose that p belongs to P and D is open dense. Then there is a direct extension p^* of p such that whenever $q \leq p^*$ belongs to D, so does $q \downarrow p^*$.*

Proof. For each $n \geq l(p)$ and each $\vec{\nu} = \langle \nu_{l(p)}, \ldots, \nu_{n-1} \rangle$ in $\prod_{l(p) \leq k < n} \kappa_k$, we will define a condition $p^{\vec{\nu}}$ which directly extends p. Let $\langle \vec{\nu}_i \mid i < \kappa \rangle$ be an enumeration of the $\vec{\nu}$. We assume that for i less than j, length $(\vec{\nu}_i)$ is at most length $(\vec{\nu}_j)$ and if these lengths are equal, then $\max \vec{\nu}_i$ is at most $\max \vec{\nu}_j$. We define a $\leq^*$-descending sequence $\langle p^i \mid i < \kappa \rangle$ of direct extensions of p and set $p^{\vec{\nu}} = p^i$ where $\vec{\nu} = \vec{\nu}_i$.

Note that if $\vec{p} = \langle p^i \mid i < \lambda \rangle$, λ limit, is a $\leq^*$-descending sequence of direct extensions of p with a $\leq^*$-lower bound, then although $\vec{p}$ may not have a *greatest* $\le^*$-lower bound, it does have a canonical *maximal* $\le^*$-lower bound q, defined by: $q_k = \bigcup_{i<\lambda} p^i_k$ for $k < l(p)$, and for $k \geq l(p)$, $f_k(q) = \bigcup_{i<\lambda} f_k(p^i)$, $a_k(q) = \bigcup_{i<\lambda} a_k(p^i) \cup \{\alpha\}$ where α is the least $\le_k$-upper bound to the elements of $\bigcup_{i<\lambda} a_k(p^i)$ and $A_k(q) = \bigcap_{i<\lambda} \pi^{-1}_{\alpha, \max a_k(p^i)}[A_k(p^i)]$.

Suppose that p^i is defined for all $i < j$ and we wish to define p^j. Let q^j be p if j equals 0, and otherwise let q^j be the canonical maximal $\le^*$-lower bound to the p^i, $i < j$. (It will be clear from the construction that the p^i, $i < j$, have a $\le^*$-lower bound.) Let n denote $l(p) + \text{length } (\vec{\nu}_j)$. If $\vec{\nu}_j$ does not belong to $\prod_{l(p) \le k < n} A_k(q^j)$ or if it does but $q^j * \vec{\nu}_j$ has no direct extension in D, then let p^j be q^j. Otherwise choose some direct extension r^j of $q^j * \vec{\nu}_j$ in D and define the direct extension p^j of q^j as follows:

(a) For k outside the interval $[l(p), n)$, $p^j_k = r^j_k$.
(b) For k inside the interval $[l(p), n)$, set $a_k(p^j) = a_k(q^j)$, $A_k(p^j) = A_k(q^j)$ and $f_k(p^j) = r^j_k \restriction (\text{Dom } (r^j_k) \setminus a_k(q^j))$.

Then note that as $p^j * \vec{\nu}_j$ is defined and equal to r^j, it follows that $p^j * \vec{\nu}_j$ belongs to D.

Let p^* be a $\le^*$-lower bound to all of these conditions p^i, $i < \kappa$. Such a $\le^*$-lower bound exists as the extension relation below p on $[0, l(p))$ is κ^+-closed and in the above construction, $a_{l(p)}(p^i)$ and $A_{l(p)}(p^i)$ never grow and for k greater than $l(p)$, $a_k(p^i)$ and $A_k(p^i)$ only grow at most κ_{k-1} times. Now if $q \le p^*$ belongs to D then choose $\vec{\nu}$ so that q is a direct extension of $p^* * \vec{\nu}$. (This $\vec{\nu}$ is $\langle \nu_{l(p)}, \ldots, \nu_{l(q)-1} \rangle$ where $\nu_k = f_k(q)(\max a_k(p^*))$ for each k.) Choose i so that $\vec{\nu}$ equals $\vec{\nu}_i$. Then as $A_k(p^i) = A_k(p^*)$ for k in $[l(p), l(q))$, $p^i * \vec{\nu}$ is a well-defined condition and therefore p^i was chosen so that $p^i * \vec{\nu}$ belongs to D. As $q \downarrow p^* = p^* * \vec{\nu}$ extends $p^i * \vec{\nu}$, it follows that $q \downarrow p^*$ also belongs to D, as desired. This proves Sublemma 51.

12.-13.Vorlesungen

We prove the following two lemmas.

Lemma 49 *(The Prikry Property) For any sentence σ, each condition in P has a direct extension that decides σ.*

Lemma 50 *P preserves κ^+.*

The main fact we need is the following.

Sublemma 51 *Suppose that p belongs to P and D is open dense. Then there is a direct extension p^* of p such that whenever $q \leq p^*$ belongs to D, so does $q \downarrow p^*$.*

Proof of Lemma 49. Suppose that the condition p has no direct extension deciding the sentence σ. Applying Sublemma 51, we may assume that whenever $q \leq p$ decides σ, then so does $q \downarrow p$. Set $l(p) = n$. We claim that $\{\nu_n \in A_n(p) \mid p * \langle \nu_n \rangle$ does not decide $\sigma\}$ must belong to the ultrafilter $U_{n,\max a_n(p)}$. Otherwise, we can thin $A_n(p)$ to $A \in U_{n,\max a_n(p)}$ so that the $p * \langle \nu_n \rangle$ for ν_n in A decide σ in the same way, and form p^* by replacing $A_n(p)$ by A. Then p^* is a direct extension of p deciding σ, contradicting our hypothesis.

Similarly, we have that whenever $p * \langle \nu_{l(p)}, \ldots, \nu_{m-1} \rangle$ is an extension of p which does not decide σ, the set $\{\nu_m \in A_m(p) \mid p * \langle \nu_{l(p)}, \ldots, \nu_{m-1}, \nu_m \rangle$ does not decide $\sigma\}$ belongs to $U_{m,\max a_m(p)}$. Therefore we can form a direct extension p^* of p such that no minimal extension $p * \vec{\nu}$ of p compatible with p^* decides σ. Now choose $q \leq p^*$ deciding σ. By choice of p, $q \downarrow p$ also decides σ. But $q \downarrow p$ is a minimal extension of p compatible with p^*, contradicting the choice of p^*. This proves Lemma 49.

Proof of Lemma 50. As κ is singular it suffices to show that if p forces $\dot{f}$ to be a function from κ_n into $(\kappa^+)^V$, then some extension q of p forces a bound on the range of $\dot{f}$. Assume that $l(p)$ is greater than n. Now using Sublemma 51, build a κ_n-sequence of direct extensions of p with lower bound p^* having the property that for each $i < \kappa_n$, if $q \leq p^*$ forces a value of $\dot{f}$ at i then so does $q \downarrow p^*$. But there are only κ-many conditions of the form $q \downarrow p^*$, and therefore p^* forces a bound on the range of $\dot{f}$. This proves Lemma 50.

Lemma 52 *P adds λ-many ω-sequences to the product of the κ_n's.*

Proof. Let G be P-generic. For each $\alpha < \lambda$ define t_α by $t_\alpha(n) = p_n(\alpha)$, where p belongs to G, $n < l(p)$ and $\alpha \in \mathrm{Dom}\ p_n$. For any $\alpha < \lambda$, either α is in the domain of $f_n(p)$ for some p in G and all $n \geq l(p)$, or α is in $a_n(p)$ for some p in G and all $n \geq l(p)$, and both cases occur cofinally in λ. In the former case, t_α belongs to V and in the latter case an easy density argument shows that t_α eventually dominates each element of the product of the κ_n's in V. We show that in the latter case, t_α also eventually dominates each t_β, $\beta < \alpha$, which does not belong to V, and therefore as the latter case must occur unboundedly in λ, λ-many new elements of the product of the κ_n's have been added.

Suppose β is less than α, t_β does not belong to V and α is in $a_n(p)$ for some p in G and all $n \geq l(p)$. Choose q in G so that β belongs to $a_n(q)$ for each $n \geq l(q)$. We may assume that q extends p. Then both β and α belong to $a_n(q)$ for each $n \geq l(q)$. By the definition of condition, we have $\pi_{\max a_n(q),\beta}(\nu) < \pi_{\max a_n(q),\alpha}(\nu)$ for each ν in $A_n(q)$. But now choose r in G so that $l(r)$ is greater than n and r extends q. Then $t_\beta(n) = r_n(\beta) = \pi_{\max a_n(q),\beta}(\nu) < \pi_{\max a_n(q),\alpha}(\nu) = r_n(\alpha) = t_\alpha(n)$, where $\nu = r_n(\max a_n(q)) \in A_n(q)$. So t_α eventually dominates t_β. $\square$

Thus after forcing with P, the GCH still holds below κ and 2^κ is at least λ, yielding a dramatic failure of the singular cardinal hypothesis.

Extender-based Prikry forcing with a single extender

In the previous section we showed how to violate the singular cardinal hypothesis at an ω-limit of cardinals with a rather high degree of strength. In this section we start with a single cardinal with much less strength and simultaneously singularise it and blow up its power set, without adding bounded subsets.

Assume GCH and suppose that κ and λ are regular with λ at least κ^{++}. We assume that κ is $H(\lambda)$-strong, which means that there is an elementary embedding $j : V \to M$ with critical point κ such that $H(\lambda)$ is contained in M and $j(\kappa)$ is greater than λ. We also make the following additional assumption:

$(*)$ λ is of the form $j(f_\lambda)(\kappa)$ for some function $f_\lambda : \kappa \to \kappa$.

$(*)$ is clearly the case if there is a formula φ such that λ is the least regular cardinal with $H(\lambda) \vDash \varphi(\kappa)$; for then we can take $f_\lambda(\bar{\kappa}) =$ the least $\bar{\lambda}$ such that $H(\bar{\lambda}) \vDash \varphi(\bar{\kappa})$. This applies for example when $\lambda = \kappa^{+n}$ for finite n or $\lambda =$ the least inaccessible greater than κ. It can also be shown that if κ is $H(\lambda)$-strong for all λ then this is necessarily witnessed by embeddings which obey $(*)$, and that for a single λ, if κ is $H(\lambda)$-strong then in a generic extension of the universe, κ is $H(\lambda)$-strong via an embedding obeying $(*)$. Thus the additional hypothesis $(*)$ should be regarded as harmless.

As in the previous section, for each $\alpha < \lambda$ we consider the κ-complete ultrafilter U_α defined by:

$X \in U_\alpha$ iff $X \subseteq \kappa$ and $\alpha \in j(X)$.

We also define the following ordering:

$\alpha \leq_j \beta$ iff $\kappa \leq \alpha \leq \beta$ and for some $f : \kappa \to \kappa$, $j(f)(\beta) = \alpha$.

Lemma 53 *For each $\alpha \in [\kappa, \lambda)$, $\kappa \leq_j \alpha$.*

Proof. Define $g : \kappa \to \kappa$ by $g(\bar{\alpha})$ = the least $\bar{\kappa}$ such that $f_\lambda(\bar{\kappa}) > \bar{\alpha}$ (if such a $\bar{\kappa}$ exists, 0 otherwise). $\square$

Lemma 54 *The partial ordering $\leq_j$ is κ^{++}-directed and in fact each $x \in [\lambda]^{\kappa^+}$ has λ-many upper bounds in $\leq_j$.*

Proof. Using GCH, let $\langle a_\alpha \mid \alpha < \kappa \rangle$ be an enumeration of ${}^{<\kappa}\kappa$ such that for each $x \in [\kappa]^{<\kappa}$, the set of α such that $x = a_\alpha$ is a cofinal subset of $(\sup x)^+$. Now note that as j is the identity below κ, $\langle a_\alpha \mid \alpha < \kappa \rangle$ is the restriction to κ of $j(\langle a_\alpha \mid \alpha < \kappa \rangle)$; let $\langle a_\alpha \mid \alpha < \lambda \rangle$ denote the restriction of $j(\langle a_\alpha \mid \alpha < \kappa \rangle)$ to λ. Then for each $x \in {}^{<\lambda}\lambda \cap M = {}^{<\lambda}\lambda$, the set of α such that $x = a_\alpha$ is a cofinal subset of $(\sup x)^+$.

Now suppose that x belongs to ${}^{\kappa^+}\lambda$; we find $\alpha < \lambda$ such that $\beta <_j \alpha$ for each $\beta \in$ Range (x). Using the fact that $\lambda \geq \kappa^{++}$ is regular, choose $\alpha < \lambda$ so that a_α equals x. Using Lemma 53, choose $g : \kappa \to \kappa$ such that $j(g)(\alpha) = \kappa$. Now for each $i < \kappa^+$ we may choose a function $f_i : \kappa \to \kappa$ such that $j(f_i)(\alpha) = i$; such an f_i can be defined by choosing $A \subseteq \kappa$ to code i and then setting $f_i(\bar{\alpha})$ = the ordinal coded by $A \cap g(\bar{\alpha})$. Now suppose that β belongs to Range (x) and choose $i < \kappa^+$ so that $\beta = x(i)$. Then $\beta = j(f)(\alpha)$, where f is defined by: $f(\bar{\alpha}) = a_{\bar{\alpha}}(f_i(\bar{\alpha}))$, the $f_i(\bar{\alpha})$-th element of $a_{\bar{\alpha}}$. So $\beta <_j \alpha$. As there are λ-many α such that x equals a_α, we are done. $\square$

Fix $\pi_{\alpha\beta}$ witnessing $\beta \leq_j \alpha$. As in the previous section we have:

Lemma 55 *Suppose that $\gamma < \beta \leq \alpha$ with $\gamma \leq_j \alpha$ and $\beta \leq_j \alpha$. Then $\{\nu < \kappa \mid \pi_{\alpha\beta}(\nu) > \pi_{\alpha\gamma}(\nu)\}$ belongs to U_α.*

Lemma 56 *Suppose that $\beta_0 \leq_j \beta_1 \leq_j \alpha$. Then there is $A \in U_\alpha$ such that $\pi_{\alpha\beta_0}$ agrees with $\pi_{\beta_1\beta_0}\pi_{\alpha\beta_1}$ on A.*

For technical reasons we make a few further demands of the projection maps $\pi_{\alpha\beta}$, $\beta \leq_j \alpha$:

Fixed projection to κ: $\pi_{\alpha\kappa}(\bar{\alpha}) = \pi_{\beta\kappa}(\bar{\alpha})$ for all $\bar{\alpha}$.
Total commutativity at κ: For $\beta \leq_j \alpha$, $\pi_{\alpha\kappa}(\bar{\alpha}) = \pi_{\beta\kappa}(\pi_{\alpha\beta}(\bar{\alpha}))$ for all $\bar{\alpha}$.
U_α is a P-point: If $\langle A_i \mid i < \kappa\rangle$ belong to U_α then for some $A \in U_\alpha$, A is almost contained in A_i for each i (i.e., modulo bounded sets).

Lemma 57 *Suppose that $\gamma < \beta \leq \alpha$ with $\gamma \leq_j \alpha$ and $\beta \leq_j \alpha$. Then $\{\nu < \kappa \mid \pi_{\alpha\beta}(\nu) > \pi_{\alpha\gamma}(\nu)\}$ belongs to U_α.*

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Fixed projection to κ: $\pi_{\alpha\kappa}(\bar{\alpha}) = \pi_{\beta\kappa}(\bar{\alpha})$ for all $\bar{\alpha}$.
Total commutativity at κ: For $\beta \leq_j \alpha$, $\pi_{\alpha\kappa}(\bar{\alpha}) = \pi_{\beta\kappa}(\pi_{\alpha\beta}(\bar{\alpha}))$ for all $\bar{\alpha}$.
U_α is a P-point: If $\langle A_i \mid i < \kappa\rangle$ belong to U_α then for some $A \in U_\alpha$, A is almost contained in A_i for each i (i.e., modulo bounded sets).

To achieve the first two of these properties, we define $\bar{X}$ to be the set of $\bar{\alpha} < \kappa$ such that for some $\bar{\kappa} \leq \bar{\alpha}$, $\bar{\kappa}$ is closed under f_λ, $\bar{\kappa}$ is inaccessible and $f_\lambda(\bar{\kappa}) > \bar{\alpha}$. Then $\bar{X}$ belongs to each of the measures U_α, $\kappa \leq \alpha < \lambda$. So we can assume that for all $\alpha \in [\kappa, \lambda)$, the projection $\pi_{\alpha\kappa}$ is defined by: $\pi_{\alpha\kappa}(\bar{\alpha}) =$ the unique $\bar{\kappa}$ witnessing $\bar{\alpha} \in \bar{X}$ (for $\bar{\alpha}$ in $\bar{X}$); $\pi_{\alpha\kappa}(\bar{\alpha}) = 0$ (for $\bar{\alpha}$ not in $\bar{X}$). This achieves the first property.

To achieve the second property, we require that $\pi_{\alpha\beta}(\bar{\alpha}) = 0$ for $\bar{\alpha}$ not in $\bar{X}$ and for $\bar{\alpha}$ in $\bar{X}$, $\pi_{\alpha\beta}(\bar{\alpha})$ is in the interval $[\pi_{\alpha\kappa}(\bar{\alpha}), \bar{\alpha}]$. As $\kappa \leq_j \beta$ for all $\beta \in [\kappa, \lambda)$, Lemma 57 implies that these requirements are vacuous on a set which belongs to each of the ultrafilters U_α, $\kappa \leq \alpha < \lambda$, and therefore can be imposed. We now have that for $\beta \leq_j \alpha$, if $\bar{\alpha}$ belongs to $\bar{X}$, then $\pi_{\alpha\beta}(\bar{\alpha})$ is in the interval $[\pi_{\alpha\kappa}(\bar{\alpha}), \bar{\alpha}]$ and therefore $\pi_{\alpha\kappa}(\bar{\alpha})$ witnesses that $\pi_{\alpha\beta}(\bar{\alpha})$ belongs to $\bar{X}$; it follows that $\pi_{\beta\kappa}(\pi_{\alpha\beta}(\bar{\alpha}))$ equals the witness $\pi_{\alpha\kappa}(\bar{\alpha})$. And if $\bar{\alpha}$ does not belong to $\bar{X}$, then $\pi_{\alpha\kappa}(\bar{\alpha}) = 0$ and $\pi_{\beta\kappa}(\pi_{\alpha\beta}(\bar{\alpha})) = \pi_{\beta\kappa}(0) = 0$. This establishes the second property.

To verify the P-point property, note that the function $\pi_{\alpha\kappa} : \kappa \to \kappa$ is non-decreasing and cofinal on $\bar{X} \in U_\alpha$, and $j(\pi_{\alpha\kappa})(\alpha) = \kappa$. Then as each A_i,

$i < \kappa$, belongs to U_α, we have that α belongs to $j(A_i) = j(\langle A_i \mid i < \kappa\rangle)_i$ for all $i < \kappa = j(\pi_{\alpha\kappa})(\alpha)$. It follows that $A = \triangle^*_{i<\kappa} A_i = \{\bar{\alpha} \in \bar{X} \mid \bar{\alpha}$ belongs to A_i for all $i < \pi_{\alpha\kappa}(\bar{\alpha})\}$ belongs to U_α, and as $\pi_{\alpha\kappa}$ is cofinal and non-decreasing on $\bar{X}$, it follows that A is almost contained in each A_i, $i < \kappa$.

For $\alpha \in [\kappa, \lambda)$ and $\nu < \kappa$ we denote $\pi_{\alpha\kappa}(\nu)$ as $\kappa(\nu)$. (This is independent of the choice of α.) A sequence $\langle \nu_0, \ldots, \nu_{n-1}\rangle$ of ordinals less than κ is κ-*increasing* iff $i < j$ implies $\kappa(\nu_i) < \kappa(\nu_j)$. For $\kappa(\nu_0) < \kappa(\nu_1)$, we have that the cardinality of $\{\nu \in \bar{X} \mid \kappa(\nu) = \kappa(\nu_0)\}$ is less than $\kappa(\nu_1)$ and therefore less than ν_1. If $\vec{\nu} = \langle \nu_0, \ldots, \nu_{n-1}\rangle$ is a sequence of ordinals less than κ, then $\kappa(\vec{\nu})$ denotes the max of the $\kappa(\nu_i)$. Note that by *total commutativity at* κ, if $\alpha \leq_j \beta$ then $\kappa(\nu) = \kappa(\pi_{\beta\alpha}(\nu))$ for each $\nu < \kappa$.

We are at last ready to define the desired forcing.

A condition p is of the form $\{(\gamma, p^\gamma) \mid \gamma \in g \backslash \{\max g\}\} \cup \{(\max g, p^{\max g}, T)\}$ where:

(a) $\kappa \in g \subseteq [\kappa, \lambda)$, g has cardinality at most κ, g has a maximal element and $\alpha \leq_j \max g$ for all $\alpha \in g$.
(b) For $\gamma \in g$, p^γ is a finite κ-increasing sequence of ordinals in $\bar{X} \subseteq \kappa$.
(c) T is a tree of κ-increasing sequences from $\bar{X}$ with trunk $p^{\max g}$. For each $\eta \geq_T p^{\max g}$, $\mathrm{Succ}_T(\eta) = \{\nu < \kappa \mid \eta * \nu \in T\}$ belongs to $U_{\max g}$ and for $\eta_1 \geq_T \eta_0 \geq_T p^{\max g}$, T_{η_1} is a subtree of T_{η_0}, where T_η denotes the set of σ such that $\eta * \sigma$ belongs to T.
(d) For $\gamma \in g$, $\kappa(p^{\max g}) \leq \kappa(p^\gamma)$.
(e) For $\nu \in \mathrm{Succ}_T(p^{\max g})$, the cardinality of $\{\gamma \in g \mid \kappa(\nu) > \kappa(p^\gamma)\}$ is at most $\kappa(\nu)$.
(f) $\pi_{\max g, \kappa}$ sends $p^{\max g}$ to p^κ.

We denote g by $\mathrm{supp}(p)$, $\max g$ by $mc(p)$ (for "maximal coordinate" of p), T by T^p and $p^{\max g}$ by p^{mc}.

For two conditions p, q as above, we say that p *extends* q iff:

1. $\mathrm{supp}(p) \supseteq \mathrm{supp}(q)$.
2. For $\gamma \in \mathrm{supp}(q)$, p^γ is an end-extension of q^γ.
3. $p^{mc(q)}$ belongs to T^q.
4. For $\gamma \in \mathrm{supp}(q)$, $p^\gamma \setminus q^\gamma$ is the range of $\pi_{mc(q),\gamma}$ on $p^{mc(q)} \setminus q^{mc(q)}$ past i,

where $i \in \text{Dom}\ (p^{mc(q)})$ is largest so that $\kappa(p^{mc(q)}(i)) \leq \kappa(q^\gamma)$.
5. $\pi_{mc(p),mc(q)}$ maps T^p to a subtree of T^q.
6. For γ in supp(q) and $\nu \in \text{Succ}_{T^p}(p^{mc})$, if $\kappa(\nu)$ is greater than $\kappa(p^\gamma)$ then $\pi_{mc(p),\gamma}(\nu) = \pi_{mc(q),\gamma}(\pi_{mc(p),mc(q)}(\nu))$.

Remark. The above properties imply that if p extends q then $\pi_{mc(p),mc(q)}(p^{mc}) = p^{mc(q)}$. See the proof of the transitivity of the extension relation below.

If in addition $p^\gamma = q^\gamma$ for each γ in supp(q), then we say that p is a *direct extension* of q. We write $p \leq q$ for p extends q and $p \leq^* q$ for p directly extends q.

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Lemma 58 *The ordering relation of P is transitive.*

Proof. Suppose that $p \leq q$ and $q \leq r$; we check that $p \leq r$. Properties 1 and 2 are clearly satisfied.

For property 3, first note that it follows from from property (f) for conditions and property 4 for extensions that if p extends q, then $\pi_{mc(p),mc(q)}(p^{mc})$ equals $p^{mc(q)}$. To see this, it suffices to show that $\pi_{mc(q),\kappa}\pi_{mc(p),mc(q)}(p^{mc})$ equals $\pi_{mc(q),\kappa}(p^{mc(q)})$, as the map $\pi_{mc(q),\kappa}$ is 1-1 on κ-increasing sequences. Now $\pi_{mc(q),\kappa}\pi_{mc(p),mc(q)}(p^{mc})$ equals $\pi_{mc(p),\kappa}(p^{mc})$ by total commutativity to κ, and by property (f) for the condition p, the latter is p^κ. And $\pi_{mc(q),\kappa}(p^{mc(q)})$ is the union of $\pi_{mc(q),\kappa}(q^{mc})$ and $\pi_{mc(q),\kappa}(p^{mc(q)} \setminus q^{mc})$. The former is q^κ by property (f) for the condition q. The latter is $p^\kappa \setminus q^\kappa$ by property 4 for the extension $p \leq q$. It follows that $\pi_{mc(q),\kappa}(p^{mc(q)})$ is also p^κ.

Now we check property 3 for the pair p, r; i.e., we check that $p^{mc(r)}$ belongs to T^r. As q extends r, $\kappa(q^{mc})$ equals $\kappa(q^{mc(r)})$ and therefore as p extends q, $p^{mc(r)} \setminus q^{mc(r)}$ is the range of $\pi_{mc(q),mc(r)}$ on $p^{mc(q)} \setminus q^{mc(q)}$. It follows from property 5 for the extension $q \leq r$ that $q^{mc(r)} * (p^{mc(r)} \setminus q^{mc(r)}) = p^{mc(r)}$ belongs to T^r, as desired.

Next we check property 4. Suppose that γ belongs to supp(r); we must show that $p^\gamma \setminus r^\gamma$ is the range of $\pi_{mc(r),\gamma}$ on $p^{mc(r)} \setminus r^{mc(r)}$ past i, where $i \in \text{Dom}\ (p^{mc(r)})$ is largest so that $\kappa(p^{mc(r)}(i)) \leq \kappa(r^\gamma)$. Since q extends r, $q^\gamma \setminus r^\gamma$ is the range of $\pi_{mc(r),\gamma}$ on $q^{mc(r)} \setminus r^{mc(r)}$ past j, where $j \in \text{Dom}\ (q^{mc(r)})$ is largest so that $\kappa(q^{mc(r)}(j)) \leq \kappa(r^\gamma)$.

First suppose that q^γ is a proper extension of r^γ, from which it follows that $\kappa(q^\gamma)$ equals $\kappa(q^{mc(r)})$ and j equals i. It suffices to show that $p^\gamma \setminus q^\gamma$ is the range of $\pi_{mc(r),\gamma}$ on $p^{mc(r)} \setminus q^{mc(r)}$, for then $p^\gamma \setminus r^\gamma = (p^\gamma \setminus q^\gamma) \cup (q^\gamma \setminus r^\gamma)$ is the range of $\pi_{mc(r),\gamma}$ on $(p^{mc(r)} \setminus q^{mc(r)}) \cup (q^{mc(r)} \setminus r^{mc(r)}$ past $i)$, which is $p^{mc(r)} \setminus r^{mc(r)}$ past i, as desired. Now since p extends q, $p^\gamma \setminus q^\gamma$ is the range of $\pi_{mc(q),\gamma}$ on $p^{mc(q)} \setminus q^{mc(q)}$ past k, where $k \in \text{Dom } (p^{mc(q)})$ is largest so that $\kappa(p^{mc(q)}(k)) \leq \kappa(q^\gamma)$. But as q extends r, $\kappa(q^{mc(q)})$ equals $\kappa(q^{mc(r)}) = \kappa(q^\gamma)$, from which it follows that k is just $\max q^{mc(q)}$. Therefore $p^\gamma \setminus q^\gamma$ is the range of $\pi_{mc(q),\gamma}$ on $p^{mc(q)} \setminus q^{mc(q)}$. Now using property 6 for the extension $q \leq r$ we have:

$$p^\gamma \setminus q^\gamma = \pi_{mc(q),\gamma}[p^{mc(q)} \setminus q^{mc(q)}] =$$
$$\pi_{mc(r),\gamma}[\pi_{mc(q),mc(r)}[p^{mc(q)} \setminus q^{mc(q)}]] =$$
$$\pi_{mc(r),\gamma}[p^{mc(r)} \setminus q^{mc(r)}].$$

The last equality holds by property 4 for the extension $p \leq q$, using the fact that $\kappa(q^{mc(r)})$ equals $\kappa(q^{mc(q)})$.

If q^γ equals r^γ, then as requirement (d) for the condition r implies that $\kappa(r^{mc(r)})$ is at most $\kappa(r^\gamma) = \kappa(q^\gamma)$, it follows that $\kappa(q^{mc(r)})$ is at most $\kappa(q^\gamma)$. As p extends q, $p^\gamma \setminus r^\gamma = p^\gamma \setminus q^\gamma$ is the range of $\pi_{mc(q),\gamma}$ on $p^{mc(q)} \setminus q^{mc(q)}$ past j, where $j \in \text{Dom } (p^{mc(q)})$ is largest so that $\kappa(p^{mc(q)}(j)) \leq \kappa(q^\gamma)$. Also, $p^{mc(r)} \setminus q^{mc(r)}$ is the range of $\pi_{mc(q),mc(r)}$ on $p^{mc(q)} \setminus q^{mc(q)}$. It follows that $p^{mc(r)} \setminus q^{mc(r)}$ past i equals the range of $\pi_{mc(q),mc(r)}$ on $p^{mc(q)} \setminus q^{mc(q)}$ past j. Using property 6 for the extension $q \leq r$ we therefore have:

$$p^\gamma \setminus r^\gamma = p^\gamma \setminus q^\gamma = \pi_{mc(q),\gamma}[p^{mc(q)} \setminus q^{mc(q)} \text{ past } j] =$$
$$\pi_{mc(r),\gamma}[\pi_{mc(q),mc(r)}[p^{mc(q)} \setminus q^{mc(q)} \text{ past } j]] =$$
$$\pi_{mc(r),\gamma}[p^{mc(r)} \setminus q^{mc(r)} \text{ past } i], \text{ as desired.}$$

We check property 5. As $\pi_{mc(p),mc(r)}(p^{mc}) = p^{mc(r)}$, it suffices to show that $\pi_{mc(p),mc(r)}$ maps $T^p_{p^{mc}}$ into $T^r_{p^{mc(r)}}$. Suppose that f σ belongs to $T^p_{p^{mc}}$. As p extends q, it follows that $\pi_{mc(p),mc(q)}(\sigma)$ belongs to $T^q_{p^{mc(q)}}$ and therefore $(p^{mc(q)} \setminus q^{mc}) * \pi_{mc(p),mc(q)}(\sigma)$ belongs to $T^q_{q^{mc}}$. As in the verification of property 4 above, $\pi_{mc(q),mc(r)}[p^{mc(q)} \setminus p^{mc}] = p^{mc(r)} \setminus q^{mc(r)}$ and therefore $\pi_{mc(q),mc(r)}\pi_{mc(p),mc(q)}(\sigma)$ belongs to $T^r_{p^{mc(r)}}$.

We claim that if ν is a component of σ then $\kappa(\nu)$ is greater than $\kappa(p^{mc(r)})$. We have $\kappa(\nu) = \kappa(\pi_{mc(p),mc(q)}(\nu))$ is greater than $\kappa(p^{mc(q)})$. If $p^{mc(r)}$ is a proper extension of r^{mc} then as a final segment of $p^{mc(r)}$ is the image under

$\pi_{mc(q),mc(r)}$ of a final segment of $p^{mc(q)}$, it follows that $\kappa(p^{mc(r)})$ equals $\kappa(p^{mc(q)})$ and therefore $\kappa(\nu)$ is also greater than $\kappa(p^{mc(r)})$. If $p^{mc(r)}$ equals r^{mc} then as $\pi_{mc(q),mc(r)}$ maps $T^q_{q^{mc}}$ into $T^r_{q^{mc(r)}}$, $\kappa(\nu) = \kappa(\pi_{mc(q),mc(r)}\pi_{mc(p),mc(q)}(\nu))$ is greater than $\kappa(q^{mc(r)}) = \kappa(p^{mc(r)})$.

Now we can apply property 6 for the extension $p \leq q$ to conclude that $\pi_{mc(q),mc(r)}\pi_{mc(p),mc(q)}(\sigma) = \pi_{mc(p),mc(r)}(\sigma) \in T^r_{p^{mc(r)}}$, as desired.

Finally, we verify property 6 for p and r. Suppose that γ belongs to $\text{supp}(r)$, ν belongs to $\text{Succ}_{T^p}(p^{mc})$ and $\kappa(\nu)$ is greater than $\kappa(p^\gamma)$. Then applying property 6 for the extension $p \leq q$ we have $\pi_{mc(p),\gamma}(\nu) = \pi_{mc(q),\gamma}\pi_{mc(p),mc(q)}(\nu)$. As ν belongs to $\text{Succ}_{T^p}(p^{mc})$, it follows from property 5 for the extension $p \leq q$ that $\pi_{mc(p),mc(q)}(\nu)$ belongs to $\text{Succ}_{T^q}(p^{mc(q)})$ and therefore to $\text{Succ}_{T^q}(q^{mc})$. As $\kappa(\pi_{mc(p),mc(q)}(\nu)) = \kappa(\nu)$ is also greater than $\kappa(p^\gamma) \geq \kappa(q^\gamma)$ and $q \leq r$, we have $\pi_{mc(p),\gamma}(\nu) = \pi_{mc(r),\gamma}\pi_{mc(q),mc(r)}\pi_{mc(p),mc(q)}(\nu)$. Recall that in the verification of property 5 for p, r we showed that $\kappa(\nu)$ is greater than $\kappa(p^{mc(r)})$; so once again applying property 6 to the extension $p \leq q$, we conclude that $\pi_{mc(p),\gamma}(\nu)$ equals $\pi_{mc(r),\gamma}\pi_{mc(p),mc(r)}(\nu)$, as desired. □

Lemma 59 *If q belongs to P and α belongs to $[\kappa, \lambda)$ then there is $p \leq^* q$ with $\alpha \in supp(p)$.*

Proof. If α belongs to $\text{supp}(q)$ then this is trivial. Suppose that α does not belong to $\text{supp}(q)$ but $\alpha \leq_j mc(q)$. Then add to q a κ-increasing sequence t^α such that $\kappa(t^\alpha) \geq \kappa(q^{mc(q)})$; the result is a direct extension of q.

Now suppose that $\alpha \not\leq_j mc(q)$. We may assume that $mc(q) <_j \alpha$, as otherwise we may choose $\beta < \lambda$ so that $\alpha, mc(q) \leq_j \beta$, find a direct extension of q whose support includes β and then by the previous paragraph add α to the support of that direct extension. The desired p will be of the form $q' \cup \{(\alpha, t, T)\}$, where q' is obtained from q by replacing the triple $(mc(q), q^{mc}, T^q)$ by $(mc(q), q^{mc})$ and t, T are defined below.

We take t to be any κ-increasing sequence such that $\pi_{\alpha,\kappa}(t) = q^\kappa$. Recall that $\pi_{\alpha,\kappa}$ is independent of α; therefore a candidate for t is $q^{mc(q)}$, which by definition projects under $\pi_{mc(q),\kappa}$ to q^κ.

A first attempt at defining T is to take T_0 = the preimage of $T^q_{q^{mc}}$ under $\pi_{\alpha,mc(q)}$, with t added as its trunk. The resulting $p_0 = q' \cup \{(\alpha, t, T_0)\}$ is a condition. The only difficulty with verifying $p_0 \leq q$ is property 6: It may be the case that for some $\gamma \in \text{supp}(q) \setminus \{mc(q)\}$ and some $\nu \in A = \text{Succ}_{T_0}(t) = \pi^{-1}_{\alpha,mc(q)}[\text{Succ}_{T^q}(q^{mc})]$, $\kappa(\nu)$ is greater than $\kappa(q^\gamma)$ but $\pi_{\alpha,\gamma}(\nu) \neq \pi_{mc(q),\gamma}\pi_{\alpha,mc(q)}(\nu)$.

To fix this problem, we shrink T_0. For $\nu \in A$ we let B_ν be the set of $\gamma \in \text{supp}(q) \setminus \{mc(q)\}$ such that $\kappa(\nu)$ is greater than $\kappa(q^\gamma)$. Then B_ν has cardinality at most $\kappa(\nu)$ as $\kappa(\nu) = \kappa(\pi_{\alpha,mc(q)}(\nu))$, $\pi_{\alpha,mc(q)}(\nu) \in \text{Succ}_{T^q}(q^{mc})$ and q is a condition. The union of the B_ν's is all of $\text{supp}(q) \setminus \{mc(q)\}$. Now for each $\nu \in A$ choose $C_\nu \in U_\alpha$ such that for γ in $B^+_\nu = \{\gamma \in \text{supp}(q) \setminus \{mc(q)\} \mid \kappa(\nu) \geq \kappa(q^\gamma)\}$, $\pi_{\alpha,\gamma}$ agrees with $\pi_{mc(q),\gamma)}\pi_{\alpha,mc(q)}$ on C_ν. Let C be the "quasi" diagonal intersection $\triangle^*_\nu C_\nu = \{\nu \mid \nu \in C_{\nu'} \text{ when } \kappa(\nu') < \kappa(\nu)\}$. Then C also belongs to U_α and we let T consist of all sequences in T_0 all of whose components (beyond the trunk) belong to C. Then $p = q' \cup \{(\alpha, t, T)\}$ is a condition which (directly) extends q, as if γ belongs to $\text{supp}(q) \setminus \{mc(q)\}$, $\nu \in \text{Succ}_T(t) = A \cap C$ and $\kappa(\nu)$ is greater than $\kappa(q^\gamma)$ then ν belongs to $C_{\kappa(q^\gamma)}$, γ belongs to $B^+_{\kappa(q^\gamma)}$ and therefore $\pi_{\alpha,\gamma}$ agrees with $\pi_{mc(q),\gamma)}\pi_{\alpha,mc(q)}$ at ν. As α belongs to the support of q, we are done. □

Lemma 60 *If q belongs to P and α belongs to $[\kappa, \lambda)$ then there is $p \leq^* q$ with $\alpha \in supp(p)$.*

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Lemma 61 *P has the κ^{++}-cc.*

Proof. Let $\{p_\alpha \mid \alpha < \kappa^{++}\}$ belong to P. We can assume that the supports of the p_α's form a Δ-system, the p_α's agree on the root of that Δ-system and also $(p^{mc}_\alpha, T^{p_\alpha})$ is independent of α. This is because the supports have size at most κ and there are only κ^+ possible pairs $(p^{mc}_\alpha, T^{p_\alpha})$. We then show that p_α, p_β are compatible for any pair α, β. The techniques for doing this are in the proof of the previous lemma: Our first candidate for a common extension of p_α and p_β is $p_\alpha \cup p_\beta$. But the support of this may not have a maximal element. So choose δ so that $mc(p_\alpha), mc(p_\beta) <_j \delta$ and let p^*_α be formed from p_α by adding δ to the support, as in the proof of the previous lemma. Then $p^* = p^*_\alpha \cup p_\beta$ is a condition, using the fact that $\kappa(p^{mc}_\alpha)$ and $\kappa(p^{mc}_\beta)$ agree. To obtain a condition extending both p_α and p_β we shrink T^{p^*} further to ensure that $\pi_{\delta,mc(p_\beta)}$ maps $T^{p^*}_{(p^*)^{mc}}$ into $T^{p_\beta}_{p^{mc}_\beta}$ and then shrink T^{p^*} again using the proof of the previous lemma to guarantee that property 6 holds for the resulting condition relative to p_β. □

Lemma 62 *The direct extension relation for P is κ-closed.*

Proof. Suppose that $\langle p_i \mid i < \delta\rangle$ is a $\leq^*$-decreasing sequence of length $\delta < \kappa$. We assume that $mc(p_i) > \kappa$ for some i, as otherwise the result follows from the analogous result for basic Prikry forcing. Choose α so that $mc(p_i) <_j \kappa$ for each $i < \delta$. Let p' be the union of the p_i's, with the trees T^{p_i} removed. Let T^* be the tree consisting of all $\langle \nu_0, \ldots, \nu_{n-1}\rangle$ in $\bigcap_{i<\delta} \pi^{-1}_{\alpha,mc(p_i)}[T^{p_i}]$ such that $\kappa(\nu_0) > \delta$, let t be p_i^{mc} for some i with $mc(p_i) > \kappa$ and let T' be the tree with trunk t, followed by the strings in T^*. Then $p' \cup \{(\alpha, t, T')\}$ is a condition: (a), (b) and (c) are easily checked; (d) holds as $\kappa(p_0^\kappa) = \kappa(p_i^{mc})$ for each $i < \delta$, (e) holds as $\kappa(\nu)$ is greater than δ for each $\nu \in \mathrm{Succ}_{T'}(t)$ and (f) holds as for some i with $mc(p_i) > \kappa$, $\pi_{\alpha\kappa}(t) = \pi_{mc(p_i),\kappa}(p_i^{mc}) = p_0^\kappa = (p')^\kappa$. Now as in the proof of Lemma 60, we can thin out T' to T_i for each $i < \delta$ so that $p' \cup \{(\alpha, t, T^i)\}$ extends p_i; finally take p to be $p' \cup \{(\alpha, t, T^*)\}$, where T^*_t is the intersection of the T^i_t's, and we have a direct extension of each p_i. $\square$

20.-21.Vorlesungen

Lemma 63 *P satisfies the Prikry property.*

Proof. We consider a strengthening of the notion of direct extension. We say that p is a *very direct extension* of q, $p \leq^{**} q$, iff p is a direct extension of q and $\mathrm{supp}(p) = \mathrm{supp}(q)$. Now if p extends q then we write $p \Downarrow q$ for the condition r obtained as follows:

(i) $\mathrm{supp}(r) = \mathrm{supp}(q)$, $r^\gamma = p^\gamma$ for $\gamma \in \mathrm{supp}(q)$.
(ii) T^r has trunk $p^{mc(q)}$ and $T^r_{p^{mc(q)}} = T^q_{p^{mc(q)}}$.

We also write $p \downarrow q$ for the condition r defined by:

(i) $\mathrm{supp}(r) = \mathrm{supp}(q)$, $r^\gamma = p^\gamma$ for $\gamma \in \mathrm{supp}(q)$.
(ii) $T^r = \pi_{mc(p),mc(q)}[T^p]$.

Note that $p \Downarrow q$ is uniquely determined by q and $p^{mc(q)}$, due to property 4 for the notion of extension, and therefore we also write $p \Downarrow q$ as $q * p^{mc(q)}$. Also note that $p \leq^* (p \downarrow q) \leq^{**} (p \Downarrow q) \leq q$.

If p extends q we also write $q \Uparrow p$ for the condition r obtained as follows:

(i) $\text{supp}(r) = \text{supp}(p)$.
(ii) $r^\gamma = q^\gamma$ for $\gamma \in \text{supp}(q)$.
(iii) $r^{mc} = \pi^{-1}_{mc(p),\kappa}(q^\kappa)$.
(iv) $r^\gamma = p^\gamma$ for $\gamma \in \text{supp}(p) \setminus \text{supp}(q)$, $\gamma \neq mc(p)$.
(v) $T^r = \{\sigma \in \pi^{-1}_{mc(p),mc(q)}[T^q] \mid p^{mc} \subseteq \sigma \to \sigma \in T^p\}$.

Note that $q \Uparrow p$ is a direct extension of q and $p \Downarrow (q \Uparrow p)$ is equal to p.

Sublemma 64 *Suppose that q_0 is a condition and D is open dense. Then q_0 has a direct extension q such that whenever p belongs to D and extends q, the condition $p \Downarrow q$ has a very direct extension which also belongs to D.*

Proof. We build a sequence $\langle q_i \mid i < \kappa \rangle$ of direct extensions of q_0. This sequence will be taken from M, an elementary submodel of $H(\lambda^+)$ of size κ^+ closed under κ-sequences and containing D as an element. Choose α so that $\beta <_j \alpha$ for each β in $M \cap [\kappa, \lambda)$ and fix an enumeration $\langle t_i \mid i < \kappa \rangle \in M$ of all κ-increasing sequences.

If q_i, t_i are defined, choose some p in $D \cap M$ extending q_i with $p^{mc} = \pi^{-1}_{mc(p),mc(q_i)} \pi_{\alpha,mc(q_i)}(t_i)$, if possible, and set q_{i+1} to be $q_i \Uparrow p$. Then p equals $p \Downarrow (q_i \Uparrow p) = (q_i \Uparrow p) * p^{mc} = q_{i+1} * \pi^{-1}_{mc(q_{i+1}),mc(q_i)} \pi_{\alpha,mc(q_i)}(t_i)$ and therefore the condition $q_{i+1} * \pi^{-1}_{mc(q_{i+1}),mc(q_i)} \pi_{\alpha,mc(q_i)}(t_i)$ belongs to D.

For limit $\lambda < \kappa$, we define $q_\lambda = r$ as follows:

(i) $\text{supp}(r) = \bigcup_{i<\lambda} \text{supp}(q_i)$ together with the least α_λ such that $\beta <_j \alpha_\lambda$ for each $\beta \in \bigcup_{i<\lambda} \text{supp}(q_i)$.
(ii) $r^\gamma = q_i^\gamma$ for $\gamma \in \text{supp}(q_i)$.
(iii) $r^{\alpha_\lambda} = \pi^{-1}_{\alpha_\lambda,\kappa}(q_0^\kappa)$.
(iv) For $\eta \in T^r$ extending $r^{mc} = r^{\alpha_\lambda}$, $\text{Succ}_{T^r}(\eta)$ is the intersection of the $\pi^{-1}_{\alpha_\lambda,mc(q_i)}[\text{Succ}_{T^{q_i}}(\pi_{\alpha_\lambda,mc(q_i)}(\eta))]$ for $i < \lambda$.

Also define $q = q_\kappa$ just as above with $\lambda = \kappa$ and $\alpha_\lambda = \alpha$, except replace (iv) by:

(v) For $\eta \in T^r$ extending $r^{mc} = r^\alpha$, $\text{Succ}_{T^r}(\eta)$ is the quasi diagonal intersection $\triangle^*_{i<\kappa} \pi^{-1}_{\alpha,mc(q_i)}[\text{Succ}_{T^{q_i}}(\pi_{\alpha,mc(q_i)}(\eta))]$.

Now suppose that p is in D and extends q. For each $i < \kappa$ let p_i be the condition r defined by:

(i) $\text{supp}(r) = \text{supp}(p)$, $r^\gamma = p^\gamma$ for $\gamma \in \text{supp}(p)$.
(ii) $T^r = \pi^{-1}_{mc(p),mc(q_i)}[T^{q_i}] \cap T^p$.

Then p_i is a very direct extension of p which extends q_i. Choose $i < \kappa$ so that t_i equals $\pi_{mc(p),\alpha}(p^{mc}) = p^\alpha$. Then p_i is an extension of q_i in D such that $\pi^{-1}_{mc(p_i),mc(q_i)}\pi_{\alpha,mc(q_i)}(t_i) = \pi^{-1}_{mc(p_i),mc(q_i)}\pi_{\alpha,mc(q_i)}(p_i^\alpha)$; as p_i extends q, this is $\pi^{-1}_{mc(p_i),mc(q_i)}(p_i^{mc(q_i)})$, and as p_i extends q_i, this is p_i^{mc}. So q_{i+1} was chosen so that $q_{i+1} * \pi^{-1}_{mc(q_{i+1}),mc(q_i)}\pi_{\alpha,mc(q_i)}(t_i) = q_{i+1} * \pi^{-1}_{mc(q_{i+1}),mc(q_i)}\pi_{\alpha,mc(q_i)}(p^\alpha)$ belongs to D. But as p extends q, this is $q_{i+1} * \pi^{-1}_{mc(q_{i+1}),mc(q_i)}(p^{mc(q_i)})$ and as p_{i+1} extends q_{i+1}, this is $q_{i+1} * p_{i+1}^{mc(q_{i+1})} = q_{i+1} * p^{mc(q_{i+1})}$. As p extends q, this equals $q_{i+1} * \pi_{\alpha,mc(q_{i+1})}(p^\alpha)$, a condition extended by $p_{i+1} \downarrow q$. Thus $p_{i+1} \downarrow q$ is a very direct extension of $p \Downarrow q$ which belongs to D. □ (Sublemma)

Now suppose that φ is a sentence and p is a condition. We wish to show that p has a direct extension deciding φ. By the sublemma, we may assume that if r is an extension of p which decides φ then so does some very direct extension of $r \Downarrow p = p * r^{mc(p)}$. We claim now that some very direct extension of p decides φ. Suppose not; we say that p is *indecisive*.

We claim that for $U_{mc(p)}$-measure one $\nu > \max(p^{mc})$ in $\text{Succ}_{T^p}(p^{mc})$, the condition $p(\nu) = p * (p^{mc} * \nu)$ is also indecisive. For, if $p(\nu)$ were decisive for $U_{mc(p)}$-measure one ν, then by thinning T^p we obtain a very direct extension of p which decides φ, contradiction. Similarly, if ν_0 belongs to $\text{Succ}_{T^p}(p^{mc})$ and $p(\nu_0)$ is indecisive, then for $U_{mc(p)}$-measure one ν_1 in $\text{Succ}_{T^p}(p^{mc} * \nu_0)$, the condition $p(\nu_0, \nu_1) = p * (p^{mc} * \nu_0 * \nu_1)$ is indecisive. Continuing in this way we can form a very direct extension q of p such that for each $\sigma \in T^q_{p^{mc}}$, the condition $p(\sigma)$ is indecisive.

Now choose $r \leq q$ which decides φ. By choice of p, a very direct extension of $r \Downarrow p$ also decides φ. But $r \Downarrow p$ is a very direct extension of a condition of the form $p(\sigma)$ where σ belongs to $T^q_{p^{mc}}$; this contradicts the choice of q. □

Lemma 65 *P preserves κ^+.*

Proof. As P forces κ to be singular, κ^+ is either preserved or given a cofinality less than κ. Thus it suffices to show that if q_0 forces $\dot{f}$ to be a function from some $\alpha < \kappa$ into κ^+ then some extension q of q_0 forces a bound on the range of $\dot{f}$. Now using Sublemma 65 form a $\leq^*$-descending sequence $\langle q_i \mid i \leq \alpha\rangle$ so that for each $i < \alpha$, q_{i+1} has the property that if p extends q_{i+1} and decides $\dot{f}$ at i, then so does a very direct extension of $p \Downarrow q_{i+1}$. Then $q = q_\alpha$ forces a bound on the range of $\dot{f}$, as if $p \leq q$ decides $\dot{f}$ at $i < \alpha$, so does a very direct extension of $p \Downarrow q \leq p \Downarrow q_{i+1}$, and there are only κ-many conditions of the form $p \Downarrow q$. $\Box$

Lemma 66 *For each α in $[\kappa, \lambda)$, $G^\alpha = \bigcup\{p^\alpha \mid p \in G\}$ is a Prikry sequence for U_α and if $\alpha < \beta$ belong to $[\kappa, \lambda)$ then G^β eventually strictly dominates G^α.*

Proof. The first conclusion follows easily from the definition of the forcing P. Suppose that $\alpha < \beta$ belong to $[\kappa, \lambda)$. Choose $\gamma < \lambda$ so that $\alpha, \beta <_j \gamma$. Then $\{\nu \mid \pi_{\gamma\beta}(\nu) > \pi_{\gamma\alpha}(\nu)$ belongs to the ultrafilter U_γ. It now follows easily from the definition of the forcing P that G^β eventually strictly dominates G^α. $\Box$

Therefore by forcing with P we obtain a model where κ has cofinality ω, GCH holds below κ and $2^\kappa = \lambda$.