Definable wellorders, Sommersemester 2009

1.-2.Vorlesungen

Introduction

In ZF, the axiom of choice is equivalent to the assertion that for every infinite cardinal κ there is a wellorder of the power set of κ. This is equivalent to saying that $H(\kappa^+)$, the set of sets whose transitive closure has size at most κ, can be wellordered for every infinite cardinal κ.

In this course we explore the possibilities for *definable* wellorders in various set-theoretic contexts. For an infinite cardinal κ we say that $H(\kappa^+)$ has a Σ_n *definable wellorder* iff there is a wellorder of $H(\kappa^+)$ which is Σ_n definable over $(H(\kappa^+), \in)$ with parameter κ. It has a Σ_n *definable wellorder with parameters* if arbitrary parameters from $H(\kappa^+)$ are allowed.

In Gödel's universe L, the situation is ideal:

Theorem 1 *Assume $V = L$. Then for each infinite cardinal κ, there is a Σ_1 definable wellorder of $H(\kappa^+)$.*

Proof. For x, y in $H(\kappa^+)$ define:

$x < y$ iff
There exists a transitive model M of $\mathrm{ZFC}^- + V = L$ of size at most κ such that x, y belongs to M and in M, $x <_L y$

This wellorder is Σ_1 over $H(\kappa^+)$ and in fact uses no parameter. □

Now what happens if we consider definable wellorders in the context of large cardinals? First consider the case $\kappa = \omega$ and make the following observation:

Proposition 2 *$H(\omega_1)$ has a Σ_n definable wellorder (with/without parameters) iff there is a Σ^1_{n+1} wellorder of the reals (with/without parameters).*

Proof. Consider the case with no parameters and $n = 1$. (The general case $n \geq 1$ (with or without parameters) follows easily from this special case.) If

$<$ is a wellorder of $H(\omega_1)$ defined by the Σ_1 formula $\varphi(x, y)$ then obtain a Σ^1_{n+1} definable wellorder of the reals as follows:

$R <^* S$ iff
There exists a real T which codes a countable transitive set M such that R, S belong to M and in M, $\varphi(R, S)$

This is Σ^1_2 as to say that T codes a countable transitive set is a Π^1_1 property.

Conversely, if $<$ is a wellorder of the reals defined by the Σ^1_2 formula $\varphi(R, S)$ then obtain a Σ_1 definable wellorder of $H(\omega_1)$ as follows:

$x <^* y$ iff
There exists a countable transitive model M of ZFC$^-$ such that x, y belong to M and in M, for some reals R, S: R codes x, S codes y and $\varphi(R, S)$

This works as for any transitive model M of ZFC$^-$, if $\varphi(R, S)$ holds in M for reals R, S in M, then in fact $\varphi(R, S)$ holds in V. $\Box$

Now we have:

Theorem 3 *(Mansfield) If there is a Σ^1_2 wellorder of the reals then every real is constructible.*

Theorem 4 *(Martin-Steel) (a) The existence of a Σ^1_{n+2} wellorder of the reals is consistent with the existence of n Woodin cardinals. (b) The existence of a Σ^1_{n+2} wellorder of the reals with parameters is inconsistent with the existence of n Woodin cardinals and a measurable cardinal above them.*

Now suppose that $\kappa = \omega_1$ and therefore we are considering definable wellorders of $H(\omega_2)$. We say that a forcing is *small* if it has size less than the least inaccessible cardinal. Note that a small forcing preserves large cardinal properties.

Theorem 5 *(F-Asperó) There is a small forcing which forces CH together with a definable wellorder of $H(\omega_2)$. In particular it is consistent with arbitrary large cardinals and CH that there is a definable wellorder of $H(\omega_2)$.*

It is not known if "definable" can be taken to be "Σ_2 definable" in the previous theorem. However Σ_1 definability is in general not possible:

Theorem 6 *(Woodin) Assume that there is a measurable Woodin cardinal and CH holds. Then there is no Σ_1 definable wellorder of $H(\omega_2)$; in fact there is no wellorder of the reals which is Σ_1 definable over $H(\omega_2)$.*

Woodin's result is optimal in the following sense:

Theorem 7 *(Avraham-Shelah) There is a small forcing which forces a wellorder of the reals which is Σ_1 definable over $H(\omega_2)$. Necessarily, CH fails in the forcing extension.*

Theorem 16 extends to all regular uncountable κ:

Theorem 8 *(F-Asperó) There is a class forcing which forces GCH, adds a definable wellorder of $H(\kappa^+)$ for all regular uncountable κ and preserves all supercompact cardinals as well as a proper class of n-huge cardinals for each n.*

It is not known if "definable" can be taken to be "Σ_1 definable" in the previous theorem, provided one restricts to regular κ greater than ω_1.

For singular κ there is a limitation in the presence of very large cardinals.

Proposition 9 *Suppose that there is a nontrivial elementary embedding from $L(H(\lambda^+)) \to L(H(\lambda^+))$ (fixing λ, with critical point less than λ). Then there is no definable wellorder of $H(\lambda^+)$ with parameters.*

The cardinal λ in this proposition has cofinality ω.

Next we consider definable wellorders in the context of forcing axioms. First suppose that κ equals ω.

Theorem 10 *(a) (Harrington, F) Martin's axiom is consistent with the existence of a Σ^1_3 wellorder of the reals. (b) (Caicedo-F) Relative to a reflecting cardinal, BSPFA is consistent with the existence of a Σ^1_3 wellorder of the reals.*

It is not known if BMM is consistent with a projective wellorder of the reals (i.e., a wellorder of the reals which is Σ^1_n with parameters for some n). Unlike BPFA, the full PFA implies that there is no such wellorder as it implies PD.

For $\kappa = \omega_1$ a surprising thing happens:

Theorem 11 *(Moore) BPFA implies that there is a definable wellorder of $H(\omega_2)$ with parameters.*

Concerning wellorders without parameters:

Theorem 12 *(Caicedo-F) Relative to a reflecting cardinal there is a model of BSPFA with a Σ_1 definable wellorder of $H(\omega_2)$.*

Theorem 13 *(Larson) Relative to enough supercompacts, there is a model of MM with a definable wellorder of $H(\omega_2)$.*

Forcing axioms have no effect on definable wellorders when κ is greater than ω_1.

One can consider definable wellorders in many other contexts. Below is a sample of open questions.

1. Is it consistent that for all infinite regular κ, GCH fails at κ and there is a definable wellorder of $H(\kappa^+)$?
2. Is the tree property at ω_2 consistent with a projective wellorder of the reals?
3. Is it consistent that the nonstationary ideal on ω_1 is saturated and there is a Σ^1_4 wellorder of the reals?
4. Is it consistent that GCH fails at a measurable cardinal κ and there is a definable wellorder of $H(\kappa^+)$?

Now we start to prove some of the results listed earlier.

Theorem 14 *(Mansfield) If there is a Σ^1_2 wellorder of the reals then every real is constructible.*

Proof. Assume that there is a nonconstructible real and let $<$ be a Σ^1_2 wellorder of the reals, which we take to be Cantor space, the set of all paths through the binary branching tree $2^{<\omega}$. For any perfect subtree T of $2^{<\omega}$, let $[T]$ denote the set of infinite paths through T, a perfect closed subset of Cantor space. For any order-preserving $f : T \to 2^{<\omega}$ we let f^* denote the induced continuous function from $[T]$ to Cantor space.

Lemma 15 *Suppose that T is constructible, $f : T \to 2^{<\omega}$ is constructible and f^* is injective. Then there is a constructible perfect $U \subseteq T$ and constructible, order-preserving $g : U \to 2^{<\omega}$ such that g^* is injective and $g^*(x) < f^*(x)$ for all $x \in [U]$.*

Proof of Lemma. As T is a perfect tree, there is a constructible $h : T \to 2^{<\omega}$ such that h^* is a bijection from $[T]$ onto Cantor space. For $s \in 2^{<\omega}$ let $\bar{s}$ be the "flip" of s, i.e., if $s = (s(0), s(1), \ldots, s(k))$ then $\bar{s} = (1 - s(0), 1 - s(1), \ldots, 1 - s(k))$. For x in Cantor space, $\bar{x}$ is defined similarly.

Let A be the set of $x \in [T]$ such that $f^*(x) > h^*(x)$ and B the set of $x \in [T]$ such that $f^*(x) > \overline{h^*(x)}$. We claim that either A or B contains a nonconstructible element: Let z be the $<$-least nonconstructible real and choose $x, y \in [T]$ so that $h^*(x) = z$, $h^*(y) = \bar{z}$. As x, y are nonconstructible and f^* is an injective, constructible function, it follows that $f^*(x), f^*(y)$ are nonconstructible and therefore $\geq z$. As $f^*(x), f^*(y)$ are distinct, either $f^*(x) > z$ or $f^*(y) > z$. But then either $f^*(x) > z = h^*(x)$ or $f^*(y) > z = \overline{h^*(y)}$, as desired.

Without loss of generality, assume that A has a nonconstructible element. Then A is Σ^1_2 with constructible parameters and therefore has a "constructible" perfect subset, i.e., $[U] \subseteq A$ for some constructible perfect tree U. If we let g be $h \upharpoonright U$ then we have satisfied the conclusion of the Lemma. $\square$ (Lemma)

Now given the Lemma we easily reach a contradiction: Let T_0 be $2^{<\omega}$ and $f_0 : T_0 \to T_0$ the identity. Successively applying the Lemma we get $T_0 \supseteq T_1 \supseteq \cdots$ and $f_0 \supseteq f_1 \supseteq \cdots$ such that $f_n^*(x) > f_{n+1}^*(x)$ for all $x \in T_{n+1}$. Since the $[T_n]$'s are compact sets, they have a nonempty intersection and if x belongs to this intersection we get $f_0^*(x) > f_1^*(x) > \cdots$, contradicting the hypothesis that $<$ is a wellorder. $\square$

3.Vorlesung

We say that a forcing is *small* if it has size less than the least inaccessible cardinal. Note that a small forcing preserves large cardinal properties.

Theorem 16 *(F-Asperó) There is a small forcing which forces CH together with a definable wellorder of $H(\omega_2)$. In particular it is consistent with arbitrary large cardinals and CH that there is a definable wellorder of $H(\omega_2)$.*

I'll begin with the following easier result.

Theorem 17 *There is a small forcing which forces CH together with a Σ_1 wellorder of $H(\omega_2)$ with parameters.*

Proof. First force CH by adding an ω_1-Cohen set. Next add an ω_2-Cohen set A. In the resulting model, $H(\omega_2)$ is $L_{\omega_2}[A]$ and CH holds. For technical reasons, we assume that $A \cap \omega_1$ is empty.

The final step is to add $B, C \subseteq \omega_1$ which "code" A in the sense that A is Δ_1 definable over $L_{\omega_2}[A, B, C]$ (the final $H(\omega_2)$) with B, C, ω_1 as parameters. This gives a Σ_1 wellorder of $L_{\omega_2}[A, B, C]$ with B, C, ω_1 as parameters: simply take the canonical wellorder with parameters A, B, C and eliminate A in favour of its Δ_1 definition with parameters B, C, ω_1.

The forcing P for adding B is a forcing to code A using "canonical functions". For each uncountable $\beta < \omega_2$ choose a bijection $f_\beta : \omega_1 \to \beta$. The set B codes A in the following way: β belongs to A iff $\mathrm{ot}(f_\beta[\gamma])$ belongs to B for a CUB set of $\gamma < \omega_1$, where "ot" stands for "ordertype". Note that if f^0_β, f^1_β are any two bijections from ω_1 onto β then the set of $\gamma < \omega_1$ where $\mathrm{ot}(f^0_\beta[\gamma])$ equals $\mathrm{ot}(f^1_\beta[\gamma])$ contains a CUB set. Thus this coding is independent of the choice of the functions f_β, $\omega_1 \leq \beta < \omega_2$.

A condition in P is a triple (p, p^*, p^{**}) where:

p is an ω_1-Cohen condition, i.e., a function from a countable ordinal $|p|$ to 2.
p^* is a countable subset of ω_2.
p^{**} is a closed, bounded subset of ω_1.
For β in p^* and γ in p^{**}, $\mathrm{ot}(f_\beta[\gamma])$ is at least γ and less than $|p|$.

We say that (q, q^*, q^{**}) extends (p, p^*, p^{**}) iff:

q end-extends p, q^* contains p^*, q^{**} end-extends p^{**}.
All elements of $q^{**} \setminus p^{**}$ are at least $|p|$.
For γ in $q^{**} \setminus p^{**}$ and β in p^*, $q(\mathrm{ot}(f_\beta[\gamma]))$ equals $A(\beta)$.

Lemma 18 *(a) For any (p, p^*, p^{**}), $\alpha \in [\omega_1, \omega_2)$ and $\delta < \omega_1$ there is an extension (q, q^*, q^{**}) of (p, p^*, p^{**}) such that α belongs to q^* and* $\max(q^{**})$ *is greater than δ.*
(b) P is ω_2-cc.
(c) P is ω-distributive.

Proof. (a) Choose γ greater than $|p|$, δ so that for distinct β_0, β_1 in $p^* \cup \{\alpha\}$, $\mathrm{ot}(f_{\beta_0}[\gamma])$, $\mathrm{ot}(f_{\beta_1}[\gamma])$ are distinct. This is possible as the set of such γ contains a CUB set. Now set $q^* = p^* \cup \{\alpha\}$, extend p to q so that $q(\mathrm{ot}(f_\beta[\gamma])$ equals $A(\beta)$ for β in $p^* \cup \{\alpha\}$ and set $q^{**} = p^{**} \cup \{\gamma\}$.
(b) Note that if $p = q$ and $p^{**} = q^{**}$ then (p, p^*, p^{**}) and (q, q^*, q^{**}) are compatible, as they are both extended by $(p, p^* \cup q^*, p^{**})$. Therefore CH gives us the ω_2-cc.

4.-5.Vorlesungen

We finish the proof of:

Theorem 19 *There is a small forcing which forces CH together with a* Σ_1 *wellorder of* $H(\omega_2)$ *with parameters.*

Lemma 20 *(a) For any* (p, p^*, p^{**}), $\alpha \in [\omega_1, \omega_2)$ *and* $\delta < \omega_1$ *there is an extension* (q, q^*, q^{**}) *of* (p, p^*, p^{**}) *such that* α *belongs to* q^* *and* $\max(q^{**})$ *is greater than* δ.
(b) P *is* ω_2*-cc.*
(c) P *is* ω*-distributive.*

Proof of (c). Suppose that $(p_0, p_0^*, p_0^{**}) \geq (p_1, p_1^*, p_1^{**}) \cdots$ is a descending ω-sequence of conditions. To obtain a lower bound (q, q^*, q^{**}) we start by taking q to be the union of the p_n's, q^* to be the union of the p_n^*'s and q^{**} to be the union of the p_n^{**} together with the supremum γ of the $\max p_n^{**}$'s. Then q must be lengthened so that for β in q^*, $q(\mathrm{ot}(f_\beta[\gamma]))$ is defined and equal to $A(\beta)$. The problem with this lengthening is that $\mathrm{ot}(f_\beta[\gamma])$ may be the same for two distinct β's in q^* at which A differs. To solve this problem, it suffices to know that for each n:

$(*)$ $\max(p_{n+1}^{**})$ belongs to a CUB set of δ's on which $\mathrm{ot}(f_\beta[\delta])$ is distinct for distinct β in p_n^*.

Then $\mathrm{ot}(f_\beta[\gamma])$ will be distinct for any two distinct β in q^*, enabling us to lengthen q as desired.

Finally, note that if $D_0, D_1, \ldots$ are open dense sets then we can build an ω-sequence $(p_0, p_0^*, p_0^{**}) \geq (p_1, p_1^*, p_1^{**}) \cdots$ below any given condition so that $(p_{n+1}, p_{n+1}^*, p_{n+1}^{**})$ belongs to D_n and obeys $(*)$. $\square$

Suppose that G is P-generic and let B be the union of the p for (p, p^*, p^{**}) in G, C the union of the p^{**} for (p, p^*, p^{**}) in G. Then for any $\beta \in [\omega_1, \omega_2)$ we have:

$(**)$ β belongs (does not belong) to A iff $\mathrm{ot}(f_\beta[\gamma])$ belongs (does not belong) to B for sufficiently large γ in C.

In fact we can write:

$(***)$ β belongs (does not belong) to A iff for some bijection $f : \omega_1 \to \beta$, $\mathrm{ot}(f[\gamma])$ belongs (does not belong) to B for sufficiently large γ in C.

This is because if β does not belong to A, $(**)$ implies that $\mathrm{ot}(f_\beta[\gamma])$ does not belong to B for sufficiently large γ in C and as $\mathrm{ot}(f_\beta[\gamma])$ equals $\mathrm{ot}(f[\gamma])$ for unboundedly many γ in C, it follows that $\mathrm{ot}(f[\gamma])$ does not belong to B for unboundedly many γ in C.

This shows that in $V[G]$, the predicate A is Δ_1 over $H(\omega_2)$ in parameters B, C and ω_1. As there is a wellorder of $H(\omega_2) = L_{\omega_2}[A, B, C]$ which is Σ_1 with parameters A, B, C it follows that there is one which is Σ_1 with parameters A, B, ω_1. $\Box$ (Theorem 27)

Theorem 21 *There is a small forcing which forces CH together with a definable wellorder of $H(\omega_2)$.*

We first prove something easier (although certainly not easy!):

Theorem 22 *Suppose that A is a subset of ω_1. Then there is a small forcing which forces CH, preserves ω_1 and forces A to be definable over $H(\omega_2)$.*

The proof uses a "weak club-guessing" property (due to Asperó, inspired by work of Avraham-Shelah). As we will need these properties later when studying $H(\kappa^+)$ for arbitrary regular uncountable κ, we present the relevant definitions in a general setting.

A *club-sequence* with *length* λ and *domain* D is a sequence $\vec{C} = \langle C_\delta \mid \delta < \lambda \rangle$, where λ is an ordinal, such that each C_δ is a subset of δ for each δ and D consists of those δ such that C_δ is a club in δ. We write D as $\mathrm{dom}(\vec{C})$. The *range* of $\vec{C}$ is the union of the C_δ, $\delta \in \mathrm{dom}(\vec{C})$.

$\vec{C}$ is a *coherent* club sequence iff there is a club-sequence $\vec{D}$ with $\mathrm{dom}(\vec{D}) \supseteq \mathrm{dom}(\vec{C})$ such that $\vec{D}, \vec{C}$ agree on $\mathrm{dom}(\vec{C})$ and whenever δ belongs to $\mathrm{dom}(\vec{D})$ and γ is a limit point of D_δ, γ also belongs to $\mathrm{dom}(\vec{D})$ and $D_\gamma = D_\delta \cap \gamma$. In this case we say that $\vec{D}$ *witnesses* the coherence of $\vec{C}$.

Suppose that $\vec{C}$ is a club sequence and there exists a fixed τ such that $\mathrm{ot}(C_\delta) = \tau$ for each δ in $\mathrm{dom}(\vec{C})$; then we say that τ is the *height* of $\vec{C}$.

Suppose that λ has uncountable cofinality and $\vec{C}$ is a club sequence of length λ. We say that $\vec{C}$ is *guessing* iff for every club C in λ there is some δ in $C \cap \mathrm{dom}(\vec{C})$ such that C_δ is almost contained in C, i.e., $C_\delta \setminus C$ is bounded in δ. We say that $\vec{C}$ is *strongly guessing* iff for every club C in λ there is a club D in λ such that C_δ is almost contained in C for all δ in $D \cap \mathrm{dom}(\vec{C})$. If $\mathrm{dom}(\vec{C})$ is stationary and $\vec{C}$ is strongly guessing then it is also guessing.

Now we weaken the concepts of guessing and strongly guessing. If X, Y are sets of ordinals then we define $X \cap^* Y$ to consist of all δ in $X \cap Y$ such that δ is not a limit point of X. (This operation is not symmetric.) Then we say that $\vec{C}$ is *type-guessing* iff for every club C in λ there is $\delta \in C \cap \mathrm{dom}(\vec{C})$ such that $\mathrm{ot}(C_\delta \cap^* C) = \mathrm{ot}(C_\delta)$. And $\vec{C}$ is *strongly type-guessing* iff for every club C in λ there is a club D in λ such that $\mathrm{ot}(C_\delta \cap^* C) = \mathrm{ot}(C_\delta)$ for every $\delta \in D \cap \mathrm{dom}(\vec{C})$.

An ordinal τ is *perfect* iff $\omega^\tau = \tau$.

Definition 23 *For κ uncountable and regular, I_κ denotes the set of perfect ordinals $\tau < \kappa$ of countable cofinality for which there is a coherent strongly type-guessing club sequence $\vec{C}$ of length κ with stationary domain and of height τ.*

To prove Theorem 28 we use:

Lemma 24 (Main Claim) *Assume GCH at $\aleph_0, \aleph_1$ and suppose that $B \subseteq \omega_1$ is a set of perfect ordinals. Then there is an ω-strategically closed, $\aleph_2$-cc forcing P which forces that I_{ω_1} equals B.*

The lemma implies that any subset of ω_1 can be made Σ_2 definable over $H(\omega_2)$ by a small forcing, a strong version of Theorem 28.

6.-7.Vorlesungen

Lemma 25 (Main Claim) *Assume GCH at $\aleph_0, \aleph_1$ and suppose that $B \subseteq \omega_1$ is an unbounded set of perfect ordinals. Then there is an ω-strategically closed, $\aleph_2$-cc forcing P which forces that I_{ω_1} equals B.*

The lemma implies that any subset of ω_1 can be made Σ_2 definable over $H(\omega_2)$ by a small forcing.

To prove the Main Claim we begin with the following lemma.

Lemma 26 *Under the assumptions of the Main Claim, write B in increasing order as $(\tau_\nu)_{\nu<\omega_1}$. Then there is an ω-closed forcing P^* of size ω_1 which forces that there are sequences $(\vec{C}^\nu)_{\nu<\omega_1}$, $(\vec{D}^\nu)_{\nu<\omega_1}$, such that $(dom(\vec{C}^\nu))_{\nu<\omega_1}$ forms a sequence of pairwise disjoint stationary subsets of ω_1 and for all $\nu < \omega_1$:*

(a) $\vec{C}^\nu$ has height τ_ν.
(b) $\vec{D}^\nu$ witnesses the coherence of $\vec{C}^\nu$.
(c) The range of $\vec{C}^\nu$ is disjoint from the domain of $\vec{C}^{\nu'}$ for all $\nu' < \omega_1$.
(d) Successor elements of $\vec{C}^\nu_\delta$ are limit ordinals for each δ in $dom(\vec{C}^\nu)$.
(e) $\vec{C}^\nu$ is a guessing club-sequence.

Proof. P^* consists of all pairs

$$p = ((\vec{C}^{p,\nu} \mid \nu < \lambda_p), (\vec{D}^{p,\nu} \mid \nu < \lambda_p))$$

(for some ordinal $\lambda_p < \omega_1$) such that for each $\nu < \lambda_p$:

(1) $\vec{C}^{p,\nu}$ and $\vec{D}^{p,\nu}$ are club sequences of length $\lambda_p + 1$.
(2) $\vec{C}^{p,\nu}$ has height τ_ν.
(3) The range of $\vec{C}^{p,\nu}$ is disjoint from the domain of $\vec{C}^{p,\nu'}$ for each $\nu' < \lambda_p$.
(4) $\vec{D}^{p,\nu}$ witnesses the coherence of $\vec{C}^{p,\nu}$.
(5) Successor elements of $\vec{C}^{p,\nu}_\delta$ are limit ordinals for each δ in $\text{dom}(\vec{C}^{p,\nu})$.

p_1 extends p_0 iff $\lambda_{p_0} \leq \lambda_{p_1}$ and for each $\nu < \lambda_{p_0}$, $\vec{C}^{p_1,\nu}$ extends $\vec{C}^{p_0,\nu}$ and $\vec{D}^{p_1,\nu}$ extends $\vec{D}^{p_0,\nu}$.

Clearly P^* has size ω_1, as we have assumed CH. To see that P^* is ω-closed, reason as follows. Suppose that $p_0 \geq p_1 \geq \cdots$ is a descending ω-sequence of conditions and we want to show that this sequence has a lower bound. We may assume that this sequence is strictly decreasing, and therefore the supremum

λ of the λ_{p_n}'s does not belong to the domain of any club-sequence mentioned by any of the p_n's. But now we can obtain a lower bound p by choosing the club-sequences $\vec{C}^{p,\nu}$ and $\vec{D}^{p,\nu}$, $\nu < \lambda$, of length $\lambda + 1$ to not include λ in their domain.

Let G be P^*-generic and for $\nu < \omega_1$ let $\vec{C}^\nu$, $\vec{D}^\nu$ respectively denote the union of the $\vec{C}^{p,\nu}$ for p in G, the union of the $\vec{D}^{p,\nu}$ for p in G.

We claim that each $\vec{C}^\nu$ is a guessing club-sequence in $V[G]$ for each $\nu < \omega_1$. Let $\dot{C}$ be a P^*-name for a club in ω_1 and let p be a condition in P^*. Let $(N_i)_{i \leq \tau_\nu}$ be a continuous chain of countable elementary substructures of some large $(H(\theta), \in, \Delta)$ (where Δ is a wellorder of $H(\theta)$) such that N_0 contains $\nu, \dot{C}$ and p and for each $i < \tau_\nu$, $(N_j)_{j \leq i}$ belongs to N_{i+1}. For $i \leq \tau_\nu$ let δ_i be $N_i \cap \omega_1$ and let $(\epsilon^i_n)_{n<\omega}$ be the Δ-least ω-sequence cofinal in δ_i.

Now choose $(q_n)_{n<\omega}$ to form a descending sequence of conditions in N_0 extending p such that for all n, λ_{q_n} is greater than ϵ^0_n and q_n forces some ordinal greater than ϵ^0_n into $\dot{C}$. Let p_0 be the lower bound to the q_n's obtained by setting $\lambda_{p_0} = \delta_0$ and $\vec{C}^{p_0,\nu'}_{\delta_0} = \vec{D}^{p_0,\nu'}_{\delta_0} = \emptyset$ for all $\nu' < \delta_0$. Then form $p_1 \leq p_0$ in a similar way, with N_0, p, $(\epsilon^0_n)_{n<\omega}$ and δ_0 replaced by N_1, p_0, $(\epsilon^1_n)_{n<\omega}$ and δ_1, respectively. Continue this for τ_ν steps to build the τ_ν-sequence $p_0 \geq p_1 \geq \cdots$, choosing lower bounds p_i at limit stages $i \leq \tau_\nu$ to obey the following:

$\vec{D}^{p_i,\nu}_{\delta_i} = \{\delta_j \mid j < i\}$
$\vec{C}^{p_{\tau_\nu},\nu}_{\delta_{\tau_\nu}} = \{\delta_j \mid j < \tau_\nu\}$.

Then $q = p_{\tau_\nu}$ is indeed a condition extending p which forces that $\vec{C}^\nu_{\delta_{\tau_\nu}}$ is a subset of $\dot{C}$. $\square$

Now to prove the Main Claim we perform an iteration with countable support $(P_\xi \mid \xi < \omega_2)$ using names $(\dot{Q}_\xi \mid \xi < \omega_2)$. The desired forcing that satisfies the Main Claim is P_{ω_2}, the direct limit of the P_ξ, $\xi < \omega_2$.

If $\vec{C}$ is a (type-) guessing club sequence of length ω_1 and $C \subseteq \omega_1$ is a club, then $P(\vec{C}, C)$ is the natural forcing for adding a club $D \subseteq \omega_1$ such that $\text{ot}(C_\delta \cap^* C) = \text{ot}(C_\delta)$ for δ in $D \cap \text{dom}(\vec{C})$. A condition in this forcing a closed, bounded subset d of ω_1 such that $\text{ot}(C_\delta \cap^* C) = \text{ot}(C_\delta)$ for all δ in $d \cap \text{dom}(\vec{C})$.

At the first stage of our iteration we force with the P^* of Lemma 26. Let $(\vec{C}^\nu)_{\nu<\omega_1}$, $(\vec{D}^\nu)_{\nu<\omega_1}$ be the club sequences added by this forcing. Let $\vec{C}$ denote the amalgamtion of the $\vec{C}^\nu$, i.e., the club sequence with domain $\bigcup_\nu \mathrm{dom}(\vec{C}^\nu)$ whose restriction to each $\mathrm{dom}(\vec{C}^\nu)$ is $\vec{C}^\nu$.

At each stage $\xi > 0$ of the iteration we pick some P_ξ-name $\dot{C}_\xi$ for a club in ω_1 and we let $\dot{Q}_\xi$ be a P_ξ-name for the forcing $P(\vec{C}, \dot{C}_\xi)$. As we have assumed CH, each P_ξ, $\xi < \omega_2$ has a dense subset of size ω_1 and the entire iteration is ω_2-cc. It follows that any club $C \subseteq \omega_1$ added by P has a P_ξ-name for some $\xi < \omega_2$. Moreover as we have assumed $2^{\omega_1} = \omega_2$, we can use a bookkeeping function to choose our names $\dot{C}_\xi$ so that every club $C \subseteq \omega_1$ added by P is named by some $\dot{C}_\xi$ and therefore we force with $P(\vec{C}, C)$ at some stage of the iteration.

8.-9.Vorlesungen

The ω_2-iteration P is ω-strategically closed: Recall that the first component of P is the forcing P^*. Suppose that $p_0 \geq p_1 \geq \cdots$ is an ω-sequence in P such that for some λ, the sup of the lengths of the p_n's on each component in the union of the supports of the p_n's equals λ. Then we can obtain a lower bound q by taking the first component of q to have length $\lambda + 1$ while assigning the empty set at λ for all of its club-sequences, and including λ into the clubs at all later components of q. The ω-strategic closure of P now follows from the fact that it is easy to form a strategy which generates sequences of p_n's as above.

It is also easy to verify that the sets added by the forcings $P(\vec{C}, C)$ are unbounded and therefore clubs; this is simply because the complement of the domain of $\vec{C}$ is stationary. It follows that P forces each $\vec{C}^\nu$ to be strongly type-guessing, as for each club $C \subseteq \omega_1$ in the extension, P explicitly adds a club D witnessing strong type-guessing for each $\vec{C}^\nu$ and C. Of course this is vacuous without knowing that the domain of $\vec{C}^\nu$ is stationary in the final model. (The positive stages of the iteration are not proper.) An argument as in the proof that P^* produces club-sequences with stationary domain verifies this last fact, and in fact shows that each $\vec{C}^\nu$ is a guessing club-sequence.

Our main and final task is now to show that if τ is perfect but not one of the desired heights, i.e., does not equal τ_ν for some $\nu < \omega_1$, then in the P-generic extension there is no strongly type-guessing club-sequence of height τ

with stationary domain. Let G be P-generic and $\vec{E}$ a club-sequence of length ω_1 with stationary domain of perfect height $\tau < \omega_1$. Choose $0 < \xi < \omega_2$ so that $\vec{E}$ belongs to $V[G_0]$ where $G_0 = G \cap P_\xi$. We work in $V[G_0]$. Let D be the club added at stage ξ of the iteration (which witnesses strong type-guessing for the club-sequence $\vec{C}$ with respect to the club C_ξ) and let $\dot{D}$ be a P/G_0-name for D. Our goal is to show that if τ is not of the form τ_η, $\eta < \omega_1$, then any condition p in P/G_0 forcing that $\dot{E}$ is a name for a club in ω_1 can be extended to a condition q forcing that for some δ in $\dot{E} \cap \mathrm{dom}(\vec{E})$, $\mathrm{ot}(E_\delta \cap^* \dot{D})$ is less than τ, the ordertype of E_δ.

Let θ be large and let $(N_i)_{i<\omega_1}$ be a continuous chain of elementary submodels of $H(\theta)$ such that N_0 contains all relevant parameters (such as p, τ, $\dot{D}$ and $\dot{E}$). Set $\delta_i = N_i \cap \omega_1$ for each $i < \omega_1$ and let D_0 be the club consisting of the δ_i's. In the final model $V[G]$, the set $\{\delta < \omega_1 \mid \delta \in \mathrm{dom}(\vec{C}) \to \mathrm{ot}(C_\delta \cap^* D_0) = \mathrm{ot}(C_\delta)\}$ contains a club. As $\mathrm{dom}(\vec{E})$ is stationary in the final model we can choose $i^* = \delta_{i^*} < \omega_1$ in $\mathrm{dom}(\vec{E})$ such that $i^* \in \mathrm{dom}(\vec{C}) \to \mathrm{ot}(C_{i^*} \cap^* D_0) = \mathrm{ot}(C_{i^*})$.

We show that some extension q of p of length i^* (i.e., with all names of clubs assigned by q on the components in its support forced to have length i^*) forces that i^* belongs to $\dot{E}$ and that $\mathrm{ot}(E_{i^*} \cap^* \dot{D})$ is less than τ, the ordertype of E_{i^*}. There are three cases.

Case 1. i^* does not belong to $\mathrm{dom}(\vec{C})$.

In this case we find an extension q of p which forces $\dot{D}$ to be disjoint from E_{i^*} above δ_0.

As i^* is greater than τ, it follows that we can choose an ω-sequence $i_0 < i_1 < \cdots$ cofinal in i^* such that $E_{i^*} \cap \delta_{i_n}$ is bounded in δ_{i_n} for each n. Now build an ω-sequence $p = p_0 \geq p_1 \geq \cdots$ of conditions such that each p_{n+1} belongs to $N_{i_{n+1}}$, forces some ordinal greater than δ_{i_n} into $\dot{E}$ and forces that the least ordinal in $\dot{D} \cap [\delta_{i_n}, \delta_{i_{n+1}})$ is greater than $\max(E_{i^*} \cap \delta_{i_{n+1}})$. Moreover we can assume that all of the components of p_{n+1} in its support are forced to have length at least δ_{i_n}. Then as i^* does not belong to the domain of $\vec{C}$ the sequence of p_n's has a greatest lower bound q which forces that $E_{i^*} \cap \dot{D}$ is bounded in i^*; in particular q forces that $\mathrm{ot}(E_{i^*} \cap^* \dot{D})$ is less than τ, as desired.

Case 2. i^* belongs to $\mathrm{dom}(\vec{C})$ and $\tau_0 = \mathrm{ot}(C_{i^*})$ is less than $\tau = \mathrm{ot}(E_{i^*})$.

In this case we find an extension q of p which forces $E_{i^*} \cap \dot{D}$ to be included in C_{i^*} above δ_0.

Denote $\mathrm{ot}(C_{i^*})$ by τ_0. The desired q will have length i^* and be obtained as the greatest lower bound of a τ_0-sequence of conditions of shorter length. To guarantee that this lower bound q exists we must ensure that the ordinal i^* can be placed into all of the clubs $\dot{D}_\eta$ for η in the support of q. As i^* now belongs to the domain of $\vec{C}$, this demands that $\mathrm{ot}(C_{i^*} \cap^* \dot{C}_\eta)$ be maximised (i.e., equal to τ_0) for each such η. In particular, the club $\dot{D} = \dot{D}_\xi$ is of the form $\dot{C}_\eta$ for some η in the support of q and therefore we must ensure that $\mathrm{ot}(C_{i^*} \cap^* \dot{D})$ is maximised, while at the same time ensuring that $\mathrm{ot}(E_{i^*} \cap^* \dot{D})$ is less than $\mathrm{ot}(E_{i^*}) = \tau$. In the present case the latter goal can be achieved by simply arranging that $E_{i^*} \cap \dot{D}$ be contained in C_{i^*} above δ_0, as C_{i^*} has ordertype τ_0 which by assumption is indeed less than τ.

Let $(\delta_{i_j})_{j<\tau_0}$ increasingly enumerate $D_0 \cap C_{i^*}$. We inductively build the p_j, $j < \tau_0$, to meet the following conditions:

1. p_0 extends p and p_j belongs to N_{i_j+1} for each j.
2. For limit j, p_j is the greatest lower bound of $(p_k)_{k<j}$.
3. Each p_{j+1} is the greatest lower bound of an ω-sequence of conditions in N_{i_j+1} and forces that $\delta_{i_{j+1}}$ belongs to $\dot{E}$.
4. For each η in the support of p_j, p_{j+1} forces that $\delta_{i_{j+1}}$ belongs to $\dot{C}_\eta$ (where $\dot{C}_\eta$ is the club considered by the iteration at stage η).
5. Each p_{j+1} forces that $E_{i^*} \cap \dot{D} \cap (\delta_{i_j}, \delta_{i_j+1})$ is empty.

As in Case 1, lower bounds are easily obtained at limit stages j less than τ_0, as C_{i^*} is disjoint from the domain of $\vec{C}$ and therefore δ_{i_j} does not belong to the domain of $\vec{C}$. Condition 4 implies that the p_j's have a greatest lower bound q at the final stage τ_0, as it implies that for each η in the union of the supports of the p_j's, a final segment of $C_{i^*} \cap^* D_0$ is forced inside $\dot{C}_\eta$, allowing us to put i^* into $\dot{D}_\eta$, the club witnessing strong type-guessing for $\vec{C}$ relative to the club $\dot{C}_\eta$. Condition 3 implies that i^* is forced into $\dot{E}$. And by condition 5, q forces that $E_{i^*} \cap \dot{D}$ above δ_{i_0} is contained in $D_0 \cap C_{i^*}$ and therefore has ordertype at most $\tau_0 < \tau$.

The conditions 1, 2 and the first part of 3 are easily arranged; to fulfill the remaining conditions, use the fact that $\tau = \mathrm{ot}(E_{i^*})$ is less than $\delta_{i_{j+1}}$ in order to meet the relevant dense sets in $N_{i_{j+1}}$ between adjacent elements of E_{i^*}.

Case 3. i^* belongs to $\mathrm{dom}(\vec{C})$ and $\tau = \mathrm{ot}(E_{i^*})$ is less than $\tau_0 = \mathrm{ot}(C_{i^*})$.

In this case we find an extension q of p which forces $\dot{D}$ to be disjoint from E_{i^*} above δ_0.

For any γ in E_{i^*} let γ^* denote the least element of E_{i^*} greater than γ. Also let $(t_k \mid k \in \omega)$ be an increasing sequence cofinal in τ_0. As τ is less than τ_0, for each k there are unboundedly many γ_k in E_{i^*} such that the ordertype of $C_{i^*} \cap^* D_0$ on the interval (γ_k, γ_k^*) is greater than t_k. Otherwise $\tau_0 = \mathrm{ot}(C_{i^*} \cap^* D_0)$ is bounded by $t_k \cdot \tau$ for some k, contradicting the assumption that τ_0 is a perfect ordinal greater than τ. Choose an increasing sequence of such γ_k's, and for each k let D_0^k consist of the first $t_k + 1$ elements of $C_{i^*} \cap D_0$ in the interval (γ_k, γ_k^*).

Let $(\delta_{i_j})_{j<\tau_0}$ increasingly enumerate the union of the D_0^k's, a club in i^*. We inductively build the p_j, $j < \tau_0$, to meet the following conditions:

1. p_0 extends p and p_j belongs to N_{i_j+1} for each j.
2. For limit j, p_j is the greatest lower bound of $(p_k)_{k<j}$.
3. Each p_{j+1} is the greatest lower bound of an ω-sequence of conditions in N_{i_j+1} and forces that $\delta_{i_{j+1}}$ belongs to $\dot{E}$.
4. For each η in the support of p_j, p_{j+1} forces that $\delta_{i_{j+1}}$ belongs to $\dot{C}_\eta$ (where $\dot{C}_\eta$ is the club considered by the iteration at stage η).
5. Each p_{j+1} forces that $E_{i^*} \cap \dot{D} \cap (\delta_{i_j}, \delta_{i_j+1})$ is empty.

As in Case 2, lower bounds exist at limit stages and i^* is forced by the final q into $\dot{E}$. By condition 5, q forces that E_{i^*} is disjoint from $\dot{D}$ above the length of p_0, and therefore has ordertype less than τ, as desired. Conditions 1-4 are easily arranged; so is condition 5 as each D_0^k is a closed set lying entirely in the open interval (γ_k, γ_k^*).

This completes the proof that there are no unintended heights of strongly type-guessing club sequences in the P-generic extension. $\square$

10.-11.Vorlesungen

Recall that we have:

Theorem 27 *There is a small forcing which forces CH together with a Σ_1 wellorder of $H(\omega_2)$ with parameters.*

Theorem 28 *Suppose that A is a subset of ω_1. Then there is a small forcing which forces CH, preserves ω_1 and forces A to be definable over $H(\omega_2)$.*

We now want to combine these results to get:

Theorem 29 *There is a small forcing which forces CH together with a definable wellorder of $H(\omega_2)$.*

Roughly speaking, in Theorem 27 we make a wellorder of $H(\omega_2)$ definable by coding it using "canonical function coding" by a subset of ω_1, and in Theorem 28 we make a subset of ω_1 definable by coding it using "club-guessing" by a subset of $H(\omega_2)$. Now we want to combine these methods to add $B \subseteq \omega_1$ and $G \subseteq H(\omega_2)$ so that:

1. B codes G using canonical function coding.
2. G codes B using club-guessing.

If we first add B and then add G then we have not achieved the desired result, as we will only get a definable wellorder of the $H(\omega_2)$ of the ground model, not of the extension. Note that we can't do this with a standard ω_2-iteration with the ω_2-cc, as then any subset of ω_1 will have appeared by some initial stage of the iteration, which makes it impossible for it to decode the generic for the entire iteration.

We need to add B and G "simultaneously". There is feedback: the forcing to add G depends on B and the forcing to add B depends on G. A condition in the desired forcing specifies partial information about B as well as partial information about G; this information is fully determined and does not depend on the ultimate choice of generic. The resulting generic produces both B and G with the desired feedback: B codes G and G codes B. The forcing has features of an iteration as G is added in ω_2 stages, however also has of a product, as conditions are completely determined in the ground model.

We now review the earlier terminology regarding canonical function coding and club guessing that will be needed for the construction.

For uncountable $\gamma < \omega_2$, a *canonical function* for γ is a function $f_\gamma : \omega_1 \to \omega_1$ such that for some surjection $\pi : \omega_1 \to \gamma$, $f_\gamma(\nu) = \mathrm{ot}(\pi[\nu])$ for all $\nu < \omega_1$. Any two canonical functions for γ agree on a club.

A *club-sequence of length λ with domain D* is a sequence $\vec{C} = (C_\delta \mid \delta < \lambda)$ where each C_δ is a subset of δ, $\lambda \leq \omega_1$ and $D = \mathrm{dom}(\vec{C})$ is the set of limit $\delta < \lambda$ such that C_δ is a club in δ. The *range* of $\vec{C}$ is the union of the C_δ, $\delta \in \mathrm{dom}(\vec{C})$. We say that $\vec{C}$ is *coherent* iff there is a club-sequence $\vec{D}$ extending $\vec{C}$ to a possibly larger domain such that $\delta \in \mathrm{dom}(\vec{D})$, γ a limit point of D_δ implies $\gamma \in \mathrm{dom}(\vec{D})$ and $D_\gamma = D_\delta \cap \gamma$. We say that $\vec{D}$ *witnesses* the coherence of $\vec{C}$.

The *height* of a club guessing sequence $\vec{C}$, if defined, is the unique τ such that $\mathrm{ot}(C_\delta) = \tau$ for all δ in $\mathrm{dom}(\vec{C})$. An ordinal τ is *perfect* iff $\omega^\tau = \tau$. If X is a set of ordinals then we let X^+ denote the set of elements of X which are not limit points of X. A club sequence $\vec{C}$ of length ω_1 with stationary domain is *strongly type guessing* iff for every club C in ω_1 there is a club D in ω_1 such that $\mathrm{ot}(C_\delta^+ \cap C) = \mathrm{ot}(C_\delta)$ for every $\delta \in \mathrm{dom}(\vec{C}) \cap D$.

The desired forcing P

Assume the GCH at $\aleph_0$ and $\aleph_1$ and fix a bookkeeping function F, i.e., a function $F : \omega_2 \to H(\omega_2)$ such that for each $a \in H(\omega_2)$, the set of α such that $F(\alpha) = a$ is unbounded in ω_2.

Choose canonical functions $(f_\gamma \mid \omega_1 \leq \gamma < \omega_2)$. We assume that $f_\gamma(\delta) \geq \delta$ for all γ and all limit $\delta < \omega_1$. Also, for distinct γ_0, γ_1 let E_{γ_0,γ_1} be a club in ω_1 of limit ordinals on which f_{γ_0} and f_{γ_1} differ.

Let A be a subset of ω_2 such that $L_{\omega_2}[A] = H(\omega_2)$ and the sequences $(f_\gamma \mid \gamma < \omega_2)$ and $(E_{\gamma_0,\gamma_1} \mid \gamma_0, \gamma_1 < \omega_2)$ are definable over $(H(\omega_2), \in, A)$.

Let $(\eta_\xi)_{\xi<\omega_1}$ increasingly enumerate the countable perfect ordinals and let $\mathcal{C}$ be the set of nonzero $\alpha \leq \omega_2$ such that $\omega_1 \cdot \alpha' < \alpha$ for all $\alpha' < \alpha$.

We will define an increasing sequence of partial orders $(P_\alpha, \leq_\alpha)$, $\alpha \in \mathcal{C}$. The desired forcing P will be $(P_{\omega_2}, \leq_{\omega_2})$.

Given $\alpha \in \mathcal{C}$ and assuming that $P_{\alpha'}$ has been defined for $\alpha' < \alpha$ in $\mathcal{C}$, conditions in P_α are of the form:

$$p = (b, C, (c_\gamma \mid \gamma \in a), ((\vec{C}^i, \vec{D}^i) \mid i < \beta), (D_\gamma \mid \gamma \in a))$$

satisfying the following conditions, where for any ordinal α, $p \restriction \alpha$ denotes $(b, C, (c_\gamma \mid \gamma \in a \cap \alpha), ((\vec{C}^i, \vec{D}^i) \mid i < \beta), (D_\gamma \mid \gamma \in a \cap \alpha))$:

1. a is a countable subset of $\bigcup_{1 \leq \rho < \alpha} [\omega_1 \cdot \rho, \omega_1 \cdot \rho + \beta)$ and γ' belongs to a whenever $\gamma' \geq \omega_1$ and $\gamma \in a$ is of the form $\omega_1 \cdot \gamma' + \zeta$ for some countable ζ.
2. C is a club in ω_1 contained in $\bigcap \{E_{\gamma,\gamma'} \mid \gamma, \gamma' \in a, \gamma \neq \gamma'\}$.
3. β is a countable ordinal closed under Gödel pairing and β belongs to C.
4. b is a subset of β of ordertype β.
5. For $\gamma \in a$, c_γ is a closed subset of β and $f_\gamma(\nu) < \beta$ for ν in c_γ.
6. Each $\vec{C}^i$ and $\vec{D}^i$ (for $i < \beta$) is a club-sequence of length $\beta + 1$, $\vec{C}^i$ has a well-defined perfect height and $\vec{D}^i$ witnesses the coherence of $\vec{C}^i$.
7. b is the set of $\xi < \beta$ such that some $\vec{C}^i$, $i < \beta$, has height η_ξ. Also, the domain of each $\vec{D}^i$ is contained in $[i + 1, \omega_1)$ and for each i, j, $\mathrm{dom}(\vec{D}^i) \cap \mathrm{dom}(\vec{D}^j) = \mathrm{dom}(\vec{D}^i) \cap \mathrm{range}(\vec{D}^j) = \mathrm{range}(\vec{D}^i) \cap \mathrm{range}(\vec{D}^j) = \emptyset$.
8. For $\gamma \in a$, D_γ is a closed subset of $\beta + 1$.
9. Suppose that γ belongs to a and there is a least α' in $\gamma \cap \mathcal{C}$ such that $F(\gamma)$ is a $P_{\alpha'}$-name for a club in ω_1. Then for each ν in $\beta \cap (\max(D_\gamma) + 1)$, $p \restriction \alpha'$ decides (in the forcing $P_{\alpha'}$) whether or not ν belongs to $F(\gamma)$. Let C_γ be the closure of the set of $\nu \in \beta \cap (\max(D_\gamma) + 1)$ such that $p \restriction \alpha'$ forces $\nu \in F(\gamma)$. Then $\mathrm{ot}((C^i_\delta)^+ \cap C_\gamma) = \mathrm{height}(\vec{C}^i)$ for each $i < \beta$ and $\delta \in D_\gamma \cap \mathrm{dom}(\vec{C}^i)$.

Clause 9 reflects our desire to code using strong type guessing. The canonical function coding is reflected in our notion of extension and makes use of components C and $(c_\gamma \mid \gamma \in a)$ above. First, for any condition p in P_α associate in a canonical way a set $\mathcal{A}(p)$ contained in $\bigcup_{1 \leq \rho < \alpha} [\omega_1 \cdot \rho, \omega_1 \cdot \rho + \beta)$ which codes $A \cap \bigcup_{\rho < \alpha} [\omega_1 \cdot \rho, \omega_1 \cdot \rho + \beta)$ on $\bigcup_{1 \leq \rho < \alpha} [\omega_1 \cdot \rho, \omega_1 \cdot \rho + \beta)$ as well as the components of p on $\bigcup_{1 \leq \rho \in a^p \alpha} [\omega_1 \cdot \rho, \omega_1 \cdot \rho + \beta)$. Then we say that the condition q *extends* p, $q \leq_\alpha p$, iff the following conditions hold (where if $q = (b, C, (c_\gamma \mid \gamma \in a), ((\vec{C}^i, \vec{D}^i) \mid i < \beta), (D_\gamma \mid \gamma \in a))$ is a condition then $b^q, C^q, c^q_\gamma, a^q \ldots$ denote $b, C, c_\gamma, a \ldots$):

a. $C^q \subseteq C^p$.
b. $\beta^p \leq \beta^q$, $a^p \subseteq a^q$ and $b^p = b^q \cap \beta^p$.

c. For $\gamma \in a^p$, $c^p_\gamma = c^q_\gamma \cap \beta^p$, $c^q_\gamma \setminus c^p_\gamma \subseteq C^p$ and $D^p_\gamma = D^q_\gamma \cap (\beta^p + 1)$.
d. $\vec{C}^{i,p} = \vec{C}^{i,1} \restriction \beta^p + 1$ and $\vec{D}^{i,p} = \vec{D}^{i,1} \restriction \beta^p + 1$ for all $i < \beta^p$.
e. For $\gamma \in a^p$ and $\nu \in c^1_\gamma \setminus c^p_\gamma$, $f_\gamma(\nu) \in b^q$ iff $\gamma \in \mathcal{A}(p)$.

The relation $\leq_\alpha$ is transitive, using the fact that if $q \leq_\alpha p$ and γ belongs to a^p then γ belongs to $\mathcal{A}(p)$ iff γ belongs to $\mathcal{A}(q)$. (The latter is verified using clause 1 in the definition of condition.)

The P_α's form an increasing sequence of partial orders and $P = P_{\omega_2}$ has size ω_2. The following are straightforward:

Lemma 30 *For $\alpha' \leq \alpha$ in $\mathcal{C}$, $p \restriction \alpha'$ belongs to $P_{\alpha'}$ for each p in P_α; furthermore, $P_{\alpha'}$ is a complete suborder of P_α.*

Lemma 31 *P has the ω_2-cc.*

Lemma 32 *P is ω_1-closed.*

If G is P-generic then $b^G = \bigcup\{b^p \mid p \in G\}$ codes G: Since the canonical function coding is built into the definition of the forcing, we have that b^G codes $\mathcal{A}(G) = \bigcup\{\mathcal{A}(p) \mid p \in G\}$; from the latter we can define the $\vec{C}^i(G)$, $\vec{D}^i(G)$, $C_\gamma(G)$, $D_\gamma(G)$ (the unions of the corresponding objects associated to $p \in G$), and this is enough to define G.

The main lemma states that in $V[G]$, b^G is definable over $(H(\omega_2), \in)$, as the set of ξ such that there is a strong type guessing club-sequence with stationary domain of height η_ξ. The argument is similar to the one used by Asperó to make any given subset of ω_1 definable over $H(\omega_2)$ in a forcing extension using strong type guessing. □

The above gives a Σ_4 definable wellorder of $H(\omega_2)$ in a small forcing extension. It is not known if this is optimal. However Woodin showed that if there is a measurable Woodin cardinal and CH holds then there is no Σ_1 definable wellorder of $H(\omega_2)$ with parameter ω_1; in fact there is no wellorder of the reals which is Σ_1 definable over $H(\omega_2)$ with parameter ω_1.a

Definable wellorders of $H(\kappa^+)$, κ large

Theorem 29 extends to all regular uncountable κ:

Theorem 33 *(F-Asperó) There is a class forcing which forces GCH, adds a definable wellorder of $H(\kappa^+)$ for all regular uncountable κ and preserves all supercompact cardinals as well as a proper class of n-huge cardinals for each n.*

For singular κ there is a limitation in the presence of very large cardinals.

Proposition 34 *Suppose that there is an elementary embedding from $L(H(\lambda^+))$ to itself fixing λ with critical point less than λ. Then there is no definable wellorder of $H(\lambda^+)$ with parameters.*

Proof of Proposition. Kunen's proof that there is no nontrivial elementary embedding $j : V \to V$ goes as follows: Let κ be the critical point of j and λ the supreumum of the $j^n(\kappa)$'s for $n \in \omega$. Then λ is the first fixed point of j greater than κ. Let F be an *ω-Jonsson function for λ*, i.e., a function F from $[\lambda]^\omega$ to λ such that whenever $X \subseteq \lambda$ has size λ then the range of F on $[X]^\omega$ is all of λ. It is not difficult to construct such a function F using the axiom of choice. Then $j(F)$ has the same property and $j[\lambda] = X$ has size λ. It follows that κ is of the form $j(F)(s)$ for some $s \in [X]^\omega$, which is impossible as s belongs to the range of j and κ does not.

Now suppose that j were an elementary embedding from $L(H(\lambda^+))$ to itself fixing λ with critical point κ less than λ. Then λ is at least the supremum $\bar\lambda$ of the $j^n(\kappa)$, $n \in \omega$. Kunen's argument shows that there cannot be an ω-Jonsson function for $\bar\lambda$ in $L(H(\lambda^+))$. Thus λ must equal $\bar\lambda$ and there is no ω-Jonsson function for λ in $L(H(\lambda^+))$. In particular, the axiom of choice must fail in $L(H(\lambda^+))$, which implies that there is no definable wellorder of $H(\lambda^+)$. □

It is not known if there is a small forcing that creates a definable wellorder of $H(\aleph_{\omega+1})$.

12.-13.Vorlesungen

Definable wellorders and forcing axioms

We first consider definable wellorders of $H(\omega_1)$, or equivalently, projective wellorders of the reals. As forcing axioms imply the negation of CH, we first show:

Theorem 35 *A projective wellorder of the reals is consistent with the negation of CH.*

I won't give the simplest proof of this result, but rather a proof which is amenable to generalisation. I begin with the following easier result:

Theorem 36 *It is consistent with the negation of CH that there is a wellorder of the reals definable in $H(\omega_2)$.*

Proof. The desired model will be obtained via an ω_1-preserving, ω_2-cc iteration over L of length ω_2 witih countable support.

Fix a sequence $(S_\alpha \mid \alpha < \omega_2)$ of pairwise almost disjoint stationary subsets of ω_1. We assume that this sequence is definable over L_{ω_2}. For any pair of reals x, y let $z = x * y$ be defined by $z = \{2n \mid n \in x\} \cup \{2n+1 \mid n \in y\}$. We will force to kill CH and create a wellorder $<$ of the reals so that:

$(*)$ $x < y$ iff for some limit α, n belongs to $x * y$ iff $S_{\alpha+n}$ is not stationary.

For the sake of later applications, we will add reals using Sacks forcing, rather than Cohen forcing. We will need a bookkeeping function, i.e., a function $F : \omega_2 \to L_{\omega_2}$ (definable over L_{ω_2}) such that for each $a \in L_{\omega_2}$, $F(\alpha) = a$ for unboundedly many $\alpha < \omega_2$.

The iteration uses the names Q_α defined as follows. Let P_α denote the first α stages of the iteration (for $\alpha \leq \omega_2$) and let G_α denote the P_α-generic. Order the reals in $L[G_\alpha]$ by: $x <_\alpha y$ iff the L-least P_α-name for x (i.e., the L-least P_α-name σ_x such that $\sigma_x^{G_\alpha} = x$) is less than the L-least P_α-name for y in the canonical wellorder of L. We assume that this is defined in such a way that if $\alpha < \beta$ are both limits then $<_\alpha$ is an initial segment of $<_\beta$.

For limit α, Q_α is trivial unless $F(\alpha)$ is a P_α-name for a pair of reals $x <_\alpha y$. In that case, Q_α is the forcing that adds a club to the complement of $S_{\alpha+n}$ for each n in $x * y$. A condition in Q_α is an ω-sequence $(c_0, c_1, \cdots)$ of closed, bounded subsets of ω_1 such that for each n in $x * y$, c_n is disjoint from $S_{\alpha+n}$.

For α equal to 0 or α successor, Q_α is Sacks forcing.

The desired forcing is $P = P_{\omega_2}$.

Lemma 37 *P is ω_2-cc.*

Proof. This follows easily, as our ground model satisfies CH, we are using countable support and each Q_α has size ω_1. $\square$

Lemma 38 *Suppose that G is P-generic and at limit stage $\alpha < \omega_2$ either Q_α is trivial or n does not belong to the real $x * y$ considered at stage α. Then $S_{\alpha+n}$ is stationary in $L[G]$. In particular, ω_1 is preserved.*

Proof. Let p be a condition in P forcing that n does not belong to the real $x * y$ considered at stage α of the iteration and forcing that $\dot{C}$ is a P-name for a club in ω_1. We want to find $q \leq p$ and i in $S_{\alpha+n}$ such that q forces i to belong to $\dot{C}$.

Let $(M_i \mid i < \omega_1)$ be a continuous chain of countable elementary submodels of some large L_θ such that M_0 contains p, α, F and $\dot{C}$. For each $i < \omega_1$ let γ_i denote $M_i \cap \omega_1$. Then $S^0_{\alpha+n} = \{i < \omega_1 \mid i = \gamma_i \text{ belongs to } S_{\alpha+n}\}$ is stationary.

Claim. There exists i in $S^0_{\alpha+n}$ such that i does not belong to S_β for any β in M_i which differs from $\alpha + n$.

Proof of Claim. Otherwise for each limit i in $S^0_{\alpha+n}$ choose $f(i) < i$ such that i belongs to S_β for some β in $M_{f(i)}$ which differs from $\alpha+n$. By Fodor, f has some constant value i_0 on a stationary subset of $S^0_{\alpha+n}$. As M_{i_0} is countable, there is a fixed β in M_{i_0} different from $\alpha + n$ such that i belongs to S_β for stationary-many i in $S^0_{\alpha+n}$. But this contradicts the fact that $S_{\alpha+n}$ and S_β are almost disjoint. $\square$ (Claim)

Choose i as in the Claim. We want to build an ω-sequence $p = p_0 \geq p_1 \geq \cdots$ with a lower bound q forcing i to belong to $\dot{C}$. Let $i_0 < i_1 < \cdots$ be an ω-sequence cofinal in i. To define p_{n+1}, choose a finite subset F_n of the support of p_n and extend p_n inside the model M_i without thinning the n-th splitting level of $p_n(\beta)$ for non-limit $\beta \in F_n$ so that p_{n+1} forces some ordinal greater than i_n to belong to $\dot{C}$. This can be done by successively considering the $(2^n)^{|F_n|}$ different choices of nodes on the n-th splitting levels of the trees specified by p_n on the non-limit components in F_n. In addition, for limit β in F_n, extend $p_n(\beta)$ to ensure that the max of this closed set is at least i_n. The

F_n's should be chosen so that their union equals the union of the supports of the p_n's.

Then the sequence of p_n's has a lower bound q: For non-limit α in the union A of the supports of the p_n's the $p_n(\alpha)$'s form a fusion sequence, so we obtain a Sacks condition when we intersect the $p_n(\alpha)$'s. As A is a subset of the model M_i, we know by the choice of i that i does not belong to S_β for any β in A which differs from $\alpha + n$. Therefore for limit β in A different from $\alpha + n$ we get a condition if we take the union of the $p_n(\beta)$'s (which has supremum i) and add i at the top. At component $\alpha + n$ we can also put i at the top as $p = p_0$ forces that n does not belong to the real $x * y$ considered at stage α of the iteration.

Finally, note that q forces i to belong to $\dot{C}$ and therefore we have proved the stationarity of $S_{\alpha+n}$. $\Box$ (Claim)

Corollary 39 *P forces the negation of CH.*

Clearly if Q_α is nontrivial at a limit stage α and n does belong to the real $x * y$ considered at stage α then $S_{\alpha+n}$ is not stationary in $L[G]$. Thus if $<$ denotes the wellorder of the reals in $L[G]$ obtained by taking the union of the $<_\alpha$'s we have:

$(*)$ $x < y$ iff for some limit $\alpha < \omega_2$, $S_{\alpha+n}$ is stationary iff n belongs to $x * y$.

As the sequence $(S_\alpha \mid \alpha < \omega_2)$ is definable over L_{ω_2}, this gives a wellorder in $L[G]$ which is definable over $L_{\omega_2}[G] = H(\omega_2)^{V[G]}$. $\Box$

Now we prove the more difficult result:

Theorem 40 *It is consistent with the negation of CH that there is a projective (indeed Σ^1_3 definable) wellorder of the reals.*

Proof. We perform an ω_2-iteration as in the previous proof, but do more at limit stages. Recall that in the previous proof we started with L and added a wellorder $<$ of ω_2-many reals such that:

$x < y$ iff for some limit $\alpha < \omega_2$, n belongs to $x * y$ iff $S_{\alpha+n}$ is nonstationary,

where $(S_\beta \mid \beta < \omega_2)$ is an L_{ω_2}-definable sequence of pairwise almost disjoint stationary subsets of ω_1. In the present proof this will be modified slightly:

(1) $x < y$ iff for some limit $\alpha < \omega_2$, $S_{\alpha+2n}$ is nonstationary for n in $x * y$ and $S_{\alpha+2n+1}$ is nonstationary for n not in $x * y$.

This small change has the advantage that not only membership, but also non-membership in $x * y$ is witnessed by the existence, rather than the non-existence, of a club.

Our goal is to express the above nonstationarity in terms of quanitification over countable models. Ideally, we would like to have (1) together with the following:

(2) If $x < y$ then there exists a real R such that for any countable transitive ZF^- model M containing R there is a limit ordinal $\bar{\alpha} < \omega_2^M$ such that $S^M_{\bar{\alpha}+2n}$ is nonstationary in M for n in $x * y$ and $S^M_{\bar{\alpha}+2n+1}$ is nonstationary in M for n not in $x * y$,

where $(S^M_\beta \mid \beta < \omega_2^M)$ denotes M's interpretation of the sequence $(S_\beta \mid \beta < \omega_2)$. We show now that (1) implies the converse of (2). It follows that (1) and (2) together give a projective wellorder of the reals, as the conclusion of (2) is first-order over $H(\omega_1)$.

Suppose that R is a real such that for any countable transitive ZF^- model M containing R there is a limit ordinal $\bar{\alpha} < \omega_2^M$ such that $S^M_{\bar{\alpha}+2n}$ is nonstationary in M for n in $x * y$ and $S^M_{\bar{\alpha}+2n+1}$ is nonstationary in M for n not in $x * y$. By Löwenheim-Skolem this holds for arbitrary transitive ZF^- models M containing R. Consider then the model $M = L_\theta[R]$ for a large regular θ and let $\alpha < \omega_2^M = \omega_2$ be the limit ordinal guaranteed the conclusion of (2) for M. As $(S_\beta \mid \beta < \omega_2)$ is definable over L_{ω_2} and θ is greater than ω_2, it follows that S^M_β equals S_β for each $\beta < \omega_2$. Thus $S_{\alpha+2n}$ is nonstationary in M for n in $x * y$ and $S_{\alpha+2n+1}$ is nonstationary in M for n not in $x * y$. It follows that these sets are nonstationary in the larger model $L[G]$ and therefore by (1), we have $x < y$.

We will not actually achieve (2) above, but a slight weakening of it. Say that a transitive ZF^- model M is *suitable* iff $M \vDash \omega_2 = \omega_2^L$ exists. We will obtain (2) restricted to suitable M. Then to establish the converse of the new version of (2), we need only observe that as our forcing preserves cardinals, $L_\theta[R]$ is indeed suitable for any large regular θ and any real R in the generic extension.

We now begin the proof. To facilitate the argument we need some extra properties of the bookkeeping function F and of the sequence $(S_\beta \mid \beta < \omega_2)$ of almost disjoint stationary subsets of ω_1.

Lemma 41 *Assume $V = L$. There is a bookkeeping function $F : \omega_2 \to L_{\omega_2}$ definable over L_{ω_2} via a formula φ and a sequence $(S_\beta \mid \beta < \omega_2)$ of almost disjoint stationary subsets of ω_1 definable over L_{ω_2} via a formula ψ such that whenever M, N are suitable transitive ZF^- models, F^M, F^N denote the interpretations of φ in M, N, respectively, $\vec{S}^M = (S^M_\beta \mid \beta < \omega^M_2)$, $\vec{S}^N = (S^N_\beta \mid \beta < \omega^N_2)$ denote the interpretations of ψ in M, N, respectively, and $\omega^M_1 = \omega^N_1$ then F^M, F^N agree on $\omega^M_2 \cap \omega^N_2$ and $\vec{S}^M$, $\vec{S}^N$ agree on $\omega^M_2 \cap \omega^N_2$. In particular, if M is suitable and $\omega^M_1 = \omega_1$ then F^M, $\vec{S}^M$ equal the restrictions of F, $\vec{S}$ to the ω_2 of M.*

Proof Sketch. For the bookkeeping function define $F(\alpha) = a$ iff via Gödel pairing α codes a pair (α_0, α_1) where a has rank α_0 in the natural wellorder of the sets in L. For the almost disjoint stationary sets, let $(D_\gamma \mid \gamma < \omega_1)$ be the canonical L_{ω_1}-definable $\diamondsuit$ sequence, for each $\alpha < \omega_2$ let A_α be the L-least subset of ω_1 coding α and define S_α to be the set of $i < \omega_1$ such that $D_i = A_\alpha \cap i$. $\square$ (Lemma 41)

14.-15.Vorlesungen

Now we describe stage α of our iteration. For non-limit $\alpha < \omega_2$ we add a Sacks real. For limit $\alpha < \omega_2$, we kill the stationarity of $S_{\alpha+2n}$ for n in $x_\alpha * y_\alpha$ and of $S_{\alpha+2n+1}$ for n not in $x_\alpha * y_\alpha$, where $x_\alpha <_\alpha y_\alpha$ are the reals chosen by the bookkeeping function F at that stage. Call this forcing Q^0_α and let H_α denote the Q^0_α-generic. Now let $X_\alpha \in L[G_\alpha * H_\alpha]$ be a subset of ω_1 which codes the ordinal α, codes a level of L in which α has size at most ω_1 and codes the generic $G_\alpha * H_\alpha$, which we can regard as an element of L_{ω_2}. We have:

$(*)$ If M is suitable and X_α belongs to M, then the limit ordinal α coded by X_α is less than ω^M_2 and $S^M_{\alpha+2n}$ is not stationary in M for n in $x_\alpha * y_\alpha$, $S^M_{\alpha+2n+1}$ is not stationary in M for n not in $x_\alpha * y_\alpha$.

This is because in any such M we can decode $G_\alpha * H_\alpha$ from X_α inside M and $S^M_{\alpha+n}$ equals $S_{\alpha+n}$ for each n.

Recall that we want to add a real which "reflects" this property into all countable, suitable models that contain it. First we force a subset Y_α of ω_1 which "localises" the above property in the following sense:

$(**)$ For any $\gamma < \omega_1$ and countable, suitable M containing $Y_\alpha \cap \gamma$ as an element: If $\gamma = \omega_1^M$ then for some limit ordinal $\bar{\alpha}$ less than ω_2^M, $S^M_{\bar{\alpha}+2n}$ is not stationary in M for n in $x_\alpha * y_\alpha$ and $S^M_{\bar{\alpha}+2n+1}$ is not stationary in M for n not in $x_\alpha * y_\alpha$.

We now describe a forcing Q^1_α to create the witness Y_α to $(**)$. A condition in Q^1_α is an ω_1-Cohen condition $r : |r| \to 2$ in $L[G_\alpha * H_\alpha]$ with the following properties:

1. The domain $|r|$ of r is a countable limit ordinal.
2. $X_\alpha \cap |r|$ is the even part of r, i.e., for $\gamma < |r|$, γ belongs to X_α iff $r(2\gamma) = 1$.
3. $(**)$ holds for all limit $\gamma \leq |r|$ with $Y_\alpha \cap \gamma$ replaced by $r \restriction \gamma$, i.e.:

$(**)_r$ For any limit $\gamma \leq |r|$ and countable, suitable M containing $r \restriction \gamma$ as an element: If $\gamma = \omega_1^M$ then for some limit ordinal $\bar{\alpha}$ less than ω_2^M, $S^M_{\bar{\alpha}+2n}$ is not stationary in M for n in $x_\alpha * y_\alpha$ and $S^M_{\bar{\alpha}+2n+1}$ is not stationary in M for n not in $x_\alpha * y_\alpha$.

As a warmup for a later argument, we pause now to consider the case $\alpha = \omega$, assume that $x_\omega <_\omega y_\omega$ are well-defined and show that the forcing $P_\omega * Q^0_\alpha * Q^1_\alpha$ preserves the stationarity of the "untouched" S_β's, i.e., of those S_β's where β is not of the form $\omega+2n$, $n \in x_\omega * y_\omega$ or of the form $\omega+2n+1$, $n \notin x_\omega * y_\omega$. Later we will show that the entire iteration preserves the stationarity of those S_β's untouched by the generic for the full ω_2-iteration P.

Suppose that (p, q^0, r) is a condition in $P_\omega * Q^0_\alpha * Q^1_\alpha$ forcing that β is not of the form $\omega + 2n$, $n \in x_\omega * y_\omega$, β is not of the form $\omega + 2n + 1$, $n \notin x_\omega * y_\omega$ and that $\dot{C}$ is a club in ω_1. We will find $(p_\omega, q^0_\omega, r_\omega)$ below (p, q^0, r) forcing i to belong to $\dot{C}$ for some i in S_β.

First note that Q^1_ω satisfies the following extendibility property: Given r and a countable limit γ greater than $|r|$, we can extend r to r^* of length γ. This is because we can take the odd part of r^* on the interval $[|r|, |r| + \omega)$ to code γ and to consist only of 0's on $[|r| + \omega, \gamma)$; then there are no new instances of requirement 3 for being a condition to check because no ZF^- model containing $r^* \restriction |r| + \omega$ can have its ω_1 in the interval $(|r|, \gamma]$.

Now let $(M_i \mid i < \omega_1)$ be a continous chain of countable elementary submodels of some large L_θ such that M_0 contains the parameters (p, q^0, r), β, $\dot{C}$, $P_\omega * Q^0_\omega * Q^1_\omega$ and a $P_\omega * Q^0_\omega$-name $\dot{X}_\omega$ for X_ω. Let i be an element of S_β such that $i = M_i \cap \omega_1$ and i does not belong to S_δ for any δ in M_i which differs from β. (We argued earlier that there must be such an i, using a Fodor argument.) Successively extend (p, q^0, r) to $(p_0, q^0_0, r_0) \geq (p_1, q^0_1, r_1) \geq \cdots$ in M_i so that for each finite n the $p_k(n)$, $k \in \omega$, form a fusion sequence and if D in M_i is a dense set for the forcing $P_\omega * Q^0_\omega * Q^1_\omega$ then for some k, (p_k, q^0_k, r_k) reduces D to the k-th splitting level of finitely many of the trees $p_k(n)$ (i.e., if finitely many of the trees $p_k(n)$ are restricted to some node on their k-th splitting level, then the resulting condition (p'_k, q^0_k, r_k) meets D). In particular, the condition (p_k, q^0_k, r_k) forces the $P_\omega * Q^0_\omega * Q^1_\omega$-generic to meet D in a condition belonging to M_i. By extendibility, the max's of the q^0_k's and the domains of the r_k's converge to i. And the (p_k, q^0_k, r_k)'s force arbitrary large ordinals less than i into $\dot{C}$.

We want to show that the (p_k, q^0_k, r_k)'s have a lower bound $(p_\omega, q^0_\omega, r_\omega)$. By fusion the p_k's have a greatest lower bound p_ω. And just as in our earlier argument, the q^0_k's have a greatest lower bound q^0_ω as i does not belong to S_δ for any δ in M_i which differs from β. We show that the condition (p_ω, q^0_ω) in $P_\omega * Q^0_\omega$ forces the union r_ω of the r_k's to be a condition in Q^1_ω. For this it suffices to force property $(**)_{r_\omega}$ when γ is equal to i, the length of r_ω. I.e., (p_ω, q^0_ω) must force:

$(***)$ For any suitable M containing r_ω: If $i = \omega_1^M$ then $S^M_{\omega+2n}$ is not stationary in M for n in $x_\omega * y_\omega$ and $S^M_{\omega+2n+1}$ is not stationary in M for n not in $x_\alpha * y_\alpha$.

Fix a generic $G_\omega * H_\omega$ below the condition (p_ω, q^0_ω). Then if D is a dense set for $P_\omega * Q^0_\omega$ belonging to M_i, by construction we have that $(G_\omega * H_\omega) \cap M_i$ meets D. Thus not only is M_i elementary in L_θ, but also $(M_i[(G_\omega * H_\omega) \cap M_i], (G_\omega * H_\omega) \cap M_i)$ is elementary in $(L_\theta[G_\omega * H_\omega], G_\omega * H_\omega)$. Let $(\bar{M}[\bar{G} * \bar{H}], \bar{G} * \bar{H})$ be the transitive collapse of $(M_i[(G_\omega * H_\omega) \cap M_i], (G_\omega * H_\omega) \cap M_i)$. As X_ω has a name in M_i, it follows that X_ω belongs to $M_i[(G_\omega * H_\omega) \cap M_i]$ and therefore $X_\omega \cap i$ belongs to $\bar{M}[\bar{G} * \bar{H}]$. As X_ω codes the generic $G_\omega * H_\omega$, it ensures the nonstationarity of $S_{\omega+2n}$ for n in $x_\omega * y_\omega$ and of $S_{\omega+2n+1}$ for n not in $x_\omega * y_\omega$ in all suitable models containing X_ω as an element; it follows that $X_\omega \cap i$ ensures the nonstationarity of $S^{\bar{M}}_{\omega+2n}$ for n in $x_\omega * y_\omega$ and of $S^{\bar{M}}_{\omega+2n+1}$ for n

not in $x_\omega * y_\omega$ in all suitable models containing $X_\omega \cap i$ as an element. Now if M is any suitable model containing r_ω as an element such that $\omega_1^M = i$, M also contains $X_\omega \cap i$ as an element (as $X_\omega \cap i$ is the even part of r_ω) and as $\omega_1^M = i = \omega_1^{\bar{M}}$, we have $S^M_{\omega+n} = S^{\bar{M}}_{\omega+n}$ for each n; it follows that $S^M_{\omega+2n}$ is nonstationary in M for n in $x_\omega * y_\omega$ and $S^M_{\omega+2n+1}$ is nonstationary in M for n not in $x_\omega * y_\omega$, establishing $(***)$.

So the (p_k, q^0_k, r_k)'s have a lower bound $(p_\omega, q^0_\omega, r_\omega)$. This condition forces unboundedly many ordinals less than i into $\dot{C}$ and therefore forces i into $\dot{C}$, where i belongs to S_β. Thus we have shown that the stationarity of S_β is preserved by the forcing $P_\omega * Q^0_\omega * Q^1_\omega$.

16.-17.Vorlesungen

To complete stage α of the iteration, we code the Q^1_α-generic Y_α by a real via the forcing $\mathcal{C}_\alpha$ defined below. This can most easily be done using a ccc almost disjoint coding with finite conditions; but for the sake of future applications we use here perfect trees to code. Note that the ground model $L[G_\alpha * H_\alpha * Y_\alpha]$ is in fact equal to $L[Y_\alpha]$ as the even part of Y_α codes $G_\alpha * H_\alpha$.

Inductively define L-countable ordinals μ_i, $i < \omega_1^L$ by: μ_i is the least $\mu > \bigcup\{\mu_j \mid j < i\}$ (this condition is vacuous if i equals 0) such that $L_\mu[Y_\alpha \cap i] \vDash \mathrm{ZF}^-$ and $L_\mu \vDash \omega$ is the largest cardinal. (There are many μ's with these properties, for example any μ such that $L_\mu[Y_\alpha \cap i]$ is an elementary submodel of $L_{\omega_1}[Y_\alpha \cap i]$). A real R *codes* Y_α *below* i iff for all $j < i$, $j \in Y_\alpha$ iff $L_{\mu_j}[Y_\alpha \cap j, R] \vDash \mathrm{ZF}^-$. For $T \subseteq 2^{<\omega}$ a perfect tree, let $|T|$ denote the least i such that $T \in L_{\mu_i}[Y_\alpha \cap i]$. A condition in $\mathcal{C}_\alpha$ is a perfect tree T such that R codes Y_α below $|T|$ whenever R is a branch through T. (Note that by absoluteness, if T is a condition then R codes Y_α below $|T|$ even for branches R through T in the generic extension; in particular this holds for the generic branch.) $\mathcal{C}_\alpha$ is ordered by: $T_0 \le T_1$ iff T_0 is a subtree of T_1. This is equivalent to $[T_0] \subseteq [T_1]$ where $[T]$ denotes the set of infinite branches through T.

Lemma 42 *(a) If T belongs to $\mathcal{C}_\alpha$ and $|T| \le i < \omega_1$ then there is a $T^* \le T$ such that $|T^*| = i$. (b) $\mathcal{C}_\alpha$ preserves ω_1.*

Proof. (a) By induction on i. We may assume that $|T|$ is less than i. If $i = j+1$ then we may also assume by induction that $|T|$ equals j and hence that T belongs to $\mathcal{A}_j = L_{\mu_j}[Y_\alpha \cap j]$. If j belongs to Y_α then we take $T^* \le T$ to have the property that R is P_T-generic over $\mathcal{A}_j$ for $R \in [T^*]$, where P_T is the forcing (isomorphic to Cohen forcing) whose conditions are the elements of T, ordered by extension. Note that T^* can be chosen in $\mathcal{A}_i = L_{\mu_i}[Y_\alpha \cap i]$ as $\mathcal{A}_j$ is a countable element of $\mathcal{A}_i$. Also $L_{\mu_j}[Y_\alpha \cap j, R] \models \mathrm{ZF}^-$ for $R \in [T^*]$, by the P_T-genericity of $R \in [T^*]$. So T^* is a condition and $|T^*| = i$. If j does not belong to Y_α then choose a real R_0 coding a well ordering of ω of ordertype μ_j, $R_0 \in \mathcal{A}_i$, and take $T^* \leq T$ to be the tree whose branches are exactly the branches R through T such that for all n, $n \in R_0$ iff R goes right at the $2n$-th splitting level of T. Then T^* belongs to $\mathcal{A}_i$ and for $R \in [T^*]$, (R, T) computes R_0 and hence $L_{\mu_j}[Y_\alpha \cap j, R]$ is *not* a model of ZF^-, since it contains R_0 as an element.

If i is a limit ordinal then choose $|T| = i_0 < i_1 < \cdots$ to be an ω-sequence cofinal in i which belongs to $\mathcal{A}_i = L_{\mu_i}[Y_\alpha \cap i]$. Define $T_0 \leq_n T_1$ iff $T_0 \leq T_1$ and T_0, T_1 have the same first n splitting levels. Now let $T_0 = T$ and for each n let $T_{n+1} \in \mathcal{C}_\alpha$ be least in $\mathcal{A}_{i_{n+1}}$ such that $|T_{n+1}| = i_{n+1}$ and $T_{n+1} \leq_n T_n$. Such T_n's exist by induction. If $T^* = \bigcap_n T_n$ then $T^* \leq T$ belongs to $\mathcal{A}_i$ and satisfies the requirement for belonging to $\mathcal{C}_\alpha$. So $T^* \leq T$, $|T^*| = i$, as desired.

(b) We say that $D \subseteq \mathcal{C}_\alpha$ is *n-dense* iff for all $T \in \mathcal{C}_\alpha$ there is $T^* \leq_n T$, $T^* \in D$. We show that if for each n, D_n is open and n-dense then for all $T \in \mathcal{C}_\alpha$ there exists $T^* \leq T$ such that T^* belongs to D_n for each n. It follows that $\mathcal{C}_\alpha$ preserves "cofinality $> \omega$," for if σ is a name for a function from ω into Ord then for each n, $D_n = \{T \in \mathcal{C}_\alpha \mid \text{For some finite } d,\ T \Vdash \sigma(n) \in d\}$ is n-dense and hence our result implies that the range of σ is covered by a set countable in the ground model.

So suppose T belongs to $\mathcal{C}_\alpha$ and D_n is open and n-dense for each n. Let M be a countable elementary submodel of some large $L_\theta[Y_\alpha]$ containing T and $\langle D_n \mid n \in \omega \rangle$ as elements and let $i = M \cap \omega_1$. Also let $i_0 < i_1 < \cdots$ be an ω-sequence cofinal in i belonging to $\mathcal{A}_i$. Note that the transitive collapse of M belongs to $\mathcal{A}_i$ as it satisfies $i = \omega_1$ whereas $L_{\mu_i} \models i$ is countable. So we can choose $T = T_0 \geq_0 T_1 \geq_1 T_2 \geq_2 \cdots$ in $\mathcal{A}_i$ so that $T_{n+1} \in D_n \cap M$ and $|T_{n+1}| \geq \alpha_{n+1}$. Then $T^* = \bigcap_n T_n$ belongs to each D_n, $T^* \leq T$ and T^* belongs to $\mathcal{C}_\alpha$ as T^* belongs to $\mathcal{A}_i$. $\square$

This completes the definition for limit $\alpha < \omega_2$ of $Q_\alpha = Q^0_\alpha * Q^1_\alpha * \mathcal{C}_\alpha$. For non-limit $\alpha < \omega_2$, Q_α is Sacks forcing. The desired forcing P is the iteration with countable support of these Q_α's.

Let R_α denote the $\mathcal{C}_\alpha$-generic real coding the Q^1_α-generic Y_α. Then $Y_\alpha \cap \omega_1^M$ can be decoded from R_α in M for any suitable M containing R_α as an element. Therefore the real R_α satisfies the following important property.

$(*)_{R_\alpha}$ For any suitable model M containing R_α as an element, there is a limit ordinal $\bar{\alpha} < \omega_2^M$ such that $S^M_{\bar{\alpha}+2n}$ is nonstationary for n in $x_\alpha * y_\alpha$ and $S^M_{\bar{\alpha}+2n+1}$ is nonstationary for n not in $x_\alpha * y_\alpha$.

We now show that the iteration P preserves the stationarity of the untouched S_β's, i.e., for P-generic G, S_β remains stationary except for β of the form $\alpha + 2n$, α limit and n in $x^G_\alpha * y^G_\alpha$ or of the form $\alpha + 2n + 1$, α limit and n not in $x^G_\alpha * y^G_\alpha$. Then as we have observed earlier, $(*)_{R_\alpha}$ for each α implies that in the P-generic extension $L[G]$, the union $<^G$ of the partial wellorders $<^G_\alpha$, $\alpha < \omega_2$ limit, has a Σ^1_3 definition:

$x <^G y$ iff there exists a real R such that for all countable, suitable M containing R as an element there is a limit $\alpha < \omega_2^M$ such that $S^M_{\alpha+2n}$ is nonstationary in M for n in $x * y$ and $S^M_{\alpha+2n+1}$ is nonstationary in M for n not in $x * y$.

Thus to complete the proof of the theorem we only need the following.

Lemma 43 *Suppose that G is P-generic. Then for $\beta < \omega_2^L$ not of the form $\alpha+2n$, α limit, $n \in x^G_\alpha * y^G_\alpha$ and not of the form $\alpha+2n+1$, α limit, $n \notin x^G_\alpha * y^G_\alpha$, S_β is stationary in $L[G]$. Moreover L and $L[G]$ have the same cardinals.*

Proof. Let p be a condition forcing that $\beta < \omega_2^L$ is not of the form $\alpha + 2n$, α limit, $n \in x^G_\alpha * y^G_\alpha$ and not of the form $\alpha + 2n + 1$, α limit, $n \notin x^G_\alpha * y^G_\alpha$, and also forcing that $\dot{C}$ is a club in ω_1^L. We want to find an extension q of p and $i < \omega_1^L$ in S_β such that q forces i to belong to $\dot{C}$.

As before let $(M_i \mid i < \omega_1^L)$ be a continuous chain of countable elementary submodels of some large L_θ such that M_0 contains all imaginable parameters, and choose $i < \omega_1^L$ in S_β so that i does not belong to S_δ for any δ in M_i other than β. Build an ω-sequence $p = p_0 \geq p_1 \geq \cdots$ of conditions below p such that for any dense set D for the forcing P in M_i, some p_k forces the generic to intersect $D \cap M_i$. Moreover ensure that for each non-limit α in the union of the supports of the p_k's, the sequence $p_k(\alpha)$ forms a fusion sequence in Sacks forcing and also that for each limit α in the union of the supports of the p_k's, if we write $p_k(\alpha) = (p_k(\alpha)^0, p_k(\alpha)^1, p_k(\alpha)^2)$, then the sequence of $p_k(\alpha)^2$'s is forced to form a fusion sequence in the coding forcing $\mathcal{C}_\alpha$. In addition, choose the sequence of p_k's to belong to the least L_μ in which $\bar{M}$, the transitive collapse of M_i, is countable.

We now produce a lower bound q to the sequence of p_k's, whose support Supp(q) is the union of the supports of the p_k's, by defining $q(\alpha)$ by induction on α in Supp(q). If α is a non-limit then we take $q(\alpha)$ to simply be the fusion of the $p_k(\alpha)$'s. Suppose then that α is a limit and $q \restriction \alpha$ is already defined as a condition in P_α. We want to define $q(\alpha) = (q(\alpha)^0, q(\alpha)^1, q(\alpha)^2)$.

For $q(\alpha)^0$, a name for a sequence of closed sets, we take the union of the closed sets in the $p_k(\alpha)^0$'s and put i at the top. This results in a condition because i is forced to not belong to any of the $S_{\alpha+2n}$, $n \in x_\alpha * y_\alpha$ or the $S_{\alpha+2n+1}$, $n \notin x_\alpha * y_\alpha$ (because such $\alpha + 2n$, $\alpha + 2n + 1$ belong to M_i or equal β) and therefore a condition will indeed result if i is added at the top. Also note that the closed sets in the $p_k(\alpha)^0$'s have maxima cofinal in i by the construction of the p_k's, so we indeed obtain closed sets when putting i at the top.

For $q(\alpha)^1$ we use the same argument that we used earlier for Q^1_ω. We take $q(\alpha)^1$ to be the union of the $p_k(\alpha)^1$'s. Fix a generic $G_\alpha * H_\alpha$ below $(q \restriction \alpha, q(\alpha)^0)$; we must show that when $q(\alpha)^1$ is interpreted by $G_\alpha * H_\alpha$ the result is a condition in Q^1_α (as interpreted by $G_\alpha * H_\alpha$). By the construction of the p_k's, M_i is not only elementary in L_θ but this remains so if we introduce $G_\alpha * H_\alpha$ as a predicate, i.e., $(M_i[(G_\alpha * H_\alpha) \cap M_i], (G_\alpha * H_\alpha) \cap M_i)$ is elementary in $(L_\theta[G_\alpha * H_\alpha], G_\alpha * H_\alpha)$. As $X_\alpha \subseteq \omega_1$ codes the generic $G_\alpha * H_\alpha$ and has a name in M_i, it follows that $X_\alpha \cap i$ belongs to the transitive collapse $\bar{M}[\bar{G} * \bar{H}]$ of $M_i[(G_\alpha * H_\alpha) \cap M_i]$. Moreover, just as X_α ensures the nonstationarity of the appropriate $S_{\alpha+n}$'s, $X_\alpha \cap i$ ensures the nonstationary of the appropriate $S^M_{\bar{\alpha}+n}$'s in any suitable M containing $X_\alpha \cap i$ such that $\omega_1^M = i$. This implies that $q(\alpha)^1$, which has $X_\alpha \cap i$ as its even part, ensures the same nonstationarity and therefore is a condition in Q^1_α.

Finally, we take $q(\alpha)^2$ to be the fusion of the $p_k(\alpha)^2$'s. To verify that this is a condition in $\mathcal{C}_\alpha$ we need to verify that it is forced to belong to the structure $\mathcal{A}_i = L_{\mu_i}[Y_\alpha \cap i]$. Recall that the sequence of p_k's belongs to the least L_μ in which $\bar{M}$, the transitive collapse of M_i, is countable. It follows that $q(\alpha)^2$ is forced to belong to $L_\mu[Y_\alpha \cap i]$ for this μ and by the definition of μ_i, we have $\mu < \mu_i$. Thus $q(\alpha)^2$ is indeed forced to belong to $\mathcal{A}_i$, as desired.

The fact that L and $L[G]$ have the same cardinals now follows from ω_1-preservation and the ω_2-cc. □

18.-20.Vorlesungen

Our next goal is to prove the following.

Theorem 44 *Relative to the consistency of a reflecting cardinal, BPFA is consistent with the existence of a Σ^1_3 wellorder of the reals.*

BPFA is the bounded forcing axiom for proper forcings. It is equivalent to the statement that any Σ_1 sentence with an element of $H(\omega_2)$ as parameter which is true in a proper forcing extension of the universe is already true. A cardinal κ is *reflecting* iff it is regular and $H(\kappa)$ is Σ_2 elementary in V.

Goldstern and Shelah showed that BPFA is consistent relative to a reflecting cardinal by starting with a reflecting cardinal in L and performing a countable support κ-iteration of proper forcings of size $< \kappa$. At each stage a proper forcing is chosen to witness a new Σ_1 fact with parameter in (the current) $H(\omega_2)$. The fact that κ is reflecting is used to show that these proper forcings can in fact be taken to have size $< \kappa$ and therefore κ will remain reflecting throughout the iteration (until the final stage). As the forcing is proper and κ-cc, it follows that ω_1 is preserved and that BPFA holds in the resulting forcing extension.

We first show:

Theorem 45 *Relative to the consisteny of a reflecting cardinal, BPFA is consistent with the existence of a wellorder of the reals which is definable over $H(\omega_2)$.*

To prove Theorem 45 we start in the same way as Goldstern-Shelah, with a reflecting cardinal κ in L, and perform a countable support iteration of length κ. A possible strategy is to code a wellorder of the reals using stationary subsets of ω_1, as in our previous proof. However this will destroy the properness of the iteration, so we take another approach, based on controlling which of certain constructible trees T have T-generic branches over L in the final model.

Lemma 46 *Assume $V = L$. Suppose that β is regular and uncountable and consider the tree $T(\beta)$ of sequences through β^+ of length less than β. Suppose that Q is a forcing such that $2^{2^{|Q|}}$ is less than β and G is Q-generic over L. Then:*
(a) $T(\beta)$, viewed as a forcing, is proper in $L[G]$.
(b) There is a proper forcing R in $L[G]$ of size β^+ which destroys the properness of $T(\beta)$; in fact, if H is R-generic over $L[G]$ then in any ω_1-preserving outer model of $L[G][H]$ there is no branch through $T(\beta)$ which is $T(\beta)$-generic over L.

Proof. (a) It suffices to show that Q is proper in $T(\beta)$-generic extensions of L. But the forcing $T(\beta)$ is β-closed and therefore does not add subsets of $2^{2^{|Q|}}$; it follows that any witness to the properness of Q in L is still a witness to its properness in any $T(\beta)$-generic extension of L.
(b) First add β^{++} Cohen reals with a finite support product over $L[G]$, producing $L[G][H_0]$. Then Lévy collapse β^+ to ω_1 with countable conditions, producing $L[G][H_0][H_1]$. As ccc and ω-closed forcings are proper, this is a proper forcing extension of $L[G]$. Now note that in $L[G][H_0][H_1]$, any β-branch through $T(\beta)$ in fact belongs to $L[G][H_0]$: Otherwise we choose a $L[G][H_0]$-name $\dot{b}$ for the new branch and build a binary ω-tree U of conditions in the Lévy collapse, each branch of which has a lower bound, such that distinct cofinal branches through U force different interpretations of the name $\dot{b}$. It follows that in $L[G][H_0]$, $T(\beta)$ has $2^{\aleph_0} = \beta^+$ nodes on a fixed level, which is impossible because GCH holds in L. Thus the tree $T(\beta)$ has at most ω_1-many branches in $L[G][H_0][H_1]$, none of which contains ordinals cofinal in β^+ and therefore none of which is $T(\beta)$-generic over L. Also, every node of $T(\beta)$ belongs to a β-branch.

Now we use Baumgartner's general method of "specialising a tree off a small set of branches".

Fact. If T is a tree of height ω_1 with at most $\aleph_1$ cofinal branches (and every node of T belongs to a cofinal branch of T) then there is a ccc forcing P such that if G is P-generic over V then in any ω_1-preserving outer model of $V[G]$, all cofinal branches through T belong to V.

Proof sketch. List the branches as $(b_i \mid i < \omega_1)$ and write T as the disjoint union of $b_i(x_i)$, where each x_i is a node on b_i and $b_i(x_i)$ denotes the tail of b_i

starting at x_i. Now add a function f with finite conditions from $\{x_i \mid i < \omega_1\}$ into ω such that if x_i is below x_j in T then $f(x_i)$ is different from $f(x_j)$. Baumgartner shows that this forcing is ccc. Now if b is a cofinal branch through T distinct from the b_i's in an outer model of $V[f]$, then b must intersect uncountably many of the $b_i(x_i)$'s and therefore contains uncountably many x_i's. But then the $f(x_i)$'s are distinct for these uncountably many x_i's, contradicting the fact that f maps into ω. $\square$ (Fact)

Now use the Fact to create a ccc extension $L[G][H_0][H_1][H_2]$ of $L[G][H_0][H_1]$ to ensure that $T(\beta)$ (viewed as a tree of height ω_1 using a cofinal ω_1-sequence through $(\beta^+)^L$) will have no new branches in any ω_1-preserving outer model. As no β-branch through $T(\beta)$ in $L[G][H_0]$ is $T(\beta)$-generic over L and all cofinal branches through $T(\beta)$ in an ω_1-preserving outer model of $L[G][H_0][H_1][H_2] = L[G][H]$ belong to $L[G][H_0]$, we are done. $\square$

Proof of Theorem 45. Let κ be reflecting in L and let C enumerate the closed unbounded subset of κ consisting of those α such that L_α is Σ_2 elementary in L_κ. (As κ is inaccessible, C is indeed unbounded in κ.) We perform a proper iteration of length κ with countable support which is nontrivial at stages α in C. The iteration $P_\alpha * Q(\alpha)$ up to and including stage α will belong to L_β where β is the least element of C greater than α. In particular, P_α has size less than κ for each $\alpha < \kappa$ and therefore κ remains reflecting throughout the iteration.

Suppose that α belongs to C; we describe the forcing $Q(\alpha)$, which is a two-step iteration $Q^0(\alpha) * Q^1(\alpha)$.

As P_α has size at most $(\alpha^+)^L$, we know that the forcing $T(\beta)$, consisting of $< \beta$ sequences through β^+, is proper in $L[G_\alpha]$ when β is regular and at least $(\alpha^{++++})^L$. In addition there is a forcing $R(\beta)$ of size β^+ which guarantees that there is no $T(\beta)$-generic over L. Now let α_n be $(\alpha^{+4(n+1)})^L$ for each finite n, and let $T(n)$ denote $T(\alpha_n)$, $R(n)$ denote $R(\alpha_n)$. Then both $T(n)$ and $R(n)$ are proper in any extension of $L[G_\alpha]$ obtained by forcing with $U(0) * U(1) * \cdots * U(n-1)$ where each $U(i)$ is either $T(i)$ or $R(i)$.

As in the earlier proofs, let $x_\alpha <_\alpha y_\alpha$ be a pair of reals in $L[G_\alpha]$ provided by the bookkeeping function and now take $Q^0(\alpha)$ to be the ω-iteration $U(0) * U(1) * \ldots$ where $U(n)$ equals $T(n)$ if n belongs to $x_\alpha * y_\alpha$ and equals $R(n)$

otherwise. This is a proper forcing and $P_\alpha * Q^0(\alpha)$ belongs to L_β, where β is the least element of C greater than α.

Now we choose a Σ_1 sentence with parameter from $L[G_\alpha]$, provided by the bookkeeping function, and ask if it holds in a proper forcing extension of $L[G_\alpha][H^0]$, where H^0 is our $Q^0(\alpha)$-generic. If so, then as κ is reflecting in $L[G_\alpha][H^0]$, there is such a proper forcing in $L_\kappa[G_\alpha][H^0]$, and also the witness to the Σ_1 sentence can be assumed to have a name in $L_\kappa[G_\alpha][H^0]$. Let β be the least element of C greater than α; then L_β is Σ_2 elementary in L_κ and therefore $L_\beta[G_\alpha][H^0]$ is Σ_2 elementary in $L_\kappa[G_\alpha][H^0]$. It follows that we can choose our proper forcing $Q^1(\alpha)$ witnessing the Σ_1 sentence to be an element of $L_\beta[G_\alpha][H^0]$, maintaining the requirement that $P_\alpha * Q(\alpha)$ belong to L_β. This completes the construction.

The iteration is proper, forces κ to be at most ω_2 and is κ-cc. It follows that κ equals ω_2 in the generic extension $L[G]$ and BPFA holds there. The desired wellorder of the reals is defined by:

$x < y$ iff
For some α in C, $(x, y) = (x^G_\alpha, y^G_\alpha)$ iff
There exists α in C such that for all n, n belongs to $x * y$ iff there is a $T(\alpha_n)$-generic over L in $L[G]$.

This works because at each stage α in C and for each n, we either forced with $T(\alpha_n)$, thereby producing a $T(\alpha_n)$-generic over (more than) L in $L[G]$, or we forced with $R(\alpha_n)$, which guaranteed that there can be no $T(\alpha_n)$-generic over L without collapsing ω_1; as ω_1 is not collapsed, there is in the latter case no $T(\alpha_n)$-generic over L in $L[G]$.

Finally, note that as C is definable over L_κ, it follows that the above gives a wellorder definable (indeed Σ_3) over the $H(\omega_2)$ of $L[G]$. □