Singular Cardinal Combinatorics, Sommersemester 2010

1.-2.Vorlesungen

These lectures are based on an article of James Cummings (*Notes on Singular Cardinal Combinatorics*, Notre Dame Journal, 2005). Topics covered include diamonds, squares, club guessing, forcing axioms and PCF theory. My intention is to follows the notes quite closely, as they are very well-written, but I will add a few proofs which James skips and pose some Questions which come to mind but are not answered in the article.

Diamonds

Let κ be regular, uncountable and S a stationary subset of κ.

$\diamondsuit_\kappa(S)$: There is $(S_\alpha \mid \alpha \in S)$ such that for any $X \subseteq \kappa$, $X \cap \alpha = S_\alpha$ for stationary-many α in S.

Theorem 1 *(Jensen) Assume $V = L$. Then $\diamondsuit_\kappa(S)$ holds for all uncountable regular κ and all stationary $S \subseteq \kappa$.*

Proof. We may assume that S consists only of limit ordinals. Define S_α by induction on $\alpha \in S$: Let (x, c) be the L-least pair such that

$(*)_\alpha$ x is a subset of α.
c is closed unbounded in α.
For β in $S \cap c$, $x \cap \beta$ does *not* equal S_β,

if such a pair (x, c) exists; set $(x, c) = (\emptyset, \emptyset)$ otherwise. We choose S_α to be x.

Now we claim that $(S_\alpha \mid \alpha \in S)$ witnesses $\diamondsuit_\kappa(S)$. If not, then let (X, C) be the L-least pair such that

$(*)_\kappa$ X is a subset of κ.
C is closed unbounded in κ.
For β in $S \cap C$, $X \cap \beta$ does *not* equal S_β.

Let M be an elementary submodel of some large L_θ which contains κ as an element and whose intersection with κ is an ordinal $\alpha < \kappa$. Then α belongs to

C and we may in fact choose M so that α also belongs to S as S is stationary (the set of candidates for α contains a closed unbounded subset of κ). Now let $\pi : M \simeq \bar{M}$ be the transitive collapse of M and notice that $\pi(X, C, \kappa)$ equals $(X \cap \alpha, C \cap \alpha, \alpha)$. It follows that $(X \cap \alpha, C \cap \alpha)$ is the $\bar{M}$-least pair satisfying $(*)_\alpha$, therefore also the L-least pair satisfying $(*)_\alpha$, and therefore $X \cap \alpha$ equals S_α. But as α belongs to $S \cap C$, this contradicts the choice of X! $\Box$

A nice consequence of $\diamondsuit_{\omega_1}$ (i.e., $\diamondsuit_{\omega_1}(S)$ where S equals all of ω_1) is the existence of a (nice) *Suslin tree*, i.e., an uncountable suborder T of ${}^{<\omega_1}2$ such that

T contains $\sigma * 0, \sigma * 1$ for each σ in T.
Each σ in T can be extended to a τ in T of any larger countable length.
T has only countable antichains.

The levels T_α of T are build by induction, the interesting case being the choice of limit levels (we cannot take all branches through lower levels, as this would give uncountably many branches and therefore an uncountable antichain). What we do is fix a $\diamondsuit_{\omega_1}$ sequence $(S_\alpha \mid \alpha < \omega_1)$ and view S_α as a subset not of α but of $T_{<\alpha}$, by enumerating the elements of $T_{<\alpha}$ in a canonical way; then form T_α by choosing branches through $T_{<\alpha}$ below each of its elements which hit S_α (if possible) and placing at level α the unions of all of the chosen branches. Now if X is a maximal antichain in the resulting tree T it follows by $\diamondsuit_{\omega_1}$ that $X \cap T_{<\alpha}$ equals S_α for some α and therefore every element of T_α lies above an element of X; it follows that X equals $X \cap T_{<\alpha}$ as any element of T of length greater than α lies above an element of T_α and therefore an element of $X \cap T_{<\alpha}$.

$\diamondsuit_\kappa(S)$ implies $\kappa^{<\kappa} = \kappa$, because if $(S_\alpha \mid \alpha \in S)$ witnesses the former then any bounded subset of κ will be equal to some S_α and there are only κ-many such S_α's. There is a partial converse:

Theorem 2 *(Gregory, Shelah) Let $\kappa = \lambda^+$, λ an uncountable cardinal, $\mu =$ cof(λ), $T = \{\alpha < \kappa \mid$ cof$(\alpha) \neq \mu\}$. Assume $2^{<\lambda} = \lambda$ and $2^\lambda = \lambda^+$ (this follows from GCH). Then $\diamondsuit_\kappa(S)$ holds for all stationary $S \subseteq T$.*

Actually we will prove something stronger than $\diamondsuit_\kappa(S)$. Consider:

$\diamondsuit'_{\lambda^+}(T)$: There is $(\mathcal{S}_\alpha \mid \alpha \in T)$ such that $\mathcal{S}_\alpha$ is a size at most λ subset of $\mathcal{P}(\alpha)$ for each $\alpha \in T$ and for all $X \subseteq \lambda^+$, $X \cap \alpha \in \mathcal{S}_\alpha$ for stationary-many α in T.
$\diamondsuit^*_{\lambda^+}(T)$ is the same as $\diamondsuit'_{\lambda^+}(T)$, except we strengthen the conclusion to: $X \cap \alpha \in \mathcal{S}_\alpha$ for all α in $C \cap T$ for some club C in λ^+.

Actually, $\diamondsuit'_{\lambda^+}(S)$ is equivalent to $\diamondsuit_{\lambda^+}(S)$ for all S: The latter clearly implies the former, so we need only show that a $\diamondsuit'_{\lambda^+}(S)$ sequence $(\mathcal{S}_\alpha \mid \alpha \in S)$ can be converted into a $\diamondsuit_{\lambda^+}(S)$ sequence $(S_\alpha \mid \alpha \in S)$. We assume that the given $\diamondsuit'_{\lambda^+}(S)$ sequence in fact guesses subsets of $\lambda \times \lambda^+$, so it specifies for each $\alpha < \lambda^+$ a sequence $(S^i_\alpha \mid i < \lambda)$ of subsets of $\lambda \times \alpha$. Now we claim that for some $i < \lambda$, $(T^i_\alpha \mid \alpha \in S)$ serves as a $\diamondsuit_{\lambda^+}(S)$ sequence, where $T^i_\alpha = \{\beta < \alpha \mid (i, \beta) \in S^i_\alpha\}$. If not, then fix for each $i < \lambda$ a pair (X_i, C_i) (with $X_i \subseteq \lambda^+$, C_i club in λ^+) such that T^i_α differs from $X_i \cap \alpha$ for α in $C_i \cap S$ and consider the intersection C of the C_i's and $X = \{(i, \alpha) \mid \alpha \in X_i\}$. But there is $\alpha \in C \cap S$ and $i < \lambda$ such that $X \cap (\lambda \times \alpha) = S^i_\alpha$ and therefore X_i equals T^i_α, contradicting the choice of (X_i, C_i).

Also note that $\diamondsuit^*_{\lambda^+}(T)$ implies $\diamondsuit'_{\lambda^+}(S)$ for all stationary $S \subseteq T$.

Proof of Theorem 2. We prove $\diamondsuit^*_{\lambda^+}(T)$. Write each $\alpha < \lambda^+$ as $\bigcup_{j<\mu} a^\alpha_j$ where the a^α_j's increase with j and have size less than λ. Also let $(x_i \mid i < \lambda^+)$ list the bounded subsets of λ^+.

For any $X \subseteq \lambda^+$, the set $C = \{\delta < \lambda^+ \mid \text{For all } \gamma < \delta \text{ there is } i < \delta \text{ such that } X \cap \gamma = x_i\}$ is club in λ^+. And for any $\alpha \in C$ we can choose $(\alpha_i \mid i < \mathrm{cof}(\alpha))$ below α such that the x_{α_i}'s are increasing under inclusion and $X \cap \alpha = \bigcup_{i<\mathrm{cof}(\alpha)} x_{\alpha_i}$.

Now suppose that α belongs to C and $\mathrm{cof}(\alpha)$ is not μ. Then we can replace the sequence $(\alpha_i \mid i < \mathrm{cof}(\alpha))$ by a subsequence, also of length $\mathrm{cof}(\alpha)$, so that it is contained in some single a^α_j, $j < \mu$.

Now we define

$$\mathcal{S}_\alpha = \{x \in \mathcal{P}(\alpha) \mid \text{For some } j < \mu \text{ and some } y \subseteq a^\alpha_j,\ x = \textstyle\bigcup_{i \in y} x_i\}.$$

Using $2^{<\lambda} = \lambda$ it follows that each $\mathcal{S}_\alpha$ has size at most λ and the above shows that it serves as a $\diamondsuit^*_{\lambda^+}(T)$ sequence, as desired. $\square$

Theorem 2 is optimal in the sense that GCH does *not* imply $\diamondsuit_{\lambda^+}(S)$ when S equals $\lambda^+ \cap \mathrm{Cof}(\lambda)$. (For example, GCH is consistent with the nonexistence of a Suslin Tree, and therefore with the negation of $\diamondsuit_{\omega_1}$.) [Question: Is GCH consistent with the failure of $\diamondsuit_{\lambda^+}(\lambda^+ \cap \mathrm{Cof}(\lambda))$ for all λ?]

The $\diamondsuit$ Ideal. For a regular uncountable κ, $I_{\diamondsuit_\kappa}$ is the set of $S \subseteq \kappa$ such that $\diamondsuit_\kappa(S)$ *fails*. It can be shown that this is a normal ideal on κ containing the nonstationary ideal. In L it equals the nonstationary ideal and if there are no Suslin trees then $I_{\diamondsuit_{\omega_1}}$ is improper, i.e., consists of the full power set of ω_1. [Question: What other possibilities are there for $I_{\diamondsuit_{\omega_1}}$ and more generally for $I_{\diamondsuit_\kappa}$?]

3.-4.Vorlesungen

First a remark about $\diamondsuit$: We showed that under GCH, $\diamondsuit_{\lambda^+}(S)$ will hold if S is a stationary subset of λ^+ disjoint from the critical cofinality $\mathrm{cof}(\lambda)$. Conversely, Shelah showed that for any uncountable cardinal λ in a model of GCH, one can force $\diamondsuit_{\lambda^+}(S)$ to fail for some stationary subset S of $\mathrm{Cof}(\lambda)$, preserving cofinalities and GCH. It is however open if one can force $\diamondsuit_{\lambda^+}(\mathrm{Cof}(\lambda))$ to fail for a singular λ under GCH.

Club Guessing

Club guessing holds for κ and $S \subseteq \kappa$ iff there exists $(C_\alpha \mid \alpha \in S)$ where C_α is club in α and for all clubs $C \subseteq \kappa$, C_α is contained in $C \cap \alpha$ for stationary many α in S. This is a weakening of $\diamondsuit_\kappa(S)$.

Theorem 3 *(Shelah) For $\lambda < \kappa$ regular with $\lambda^+ < \kappa$, club guessing holds for κ and any stationary $S \subseteq \kappa \cap \mathit{Cof}(\lambda)$.*

Proof. For E, F clubs in α define $\mathrm{pd}(E, F) = \{\sup(F \cap \gamma) \mid \gamma \in E, F \cap \gamma \neq \emptyset\}$. (We "push down" E onto F.)

Now start with an arbitrary sequence $(C_\alpha \mid \alpha \in S)$ with C_α club in α of ordertype λ. We define a decreasing sequence of clubs E_i in κ (of some length $\leq \lambda^+$) and define $(C^i_\alpha \mid \alpha \in \mathrm{Lim}\ (E_i) \cap S)$ by setting $C^i_\alpha = \mathrm{pd}(C_\alpha, E_i \cap \alpha)$. We start with $E_0 = \kappa$. Given E_i, we assume that $(C^i_\alpha \mid \alpha \in \mathrm{Lim}\ (E_i) \cap S)$ fails to have the club guessing property and choose $E_{i+1} \subseteq \mathrm{Lim}\ (E_i)$ to be a club so that for all α in $E_{i+1} \cap S$, C^i_α is not contained in $E_{i+1} \cap \alpha$. For limit λ we take E_λ to be the intersection of the E_i, $i < \lambda$.

We claim that E_{i+1} is undefined for some $i < \lambda^+$ (and hence the theorem is proved). Otherwise let E be the intersection of the E_i's and fix some α in $E \cap S$. Then α belongs to Lim $(E_i) \cap S$ and $C^i_\alpha = \mathrm{pd}(C_\alpha, E_i \cap \alpha)$ for each $i < \lambda^+$. For each $\gamma \in C_\alpha$, the sequence of suprema $(\sup(E_i \cap \gamma) \mid i < \lambda^+)$ is nonincreasing so must stabilise. Since C_α has ordertype λ we can find $i < \lambda^+$ large enough so that this stabilisation has occurred for every $\gamma \in C_\alpha$, so $C^i_\alpha = C^{i+1}_\alpha \subseteq E_{i+1}$. But since α belongs to $E_{i+1} \cap S$, this contradicts our choice of E_{i+1}. $\Box$

Cummings says that there are many interesting variants of club guessing, and refers to work of Ishiu for further information about them. [Question: Is it consistent for Club Guessing to fail for λ^+ and $\mathrm{Cof}(\lambda) \cap \lambda^+$ for all regular λ?.]

Squares

For an uncountable cardinal μ, $\Box_\mu$ asserts that there exists $(C_\alpha \mid \alpha < \mu^+$, α limit) such that each C_α is a club in α of ordertype at most μ and the C_α's *cohere* with each other: $\bar\alpha \in$ Lim (C_α) implies $C_{\bar\alpha} = C_\alpha \cap \bar\alpha$.

It is worth noting that without added strength we can impose a further requirement on the $\Box_\mu$ sequence: If α has cofinality less than μ then C_α has ordertype less than μ. To achieve this, fix a club C in μ of ordertype $\mathrm{cof}(\mu)$ and whenever $\mathrm{ot}(C_\alpha)$ belongs to $C \cup \{\mu\}$, replace C_α by $\{\beta \in C_\alpha \mid \mathrm{ot}(C_\alpha \cap \beta) \in C\}$; also, whenever $\mathrm{ot}(C_\alpha)$ does not belong to $C \cup \{\mu\}$, replace C_α by $C_\alpha \setminus (\beta+1)$, where β is the largest element of C_α with $\mathrm{ot}(C_\beta)$ in C.

In L, $\Box_\mu$ holds for every μ. To kill $\Box_\mu$ for a regular μ one only needs to Lévy collapse the least Mahlo cardinal greater than μ to become μ^+. Killing $\Box_\mu$ for a singular μ requires much stronger large cardinal hypotheses. It is typically done by starting with a supercompact.

Proposition 4 *Suppose that κ is λ^+-supercompact, $\kappa \leq \lambda$. Then $\Box_\lambda$ fails.*

Proof. First note that if $S \subseteq \lambda^+$ is a stationary subset of $\mathrm{Cof}(< \kappa) \cap \lambda^+$ then for some $\alpha < \lambda^+$ of uncountable cofinality, $S \cap \alpha$ is stationary in α: Choose a regular $\gamma < \kappa$ so that $S \cap \mathrm{Cof}(\gamma)$ is stationary. Then as γ is less than the critical point of j, it follows that j is continuous at ordinals of cofinality γ and therefore $j[S]$ is a stationary subset of $\alpha = \sup(j[\lambda^+]) < j(\lambda^+)$. Also by the λ^+ supercompactness of j, $j[S]$ belongs to M and therefore in M there is an $\alpha < j(\lambda^+)$ such that $j(S) \cap \alpha$ is stationary. By elementarity it follows that in V there is $\alpha < \lambda^+$ such that $S \cap \alpha$ is stationary.

Now note the following general fact: If $\square_\lambda$ holds then any stationary subset of λ^+ contains a nonreflecting stationary subset of λ^+, i.e., a stationary subset T of λ^+ such that $T \cap \alpha$ is nonstationary for all $\alpha < \lambda^+$ of uncountable cofinality. For, given any stationary $S \subseteq \lambda^+$, by Fodor we can choose a stationary $T \subseteq S$ and β such that $\mathrm{ot}(C_\alpha) = \beta$ for all α in T. If $\alpha < \lambda^+$ has uncountable cofinality, then $T \cap \alpha$ can contain at most one limit point of C_α by coherence, so $T \cap \alpha$ is nonstationary in α. For future use also note that if we set $D_\alpha = C_\alpha$ when $\mathrm{ot}(C_\alpha) \leq \beta$ and $D_\alpha = \{\gamma \in C_\alpha \mid \mathrm{ot}(C_\alpha \cap \gamma) > \beta\}$ otherwise, then we get a $\square_\mu$ sequence $(D_\alpha \mid \alpha < \mu^+)$ with the added property that Lim $(D_\alpha) \cap \alpha$ is disjoint from T for all α. $\square$

By the last part of the previous proof together with our work on $\diamondsuit$, we now get the following:

Proposition 5 *Assume GCH and $\square_\mu$ for an uncountable cardinal μ. Then there is a μ^+-Suslin tree.*

Proof Sketch. By our earlier work there is a $\diamondsuit_{\mu^+}(T)$ sequence for some stationary $T \subseteq \mu^+ \cap \mathrm{Cof}(\gamma)$, where $\gamma < \mu$ is regular and different from $\mathrm{cof}(\mu)$, together with a $\square_\mu$ sequence $(C_\alpha \mid \alpha < \mu^+)$ with the property that Lim $(C_\alpha) \cap \alpha$ is disjoint from T for all α.

Now build a μ^+ tree in stages, using the $\diamondsuit_{\mu^+}(T)$ sequence to guess at maximal antichains at stages $\alpha \in T$, and using the $\square_{\mu^+}$ sequence (which "avoids T") to obtain for each x in the tree and each higher tree level α a canonical branch $b(x, \alpha)$ containing x cofinal in the α-th level. For α not in T all canonical branches are continued to level α and for α in T only those canonical branches which pass through the guess at a maximal antichain are continued. The "T-avoidance" of the $\square$ sequence is used to show that the canonical branches $b(x, \alpha)$ leading to level α are indeed cofinal. $\square$

Weak squares

We consider the following weakening of $\square$:

$\Box_{\mu,\lambda}$ says that there exists $(\mathcal{C}_\alpha \mid \alpha < \mu^+,\ \alpha \text{ limit})$ such that each $\mathcal{C}_\alpha$ is a nonempty and size $\leq \lambda$ set of clubs in α, each of which has ordertype at most μ, and whenever C belongs to $\mathcal{C}_\alpha$ and β is a limit point of C, we have $C \cap \beta$ belongs to $\mathcal{C}_\beta$.

$\Box_{\mu,\mu^+}$ is provable, as we can simply choose clubs of ordertype at most μ through each limit $\alpha < \mu^+$ and take $\mathcal{C}_\alpha$ to consist of all intersections with α of such clubs. Jensen showed that $\Box_{\mu,\mu}$, also denoted by $\Box^*_\mu$ and called "Weak Square at μ", is equivalent to the existence of a special μ^+-Aronszajn tree, i.e., a tree T of height μ^+ with levels of size at most μ such that for some $f : T \to \mu$, $f(x)$ is different from $f(y)$ whenever x, y are comparable in T.

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Proposition 6 *Weak Square at μ is equivalent to the existence of a special μ^+-Aronszajn tree.*

Proof. Suppose that $(\mathcal{C}_\alpha \mid \alpha < \mu^+,\ \alpha \text{ limit})$ witnesses Weak Square at μ. Note that for any $\gamma < \mu^+$ the set of $C \cap \gamma$ for C in $\bigcup_\alpha \mathcal{C}_\alpha$ has size at most μ: If C belongs to $\mathcal{C}_\alpha$ then $C \cap \gamma$ is either finite or the union of $C \cap \delta$ for some limit point of δ of C together with a finite set; in the latter case $C \cap \delta$ belongs to $\mathcal{C}_\delta$ so we get at most μ possibilities. Thus for our tree we can simply take all initial segments of elements of $\bigcup_\alpha \mathcal{C}_\alpha$, ordered by end-extension, with ot as the specialising function.

Conversely, suppose that we are given a special μ^+ Aronszajn tree T with specialising function $f : T \to \mu$. We will associate to each $x \in T$ of limit T-height $|x|$ an unbounded $A_x \subseteq |x|$ of ordertype at most μ such that if $\delta < |x|$ is a limit point of A_x then $A_x \cap \delta$ is A_y where y is the T-predecessor of x of T-height δ. This suffices, as then we can take C_x to be the closure of A_x (without $|x|$) for each x of limit T-height and get a Weak Square sequence by setting $\mathcal{C}_\alpha$ to be the set of C_x for x of T-height α.

For $x \in T$ of limit T-height define $A_x = \{\gamma_i \mid i < j_x\}$ as follows: Let y_i be the T-predecessor y of x of T-height greater than the the γ_j, $j < i$, with least possible specialising value $f(y)$; then γ_i is the T-height of y_i. This definition continues until one generates an unbounded subset of $|x|$, which we take to be A_x. Clearly A_x has ordertype at most μ. It is routine to check the coherence property: if y is the T-predecessor of x of height γ_λ, λ limit, then A_y will be defined in exactly the same way as A_x until it becomes cofinal in the height of y. $\square$

Remarks. Without introducing any extra strength, we can arrange that a Weak Square sequence $(\mathcal{C}_\alpha \mid \alpha \text{ limit}, \alpha < \mu^+)$ at μ has the following two additional properties: (i) For all limit $\alpha < \mu^+$ there is $C \in \mathcal{C}_\alpha$ of ordertype $\mathrm{cof}(\alpha)$. (ii) If μ is singular, then the club sets appearing in the $\mathcal{C}_\alpha$'s each have ordertype less than μ.

The extent of Square

Weak Square will hold at μ if μ is regular and there are only μ bounded subsets of μ: If α has cofinality less than μ then take $\mathcal{C}_\alpha$ to consist of all clubs in α of ordertype less than μ (there are only μ many); if α has cofinality μ then take $\mathcal{C}_\alpha$ to consist of one club in α of ordertype μ.

On the other hand, GCH is not sufficient to imply $\square_\mu$ for an uncountable regular μ: If $\kappa > \mu$ is Mahlo then $\square_\mu$ will fail after applying Coll $(\mu, <\kappa)$, the forcing that collapses every ordinal less than κ to μ with conditions of size $< \mu$ (and turns κ into μ^+). The idea is that if $(C_\alpha \mid \alpha \text{ limit}, \alpha < \kappa)$ were a $\square_\mu$ sequence in the extension then using Mahloness there is a $\bar{\kappa} < \kappa$ regular in the ground model such that $(C_\alpha \mid \alpha \text{ limit}, \alpha < \bar{\kappa})$ belongs to the intermediate model $V[\bar{G}]$ obtained by restricting the generic to Coll $(\mu, <\bar{\kappa})$; but then $C_{\bar{\kappa}}$ was added over this model by a μ-closed forcing, which is impossible since all of its initial segments belong to $V[\bar{G}]$.

Dropping GCH, Weak Square can fail at a regular μ: Mitchell found a way of turning a Mahlo $\kappa > \mu$ into μ^+, in such a way that κ-many bounded subsets of μ are added and Weak Square will fail at μ.

As we have mentioned, killing Square at a singular μ is much harder and typically uses a supercompact. In fact a strong compact κ is sufficient to imply the failure of Weak Square at any singular μ of cofinality less than κ. [Question: Is a strong compact consistent with Weak Square at a singular cardinal above it?]

Approachability and $I[\lambda]$

Let κ be regular and $(a_\alpha \mid \alpha < \kappa)$ a sequence of bounded subsets of κ. A limit ordinal $\gamma < \kappa$ is *approachable* relative to this sequence iff there is an unbounded $A \subseteq \gamma$ of ordertype $\mathrm{cof}(\gamma)$ such that each proper initial segment of A is of the form a_α for some $\alpha < \gamma$. The *approachability ideal* $I[\kappa]$ consists of all $S \subseteq \kappa$ such that for some sequence $(a_\alpha \mid \alpha < \kappa)$ of bounded subsets of κ, almost all elements of S are approachable relative to $(a_\alpha \mid \alpha < \kappa)$ (i.e., for some club $C \subseteq \kappa$, all elements of $S \cap C$ are approachable relative to $(a_\alpha \mid \alpha < \kappa)$).

The set $\mathrm{Cof}(\omega) \cap \kappa$ belongs to $I[\kappa]$ because we can choose as our sequence $(a_\alpha \mid \alpha < \kappa)$ an enumeration of the finite subsets of κ. Also note that if $\lambda < \kappa$ is regular and $\kappa^{<\lambda}$ equals κ then there is a single "universal" sequence $(a_\alpha \mid \alpha < \kappa)$ which witnesses the approachability of each subset of $\mathrm{Cof}(\lambda)$ in $I[\kappa]$; any sequence $(a_\alpha \mid \alpha < \kappa)$ which enumerates each element of $[\kappa]^{<\lambda}$ unboundedly often will suffice. Also under the assumption $\kappa^{<\lambda} = \kappa$ there is a maximal subset of $\mathrm{Cof}(\lambda)$ in $I[\kappa]$ (maximal modulo the nonstationary ideal): take $S(\lambda)$ to be the set of ordinals of cofinality λ which are approachable with respect to the universal sequence. We can refer to $S(\lambda)$ as the set of "approachable points of cofinality λ in κ". (In extreme cases, $S(\lambda)$ will be nonstationary.)

It is also easy to see that $I[\kappa]$ is an ideal, because if S_0, S_1 belong to $I[\kappa]$, witnessed by sequences $(a^0_\alpha \mid \alpha < \kappa), (a^1_\alpha \mid \alpha < \kappa)$ and clubs C_0, C_1, then $S_0 \cup S_1$ is witnessed to belong to $I[\kappa]$ by $(a_\alpha \mid \alpha < \kappa)$ and club $C_0 \cap C_1$, where $a_{2\alpha+i}$ equals a^i_α. The same argument shows that the diagonal union $\{\alpha \mid \alpha \in S_i \text{ for some } i < \alpha\}$ of sets $(S_i \mid i < \kappa)$ in $I[\kappa]$ also belongs to $I[\kappa]$, using the "join" $a_{\langle\alpha,i\rangle} = a^i_\alpha$ of sequences $(a^i_\alpha \mid \alpha < \kappa)$ and the diagonal intersection of clubs $(C_i \mid i < \kappa)$ witnessing the membership of S_i in $I[\kappa]$.

So $I[\kappa]$ is a normal ideal. But it need not be a *proper* ideal, i.e., it could be that κ itself belongs to $I[\kappa]$. For example, if Weak Square holds at μ then μ^+ belongs to $I[\mu^+]$: We can witness Weak Square with a sequence $(\mathcal{C}_\alpha \mid \alpha$ limit, $\alpha < \mu^+)$ such that each $\mathcal{C}_\alpha$ contains a club of ordertype $\mathrm{cof}(\alpha)$. Let $(a_\beta \mid \beta < \mu^+)$ enumerate $\bigcup_\alpha \mathcal{C}_\alpha$; then almost all $\alpha < \mu^+$ are approachable relative to $(a_\beta \mid \beta < \mu^+)$. In particular, $\mu^{<\mu} = \mu$ implies that μ^+ belongs to $I[\mu^+]$. Without any cardinal arithmetic assumption we have:

Proposition 7 *If μ is regular then Cof$(<\mu) \cap \mu^+$ belongs to $I[\mu^+]$.*

To prove Proposition 9 we introduce *partial squares*.

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Partial Squares

In the Shelah tradition of proving weakenings of combinatorial principles in ZFC, we look at partial squares. (This will also be useful when studying the next topic, approachability.) Let S be a subset of $\{\alpha < \mu^+ \mid \mathrm{cof}(\alpha) = \lambda\}$. We say that S *carries a partial square* iff there is $(C_\alpha \mid \alpha \in S)$ such that each C_α is club in α of ordertype λ and whenever β is a common limit point of $C_{\alpha_0}, C_{\alpha_1}$ for two α_0, α_1 in S, then we have $C_{\alpha_0} \cap \beta = C_{\alpha_1} \cap \beta$.

Theorem 8 *(Shelah) If $\lambda < \mu$ are regular then $\{\alpha < \mu^+ \mid cof(\alpha) = \lambda\}$ is the union of μ sets, each of which carries a partial square.*

Proof. We may assume that λ is uncountable. Fix some large regular θ and let $\mathcal{M}$ be the structure $(H(\theta), \in, <_\theta)$ where $<_\theta$ is a wellorder of $H(\theta)$. For $\alpha < \mu^+$ of cofinality λ and $\zeta < \mu$ we let $M(\alpha, \zeta)$ be the Skolem hull of $\{\alpha\} \cup \zeta$ in $\mathcal{M}$.

For each $\alpha < \mu^+$ of cofinality λ choose $\zeta(\alpha)$ to be the least $\zeta \geq \lambda$ such that $M(\alpha, \zeta) \cap \mu$ is an ordinal of uncountable cofinality. Set $N_\alpha = M(\alpha, \zeta(\alpha))$ and note that $E_\alpha = N_\alpha \cap \alpha$ is unbounded in α.

We claim that E_α is ω-closed. To see this, let $x \subseteq E_\alpha$ have ordertype ω, let β be the sup of x and let γ be the least element of $E_\alpha \setminus \beta$. We want to show that $\beta = \gamma$. Note that as γ belongs to N_α, N_α contains an increasing cofinal map f from $\mathrm{cof}(\gamma)$ into γ and f restricted to $N_\alpha \cap \mathrm{cof}(\gamma)$ is cofinal into $N_\alpha \cap \gamma$. Now as $N_\alpha \cap \mu$ is an ordinal, if $\mathrm{cof}(\gamma) < \mu$ then the range of f is contained in N_α, and so $N_\alpha \cap \gamma = E_\alpha \cap \gamma$ is cofinal in γ, as desired. So $\mathrm{cof}(\gamma)$ equals μ. Thus $\mathrm{cof}(N_\alpha \cap \gamma) = \mathrm{cof}(N_\alpha \cap \mu)$, which is impossible as $\mathrm{cof}(N_\alpha \cap \mu)$ is greater than ω while $\mathrm{cof}(N_\alpha \cap \gamma) = \mathrm{cof}(N_\alpha \cap \beta) = \omega$.

Let D_α be the closure of E_α as a set of ordinals. For $\rho, \sigma < \mu$ we define $S(\rho, \sigma) = \{\alpha \mid N_\alpha \cap \mu = \rho,\ \mathrm{ot}(D_\alpha) = \sigma\}$. We show that each of these sets carries a partial square, witnessed by clubs C_α contained in D_α.

We first show that if α, α^* belong to $S(\rho, \sigma)$ and γ is a common limit point of D_α, D_{α^*} then $D_\alpha \cap \gamma$ equals $D_{\alpha^*} \cap \gamma$. If γ has cofinality ω then by the above γ belongs to $N_\alpha \cap N_{\alpha^*}$; as γ has size at most μ and $N_\alpha \cap \mu$ equals $N_{\alpha^*} \cap \mu$ it follows that $N_\alpha \cap \gamma$ equals $N_{\alpha^*} \cap \gamma$. If γ has uncountable cofinality then γ is a limit of η of cofinality ω which are common limits of D_α, D_{α^*}; we then just showed $N_\alpha \cap \eta = N_{\alpha^*} \cap \eta$ for such η and therefore $N_\alpha \cap \gamma$ equals $N_{\alpha^*} \cap \gamma$.

Finally, fix $C \subseteq \sigma$ a club of ordertype λ and set $C_\alpha = \{\gamma \in D_\alpha \mid \mathrm{ot}(D_\alpha \cap \gamma) \in C\}$. This thinning of the D_α sequence preserves coherence and therefore we have a partial square on $S(\rho, \sigma)$, as desired. $\Box$

Now we show:

Proposition 9 *If μ is regular then Cof($< \mu$) $\cap\, \mu^+$ belongs to $I[\mu^+]$.*

Proof of Proposition 9. It suffices to show that for each regular $\lambda < \mu$, $\mathrm{Cof}(\lambda) \cap \mu^+$ belongs to $I[\mu^+]$. Recall that $\mathrm{Cof}(\lambda) \cap \mu^+$ is the union of μ sets, each of which carries a partial square. It suffices to show that if S is one of these sets then S belongs to $I[\mu^+]$. Let $(C_\alpha \mid \alpha \in S)$ be a partial square sequence on S. For γ a limit point of some C_α let D_γ be $C_\alpha \cap \gamma$ for all such α. Then the sequence $(D_\gamma \mid \gamma < \mu^+)$ witnesses the approachability of S. $\Box$

Thus for regular μ, the ideal $I[\mu^+]$ is only interesting on $\mathrm{Cof}(\mu)$. As mentioned earlier, Mitchell constructed a model using a Mahlo cardinal in which Weak Square fails at ω_1; it can be verified that also in this model, μ^+ does not belong to $I[\mu^+]$.

For singular μ, we'll show that $I[\mu^+]$ does contains stationary sets on any cofinality (however it need not be the case that $\mathrm{Cof}(\lambda)$ belongs to $I[\mu^+]$ for each regular $\lambda < \mu$). This follows from the following more general result of Shelah:

Theorem 10 *Let $\kappa < \kappa^+ < \theta < \lambda$ be regular. Then there is a subset A of Cof(κ) $\cap\, \lambda$ which belongs to $I[\lambda]$ such that $A \cap \delta$ is stationary in δ for stationary many δ in Cof(θ) $\cap\, \lambda$ (and in particular A is stationary).*

Proof. We use the concept of *internally approachable (IA) chain.* If $\mathcal{A}$ is a structure $(H(\epsilon), \in, <_\epsilon, \ldots)$ with ϵ large and regular, $<_\epsilon$ a wellorder of $H(\epsilon)$ and ... countably many additional constants, functions and relations and γ is a limit ordinal, then an *IA chain of substructures of $\mathcal{A}$ of length γ* is a continuous and increasing sequence $(M_i \mid i < \gamma)$ of elementary substructures of $\mathcal{A}$ such that $(M_i \mid i <\leq j)$ belongs to M_{j+1} for each $j < \gamma$. Of course for uncountable regular γ and any subset x of $\mathcal{A}$ of size $< \gamma$ it is easy to build an IA chain of length γ of substructures of $\mathcal{A}$ of size less than γ containing x as a subset.

Now let $(C_\xi \mid \xi \in \mathrm{Cof}(\kappa) \cap \theta)$ be a club guessing sequence, which exists because κ^+ is less than θ. Let $(M_i \mid i < \lambda)$ be an IA chain such that $(C_\xi \mid \xi \in \mathrm{Cof}(\kappa) \cap \theta)$ is a subset of M_0 and each M_i has size less than λ. Let $(a_i \mid i < \lambda)$ enumerate the bounded subsets of λ in $\bigcup_{i<\lambda} M_i$. The desired set A is the set of $\gamma < \lambda$ of cofinality κ which are approachable relative to the sequence $(a_i \mid i < \lambda)$. We must show that $A \cap \delta$ is stationary in δ for stationary many $\delta \in \mathrm{Cof}(\theta) \cap \lambda$.

If not then $A \cap \delta$ is nonstationary in δ for almost all $\delta < \lambda$ of cofinality θ. It follows that we can build an IA chain $(N_j \mid j < \theta)$ of substructures of size θ such that N_0 contains $(C_\xi \mid \xi \in \mathrm{Cof}(\kappa) \cap \theta)$ as a subset and $(M_i \mid i < \lambda)$ as an element and setting $\delta = \sup(\bigcup_{j<\theta} N_j \cap \lambda)$, we have $A \cap \delta$ is nonstationary in δ.

For $j < \theta$ let α_j be $\sup(N_j \cap \lambda)$; then the sequence of α_j's is continuous and cofinal in δ. Also choose $(\beta_j \mid j < \theta)$ in $M_{\delta+1}$ to be continuous and cofinal in δ. Then $e = \{j < \theta \mid \alpha_j = \beta_j\}$ is club in θ so by club-guessing, C_ξ is contained in e for stationary many $\xi \in \mathrm{Cof}(\kappa) \cap \theta$. For such a ξ let $c = \{\alpha_j \mid j \in C_\xi\} = \{\beta_j \mid j \in C_\xi\}$. The proper initial segments of c lie both in N_ξ and in $M_{\delta+1}$. If x is such a proper initial segment of c then N_ξ sees that x belongs to some M_i and so x bleongs to M_i for some i in $N_\xi \cap \lambda$; hence x belongs to M_{α_ξ}. Thus by definition of A, c witnesses that α_ξ belongs to A. As this holds for stationary many ξ, we have shown that $A \cap \delta$ is stationary in δ, contradiction! $\square$

To summarise: For regular μ, $I[\mu^+]$ contains $\mathrm{Cof}(\lambda) \cap \mu^+$ for regular $\lambda < \mu$; it may fail to contain $\mathrm{Cof}(\mu) \cap \mu^+$. For singular μ, $I[\mu^+]$ contains stationary subsets of $\mathrm{Cof}(\lambda) \cap \mu^+$ for any regular $\lambda < \mu$. For weakly inaccessible κ, $I[\kappa]$ contains stationary subsets of $\mathrm{Cof}(\lambda) \cap \kappa$ for any regular $\lambda < \kappa$. [Questions: For uncountable regular μ is it possible that $I[\mu^+]$ contain only nonstationary subsets of $\mathrm{Cof}(\mu)$? For singular μ is it possible that $I[\mu^+]$ fails to contain $\mathrm{Cof}(\lambda) \cap \mu^+$ for all uncountable regular $\lambda < \mu$? For weakly inaccessible μ is it possible that $I[\mu]$ fails lto contain $\mathrm{Cof}(\lambda) \cap \mu$ for all uncountable regular $\lambda < \mu$?]

9.-10.Vorlesungen

Approachability and Forcing

Shelah originally introduced approachability to answer the question of when μ^+-closed forcing preserves the stationarity of subsets of $\mathrm{Cof}(\mu)$. For this purpose it is convenient to note that approachability can be formulated in an equivalent way using elementary submodels. Let θ denote a large regular cardinal and $\mathcal{A}$ a structure of the form $(H(\theta), \in, <_\theta, \ldots)$ where $<_\theta$ is a wellorder of $H(\theta)$ and $\ldots$ represents countably many additional functions, relations and constants. Then $\gamma < \kappa$ is *approachable relative to* $\mathcal{A}$ if there is an unbounded $A \subseteq \gamma$ of ordertype $\mathrm{cof}(\gamma)$ such that each proper initial segment of A belongs to $\mathrm{Sk}^{\mathcal{A}}(\gamma)$ (the set of elements of $H(\theta)$ which are definable in $\mathcal{A}$ from parameters less than γ; Sk stands for "Skolem hull"). It is easy to see that $S \subseteq \kappa$ belongs to $I[\kappa]$ iff for some $\mathcal{A}$ as above, almost all elements of S are approachable relative to $\mathcal{A}$.

Proposition 11 *Suppose that κ is regular and uncountable.*
(a) If $S \subseteq \kappa \cap Cof(\omega)$ is stationary then countably closed forcing preserves the stationarity of S.
(b) More generally, if $S \subseteq \kappa \cap Cof(\mu)$ is stationary and belongs to $I[\kappa]$ then μ^+-closed forcing preserves the stationarity of S.

Proof. We prove (b). Let the structure $\mathcal{A} = (H_\theta, \in, <_\theta, \ldots)$ witness $S \in I[\kappa]$ for some large θ and let P be a μ^+-closed forcing, p a condition in P forcing $\dot{C}$ to be a club in κ. Expand $\mathcal{A}$ to $\mathcal{A}^*$ so as to include $P, p, \dot{C}$. Now consider the club C of all $\gamma < \kappa$ such that $\gamma = \kappa \cap \mathrm{Sk}^{\mathcal{A}^*}(\gamma)$ and choose γ in $C \cap S$. Also let $A \subseteq \gamma$ be unbounded of ordertype μ such that all proper initial segments of A belong to $\mathrm{Sk}^{\mathcal{A}^*}(\gamma)$. Now the point is that if we successively extend p in μ steps in the $<_\theta$-least way, at step i forcing an ordinal greater than the i-th element of A into $\dot{C}$, then the resulting conditions belong to $\mathrm{Sk}^{\mathcal{A}^*}(\gamma)$ by the choice of A. Therefore a lower bound to these conditions forces that $\dot{C}$ is unbounded below γ. It follows that p has an extension forcing $\gamma \in S$ into $\dot{C}$, proving that the stationarity of S is preserved. $\square$

There is a kind of converse to this result: Suppose that $\kappa^{<\mu} = \kappa$ and S is the set of points of cofinality μ approachable with respect to a universal enumeration of $[\kappa]^{<\mu}$. Also suppose that $\mu^{<\mu} = \mu$. Now consider the forcing whose conditions are closed bounded subsets c of κ of ordertype less than μ^+

such that $c \cap \mathrm{Cof}(\mu)$ is contained in S and the bounded subsets of c of size less than μ appear in the universal enumeration before stage $\max(c)$. Then this forcing is μ^+-closed and kills the stationarity of $\mathrm{Cof}(\mu) \cap (\kappa \setminus S)$. (We'll see later that it is indeed possible for $\mathrm{Cof}(\mu) \cap (\kappa \setminus S)$ to be stationary, even when GCH holds and $\mu = \omega_1$, $\kappa = \aleph_{\omega+1}$.)

Scales, good points and exact upper bounds

Given an index set X and an ideal I on X we can order the functions from X into Ord by: $f <_I g$ iff $\{x \mid f(x) \geq g(x)\} \in I$. Define $=_I$ and $\leq_I$ in the obvious way. A $<_I$ increasing sequence $(f_i \mid i < \alpha)$ has an *exact upper bound (eub)* iff there is an f such that $f_i <_I f$ for all i and every $g <_I f$ satisfies $g <_I f_i$ for some i. If f exists then of course it is unique modulo the ideal I.

Suppose $f : X \to \mathrm{Ord}$. A *scale of length* α *in* $\prod_X f(x)/I$ is a $<_I$ increasing sequence $(f_i \mid i < \alpha)$ in $\prod_X f(x)$ which is cofinal in $\prod_X f(x)$ under the relation $<_I$. In this case it follows that f is an eub for $(f_i \mid i < \alpha)$ and conversely, if f is an eub for $(f_i \mid i < \alpha)$ then $(f_i^* \mid i < \alpha)$ forms a scale in $\prod_X f(x)$, where $f_i^*(x) = f_i(x)$ if the latter is less than $f(x)$, 0 otherwise.

Weaker then *eub* is *lub (least upper bound)*. f is an lub for $(f_i \mid i < \alpha)$ iff $f_i \leq_I f$ for each i and every function which is below f on an I*-positive* set is below some f_i on an I*-positive* set.

Also note the following: If $\vec{f} = (f_i \mid i < \gamma)$ and $\vec{g} = (g_j \mid j < \delta)$ are *cofinally interleaved* in the sense that $\{h \mid h <_I f_i \text{ for some } i\} = \{h \mid h <_I g_j \text{ for some } j\}$ then $\vec{f}$ has an eub iff $\vec{g}$ has an eub and these eub's are equal modulo I.

Our goal is to use the nontriviality of the approachability ideal to build scales, or equivalently, to build sequences with eub's. A key concept for achieving this is that of a *good point*. Suppose that I is an ideal on X and let $(f_i \mid i < \gamma)$ be $<_I$ increasing. A limit ordinal $\alpha \leq \gamma$ is a *good point* iff $\mathrm{cof}(\alpha) > \mathrm{card}(X)$ and there is an eub h for $(f_i \mid i < \alpha)$ such that $\mathrm{cof}(h(x)) = \mathrm{cof}(\alpha)$ for all x. Equivalently: There is a *pointwise increasing* sequence $(h_j \mid j < \mathrm{cof}(\alpha))$ cofinally interleaved modulo I with $(f_i \mid i < \alpha)$.

Theorem 12 *Let card$(X) < \kappa < \lambda$ with κ and λ regular. Suppose that $(f_i \mid i < \lambda)$ is a $<_I$ increasing sequence with stationarily many good points of cofinality κ. Then there exists an eub h such that cof$(h(x)) > \kappa$ for all x.*

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Proof. First we construct an lub and then show that this lub is in fact an eub.

Step 1. By induction we construct functions g_j such that $f_i <_I g_j$ for all i and for $j_1 < j_2$, $g_{j_2} \leq_I g_{j_1}$, $g_{j_2} \neq_I g_{j_1}$. Start by choosing g_0 to be any upper bound, and for all j if g_j fails to be an lub for $(f_i \mid i < \lambda)$ we choose g_{j+1} to witness this failure.

For limit μ set $S_\mu(x) = \{g_j(x) \mid j < \mu\}$ and define $h^i_\mu(x) = \min(S_\mu(x) \setminus f_i(x))$. We claim that for $\mu < \mathrm{card}(X)^+$, $(h^i_\mu \mid i < \lambda)$ is eventually constant modulo I. If not, we find γ good of cofinality κ such that h^i_μ does not stabilise for large $i < \gamma$ and fix $(H_\zeta \mid \zeta < \kappa)$ pointwise increasing and cofinally interleaved with $(f_i \mid i < \gamma)$. The function $x \mapsto \min(S_\mu(x) \setminus H_\zeta(x))$ cannot stabilise for large $\zeta < \kappa$, but this is impossible because $\mathrm{card}(S_\mu(x)) \leq \mathrm{card}(X) < \kappa$. We now choose g_μ so that $g_\mu =_I h^i_\mu$ for all large i.

We show that this construction stops in fewer than $\mathrm{card}(X)^+$ steps. Suppose not. For each x and each i the value of $h^i_\mu(x)$ will stabilise for large limit $\mu < \mathrm{card}(X)^+$ since the smallest value which will ever appear must turn up at some point. So for each $i < \lambda$ the function h^i_μ stabilises for large limit μ. Thus there is an unbounded $B \subseteq \lambda$ and a fixed ν such that for $i \in B$, h^i_μ is constant for limit $\mu \geq \nu$. If $\nu \leq \mu_1 < \mu_2$ we may choose $i \in B$ large enough so that $g_{\mu_1} =_I h^i_{\mu_1}$ and $g_{\mu_2} =_I h^i_{\mu_2}$, yielding $g_{\mu_1} =_I g_{\mu_2}$, contradicting the choice of the functions g_j.

So the construction halts at some stage before $\mathrm{card}(X)^+$, producing an lub g.

Step 2. Suppose now that our lub g from Step 1 is not an eub. Then we may find $h <_i g$ such that the set $S_i = \{x \mid f_i(x) \leq h(x)\}$ is I-positive for all i. We claim that this sequence of sets is eventually constant modulo I. If not, then we find a good point γ of cofinality κ such that S_i does not stabilise modulo I for large $i < \gamma$ and fix $(H_\zeta \mid \zeta < \kappa)$ pointwise increasing and cofinally interleaved with $(f_i \mid i < \lambda)$. If $D_\zeta = \{x \mid H_\zeta(x) \leq h(x)\}$ then D_ζ cannot stabilise for large ζ, but this is impossible because D_ζ decreases with ζ and $\mathrm{card}(X) < \kappa$.

Let S be such that $S_i =_I S$ for large i and define g^* so that g^* agrees with h on S and with g on the complement of S. Then by construction, g^* is an upper bound for the f_i's and g^* is below g on an I-positive set, which is impossible since g is an lub.

To finish we must check that $\mathrm{cof}(g(x)) > \kappa$ for almost all x. This follows from an argument similar to that we gave in Step 1 that h^i_μ stabilises for large i. $\square$

Remark. The converse of the above result is also true: Let C be a cub subset of λ and build an IA chain $(M_j \mid j < \kappa)$ of structures of size less than κ with union M such that $\gamma = \sup(M \cap \lambda)$ belongs to C. Then the function $\bar{h}$ given by $x \mapsto \sup(M \cap h(x))$ is an eub for the f_i's with $\mathrm{cof}(\bar{h}(x)) = \kappa$ for each x.

Building scales, goodness and approachability

As a first application of the nontriviality of the approachability ideal and the previous result about eub's, we prove:

Theorem 13 *There is an infinite $A \subseteq \omega$ and a scale of length $\aleph_{\omega+1}$ in $\prod_{n\in A} \aleph_n/Fin$.*

Proof. Choose some large θ and build an internally approachable chain $(M_\alpha \mid \alpha < \aleph_{\omega+1})$ in $(H(\theta), \in, <_\theta)$ consisting of structures of size $\aleph_\omega$ with $M_\alpha \cap \aleph_{\omega+1}$ an ordinal. Let g_α be the $<_\theta$ least function which dominates modulo finite all functions in $M_\alpha \cap \prod_{n\in\omega} \aleph_n$.

Recall that $I[\aleph_{\omega+1}]$ contains a stationary subset of $\mathrm{Cof}(\aleph_k)$ for each finite k. Now note the following:

Lemma 14 *Let $\mathcal{A}$ denote $(H(\theta), \in, <_\theta)$. Let S belong to $I[\aleph_{\omega+1}]$, $S \subseteq Cof(\aleph_k)$. Then for almost all γ in S there is an internally approachable chain $(N_i \mid i < \aleph_k)$ of substructures of $\mathcal{A}$ with union $N \subseteq M_\gamma$ such that card$(N_i) < \aleph_k$ for all i, $\sup(N \cap \aleph_{\omega+1}) = \gamma$ and $M_\alpha \in N$ for cofinally many $\alpha < \gamma$.*

Proof of Lemma. Expand $\mathcal{A}$ to $\mathcal{B}$ by adding a predicate for a sequence of bounded subsets of $\aleph_{\omega+1}$ witnessing that S belongs to $I[\aleph_{\omega+1}]$ and build an IA chain $(M^*_\alpha \mid \alpha < \aleph_{\omega+1})$ of substructures of $\mathcal{B}$ with $\mathrm{card}(M^*_\alpha) = \aleph_\omega$ and $M^*_\alpha \cap \aleph_{\omega+1}$ an ordinal. Also assume that $(M_\alpha \mid \alpha < \aleph_{\omega+1})$ belongs to M^*_0. Now choose γ in S so that $\gamma = M^*_\gamma \cap \aleph_{\omega+1}$. Fix a sequence $(\gamma_j \mid j < \aleph_k)$ continuous and cofinal in γ such that $(\gamma_i \mid i \leq j)$ belongs to $M^*_{\gamma_j+1}$ for each j. Then take N_i to be the Skolem hull in M^*_γ of the set of parameters $\{P_j \mid j < i\}$ where $P_j = (M^*_{\gamma_i} \mid i \leq j)$. Then the N_i's have the desired properties. $\square$ (Lemma)

Now fix a finite k and apply the Lemma to almost all $\gamma \in S$ to obtain the sequence $(N_i \mid i < \aleph_k)$. Define h_i to be the function $m \mapsto \sup(N_i \cap \aleph_m)$ and h to be the function $m \mapsto \sup(N \cap \aleph_m)$. Then h is an eub for $(h_i \mid i < \aleph_k)$ as $\mathrm{cof}(h(m)) = \aleph_k$ for all m.

We claim that $(h_i \mid i < \aleph_k)$ and $(g_\alpha \mid \alpha < \gamma)$ are cofinally interleaved. On the one hand, each h_i is defined from the corresponding N_i, so h_i belongs to M_γ and hence for some $\beta < \gamma$, h_i belongs to M_β and is dominated modulo finite by g_β. Conversely, for cofinally many $\alpha < \gamma$, M_α belongs to N so g_α belongs to N_i for some $i < \aleph_k$ and is dominated everywhere by h_i. It follows that h is an eub for $(g_\alpha \mid \alpha < \gamma)$.

Applying Theorem 12 our sequence $(g_\alpha \mid \alpha < \aleph_{\omega+1})$ is increasing modulo finite and for each k has an eub g_k such that $g_k(n)$ has cofinality greater than $\aleph_k$ for each n. Let g be an eub for $(g_\alpha \mid \alpha < \aleph_{\omega+1})$; it follows that for each k the set $\{n \mid \mathrm{cof}(g(n)) = \aleph_k\}$ is finite. Let A be the set of k such that $\mathrm{cof}(g(n)) = \aleph_k$ for some n. For each $k \in A$ and each n with $\mathrm{cof}(g(n))) = \aleph_k$ fix $(\beta^n_i \mid i < \aleph_k)$ increasing and cofinal in $g(n)$. If we now define $f_\alpha(k)$ to be the least i such that $g_\alpha(n) < \beta^n_i$ for all n with $\mathrm{cof}(g(n)) = \aleph_k$, then the sequence of f_α's can be thinned out to give a scale of length $\aleph_{\omega+1}$ in $\prod_{k\in A} \aleph_k/Fin$. For future use note that if X is the set of good points for the sequence $(g_\alpha \mid \alpha < \aleph_{\omega+1})$ then the resulting scale is good at almost every point in X. $\square$

13.Vorlesung

Square principles and scale properties

Recall that a *scale of length* α in $\prod_{x\in X} f(x)/I$ is a $<_I$ increasing sequence $(f_i \mid i < \alpha)$ in $\prod_{x\in X} f(x)$ which is cofinal in $\prod_{x\in X} f(x)$ under the relation $<_I$. Equivalently, f is an *exact upper bound* for the sequence $(f_i \mid i < \alpha)$: $f_i <_I f$ for each $i < \alpha$ and whenever $g <_I f$, we have $g <_I f_i$ for some i. A limit ordinal $\beta \leq \alpha$ is a *good point* of $(f_i \mid i < \alpha)$ iff $\mathrm{cof}(\beta) > \mathrm{card}(X)$ and there is an exact upper bound f for $(f_i \mid i < \beta)$ such that $\mathrm{cof}(f(x)) = \mathrm{cof}(\beta)$ for each x.

We focus now on scales of length $\aleph_{\omega+1}$ in products $\prod_{n\in A}\aleph_n/\mathrm{Fin}$ where A is an infinite subset of ω and Fin is the ideal of finite sets. We proved:

Theorem 15 *There is a stationary subset X of $Cof(\aleph_k)\cap\aleph_{\omega+1}$ in $I[\aleph_{\omega+1}]$ (even concentrating on a single cofinality $\aleph_k$). And for any such X there is an $A\subseteq\omega$ and a scale of length $\aleph_{\omega+1}$ in $\prod_{n\in A}\aleph_n/Fin$ for which almost every element of X is a good point.*

If we assume *approachability*, i.e., that the entire $\aleph_{\omega+1}$ belongs to $I[\aleph_{\omega+1}]$, then we can do better: we get a *good scale*, i.e., a scale as above for which every element of $\mathrm{Cof}(>\omega)\cap\aleph_{\omega+1}$ is good. The reason is as follows: Theorem 15 gives us a scale $(f_\alpha \mid \alpha<\aleph_{\omega+1})$ which is good at almost every point of uncountable cofinality. Fix a club C such that every limit point of C of uncountable cofinality is good and enumerate C as $(\alpha_i \mid i<\aleph_{\omega+1})$. Now consider the new scale given by $g_i=f_{\alpha_i}$. If i has uncountable cofinality then α_i is good and we may fix an EUB h for $(f_\alpha \mid \alpha<\alpha_i)$ such that $\mathrm{cof}(h(n))=\mathrm{cof}(i)$ for all n; the sequence $(g_j \mid j<i)$ is cofinal in $(f_\alpha \mid \alpha<\alpha_i)$ so h is also an EUB for $(g_j \mid j<i)$ and thus i is a good point for $(g_j \mid j<\aleph_{\omega+1})$.

If we assume $\square_{\aleph_\omega}$, a hypothesis stronger than approachability, we can obtain a *very good scale*. $(f_\alpha \mid \alpha<\aleph_{\omega+1})$ is *very good* iff for every limit $\alpha<\aleph_{\omega+1}$ of uncountable cofinality there is a club $C\subseteq\alpha$ such that for some n, $(f_\alpha(m) \mid \alpha\in C)$ is strictly increasing for all $m\geq n$. If $(C_\alpha \mid \alpha<\aleph_{\omega+1})$ is a square sequence then start with an arbitrary scale $(g_\alpha \mid \alpha<\aleph_{\omega+1})$. We may assume that each C_α has ordertype less than $\aleph_\omega$. Now construct f_α to dominate g_α pointwise and arrange that for limit α, $f_\alpha(m)>f_\beta(m)$ for $\beta\in\mathrm{Lim}\,(C_\alpha)$ and for m such that $\mathrm{ot}(C_\alpha)<\aleph_m$. A similar construction works assuming only $\square_{\aleph_\omega,\aleph_n}$ for some finite n.

Weak square, i.e. the principle $\square_{\aleph_\omega,\aleph_\omega}$, is sufficient to obtain a *better scale*, a notion between good and very good. A scale $(f_\alpha \mid \alpha<\aleph_{\omega+1})$ is *better* iff for limit $\alpha<\aleph_{\omega+1}$ thre is a club $C\subseteq\alpha$ such that for every $\beta\in C$ there is an m such that $f_\gamma(n)<f_\beta(n)$ for all $\gamma\in C\cap\beta$ and $n\geq m$. Suppose that $(\mathcal{C}_\alpha \mid \alpha<\aleph_{\omega+1})$ is a weak square sequence and assume $\mathrm{ot}(C)<\aleph_\omega$ for each club C in some $\mathcal{C}_\alpha$. At stage α we form f_C for each $C\in\mathcal{C}_\alpha$ by defining $f_C(m)=\sup\{f_\beta(m)\mid\beta\in C\}$ when $\mathrm{ot}(C)$ is less than $\aleph_m$. Then choose f_α to dominate mod finite all f_C, $C\in\mathcal{C}_\alpha$.

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We show that MM (Martin's Maximum) implies that there is no good scale. So MM refutes the approachability property $\aleph_{\omega+1} \in I[\aleph_{\omega+1}]$, which can be viewed as a very weak square principle.

Theorem 16 *MM implies that there is no scale which is good at almost every point of cofinality* $\aleph_1$.

Proof. We define a Namba-like forcing P which adds a new function to $\prod_n \aleph_n$. A condition is a tree T such that each $t \in T$ is a finite sequence with $t(n) \in \aleph_{n+2} \cap \mathrm{Cof}(\omega)$ for $n <$ length (t). A condition is required to have a "stem" s such that every $t \in T$ is comparable with s and if t extends s then $\{\alpha \mid t * \alpha \in T\}$ is stationary in $\aleph_{\text{length }(t)+2}$. P is ordered by inclusion.

For T_1, T_2 in P, we say that T_1 is a *direct extension* of T_2, and write $T_1 \leq^* T_2$, iff T_1 extends T_2 and has the same stem. Clearly if S extends T then $S \leq^* T_s$ where s is the stem of S and T_s consists of the elements of T comparable with s.

Claim 1. If τ is a name for a countable ordinal and S is a condition then S has a direct extension which evaluates τ.

Proof. Let s be the stem of S, of length n. If the claim fails, then for stationarily many $\alpha \in \aleph_{n+2}$ we have $s * \alpha$ in S and no direct extension of $S_{s*\alpha}$ evaluating τ (otherwise by $\aleph_2$-completness, S would have a direct extension evaluating τ). Repeating this argument, we may work up the tree to build a direct extension U of S such that for every $t \in U$ extending s, there is no direct extension of U_t evaluating τ. But this is impossible as some extension of U evaluates τ and is a direct extension of U_t for some t extending s. □

Similarly, if S has a stem of length n and τ is a name for an element of $\aleph_{n+1}$ then there is a direct extension of S which evaluates τ.

Let $f \in \prod_n \aleph_{n+2}$ be the generic function added by P and let $\dot{f}$ be a P-name for it.

Claim 2. If S forces $\dot{g} < \dot{f}$ then there is a direct extension T of S and a function h in $\prod_n \aleph_{n+2} \cap V$ such that T forces $\dot{g} < h$.

Proof. For simplicity of notation, assume that the stem of S is empty. For each α such that $\langle\alpha\rangle$ belongs to S choose a direct extension of $S_{\langle\alpha\rangle}$ evaluating $\dot{g}(0)$. By Fodor we may thin out to obtain a direct extension of S evaluating $\dot{g}(0)$. Working up the tree level by level we build a direct extension T of S such that for every $t \in T$, T_t evaluates $\dot{g}$ (length (t)) $< \aleph_{\text{length }(t)+2}$. As there are only $\aleph_{n+1}$ nodes t of length n, it follows that T produces a function in the ground model bounding $\dot{g}$. $\square$

Claim 3. P is stationary-preserving.

16.Vorlesung

Proof of Claim 3. Fix A a stationary subset of ω_1, $\dot{C}$ a name for a club and S a condition. We find $U \leq S$ and $\delta \in A$ such that U forces $\delta \in \dot{C}$. To ease notation, assume that S is the trivial condition.

We can assign to each $\langle\alpha\rangle$ in S an ordinal $\gamma_{\langle\alpha\rangle}$ so that for each $i < \omega_1$ there are stationarily many $\alpha \in \omega_2 \cap \mathrm{Cof}(\omega)$ such that $\gamma_{\langle\alpha\rangle} = i$. Now using Claim 1, find an extension S' of S with the same first level as S such that $S'_{\langle\alpha\rangle}$ evaluates $\min(\dot{C} \setminus \gamma_{\langle\alpha\rangle})$ to some ordinal $\delta_{\langle\alpha\rangle}$.

Repeating this, we thin out level by level to obtain a direct extension T of S together with an assignment of γ_t and δ_t to t in T such that T_t forces that $\min(\dot{C} \setminus \gamma_t) = \delta_t$, and for every t and $i < \omega_1$ there are stationarily many α with $\gamma_{t*\alpha} = i$.

Now for each countable δ consider the game G_δ in which the players build a branch through T: At round n player I chooses a nonstationary set $A_n \subseteq \aleph_{n+2}$ and a countable ordinal $\beta_n < \delta$; player II responds with $\alpha_n \notin A_n$. Player II loses immediately if $(\alpha_0, \ldots, \alpha_n) \notin T$ or $\gamma_{\langle\alpha_0,\ldots,\alpha_n\rangle} \leq \beta_n$ or $\delta_{\langle\alpha_0,\ldots,\alpha_n\rangle} \geq \delta$.

The game G_δ is open so is determined. Let X be the set of δ such that I has a winning strategy in G_δ and fix such strategies τ_δ, $\delta \in X$.

We claim that X is nonstationary. If not, choose a countable N elementary in some large $H(\theta)$ such that N contains all relevant parameters and $\delta = N \cap \omega_1$ belongs to X. We will describe a run of the game G_δ in which player I plays according to his (supposedly winning) strategy τ_δ while player II plays ordinals from N and never loses.

This run is described as follows: If II has played $\alpha_0, \ldots, \alpha_{k-1}$ then let β_k be the ordinal part of the strategy τ_δ's response. We consider the union over all $\gamma \in X$ of the nonstationary sets provided by the various strategies τ_γ in response to $\alpha_0, \ldots, \alpha_{k-1}$; this union is a nonstationary subset of $\aleph_{k+2}$ lying in N, and as β_k belongs to N we may choose a suitable α_k in N. The key point is that the map $s \mapsto \delta_s$ belongs to N so the requirement $\delta_{\langle \alpha_0, \ldots, \alpha_n \rangle} < \delta$ is guaranteed.

Now choose δ in the stationary set A so that player II has a winning strategy ρ in the game G_δ. Choosing $(\delta_n \mid n \in \omega)$ to be increasing and cofinal in δ we use ρ to thin out T to $U \leq^* T$ such that for all $u \in U$ of length n we have $\delta_n < \gamma_u \leq \delta_u < \delta$. Then U forces that δ is a limit point of $\dot{C}$ and therefore belongs to $\dot{C}$. $\square$

Now we show that MM kills good scales. Let λ denote $\aleph_\omega$. Suppose that $(g_\alpha \mid \alpha < \lambda^+)$ were a good scale in $\prod_{n \in A} \aleph_n$ (A infinite). For simplicity we assume that A equals ω. We apply MM to the poset $Q = P * \mathrm{Coll}\,(\omega_1, \lambda^+)$, whose second factor is the ω-closed forcing that collapses λ^+ to ω_1. Then Q is stationary-preserving and forces:

1. $\mathrm{cof}((\lambda^+)^V) = \mathrm{card}((\lambda^+)^V) = \omega_1$.
2. There is a function $h \in \prod_{n \in \omega} (\aleph_n \cap \mathrm{Cof}(\omega)^V)$ which is an exact upper bound for $(g_\alpha \mid \alpha < (\lambda^+)^V)$.

In a Q-generic extension we may choose for each n a countable set $S_n \in V$ which is cofinal in $h(n)$. We may also choose an increasing and cofinal sequence $(\alpha_i \mid i < \omega_1)$ in $(\lambda^+)^V$ such that for each i there is $H_i \in \prod_n S_n$ with $g_{\alpha_i} <^* H_i <^* g_{\alpha_{i+1}}$.

Now go back to V. Let C be a club in λ^+. Applying MM we may obtain countable sets $S_n^* \subseteq \aleph_n$ and increasing $(\alpha_i^* \mid i < \omega_1)$ with $\delta = \sup_i \alpha_i^*$ in C, and functions $H_i^* \in \prod_n S_n^*$ such that $g_{\alpha_i^*} <^* H_i^* <^* g_{\alpha_{i+1}^*}$ for each i. But then δ cannot be a good point, i.e., there can be no unbounded $A \subseteq \delta$ and $m < \omega$ such that the sequence $(g_\alpha(n) \mid \alpha \in A)$ is strictly increasing for $n \geq m$: For, given such an A and m, we may find an increasing sequence $(\beta_i \mid i < \omega_1)$ from A and corresponding $(H_{j_i}^* \mid < \omega_1)$ from $\prod_n S_n^*$ such that $g_{\beta_i} <^* H_{j_i}^* <^* g_{\beta_{i+1}}$. But then we can fix $n > m$ and $B \subseteq \omega_1$ unbounded such that $g_{\beta_i}(n) < H_{j_i}^*(n) < g_{\beta_{i+1}}(n)$ for all i in B. This contradicts the fact that $H_{j_i}^*(n)$ belongs to the countable set S_n^* for each i!

So we have shown that there are stationarily many bad points δ of cofinality ω_1, as desired. $\Box$