Trees, Reflection and Approachability

1.-2.Vorlesungen

Three combinatorial properties that have played an influential role in modern set theory are the following. Let κ denote a regular, uncountable cardinal (normally greater than ω_1).

TP: The Tree Property. Any κ-tree has a κ-branch.

RP: The Reflection Property. Certain stationary subsets S of κ reflect to an ordinal α with a certain uncountable cofinality (i.e. $S \cap \alpha$ is stationary in α).

AP: The Approachability Property. Certain stationary subsets S of κ are *approachable*, i.e. there is an enumeration $(a_i \mid i < \kappa)$ of (some of the) bounded subsets of κ such that for almost all ordinals α in S, there is a club in α of ordertype $\mathrm{cof}(\alpha)$ all of whose proper initial segments are enumerated as a_i for some $i < \alpha$.

In the *AP*, "almost all ordinals in S" means all ordinals in $S \cap C$ for some club C.

The tree property is a natural generalisation of König's Tree Lemma: An infinite, finitely branching tree of height ω has an infinite branch. The Reflection Property was first studied by Jensen, who proved the equivalence of a form of it with weak compactness, assuming $V = L$. The Approachability Property was introduced by Shelah in his study of when sufficiently-closed forcing notions preserve stationarity.

In this course I'll discuss these properties individually and then look at how they interact. We'll arrive at the *Eightfold Way Theorem*, asserting that all eight Boolean combinations of TP, RP and AP are possible at double successor cardinals. The proof of this makes use of variants of Mitchell's forcing to make TP hold, Kunen's trick to kill TP by adding a homogeneous Suslin tree and resurrect it by forcing with that tree, and the forcing to add a nonreflecting stationary set. Also to achieve the result at the double successor of a singular cardinal we use a "Prikry-ised" version of Mitchell's forcing to ensure the desired properties at the double successor of a measurable cardinal that has been made singular with Prikry forcing.

I'll now state the Eightfold Way result more precisely. Let $\mu = \kappa^+$ be a successor cardinal and simply write:

TP: μ^+ has the tree property.
RP: Every stationary subset of $\mu^+ \cap \text{cof}(< \mu)$ reflects to an ordinal of cofinality μ.
AP: The entire set μ^+ is approachable.

Theorem 1 *Suppose $\mu = \kappa^+$ where $\kappa^{<\kappa} = \kappa$. Then assuming a weak compact above μ, each Boolean combination of TP, RP, AP holds in a generic extension in which cardinals up to and including μ are preserved.*

Theorem 2 *If in the previous theorem we add the hypothesis that κ is measurable, then we can in addition require that κ have cofinality ω in the generic extension.*

These theorems are not the end of the story. For example, one should prove that the Eightfold Way can also be achieved at the double successor of small singular cardinals like $\aleph_{\omega+2}$. And the case of successors (as opposed to double successors) of singular strong limit cardinals is not fully understood; in particular it is not known if it is possible to have both the TP and the AP simultaneously at such a cardinal (or both the TP and the RP simultaneously at $\aleph_{\omega+1}$).

The Tree Property

A *tree* is a partial ordering $T = (T, \leq_T)$ with the property that for each $t \in T$, $T_t =$ the set of $\leq_T$-predecessors of t is well-ordered by $\leq_T$. The *α-th level of T* is $T_\alpha = \{t \in T \mid T_t \text{ is well-ordered by } \leq_T \text{ with ordertype } \alpha\}$. The *height* of T is the supremum of $\{\alpha + 1 \mid T_\alpha \text{ is nonempty}\}$.

Let κ be an infinite regular cardinal. T is a *κ-tree* iff T has height κ and for $\alpha < \kappa$, T_α has cardinality less than κ. A κ-tree T is *κ-Aronszajn* iff it has no κ-branch, i.e., there is no subset of T well-ordered by $\leq_T$ with ordertype κ.

κ has the *tree property (TP)* iff there is no κ-Aronszajn tree. $\aleph_0$ has the TP as by König's Lemma, a finitely-branching tree of height ω must have an infinite branch. Here is a brief summary of some of the many results about the TP:

(Aronszajn) ω_1 does not have the TP.
(Specker) More generally, if $\tau^{<\tau} = \tau$ then the TP fails at τ^+.
(Erdös-Tarski) A strongly inaccessible cardinal has the TP iff it is weak compact.
(Mitchell) The TP can hold at ω_2. More generally, if $\tau^{<\tau} = \tau$ and there is a weak compact above τ one can force the TP at τ^{++}.
(Abraham) Given a supercompact and a weak compact above it one can force the TP at both ω_2 and ω_3.
(Cummings-Foreman) Given infinitely-many supercompacts one can force TP at all of the $\aleph_{2+n}$'s, $n < \omega$.
(Magidor-Shelah) The TP holds at the successor of the supremum of ω-many strong compacts.
(Magidor-Shelah) Assuming hugeness, the TP can be forced at $\aleph_{\omega+1}$ with $\aleph_\omega$ strong limit.
(Sinapova, improving Magidor-Shelah) Assuming infinitely-many supercompacts one can force the TP at $\aleph_{\omega+1}$ with $\aleph_\omega$ strong limit.
(Neeman) Assume infinitely-many supercompacts one can force the TP at every $\aleph_{2+n}$, $n < \omega$, and also at $\aleph_{\omega+1}$ with $\aleph_\omega$ strong limit.
(Friedman-Dobrinen) Assuming a weak compact hypermeasurable one can force the TP at the double successor of a measurable.
(Friedman-Halilovic) Assuming a weak compact hypermeasurable one can force the TP at $\aleph_{\omega+2}$ with $\aleph_\omega$ strong limit.
(Gitik) One can force the TP at $\aleph_{\omega+2}$ with $\aleph_\omega$ strong limit from optimal assumptions (weaker than a weak compact hypermeasurable).
(Fontanella-Friedman) Assuming infinitely-many supercompacts with a weak compact above one can force the TP to hold at both $\aleph_{\omega+1}$ and $\aleph_{\omega+2}$ simultaneously ($\aleph_\omega$ is *not* strong limit).
(Friedman-Honzik) Assuming a strong cardinal with a measurable λ of Mitchell order λ^{++} above it, one can force the TP at each $\aleph_{2+2n}$, $n < \omega$ together with $\aleph_\omega$ strong limit and $2^{\aleph_\omega} = \aleph_{\omega+2}$.
(Golshani-Hayut) Assuming a Woodin for supercompactness, for any $\beta < \omega_1$ one can force the TP at all of the $\aleph_{\omega\cdot\alpha+1}$'s, $\alpha < \beta$ simultaneously.

So the story goes on. The natural goal was suggested by Magidor: Get the TP to hold simultaneously at all regular cardinals greater than $\aleph_1$.

Now back to earth. We prove some basic facts about the TP.

Theorem 3 *There is an ω_1-Aronszajn tree.*

Proof. We construct an ω_1-tree T whose elements are bounded, increasing, well-ordered sequences of rational numbers, ordered by end-extension. It is clear that such a tree has no ω_1-branch, as that would give an increasing sequence of rationals of length ω_1, which is impossible.

We construct the α-th level T_α of T by induction on $\alpha < \omega_1$. We inductively maintain the following property:

$(*)$ T_α is countable and if x belongs to T_β, $\beta < \alpha$ and q is a rational greater than $\sup(x)$ then x is extended by some $y \in T_\alpha$ with $\sup(y) < q$.

T_0 consists only of the empty sequence (we take $\sup(\emptyset)$ to be $-\infty$). To define $T_{\alpha+1}$ from T_α, simply extend each $x \in T_\alpha$ with each rational $q > \sup(x)$. It is clear that property $(*)$ is preserved. If α is a limit ordinal then for each x in some T_β, $\beta < \alpha$, and each rational $q > \sup(x)$, we extend x to $x_1 \subseteq x_2 \subseteq \cdots$ so that $\sup(x_n) < q$ for each n and the levels of the x_n's are cofinal in α; then put the resulting sequence $\bigcup_n x_n$ into T_α. It follows that T_α is countable and that for each $x \in \bigcup_{\beta<\alpha} T_\beta$ and $q > \sup(x)$, x has an extension y in T_α with $\sup(y) \leq q$; by choosing q' between q and $\sup(x)$ we can in fact guarantee $\sup(y) < q$, which gives $(*)$ for α. □

The previous proof generalises. For an infinite cardinal λ, let Q_λ be the set of eventually-zero λ sequences of 0's and 1's, ordered lexicographically. Then λ can be order-preservingly embedded into any open interval of Q_λ. Now the cardinality of Q_λ is $\lambda^{<\lambda}$; if this is λ, then we can replace the rationals by Q_λ in the previous proof, obtaining:

Theorem 4 *If $\lambda^{<\lambda} = \lambda$ then there is a λ^+-Aronszajn tree. In particular if GCH holds and λ is regular, there is a λ^+-Aronszajn tree.*

The consistency strength of the existence of an uncountable κ with the tree property is that of a weak compact ($= \Pi^1_1$ reflecting) cardinal:

Theorem 5 *(1) A strongly inaccessible cardinal has the tree property iff it is weak compact.*
(2) (Jensen) If κ has the tree property then κ is weak compact in L. In particular, weak compactness and the tree property are the same in L.

Can ω_2 have the tree property? By the above we will need to use a weak compact cardinal and kill CH to obtain the consistency of this.

Mitchell's Forcing

Theorem 6 *(Mitchell) Suppose that κ is weak compact. Then in some forcing extension, $\kappa = \omega_2$, $2^\omega = \omega_2$ and ω_2 has the tree property.*

Proof. Mitchell's forcing first adds κ-many Cohen reals (with finite support) and then "slowly" collapses each ordinal less than κ to ω_1, using an iteration of Lévy collapses but where the α-th Lévy collapse only uses conditions provided by the first α-many Cohen reals.

More precisely, let $\mathbb{P}$ denote $\mathrm{Add}(\omega, \kappa)$, the forcing for adding κ-many Cohen reals with finite support and for each $\alpha < \kappa$, let $\mathbb{P}_\alpha$ denote the subforcing $\mathrm{Add}(\omega, \alpha)$. Also let $\mathbb{R}_\alpha$ denote the forcing $\mathrm{Coll}(\omega_1, \alpha)$ of the model $V[\mathbb{P}_\alpha]$.

Then the desired forcing $\mathbb{Q}$ consists of pairs (p, f) where $p \in \mathbb{P}$, f is a partial function on κ with countable domain and for each $\alpha \in \mathrm{Dom}(f)$, $f(\alpha)$ is a $\mathbb{P}_\alpha$-name for a condition in $\mathbb{R}_\alpha$. Extension is defined in the natural way: (q, g) extends (p, f) iff q extends p and for all $\alpha \in \mathrm{Dom}(f)$, $q|\alpha$ forces that $g(\alpha)$ extends $f(\alpha)$.

Thus for each $\alpha < \kappa$, a $\mathbb{Q}$-generic adds a collapse of α to ω_1 using conditions from the model $V[\mathbb{P}_\alpha]$; it is a kind of "diagonal" Lévy collapse of each ordinal less than κ to ω_1. $\mathbb{Q}$ has size κ so cardinals above κ are preserved.

3.-4.Vorlesungen

Also κ itself is preserved:

Lemma 7 *$\mathbb{Q}$ is κ-cc.*

Proof. Suppose that A is a maximal antichain in $\mathbb{Q}$. Using the inaccessibility of κ, let M be an ω-closed sufficiently elementary submodel of V containing $\mathbb{Q}, A$ as elements which is transitive below κ, and let α be $M \cap \kappa$. Then α has uncountable cofinality as it is regular in M and M is ω-closed. And $A \cap M$ is a maximal antichain in $\mathbb{Q} \cap M$, by elementarity. But then $A \cap M$ is in fact a maximal antichain in $\mathbb{Q}$ and therefore equals A: If q belongs to $\mathbb{Q}$ then $q|\alpha$ belongs to M by the ω-closure of M and therefore $q|\alpha$ is compatible with some element a of $A \cap M$; then q is also compatible with a. So we conclude that maximal antichains have size less than κ. $\square$

More interesting is the fact that $\mathbb{Q}$ preserves ω_1 (it is neither ccc nor countably closed). To prove this it is helpful to use *projections* and *term forcing*.

Definition 8 *Suppose that $\mathbb{P}$ and $\mathbb{Q}$ are arbitrary partial orders and $\pi : \mathbb{Q} \to \mathbb{P}$. Then π is a* projection *if it is order-preserving and for all $q \in \mathbb{Q}$ and $p \leq \pi(q)$ there is $q^* \leq q$ such that $\pi(q^*) \leq p$.*

Lemma 9 *Suppose that $\pi : \mathbb{Q} \to \mathbb{P}$ is a projection and H is $\mathbb{Q}$-generic. Then $\pi[H]$ generates a $\mathbb{P}$-generic.*

Proof. Suppose that D is open dense on $\mathbb{P}$. Then $\pi^{-1}[D]$ is dense on $\mathbb{Q}$, as given $q \in \mathbb{Q}$ we can choose p in D below $\pi(q)$ and then $q^* \leq q$ with $\pi(q^*) \in D$. It follows that $\pi[H]$ meets D and as π is order-preserving, any two conditions in $\pi[H]$ are compatible. So $\pi[H]$ generates a $\mathbb{P}$-generic. $\square$

Now to apply this to the current context let $\mathbb{R}$ consist of all conditions (p, f) in $\mathbb{Q}$ where $p = 1_{\mathbb{P}}$ is the trivial condition of $\mathbb{P}$ and order $\mathbb{R}$ as a suborder of $\mathbb{Q}$. Define the map $\pi : \mathbb{P} \times \mathbb{R} \to \mathbb{Q}$ by sending $(p, (1_{\mathbb{P}}, f))$ to (p, f).

Lemma 10 *π is a projection from $\mathbb{P} \times \mathbb{R}$ to $\mathbb{Q}$.*

Proof. π is clearly order-preserving. Suppose that $(p^*, f^*) \leq \pi(p, (1_{\mathbb{P}}, f)) = (p, f)$. Define f^{**} to have the same domain as f^* and such that for $\alpha \in \text{Dom}(f^*)$, $f^{**}(\alpha)$ is a $\mathbb{P}_\alpha$-name which equals $f^*(\alpha)$ if this extends $f(\alpha)$ and equals $f(\alpha)$ otherwise. Then $(p^*, (1_{\mathbb{P}}, f^{**}))$ is a condition extending $(p, (1_{\mathbb{P}}, f))$ which projects to $(p^*, f^{**}) \leq (p^*, f^*)$. $\square$

Therefore by Lemma 9, a generic for $\mathbb{P} \times \mathbb{R}$ yields a generic for $\mathbb{Q}$. So to show that $\mathbb{Q}$ preserves ω_1 it suffices to show that $\mathbb{P} \times \mathbb{R}$ does. But recall:

Lemma 11 *(Easton's Lemma) If $\mathbb{P}$ is ccc and $\mathbb{R}$ is countably closed then:*
(a) $\mathbb{R}$ is countably distributive in $V[\mathbb{P}]$, i.e. any ω-sequence of ordinals in $V[\mathbb{P} \times \mathbb{R}]$ belongs to $V[\mathbb{P}]$.
(b) $\mathbb{P}$ is ccc in $V[\mathbb{R}]$.

Now in our context, the forcings $\mathbb{P}$ and $\mathbb{R}$ that we are using are indeed ccc and countably closed, respectively. It then follows either from (a) or from (b) that ω_1 is preserved.

Proof of Easton's Lemma. (a) Suppose that $\dot{f}$ is a $\mathbb{P} \times \mathbb{R}$-name for a function from ω into Ord and choose (p_0, r_0) in $\mathbb{P} \times \mathbb{R}$ deciding a value for $\dot{f}(0)$. Then choose (p_1, r_1) with $r_1 \leq r_0$ deciding a different value for $\dot{f}(0)$ (note that p_0, p_1 are incompatible). Continue doing this through countable ordinal stages, noting that as $\mathbb{R}$ is countably closed one can take a lower bound r_λ of the r_i's, $i < \lambda$, at countable limit stages λ. The induction must stop at some countable ordinal stage with some condition r^0, as the p_i's form an antichain. Then extend r^0 to r^1 in the same way to decide possibilities for $\dot{f}(1)$. After ω steps one arrives at a condtion r^* in $\mathbb{R}$ which forces $\dot{f}$ to have a $\mathbb{P}$-name. So any ω-sequence of ordinals in $V[\mathbb{P} \times \mathbb{R}]$ belongs to $V[\mathbb{P}]$.
(b) Suppose that $(\dot{p}_i \mid i < \omega_1)$ were a $\mathbb{R}$-name for an antichain on $\mathbb{P}$ in $V[\mathbb{R}]$. Then build a descending ω_1-sequence $(r_i \mid i < \omega_1)$ of conditions in $\mathbb{R}$ such tha r_{i+1} forces a value p_i for $\dot{p}_i$. Then the p_i's form an uncountable antichain in V, contradicting the fact that $\mathbb{P}$ is ccc in V. $\square$

Another useful fact derivable from the projection analysis is that the forcing $\mathbb{Q}$ does not add new branches to certain trees.

Lemma 12 *Suppose that T is a tree whose height has uncountable cofinality.*
(a) If $\mathbb{P} \times \mathbb{P}$ is ccc then $\mathbb{P}$ does not add a new cofinal branch through T.
(b) If $\mathbb{P}$ is $\omega -$ closed and the levels of T have size less than 2^ω then $\mathbb{P}$ does not add a new cofinal branch through T.

Proof. (a) We show that for any uncountable regular κ, if $\mathbb{P}$ adds a subset x of κ such that $x|\alpha$ is in the ground model for all $\alpha < \kappa$ but x itself is not in the ground model then $\mathbb{P} \times \mathbb{P}$ is not ccc. It suffices to show that $\mathbb{P}$ is not ccc in $V[G]$ where G is $\mathbb{P}$-generic.

Choose a sequence of conditions $(p_i \mid i < \kappa)$ in G and an increasing sequence $(\alpha_i \mid i < \kappa)$ of ordinals less than κ such that p_i fixes $x|\alpha_i$ (to be a specific element of V) but does not fix $x|\alpha_{i+1}$. Then choose q_{i+1} extending p_i to disagree with p_i about $x|\alpha_{i+1}$. But then the q_{i+1}'s form an antichain as any condition extending q_{i+1} disagrees with p_{i+1} (and therefore with p_j for all $j > i$) about x_{i+1} and therefore cannot extend q_{j+1} for any $j > i$, as q_{j+1} extends p_j.

Now note that a new branch through T would yield a subset of $\kappa =$ the cofinality of the height of T not in the ground model with all proper initial segments in the ground model, contradicting the above.

5.-6.Vorlesungen

(b) Suppose that $\dot{b}$ were a $\mathbb{P}$-name for a new branch through T. Build a binary tree $(p_s \mid s \in 2^{<\omega})$ of conditions in $\mathbb{P}$ such that p_t extends p_s when t extends s and p_{s*0}, p_{s*1} force different values for $\dot{b}$ at some level α_s. Arrange that the sup α of the α_s's along any infinite branch through $2^{<\omega}$ is independent of the choice of the infinite branch. But then as $\mathbb{P}$ is ω-closed, we get 2^ω many possible nodes of T on level α, contrary to hypothesis. $\Box$

The previous "No New Branches Lemma" will be important for the proof that the tree property holds in the Mitchell forcing extension.

We have that in $V[G]$, where G is P-generic, κ equals ω_2 and there are ω_2-many reals. For the sake of contradiction, suppose that T were a κ-Aronszajn tree in $V[G]$. Let $\dot{T}$ be a name for T and assume that (the trivial condition of) $\mathbb{Q}$ forces that $\dot{T}$ is κ-Aronszajn. As κ is weak compact it is Π^1_1 reflecting, and therefore there is an $\alpha < \kappa$ such that $\mathbb{Q}_\alpha = (\mathbb{Q}$ restricted to $\alpha)$ forces that $\dot{T}_\alpha = (\dot{T}$ restricted to $\alpha)$ is a name for an α-Aronszajn tree in $V[G_\alpha]$, where G_α denotes the $\mathbb{Q}_\alpha$-generic. In particular $T_\alpha = \dot{T}_\alpha^{G_\alpha}$ is an α-Aronszajn tree in $V[G_\alpha]$ and therefore has no cofinal branch in that model.

But notice the following: The tree T_α surely does have a cofinal branch in the larger model $V[G]$, as T_α is an initial segment of the tree $T = \dot{T}^G$ and so we get a cofinal branch through T_α simply by choosing any node of T at level α and considering the cofinal branch through T_α consisting of the T-predecessors of that node. In other words, if we factor $V[G]$ as $V[G_\alpha][G/G_\alpha]$ then the quotient-generic G/G_α is responsible for adding a cofinal branch through the tree T_α.

We next show that the extension from $V[G_\alpha]$ to $V[G] = V[G_\alpha][G/G_\alpha]$ can be achieved by a forcing which looks just like $\mathbb{Q}$ and therefore like $\mathbb{Q}$ is the projection of a Cohen product times an ω-closed "term forcing". This will then imply via the No New Branches Lemma that in fact G/G_α could not be blamed for adding a cofinal branch through T_α, the desired contradiction.

Recall that $\mathbb{Q}_\alpha$ consists of conditions of the form (p, f) where p belongs to the Cohen product $\mathbb{P}_\alpha = \mathrm{Add}(\omega, \alpha)$ and f is a countable function with domain contained in α which to each β in its domain assigns a $\mathbb{P}_\beta$-name $f(\beta)$ for a condition in the $\mathrm{Coll}(\omega_1, \alpha)$ of $V[\mathbb{P}_\alpha]$. In particular G_α, the generic for $\mathbb{Q}_\alpha$, adds a generic for $\mathbb{P}_\alpha = \mathrm{Add}(\omega, \alpha)$, which we denote by $G(\mathbb{P}_\alpha)$.

We define an embedding π of $\mathbb{Q}$ into a two-step iteration $\mathbb{Q}_\alpha * \mathbb{S}$. Working in $V[G_\alpha]$, $\mathbb{S}$ consists of pairs (p, f) where p belongs to $\mathrm{Add}(\omega, [\alpha, \kappa))$ and f is a countable function with domain contained in $[\alpha, \kappa)$ which assigns to each β in its domain a $\mathbb{P}_{[\alpha,\beta)}$-name for a condition in the $\mathrm{Coll}(\omega_1, \beta)$ of $V[G_\alpha][\mathbb{P}_{[\alpha,\beta)}]$. The embedding π is defined by sending (p, f) in $\mathbb{Q}$ to the pair whose first entry is $(p|\alpha, f|\alpha)$ and whose second entry is $(p|[\alpha, \kappa), f^*)$ where f^* has domain $\mathrm{Dom}(f) \cap [\alpha, \kappa)$ and for β in its domain, $f^*(\beta)$ is the translation of the $\mathbb{P}_\beta$-name $f(\beta)$ into a $\mathbb{Q}_\alpha$-name for a $\mathbb{P}_{[\alpha,\beta)}$-name.

Lemma 13 *π is an order-preserving embedding with dense range.*

Proof. Clearly π is order-preserving. To see that its range is dense in $\mathbb{Q}_\alpha * \mathbb{S}$, let $((p_0, f_0), (p_1, f_1))$ be a condition in the latter forcing. Let p be the union of p_0 and p_1; then p belongs to $\mathbb{P} = \mathrm{Add}(\omega, \kappa)$. The domain of f_1 is forced by $\mathbb{Q}_\alpha$ to be countable; but as any countable set in $V[\mathbb{Q}_\alpha]$ is covered by a countable set in V, we may assume that $\mathrm{Dom}(f_1)$ is a countable set in V. And for each β in $\mathrm{Dom}(f_1)$, $f_1(\beta)$ is a $\mathbb{Q}_\alpha$-name for a $\mathbb{P}_{[\alpha,\beta)}$-name for a condition in the $\mathrm{Coll}(\omega_1, \beta)$ of $V[G_\alpha][\mathbb{P}_{[\alpha,\beta)}]$. But as the $\mathrm{Coll}(\omega_1, \beta)$ of $V[G_\alpha * \mathbb{P}_{[\alpha,\beta)}]$ is the same as the $\mathrm{Coll}(\omega_1, \beta)$ of $V[\mathbb{P}_\alpha * \mathbb{P}_{[\alpha,\beta)}] = V[\mathbb{P}_\beta]$, we may regard $f_1(\beta)$ as a $\mathbb{P}_\beta$-name for a condition in the $\mathrm{Coll}(\omega_1, \beta)$ of $V[\mathbb{P}_\beta]$. Now define f with domain $\mathrm{Dom}(f_0) \cup \mathrm{Dom}(f_1)$ by taking $f(\beta)$ to be $f_0(\beta)$ for $\beta < \alpha$ and to be $f_1(\beta)$ for $\beta \in \mathrm{Dom}(f_1)$, regarded as a $\mathbb{P}_\beta$-name. Then with these definitions of p and f, $\pi((p, f))$ extends $((p_0, f_0), (p_1, f_1))$, as desired. □

It is clear that $\mathbb{S}$ is a forcing in $V[G_\alpha]$ which looks in $V[G_\alpha]$ just as $\mathbb{Q}$ looks in V. Therefore in $V[G_\alpha]$ we can write $\mathbb{S}$ as the projection of a product $\mathbb{P}^* \times \mathbb{R}^*$ where $\mathbb{P}^* = \mathrm{Add}(\omega, [\alpha, \kappa))$ and $\mathbb{R}^*$ is countably closed. Note that $2^\omega = \omega_2 = \alpha$ in $V[G_\alpha]$. It then follows by the No New Branches Lemma (b) that $\mathbb{R}^*$ could not add a cofinal branch through the hypothesised α-Aronszajn tree T_α. Also, $\mathbb{P}^* \times \mathbb{P}^*$ is ccc in $V[G_\alpha]$ and therefore also in $V[G_\alpha][\mathbb{R}^*]$, by Easton's Lemma (b). It then follows by the No New Branches Lemma (a) that $\mathbb{P}^*$ could not add a cofinal branch through T_α over $V[G_\alpha][\mathbb{R}^*]$, finishing the proof of Mitchell's theorem. □

The previous proof generalises to show that if $\lambda < \mu < \kappa$ are regular cardinals in a model of GCH and κ is weak compact, then in some forcing extension, cardinals up to μ are preserved and $\kappa = \mu^+ = 2^\lambda$ has the TP (tree property). For example, if we take λ to be any regular cardinal and μ to be λ^+ then we get the TP at λ^{++}. Or if we take λ to be ω and μ to be $\aleph_{\omega+1}$ then we get the TP at $\aleph_{\omega+2} = 2^\omega$. Getting the TP at $\aleph_{\omega+2}$ with $\aleph_\omega$ strong limit is harder and requires more than a weak compact, as it entails the failure of GCH at a singular strong limit cardinal. Halilović and I got this result using a "weak compact hypermeasurable"; Gitik improved this to the weakest possible large cardinal hypothesis.

Tree Property Resurrection

Kunen introduced a clever way to kill the tree property at a weak compact and then resurrect it with further forcing. Kunen's method will be needed for the proof of the Eightfold Way result.

Theorem 14 *(Kunen) Let κ be weak compact. Then there is a 2-step iteration $\mathbb{Q} * \mathbb{R}$ such that κ is inaccessible but not weak compact in $V[\mathbb{Q}]$ however weak compact in $V[\mathbb{Q} * \mathbb{R}]$.*

Proof. Suppose that κ is weakly compact. Let $\mathbb{P}_\kappa$ be the reverse Easton iteration of length κ which at inaccessible $\alpha < \kappa$ adds an α-Cohen set. Let G_κ be $\mathbb{P}_\kappa$-generic.

Now over $V[G_\kappa]$, consider the following forcing $\mathbb{Q}$ for adding a κ-Suslin tree:

For an ordinal $\alpha < \kappa$, a *tree of height* $\alpha + 1$ or $\alpha + 1$-*tree* is a nonempty subtree T of $2^{\leq \alpha}$ with the property that each node of T of length less than α can be extended to a node of T of length α. If T^* is an $\alpha^* + 1$-tree, $\alpha^* \geq \alpha$, then T^* extends T if T equals $T^* \cap 2^{\leq \alpha}$. $\mathbb{Q}$ consists of all $\alpha + 1$-trees, $\alpha < \kappa$, ordered by extension.

Claim 1. $\mathbb{Q}$ adds a κ-Suslin tree.

Proof. Clearly $\mathbb{Q}$ adds a κ-tree $T_\mathbb{Q}$. Our strategy to prove that $T_\mathbb{Q}$ is κ-Suslin is as follows: Suppose that $T \in \mathbb{Q}$ forces that $\dot{A}$ is a maximal antichain in $T_\mathbb{Q}$. Set $T_0 = T$ and choose a $T_1 \leq T_0$ of height $\alpha_1 + 1$ which forces that each node of T_0 is compatible with some node in $\dot{A} \cap T_1$. Then choose $T_2 \leq T_1$ of height $\alpha_2 + 1$ which forces that each node of T_1 is compatible with some node in $\dot{A} \cap T_2$. Continue for ω steps to get $T_\omega^- =$ the union of the T_n's of height α_ω with the property that any of its nodes s is forced by some T_n to be compatible with some node s^* in $\dot{A} \cap T_\omega^-$. Then define T_ω by choosing for each node s in T_ω^- a cofinal branch $b(s)$ containing s and s^* and taking level α_ω of T_ω to consist of the unions of these branches $b(s)$. Then T_ω forces that every node of $T_\mathbb{Q}$ of height at least α_ω extends a node of $\dot{A}$ on some level $< \aleph_\omega$ and therefore $\dot{A} = \dot{A} \cap T_\omega^-$ has size less than κ.

The above strategy would work if $\mathbb{Q}$ were κ-closed, but unfortunately $\mathbb{Q}$ is not κ-closed, as if $T_0 \geq T_1 \geq \cdots$ is a descending sequence of α_i-trees of limit length $\lambda < \kappa$, the union of the T_i's may have no path of length $\alpha = \sup_{i<\lambda} \alpha_i$ and therefore not be extendible to a condition.

But this problem is easily fixed. If in addition to the T_i's we have for each node $s \in T_i$ of length $< \alpha_i$ a node $s_i(s) \in T_i$ of length α_i extending s such that $i < j \to s_i(s) \subseteq s_j(s)$ for all $s \in T_i$ of length $< \alpha_i$, then the union $s_\lambda(s)$ of the $s_i(s)$'s forms a path through the union of the T_i's of length α_λ and we can extend this union to a condition T_λ whose α_λ-th level consists of the unions $s_\lambda(s)$ of the $s_i(s)$, $i < \lambda$, for s of length $< \alpha_\lambda$.

So if we want to hit open dense sets D_i, $i < \lambda$ below a condition we can do so by building a descending λ-sequence of conditions, hitting D_i at stage $i + 1$ and carrying along assignments $s_i(s) \in T_i$ for $s \in T_i$ of length $< \alpha_i$ as above in order to facilitate the definition of T_β for limit β. It follows that the above strategy works, provided we "help" the construction of our descending sequence of trees by simultaneously selecting branches of length α_i in T_i below each node in $T_i \cap 2^{<\alpha_i}$ to facilitate the existence of a lower bound at limit stages. $\Box$ (*Claim 1*)

7.-8.Vorlesungen

Now in the model $V[G_\kappa][T_\mathbb{Q}]$ define the forcing $\mathbb{R}$ as follows: A condition is a pair (α, s_α) where $\alpha < \kappa$ and s_α is a function which assigns to each node $s \in T_\mathbb{Q}$ of length less than α a node $s_\alpha(s)$ in $T_\mathbb{Q}$ of length α extending s. And (β, s_β) extends (α, s_α) if $\beta \geq \alpha$ and for s in $T_\mathbb{Q}$ of length less than α, $s_\beta(s)$ extends $s_\alpha(s)$.

Claim 2. In the model $V[G_\kappa]$, the 2-step iteration $\mathbb{Q} * \mathbb{R}$ is equivalent to κ-Cohen.

Proof. The forcing $\mathbb{Q} * \mathbb{R}$ has $\{(T, (\alpha, s_\alpha)) \mid T$ has height $\alpha + 1$ and s_α assigns to each node $s \in T$ of length $< \alpha$ an extension of s of length α in $T\}$ as a dense subforcing. But this dense subforcing is κ-closed of size κ and therefore equivalent to κ-Cohen. $\square$ (*Claim 2*)

It follows that $\mathbb{P}_\kappa * \mathbb{Q} * \mathbb{R}$ is equivalent to $\mathbb{P}_\kappa * \kappa$-Cohen. Finally we show:

Claim 3. $\mathbb{P}_\kappa * \kappa$-Cohen preserves the weak compactness of κ.

Proof. $\mathbb{P}_\kappa * \kappa$-Cohen is an iteration $\mathbb{P}_{\kappa+1}$ of length $\kappa + 1$ where at inaccessible α we force with α-Cohen. Let $G_{\kappa+1}$ be generic for $\mathbb{P}_{\kappa+1}$. We must show that in $V[G_{\kappa+1}]$, if M^* is a $< \kappa$-closed transitive model of ZF^- of size κ then there is another such model N^* and an elementary embedding $j^* : M^* \to N^*$ with critical point κ. We may assume that M^* is of the form $M[G_{\kappa+1}]$ where M is a $< \kappa$-closed transitive model of ZF^- of size κ in V (as any M^* is an element of one of this form).

By the weak compactness of κ in V we have an elementary embedding $j : M \to N$ with critical point κ where N is a $< \kappa$-closed transitive model of ZF^- of size κ. We just have to show that j can be lifted to some $j^* : M[G_{\kappa+1}] \to N[H_{j(\kappa)+1}]$ where $H_{j(\kappa)+1}$ is generic over N for the iteration $j(\mathbb{P}_{\kappa+1})$. We can simply take $H_{\kappa+1}$ to equal $G_{\kappa+1}$ and then build $H_{[\kappa+1, j(\kappa)+1]}$ to be any generic for the forcing $j(\mathbb{P})_{[\kappa+1, j(\kappa)+1]}$ over the model $N[H_{\kappa+1}]$ such that $H(j(\kappa))$ extends $H(\kappa) = G(\kappa)$; such a generic exists because the forcing is $< \kappa$-closed and we need only meet κ-many dense sets. $\square$ (*Claim 3.*)

Now let H be $\mathbb{Q}$-generic over $V[G_\kappa]$. Then in $V[G_\kappa][H]$, κ is not weakly compact as there is a κ-Suslin tree. However after further forcing with $\mathbb{R}$, we recover the weak compactness, and therefore the tree property, at κ. $\square$

Can the tree property hold at $\aleph_{\omega+2}$ with $\aleph_\omega$ strong limit? Recall that the tree property cannot hold at κ^+ if $\kappa^{<\kappa} = \kappa$, so for this we need GCH to fail at $\aleph_\omega$ with $\aleph_\omega$ strong limit, a violation of the singular cardinals hypothesis.

Halilović and I were able to get this by starting with a strong enough cardinal κ (“weak compact hypermeasurable”), using a result of Dobrinen and myself to get the tree property at κ^{++} keeping κ measurable, and then applying a "Prikry collapse forcing" to turn κ into $\aleph_\omega$, preserving the tree property at $\kappa^{++} = \aleph_{\omega+2}$ and keeping $\aleph_\omega$ strong limit. This approach used an iteration of κ-Sacks forcing.

Instead I'll give a proof here using an appropriate variant of Mitchell forcing, as this will be useful later when we combine the TP with the RP and AP.

First consider the easier problem of getting the TP at κ^{++} for some singular strong limit cardinal κ of cofinality ω (not necessarily for $\kappa = \aleph_\omega$). The idea is to start with a model V where κ is measurable, $\lambda > \kappa$ is weakly compact and κ remains measurable after forcing with $\mathbb{P} = \text{Add}(\kappa, \lambda)$. Any model in which κ is a "Laver-prepared supercompact" has this property, as in such models the supercompactness of κ is preserved by any further κ-directed closed forcing such as $\mathbb{P}$. (Actually much less than supercompactness is needed, we only need that κ is "weakly compact hypermeasurable", as verified by Halilović and myself using Woodin's "surgery" method.)

Then we apply a *Prikry-ised* version of Mitchell's forcing, defined as follows: Fix a normal measure U on κ in $V[G(\mathbb{P})]$ where $G(\mathbb{P})$ is $\mathbb{P}$-generic over V. For $\alpha < \lambda$ let $G(\mathbb{P}_\alpha)$ denote the restriction of $G(\mathbb{P})$ to $\text{Add}(\kappa, \alpha)$. We say that α is *good* if U_α ($= U$ restricted to the model $V[G(\mathbb{P}_\alpha)]$) belongs to $V[G(\mathbb{P}_\alpha)]$) and is a normal measure there. The set of good $\alpha < \lambda$ forms an unbounded subset of λ which contains all of its limit points of cofinality greater than κ.

Recall that a condition in ordinary Mitchell (to get the TP at κ^{++} for regular κ) was a pair (p, f) where p belongs to $\mathbb{P}$ and f is a function of size at most κ which for inaccessible $\alpha < \lambda$ chooses a $\mathbb{P}_\alpha$-name for a condition in $\text{Coll}(\kappa^+, \alpha)$ of the model $V[\mathbb{P}_\alpha]$. A condition in $\mathbb{Q} =$ Prikry-ised Mitchell is a triple (p, r, f) where (p, r) belongs to $\mathbb{P} * \text{Prikry}(U)$ and f is a function of size at most κ which chooses for good, inaccessible $\alpha < \lambda$ a $\mathbb{P}_\alpha * \text{Prikry}(U_\alpha)$-name for a condition in $\text{Coll}(\kappa^+, \alpha)$.

As before $\mathbb{Q} =$ Prikry-ised Mitchell is λ-cc and therefore λ is not collapsed. Also, in analogy with ordinary Mitchell forcing, $\mathbb{Q}$ is the projection of a product $(\text{Add}(\kappa, \lambda) * \text{Prikry}(U)) \times \mathbb{R}$ where $\mathbb{R}$ is κ^+-closed. As $\text{Add}(\kappa, \lambda) * \text{Prikry}(U)$ has the κ^+-cc and does not add bounded subsets of κ, it follows by Easton's Lemma that cardinals up to and including κ^+ are preserved. Cardinals between κ^+ and λ are collapsed to κ^+ so after forcing with $\mathbb{Q}$, λ becomes κ^{++}. And by virtue of the use of Prikry(U), κ becomes a strong limit cardinal of cofinality ω.

Now we want to verify the TP at κ^{++} in the $\mathbb{Q}$-generic extension. As before, for good inaccessible α, $\mathbb{Q}$ is equivalent to $\mathbb{Q}_\alpha * \mathbb{S}$, where $\mathbb{Q}_\alpha$ is $\mathbb{Q}$ restricted to α and $\mathbb{S}$ is the quotient forcing, which is the projection of the product of the Add*Prikry quotient $(\mathbb{P} * \text{Prikry}(U))/G(\mathbb{Q}_\alpha) = (\mathbb{P} * \text{Prikry}(U))/G(\mathbb{P}_\alpha * \text{Prikry}(U_\alpha))$ with a κ^+-closed forcing. By the earlier No New Branches Lemma, it then suffices to show:

Lemma 15 *Let $\mathbb{T}$ denote the quotient $(\mathbb{P} * \textit{Prikry}(U))/G(\mathbb{P}_\alpha * \textit{Prikry}(U_\alpha))$. Then $\mathbb{T} \times \mathbb{T}$ has the κ^+-cc.*

9.-10.Vorlesungen

Proof. Recall that the above quotient $\mathbb{T}$ consists of those conditions $(q, (t, B))$ in $\mathbb{P}*$Prikry(U) which are compatible with every condition in $G(\mathbb{P}_\alpha*\text{Prikry}(U_\alpha))$.

First we show that $\mathbb{T}$ is κ^+-cc; a similar argument will show that its square is also κ^+-cc. Suppose not and let $(p, (s, A))$ be a condition in $\mathbb{P}_\alpha * \text{Prikry}(U_\alpha)$ which forces that $((q_i, (t_i, B_i)) \mid i < \kappa^+)$ is an antichain in $\mathbb{T}$. Each t_i is an initial segment of s and each $q_i|\alpha$ is extended by p, as otherwise some extension of $(p, (s, A))$ forces $(q_i, (t_i, B_i))$ out of the quotient $\mathbb{T}$. And for each i, as $(q_i, (t_i, B_i))$ is compatible with $(p, (s, A))$ we can choose $(q{-}i \cup p, (s, B_i^*))$ extending both $(q_i, (t_i, B_i))$ and $(p, (s, A))$. However note that $(q_i \cup p, (s, B_i^*))$ need not be forced into $\mathbb{T}$. Let q_i^* denote $q_i \cup p$.

We can thin the set of $(q_i^*, (s, B_i^*))$'s to ensure that the q_i^*'s are pairwise compatible.

Now choose any two distinct i and j and consider $(q_i^*, (s, B_i^*))$ and $(q_j^*, (s, B_j^*))$. These conditions have the obvious lower bound $(q_i^* \cup q_j^*, (s, B_i^* \cap B_j^*))$ in $\mathbb{P} * \text{Prikry}(U)$, however once again we don't know that this condition is forced into $\mathbb{T}$; for this we would like that for each finite x contained in $s \cup A$, some extension of $q_i^* \cup q_j^*$ forces that x is contained in $B_i^* \cap B_j^*$. But consider the partition $f : [\kappa]^{<\omega} \to 2$ defined by $f(x) = 0$ iff $q_0^* \cup q_1^*$ can be extended to force x into $B_i^* \cap B_j^*$; this partition has a measure one homogeneous set A^*. As $B_i^* \cap B_j^*$ has measure one, f takes the value 0 on A^*. Thus the condition $((p,(s,A^*))$ forces that $(q_i^* \cup q_j^*, (s, B_i^* \cap B_j^*))$ belongs to $\mathbb{T}$ and this contradicts the assumption that $(p,(s,A))$ forces $(q_i,(t,B_i))$ and $(q_j,(t,B_j))$ to be incompatible.

Now suppose that we instead work not with $\mathbb{T}$ but with $\mathbb{T}^2$ and therefore must consider pairs $((q_i^0,(t_i^0,B_i^0)),(q_i^1,(t_i^1,B_i^1)))$ for $i < \kappa^+$. As before we can choose $(q_i^{0,*},(s,B_i^{0,*}))$'s and $(q_i^{1,*},(s,B_i^{1,*}))$'s to witness compatibility with $(p,(s,A))$ and thin out the set of pairs to ensure that the $q_i^{0,*}$'s are pairwise compatible and the $q_i^{1,*}$'s are pairwise compatible. Then choose distinct i and j and form the pair with first component $(q_i^{0,*} \cup q_j^{0,*},(s,B_i^{0,*} \cap B_j^{1,*}))$ and second component $(q_i^{1,*} \cup q_j^{1,*},(s,B_i^{1,*} \cap B_j^{1,*}))$; as before we can choose a measure one homogeneous set A^* to ensure that some extension $(p,(s,A^*))$ of $(p,(s,A))$ forces this pair into $\mathbb{T}^2$, yielding a contradiction that proves the lemma. □

Remark. The proof above actually shows that given κ^+-many conditions in $\mathbb{T}$ and $\alpha < \kappa$, α-many of these conditions have a common lower bound. The same applies to $\mathbb{T}^2$.

Reflection

Recall that we want to study the interaction between the tree property and two other properties, *reflection* and *approachability*. We take a look now at reflection.

Suppose that μ is an uncountable regular cardinal and S is a stationary subset of μ^+ consisting of ordinals of cofinality less than μ. Then S *reflects* if there is an ordinal $\alpha < \lambda$ of cofinality μ such that $S \cap \alpha$ is stationary in α. *Stationary reflection* holds at μ^+ if every such S reflects.

To obtain reflection at μ^+ we begin with a weak compact cardinal λ greater than μ and then apply a forcing (such as Mitchell forcing) to turn λ into μ^+.

Theorem 16 *(a) Suppose that λ is weak compact. Then for every stationary subset S of λ there is an inaccessible $\alpha < \lambda$ such that $S \cap \alpha$ is stationary in α.*

(b) However, the latter form of reflection at an inaccessible does not imply weak compactness.

Proof. (a) Suppose that S is a stationary subset of λ. The statement that λ is inaccessible and S is stationary is Π^1_1. So by Π^1_1 reflection, There is an inaccessible $\alpha < \lambda$ such that $S \cap \alpha$ is stationary in α.
(b) We use Kunen resurrection. Recall that we can get a sequence of models $V \subseteq V[T] \subseteq V[T, B]$ where λ is weak compact in V, T is a λ-Suslin tree in $V[T]$ and λ is again weak compact in $V[T, B]$ (resurrection of weak compactness). Moreover these three models share the same bounded subsets of λ and the forcing that adds B over $V[T]$ is λ-cc. Now suppose that S is a stationary subset of λ in $V[T]$ (where λ is not weak compact). If S remains stationary in $V[T, B]$ then S reflects in that model, as λ is weak compact there, and therefore also in $V[T]$, as $V[T]$ and $V[T, B]$ have the same bounded subsets of λ. So it suffices to prove:

Lemma 17 *Suppose that S is a stationary subset of λ and $\mathbb{P}$ is a λ-cc forcing. Then S remains stationary in $V[\mathbb{P}]$.*

Proof of Lemma. We show that any club (in λ) in $V[\mathbb{P}]$ contains a club in V. Suppose that $\dot{C}$ is a name for a club in λ and for each $\alpha < \lambda$ choose a maximal antichain A_α in $\mathbb{P}$ of conditions choosing a value for the α-th element of $\dot{C}$. By the λ-cc there are fewer than λ-many possibilities for the α-th element of $\dot{C}$ for each $\alpha < \lambda$. Now let C consist of those $\alpha < \lambda$ such that if β is less than α then the possibilities for the β-th element of $\dot{C}$ are all less than α. Then C is forced to be contained in the set of limit points of $\dot{C}$ and therefore is forced to be a subset of $\dot{C}$.

This completes the proof of the Theorem. □

Next we show how to get stationary reflection at the successor μ^+ of a regular cardinal μ.

Theorem 18 *Assume GCH and suppose that λ is weak compact, $\mu < \lambda$ is regular and $\mathbb{P}$ is the forcing Coll($\mu, < \lambda$) for turning λ into μ^+. Then stationary reflection holds at μ^+ in $V[\mathbb{P}]$.*

Proof. Suppose that $\dot{S}$ is a $\mathbb{P}$-name for a stationary subset of λ consisting of ordinals of cofinality less than μ. Then by Π^1_1 reflection there is some $\alpha < \mu^+$

which is inaccessible in V such that $\dot{S}_\alpha \cap \alpha$ is stationary in α in the model $V[\mathbb{P}_\alpha]$, where $\dot{S}_\alpha$ is the restriction of the name $\dot{S}$ to $\mathbb{P}_\alpha = \mathrm{Coll}(\mu, < \alpha)$. We are done if we can show that $\dot{S}_\alpha$ remains stationary in the larger model $V[\mathbb{P}]$. Let $\mathbb{P}^\alpha$ denote the tail forcing $\mathrm{Coll}(\mu, [\alpha, \lambda))$ for collapsing ordinals at least α to μ. Note that α is the μ^+ of the model $V[\mathbb{P}_\alpha]$.

In $V[\mathbb{P}_\alpha]$ build a continuous chain $(M_i \mid i < \mu^+)$ of sufficiently elementary submodels of size less than μ^+ containing p and $\dot{C}$ which are transitive below μ^+ with the property that for each i, $(M_j \mid j \leq i)$ belongs to M_{i+1} and M_{i+1} is μ-closed. Also assume that the M_i's are endowed with wellorders so that it makes sense to talk about "the least element of M_i" with any given property. As S_α is stationary we can choose a limit i of cofinality less than μ such that $M_i \cap \mu$ belongs to S_α. Let $(i_\eta \mid \eta < \mathrm{cof}(i))$ be increasing, continuous and cofinal in i. Now build a descending sequence of conditions $(p_j \mid j < \mathrm{cof}(i))$ below p such that p_j belongs to M_{i_j+1} and p_{i_j+1} forces some ordinal greater than $M_{i_j} \cap \mu^+$ into $\dot{C}$. Then the greatest lower bound of the p_j's forces that S_α intersects $\dot{C}$, showing that the stationarity of S is preserved. □

11.-12.Vorlesungen

Approachability

The proof of the previous result concerning reflection after the Lévy collapse of a weak compact raises a general question:

Question. Suppose that λ is regular and S is a stationary subset of λ consisting of ordinals of cofinality ν. Is the stationarity of S preserved by ν^+-closed forcing?

Shelah gave a fairly thorough answer to this question: Roughly speaking, the stationarity of S will be preserved by arbitrary ν^+-closed forcings iff S belongs to the *approachability ideal.*

Let θ denote a large regular cardinal and $\mathcal{A}$ a structure of the form $(H(\theta), \in, <_\theta, \ldots)$ where $<_\theta$ is a wellorder of $H(\theta)$ and ... represents countably many additional functions, relations and constants. Then $\gamma < \lambda$ is *approachable relative to* $\mathcal{A}$ if there is an unbounded $A \subseteq \gamma$ of ordertype $\mathrm{cof}(\gamma)$ such that each proper initial segment of A belongs to $\mathrm{Sk}^{\mathcal{A}}(\gamma)$ (the set of elements of $H(\theta)$ which are definable in $\mathcal{A}$ from parameters less than γ; Sk stands for

"Skolem hull"). Then $S \subseteq \lambda$ belongs to the *approachability ideal* $I[\lambda]$ iff for some $\mathcal{A}$ as above, almost all elements of S are approachable relative to $\mathcal{A}$ (where "almost all" means "on a club").

Proposition 19 *Suppose that λ is regular and uncountable. If $S \subseteq \lambda \cap$ Cof(ν) is stationary and belongs to $I[\lambda]$ then ν^+-closed forcings preserve the stationarity of S.*

Proof. Let the structure $\mathcal{A} = (H_\theta, \in, <_\theta, \ldots)$ witness $S \in I[\lambda]$ for some large θ and let P be a ν^+-closed forcing, p a condition in P forcing $\dot{C}$ to be a club in λ. Expand $\mathcal{A}$ to $\mathcal{A}^*$ so as to include $P, p, \dot{C}$. Now consider the club C of all $\gamma < \lambda$ such that $\gamma = \lambda \cap \mathrm{Sk}^{\mathcal{A}^*}(\gamma)$ and choose γ in $C \cap S$. Also let $A \subseteq \gamma$ be unbounded of ordertype ν such that all proper initial segments of A belong to $\mathrm{Sk}^{\mathcal{A}^*}(\gamma)$. Now the point is that if we successively extend p in ν steps in the $<_\theta$-least way, at step i forcing an ordinal greater than the i-th element of A into $\dot{C}$, then the resulting conditions belong to $\mathrm{Sk}^{\mathcal{A}^*}(\gamma)$ by the choice of A. Therefore a lower bound to these conditions forces that $\dot{C}$ is unbounded below γ. It follows that p has an extension forcing $\gamma \in S$ into $\dot{C}$, proving that the stationarity of S is preserved. $\square$

For our purposes the main fact we need about the approachability ideal is the following, which we state without proof:

Fact. If $\lambda = \mu^+$ with μ regular then $\lambda \cap$ Cof$(< \mu)$ belongs to $I[\lambda]$.

Armed with the above information, we can verify that we get stationary reflection in the Mitchell and Prikry-ised Mitchell models.

Proposition 20 *Let $\mathbb{M}$ denote the Mitchell collapse to turn a weak compact λ into κ^{++}, obtaining the tree property at κ^{++}. Then stationary reflection holds in $V[\mathbb{M}]$: For every stationary subset S of κ^{++} consisting of ordinals of cofinality at most κ there is $\alpha < \kappa^{++}$ of cofinality κ^+ such that $S \cap \alpha$ is stationary. And the same holds for the Prikry-ised version of Mitchell forcing.*

Proof. As in the case of the Lévy collapse we let $\dot{S}$ be a $\mathbb{M}$-name for a stationary subset of λ consisting of ordinals of cofinality at most κ and apply weak compactness to get an inaccessible $\alpha < \lambda$ so that $\mathbb{M}_\alpha$, the Mitchell forcing below α, forces $\dot{S}_\alpha$, the restriction of the name $\dot{S}$ to α, to be stationary. Work in the model $V[G(\mathbb{M}_\alpha)]$ where $G(\mathbb{M}_\alpha)$ is $\mathbb{M}_\alpha$-generic and let S_α denote the interpretation of $\dot{S}_\alpha$. We need only show that S_α remains stationary after forcing with the Mitchell quotient $\mathbb{M}/G(\mathbb{M}_\alpha)$. Recall that this quotient is the projection of the product of the κ^+-cc forcing $\mathrm{Add}(\kappa, [\alpha, \lambda))$ and a κ^+-closed term forcing $\mathbb{T}$. As S_α consists of ordinals of cofinality at most κ, it belongs to $I[\alpha]$ and therefore by the *Fact* mentioned above, its stationarity is preserved by the κ^+-closed forcing $\mathbb{T}$. After forcing with $\mathbb{T}$, α gets cofinality κ^+ and by Easton's lemma, $\mathrm{Add}(\kappa, [\alpha, \lambda))$ is still κ^+-cc. So again the stationarity of S_α is preserved.

The same proof applies to Prikry-ised Mitchell, as (after a harder argument) it is again the case that the quotient is the projection of the product of a κ^+-cc and a κ^+-closed forcing. □

Note that even though approachability holds on $\mathrm{Cof}(\leq \kappa)$ in the Mitchell model where κ^{++} has the tree property, it is not clear whether it holds on $\mathrm{Cof}(\kappa^+)$. In fact we'll see that $\mathrm{Cof}(\kappa^+) \cap \kappa^{++}$ does belong to $I[\kappa^{++}]$ for the usual version of Mitchell forcing, but there is a variant, which still gives the tree property and reflection at κ^{++}, but for which approachability fails on $\mathrm{Cof}(\kappa^+) \cap \kappa^{++}$. Again, the same applies to Prikry-ised Mitchell.

A further remark about stationary reflection

We used a weak compact to get stationary reflection at ω_2. Actually, Harrington-Shelah improved this:

Theorem 21 *Suppose that κ is Mahlo. Then in a forcing extension, κ equals ω_2 and stationary reflection holds at ω_2, i.e. for every stationary subset S of ω_2 consisting of ordinals of cofinality ω, $S \cap \alpha$ is stationary for some $\alpha < \omega_2$ of cofinality ω_1.*

Proof sketch. First apply $\mathbb{P} = \mathrm{Coll}(\omega_1, < \kappa)$ to turn κ into ω_2 with a Lévy collapse. Then iterate for ω_3 steps with supports of size at most ω_1 where at each step β one chooses a subset X_β of $\mathrm{Cof}(\omega) \cap \omega_2$ which does not reflect (i.e. such that $X_\beta \cap \alpha$ is nonstationary for all $\alpha < \omega_2$ of cofinality ω_1) and then adds a club disjoint from X_β using bounded closed conditions disjoint from X_β. Let $\mathbb{Q}$ denote this latter iteration. The desired forcing is $\mathbb{P} * \mathbb{Q}$.

$\mathbb{P} * \mathbb{Q}$ is ω_3-cc, so the key lemma is that $\mathbb{Q}$ is ω_2-distributive, i.e. does not add new ω_1-sequences of ordinals, over $V[\mathbb{P}]$. This completes the argument, as in the final model any subset X of ω_2 appears at some stage $\beta < \omega_3$ and if it does not reflect has its stationarity killed. ω_1 and cardinals at least κ are preserved as $\mathbb{P}$ is ω-closed and κ-cc, and $\mathbb{Q}$ is an ω_2-distributive, ω_3-cc forcing in $V[\mathbb{P}]$.

Let G be $\mathbb{P}$-generic. Suppose that in $V[G]$ we are given ω_1-many dense sets $(D_i \mid i < \omega_1)$ on $\mathbb{Q}$ and a condition q in $\mathbb{Q}$. We want to extend q to meet all of the D_i's. Using the Mahloness of κ in V, let M be an ω-closed elementary submodel of some large $H(\theta)^{V[G]}$ containing all the relevant information such that $\bar\kappa = M \cap \kappa$ is the ω_2 of $V[\bar G]$, where $\bar{\mathbb{P}}$ denotes the forcing $\mathbb{P}$ restricted to $\bar\kappa$ and $\bar G = G \cap \bar{\mathbb{P}}$ is $\bar{\mathbb{P}}$-generic. Also let $\bar{\mathbb{Q}}$ denote the image of $\mathbb{Q}$ when transitively collapsing M to $\bar M$. Then it suffices to show that $\bar{\mathbb{Q}}$ is (equivalent to) an ω_2-closed forcing in $V[\bar G]$, as then we can form an ω_1-descending sequence of conditions hitting the D_i's, and take a lower bound.

Note that it suffices to work with proper initial segments $\mathbb{Q}_\beta$ of the iteration $\mathbb{Q}$, as any maximal antichain in $\mathbb{Q}$ is contained in $\mathbb{Q}_\beta$ for some $\beta < \kappa^+ = \omega_3$. Now we argue inductively: To know that $\bar{\mathbb{Q}}_\beta$ is (equivalent to) an ω_2-closed forcing in $V[\bar G]$, it suffices to know that for $\gamma < \beta$ in M, $\bar X_\gamma = X_\gamma \cap \bar\kappa$ is nonstationary in $V[\bar G][\bar{\mathbb{Q}}_\gamma]$, because then we can form the ω_2-closed dense subset consisting of conditions which on component $\bar\gamma$ have max in $\bar C_\gamma$, where the latter is a club disjoint from $\bar X_\gamma$. And to complete the (double) induction we can verify the nonstationarity of $\bar X_\beta$ using the ω_2-closure of $\bar{\mathbb{Q}}_\beta$: We build a generic for this forcing with a lower bound q^* and note that q^* forces $\bar X_\beta = X_\beta \cap \bar\kappa$ to be nonstationary (by the definition of the X_β's). We know inductively that $\mathbb{Q}_\beta$ is ω_2-distributive in $V[G]$ so $\bar X_\beta$ is nonstationary already in $V[G]$ and therefore as the forcing $\bar{\mathbb{P}} * \bar{\mathbb{Q}}_\beta$ is a regular subforcing of $V[\mathbb{P}]$ with an ω-closed quotient, $\bar X_\beta$ is already nonstationary in $V[\bar G][\bar{\mathbb{Q}}_\beta]$, completing the induction. $\square$

13.-14.Vorlesungen

The Eightfold Way

Let μ be a regular uncountable cardinal. Then we consider all eight Boolean combinations of the following three properties of μ^+.

TP (Tree Property): There is no μ^+-Aronszajn tree.
RP (Reflection Property): If S is a stationary subset of $\mu^+ \cap \mathrm{Cof}(< \mu)$ then S reflects, i.e. $S \cap \alpha$ is stationary in α for some $\alpha < \mu^+$ of cofinality μ.

AP (Approachability Property): μ^+ belongs to the approachability ideal $I[\mu^+]$, i.e. there exists a list $(x_i \mid i < \mu^+)$ of bounded subsets of μ^+ such that almost all $\alpha < \mu^+$ are approachable relative to $\vec{x}$, i.e. there is a club c in α all of whose proper initial segments belong to $\{x_i \mid i < \alpha\}$.

Theorem 22 *Let κ be a regular cardinal with $\kappa^{<\kappa} = \kappa$ and let μ be κ^+. Then (given a weak compact above κ) for each Boolean combination Φ of* TP, RP *and* AP *there is a generic extension preserving cardinals up to μ in which Φ holds. If κ is measurable then we can also ensure that κ has cofinality ω in these generic extensions.*

Below we establish the first conclusion (where κ need not be measurable) and later observe that our arguments also work to handle the second conclusion.

Easiest Cases

Suppose that we add a κ^+-Cohen set. Then we preserve cardinals up to $\mu = \kappa^+$ and get $2^\kappa = \kappa^+$, hence $\mu^{<\mu} = \mu$. From this it follows that there is a μ^+-Aronszajn tree and therefore TP fails at μ^+. Also we have AP:

Lemma 23 *Suppose that $\mu^{<\mu} = \mu$. Then AP holds at μ^+.*

Proof. The hypothesis implies that $(\mu^+)^{<\mu} = \mu^+$ so we can list all subsets of μ^+ of size less than μ as $\vec{x} = (x_i \mid i < \mu^+)$. As $\mu^{<\mu} = \mu$ there are club-many $\alpha < \mu^+$ so that all bounded subsets of α of size $< \mu$ are listed as x_i for some $i < \alpha$. Suppose that c is a club in such an α of ordertype $\mathrm{cof}(\alpha)$; then each proper initial segment of c is a bounded subset of α of size $< \mu$ and therefore we see that α is approachable relative to $\vec{x}$ for club-many α's. $\square$

So by adding a μ-Cohen set we get both $\sim$ TP and AP. To get $\sim$ RP we then force a non-reflecting stationary set:

Lemma 24 *Suppose that λ is regular and uncountable. Then there is a forcing $\mathbb{P}$ that adds no bounded subsets of λ and adds a subset S of λ such that $S \cap \alpha$ is nonstationary for each $\alpha < \lambda$ of uncountable cofinality and $S \cap Cof(\delta)$ is stationary for each regular $\delta < \lambda$.*

Proof. We force with "everywhere nonstationary" bounded subsets of λ: A condition in $\mathbb{P}$ is $p : |p| \to 2$ where $|p| < \lambda$ and for all $\alpha \leq |p|$ of uncountable cofinality, $p(\bar{\alpha}) = 0$ for club-many $\bar{\alpha} < \alpha$. Clearly $S = \{\beta < \lambda \mid p(\beta) = 1$ for some $p \in G\}$ (where G is the generic) has the property that $S \cap \alpha$ is nonstationary for each $\alpha < \lambda$; we must check that $\mathbb{P}$ does not add bounded subsets of λ and that $S \cap \mathrm{Cof}(\delta)$ is stationary for each regular $\delta < \lambda$.

Suppose that p is a condition, $\delta < \lambda$ is regular, $(D_i \mid i < \delta)$ are dense and $\dot{C}$ is a name for a club in λ. We show that p can be extended to meet all of the D_i's and to force some ordinal in $\dot{C}$ of V-cofinality δ into $S \cap \dot{C}$.

Let $(M_i \mid i < \delta)$ be a continuous chain of elementary submodels of some large $(H(\theta), <_\theta)$ (where $<_\theta$ is a wellorder of $H(\theta)$) such that $M_i \cap \lambda = \lambda_i$ is an ordinal and $(M_j \mid j \leq i)$ belongs to M_{i+1} for each $i < \delta$. We also assume that M_0 contains all of the relevant parameters (p, $\dot{C}$, $(D_i \mid i < \delta)$, ...). Then we can build a descending sequence $(p_i \mid i < \delta)$ of conditions below p such that p_i is an element of M_{i+1} for each i and p_{i+1} both belongs to D_i and forces some ordinal at least λ_i to belong to $\dot{C}$. To ensure that we can take lower bounds at limits, we also require that p_i take the value 0 at λ_i; this ensures that $p_i = \bigcup_{j<i} p_j$ takes the value 0 on a closed unbounded subset of λ_i for limit i. Extend the union of the p_i, $i < \delta$, to a function of length $\lambda_\delta + 1$ by assigning the value 1 at λ_δ to ensure that λ_δ belongs to S. Then this is a condition extending p which meets all of the D_i, $i < \delta$, and forces λ_δ into $\dot{C}$. □

Analogously to Kunen's result that λ-Cohen is equivalent to a two-step iteration where one adds a λ-Suslin tree and then adds branches to it, we have the following result, which we'll need later:

Lemma 25 *Let $\mathbb{P}$ be the forcing above to add a nonreflecting stationary subset S of the uncountable cardinal λ, where $\lambda^{<\lambda} = \lambda$. Let $\mathbb{Q}$ be the forcing that adds a club disjoint from S using bounded, closed conditions disjoint from S. Then $\mathbb{P} * \mathbb{Q}$ is equivalent to λ-Cohen forcing.*

Proof. Consider the dense subset of $\mathbb{P} * \mathbb{Q}$ consisting of pairs (p, c) where p belongs to $\mathbb{P}$, c is closed and bounded, $|p| = \max(c)$ and p takes the value 0 on c. This dense subset is λ-closed and as $\lambda^{<\lambda} = \lambda$, it has size λ. It follows that $\mathbb{P} * \mathbb{Q}$ is equivalent to λ-Cohen forcing. □

Case 1: $\sim$ TP, AP and $\sim$ RP. We add a κ^+-Cohen set to guarantee $\sim$ TP and AP at $\mu^+ = \kappa^{++}$. Then add a nonreflecting stationary set S to μ^+ to get $\sim$ RP. The latter forcing does not add subsets of μ and therefore we still have $\mu^{<\mu} = \mu$ and therefore $\sim$ TP and AP.

Case 2: $\sim$ TP, AP and RP. Again we add a κ^+-Cohen set to guarantee $\sim$ TP and AP. Now assuming that we have a weak compact λ greater than κ, we follow Baumgartner and use $\mathrm{Coll}(\mu, < \lambda)$ to ensure the RP at μ^+. As this forcing does not add bounded subsets of μ we still have $\mu^{<\mu} = \mu$ and therefore $\sim$ TP and AP.

15.Vorlesung

The Eightfold Way

Less easy cases

Case 3: TP, AP and RP.

We verify these in the standard Mitchell model. Recall that this model is obtained by starting with $\kappa < \lambda$, $\kappa = \kappa^{<\kappa}$ and λ weak compact and using Mitchell's forcing $\mathbb{Q}$ to turn λ into κ^{++} and obtain the TP at κ^{++}. Conditions in $\mathbb{Q}$ are pairs (p, f) where p belongs to $\mathrm{Add}(\kappa, \lambda)$ and f is a function of size at most κ whose domain consists of inaccessible α between κ and λ, and where $f(\alpha)$ is an $\mathrm{Add}(\kappa, \alpha)$-name for a condition in the $\mathrm{Coll}(\kappa^+, \alpha)$ of $V[\mathrm{Add}(\kappa, \alpha)]$. We have verified that TP and RP hold in Mitchell's model; we now verify that AP does too.

To do so we make just a very small modification to Mitchell's forcing: We allow the domains of the f's to consist of arbitrary strong limit cardinals less than λ of cofinality greater than κ, not only of inaccessibles between κ and λ.

Now note that for strong limit $\alpha < \lambda$ of cofinality greater than κ the forcing $\mathbb{Q}$ adds a function g_α from κ^+ cofinally into α such that $g_\alpha | i$ belongs to $V[\mathrm{Add}(\kappa, \alpha)]$ for each $i < \kappa^+$. This is because it adds a $\mathrm{Coll}(\kappa^+, \alpha)$-generic over $V[\mathrm{Add}(\kappa, \alpha)]$. And if $\vec{x} = (x_i \mid i < \lambda)$ is a list of all size at most κ subsets of λ, then for almost all strong limit $\alpha < \lambda$ of cofinality greater than κ (i.e. for all such in a club), the size at most κ subsets of α in $V[\mathrm{Add}(\kappa, \alpha)]$ are exactly the $(x_i \mid i < \alpha)$. It follows that we get approachability relative to $\vec{x}$ for almost all ordinals less than $\kappa^{++} = \lambda$ of cofinality κ^+. Approachability for ordinals of cofinality at most κ follows from Shelah's general result that $\mathrm{Cof}(\leq \kappa) \cap \kappa^{++}$ must belong to $I[\kappa^{++}]$.

Case 4: TP, AP and $\sim$ RP. Consider the two-step iteration $\mathbb{Q} * \mathbb{R}$ where $\mathbb{Q}$ is the (mild variant of) Mitchell forcing above and $\mathbb{R}$ is the forcing that adds a nonreflecting stationary subset S of κ^{++} (i.e. S is a stationary set of ordinals of cofinality at most κ such that $S \cap \alpha$ is not stationary in α for $\alpha < \kappa^{++}$ of uncountable cofinality). Clearly $\sim$ RP holds in $V[\mathbb{Q} * \mathbb{R}]$. Also AP holds in this model as it is upwards-absolute to models that don't change cardinals. It remains to verify TP.

Our strategy is to first show that the TP holds in $V[\mathbb{Q} * \mathrm{Add}(\kappa^{++}, 1)]$ where $\mathrm{Add}(\kappa^{++}, 1)$ is κ^{++}-Cohen forcing. Then we use:

Lemma 26 *Suppose $\kappa^{<\kappa} = \kappa$ and $2^\kappa = \kappa^{++}$. Let S be a nonreflecting stationary subset of κ^{++} and T a κ^{++}-tree. Then the forcing $\mathbb{S}$ that kills the stationarity of S does not add a new cofinal branch through T.*

This lemma implies that TP holds in $V[\mathbb{Q} * \mathbb{R}]$, as if T were a κ^{++}-Aronszajn tree in this model, the lemma implies that T is still κ^{++}-Aronszajn in $V[\mathbb{Q} * \mathbb{R} * \mathbb{S}] = V[\mathbb{Q} * \mathrm{Add}(\kappa^{++}, 1)]$, contradicting the fact that the TP holds in this last model.

Proof of Lemma. Suppose that $\dot{b}$ were an $\mathbb{S}$-name for a new cofinal branch through T. Choose M to be an elementary submodel of some large $H(\theta)$ which is $< \kappa$-closed of size κ^+ and whose intersection with κ^{++} is an ordinal α of cofinality κ^+. Also assume that M contains all relevant parameters.

As S is non-reflecting we may choose a club C in α of ordertype κ^+ disjoint from S. Now build a tree of conditions $(q_s \mid s \in 2^{<\kappa})$ in $\mathbb{S}$ together with a tree of nodes $(t_s \mid s \in 2^{<\kappa})$ such that:

(a) q_s, t_s are in M for all s.
(b) q_s forces that t_s lies on $\dot{b}$.
(c) $\max(q_s)$ belongs to C.
(d) If s^* extends s then $q_{s^*} \leq q_s$ and t_{s^*} is a node extending t_s.
(e) For all s, t_{s*0} and t_{s*1} are incomparable in T.

The construction is straightforward using the facts that M is $< \kappa$-closed, C is disjoint from S and $\dot{b}$ is forced to be a new branch through T.

Now for each $x \in 2^\kappa$ we get a condition q_x below the $q_{x|i}$, $i < \kappa$, and then can extend q_x to r_x deciding where $\dot{b}$ hits level $\alpha = M \cap \kappa^{++}$ of T. This gives $2^\kappa = \kappa^{++}$-many distinct nodes on level α of T, contradicting the fact that T is a κ^{++}-tree. $\square$

Remark. The same proof yields a similar result when κ is singular of cofinality ω: Suppose that κ is singular of cofinality ω, $2^\kappa = \kappa^\omega = \kappa^{++}$, S is a non-reflecting stationary subset of κ^{++}, T is a κ^{++}-tree and $\mathbb{S}$ is the forcing that kills the stationarity of S. Then $\mathbb{S}$ does not add a new cofinal branch through T. This is needed to handle *Case 4* when κ is made singular of cofinality ω using Prikry forcing.

16.-17.Vorlesungen

To complete *Case 4* we have to show that the TP holds at κ^{++} in $V[\mathbb{Q} * \text{Add}(\kappa^{++}, 1)]$. To obtain this we have to modify Mitchell forcing $\mathbb{Q}$ again, to a forcing which we call $\mathbb{Q}^*$. The idea is to "fold into" $\mathbb{Q}$ approximations $\text{Add}(\alpha, 1)$ to $\text{Add}(\lambda, 1)$ for inaccessible $\alpha < \lambda$. This will enable us to verify the TP in the same way as for $\mathbb{Q}$, by writing the quotients as the projection of nice products.

Recall that conditions in $\mathbb{Q}$ are pairs (p, f) where p belongs to $\text{Add}(\kappa, \lambda)$ and f is a function of size at most κ which at strong limit $\alpha < \lambda$ of cofinality greater than κ in its domain assigns an $\text{Add}(\kappa, \alpha)$-name $f(\alpha)$ for a condition in $\text{Coll}(\kappa^+, \alpha)$. Now we add a third component: Conditions in $\mathbb{Q}^*$ are triples (p, f, q) where (p, f) belongs to $\mathbb{Q}$ and q is a function with Easton domain which at inaccessible $\alpha < \lambda$ assigns a $\mathbb{Q}^*_\alpha$-name for a condition $q(\alpha)$ in $\text{Add}(\alpha, 1)$ (where $\mathbb{Q}^*_\alpha$ denotes the inductively-defined set of triples $(\bar{p}, \bar{f}, \bar{q})$ in $\mathbb{Q}^*$ such that $\bar{p}$ belongs to $\text{Add}(\kappa, \alpha)$ and $\bar{f}$, $\bar{q}$ have domains contained in α).

By *Easton domain* we mean that the domain of f is bounded below any inaccessible, including λ. Thanks to this restriction the forcing $\mathbb{Q}^*$ is λ-cc and therefore preserves λ.

We now verify the TP in $V[\mathbb{Q}^* * \text{Add}(\lambda, 1)]$. Let $G(\mathbb{Q}^*) * g$ be $\mathbb{Q}^* * \text{Add}(\lambda, 1)$-generic and suppose that T were a λ-Aronszajn tree in $V[G(\mathbb{Q}^*) * g]$. Using the weak compactness of λ, choose an inaccessible $\alpha < \lambda$ such that $T|\alpha$ is an α-Aronszajn tree in $V[G(\mathbb{Q}^*|\alpha) * g|\alpha]$ where $G(\mathbb{Q}^*|\alpha)$ is the restriction of $G(\mathbb{Q}^*)$ to the forcing $\mathbb{Q}^*|\alpha$. Now $T|\alpha$ has a branch in $V[G(\mathbb{Q}^*) * g]$ (as any node of T at level α determines such a branch) and since $\mathrm{Add}(\lambda, 1)$ does not add bounded subsets of λ, $T|\alpha$ has a branch in $V[G(\mathbb{Q}^*)]$. We can view $V[G(\mathbb{Q}^*)]$ as a $\mathbb{Q}^*/G(\mathbb{Q}^*|\alpha) * g|\alpha)$-generic extension of $V[G(\mathbb{Q}^*|\alpha) * g|\alpha)]$, as the forcing $\mathbb{Q}^*$ adds a generic for $\mathrm{Add}(\alpha, 1)$ over $V[G(\mathbb{Q}^*|\alpha)]$. Note that the tree $T|\alpha$ lives in the model $V[G(\mathbb{Q}^*|\alpha) * g|\alpha)]$.

Now apply Avraham's trick using term-forcings to argue that the quotient $\mathbb{Q}^*/G(\mathbb{Q}^*_\alpha) * g|\alpha$ is the projection of a product $\mathrm{Add}(\kappa, [\alpha, \lambda)) \times \mathbb{T} \times \mathbb{R}$ where $\mathbb{T}$ and $\mathbb{R}$ are κ^+-closed. And the square of $\mathrm{Add}(\kappa, [\alpha, \lambda))$ is κ^+-cc, so by earlier lemmas, the quotient $\mathbb{Q}^*/G(\mathbb{Q}^*|\alpha) * g|\alpha)$ cannot add a branch through $T|\alpha$, a contradiction.

Remark. Of course in the last step of the above proof we know that $\mathrm{Add}(\kappa, [\alpha, \lambda))$ is in fact κ^+-Knaster, but in the case of Prikry-ised Mitchell we only get that the square is κ^+-cc, which suffices for the desired contradiction.

This completes *Case 4* of the Eightfold Way.

We are left with the 4 cases in which AP fails. For these cases we need a variant of Mitchell forcing $\mathbb{Q}^+$ which kills the AP. A condition in $\mathbb{Q}^+$ is a pair (p, f) as in $\mathbb{Q}$ where p belongs to $\mathrm{Add}(\kappa, \lambda)$ and f is a function of size at most κ which for $\alpha < \lambda$ in its domain assigns an $\mathrm{Add}(\kappa, \alpha)$-name for a condition in $\mathrm{Coll}(\kappa^+, \alpha)$; however now we require that the domain of f consist solely of successor inaccessibles (i.e. inaccessibles which are not limits of inaccessibles) less than λ.

The proofs of TP and RP at $\lambda = \kappa^{++}$ in $V[\mathbb{Q}^+]$ are just as with the standard Mitchell forcing $\mathbb{Q}$. The only significant difference between $\mathbb{Q}$ and $\mathbb{Q}^+$ is that the former gives approachability whereas the latter does not.

To see this, first note that in $V[G(\mathbb{Q}^+)]$ there is a largest subset S of $\lambda \cap \mathrm{Cof}(\kappa^+)$ in the approachability ideal $I[\lambda]$ (modulo clubs): Define S to be the set of α of cofinality κ^+ such that there is a club c in α of ordertype κ^+ all of whose proper initial segments belong to $V[G(\mathrm{Add}(\kappa, \alpha))]$. If $(x_i \mid i < \lambda)$ witnesses that T is a subset of $\lambda \cap \mathrm{Cof}(\kappa^+)$ in $I[\lambda]$ then for almost all $\alpha < \lambda$ of cofinality κ^+, all x_i of size at most κ with $i < \alpha$ belongs to $V[G(\mathrm{Add}(\kappa, \alpha))]$

and therefore T is contained in S modulo a club. And S is witnessed into $I[\lambda]$ by any sequence $(x_i \mid i < \lambda)$ which enumerates the size at most κ subsets of λ.

So to show that approachability fails, it suffices to find a stationary subset B of $\lambda \cap \mathrm{Cof}(\kappa^+)$ which is disjoint from S. We take B to be the stationary set of $\alpha < \lambda$ such that α is a limit of inaccessibles and $2^\kappa = \kappa^{++} = \alpha$ in $V[\mathbb{Q}^+|\alpha]$.

Claim. B is disjoint from S.

Proof of Claim. Suppose that α belongs to B. If α also belongs to S then there is a club c in α of ordertype κ^+, all of whose proper initial segments belongs to $V[G(\mathrm{Add}(\kappa, \alpha))]$. Equivalently, there is an increasing, cofinal function g from κ^+ into α such that for $i < \kappa^+$, $g|i$ belongs to $V[G(\mathbb{Q}^+|\alpha)]$. Of course g itself does not belong to this model as α is κ^{++} there. So the function g is added over $V[G(\mathbb{Q}^+|\alpha)]$ by the quotient forcing $\mathbb{Q}^+/G(\mathbb{Q}^+|\alpha)$.

Now note that α is a limit of inaccessibles and therefore not in the domain of f for any condition (p, f) in $\mathbb{Q}^+$. So if we let x denote $G(\mathrm{Add}(\kappa, \lambda))(\alpha)$, the κ-Cohen with index α added by $G(\mathrm{Add}(\kappa, \lambda))$, then α is still κ^{++} in $V[G(\mathbb{Q}^+|\alpha) * x]$ and g is added by the quotient $\mathbb{Q}^+/G(\mathbb{Q}^+|\alpha) * x$. (I.e., a κ-Cohen is added over $V[G(\mathbb{Q}^+|\alpha)]$ before α gets collapsed.)

Now as before the quotient $\mathbb{Q}^+/G(\mathbb{Q}^+|\alpha) * x$ is the projection of a product $\mathrm{Add}(\kappa, [\alpha+1, \lambda)) \times \mathbb{T}$ where $\mathbb{T}$ is a κ^+-closed forcing. First note that $\mathbb{T}$ cannot add g over $V[G(\mathbb{Q}^+|\alpha) * x]$, as if $\dot{g}$ were a $\mathbb{T}$-name for g then we could build a tree of height κ of conditions in $\mathbb{T}$ such that lower bounds of distinct branches force different information about $\dot{g}$; then a lower bound of some branch of this tree forces that x can be calculated from $\dot{g}|i$ for some $i < \kappa^+$, contradicting the hypothesis that $\dot{g}|i$ is forced to belong to $V[G(\mathbb{Q}^+|\alpha)]$. But also $\mathrm{Add}(\kappa, [\alpha+1, \lambda))$ cannot add g over $V[G(\mathbb{Q}^+|\alpha) * x * G(\mathbb{T})]$, as its square is κ^+-cc. So α cannot belong to S. $\square$

We can now handle the $\sim$ AP cases.

Case 5. $\sim$ AP, TP, RP.
Force with $\mathbb{Q}^+$.

Csae 6. $\sim$ AP, $\sim$ TP, RP.
First prepare with a reverse Easton iteration that adds an α-Cohen set to inaccessible $\alpha < \lambda$; the point of this is that if we then add a λ-Cohen we ensure the weak compactness of λ. Also, even if we don't add the λ-Cohen, λ is nonetheless Mahlo.

Now force with $\mathbb{T} \times \mathbb{Q}^+$ where $\mathbb{T}$ adds a λ-Suslin tree. This kills the TP, as the λ-Suslin tree added by $\mathbb{T}$ has no cofinal branch after forcing with $\mathbb{Q}^+$ as $(\mathbb{Q}^+)^2$ is λ-cc. As λ is still Mahlo after forcing with $\mathbb{T}$, we also see that $\sim$ AP holds (as the argument that $\mathbb{Q}^+$ kills the AP only used the Mahloness of λ). To see that the RP holds, note that if we force to add a branch through the tree added by $\mathbb{T}$, the three-step iteration is equivalent to $\mathrm{Add}(\lambda, 1) \times \mathbb{Q}^+$ and as λ is weakly compact after forcing with $\mathrm{Add}(\lambda, 1)$, we get the RP in this model. It follows that we also have the RP in the smaller model $V[\mathbb{T} \times \mathbb{Q}^+]$, as the forcing to add a branch is λ-cc and therefore preserves stationary subsets of λ.

Case 7. $\sim$ AP, TP, $\sim$ RP.
Force with $\mathbb{Q}^+ * \mathbb{S}$ where $\mathbb{S}$ adds a nonreflecting stationary set. Clearly we get $\sim$ RP. And as $\mathbb{S}$ is stationary-preserving (it is a regular subforcing of λ-Cohen, which is λ-closed), the witness to $\sim$ AP added by $\mathbb{Q}^+$ is still stationary, so we have $\sim$ AP. Finally, note that $\mathbb{Q}^+$ was designed to ensure the TP after forcing with λ-Cohen and as the forcing to kill the stationarity of the set added by $\mathbb{S}$ does not add branches through λ-trees, we also have the TP in $V[\mathbb{Q}^+ * \mathbb{S}]$.

Case 8. $\sim$ AP, $\sim$ TP, $\sim$ RP.
Force with $\mathbb{Q}^+ \times \mathbb{S} \times \mathbb{T}$ where $\mathbb{S}$ adds a nonreflecting stationary subset of λ and $\mathbb{T}$ adds a λ-Suslin tree. This can be viewed as an iteration in any order so we get $\sim$ RP and $\sim$ TP. As λ is still Mahlo after forcing with $\mathbb{S} \times \mathbb{T}$ we also get $\sim$ AP.

This completes the Eightfold Way.

18.Vorlesung

The Eightfold Way: Further work

We showed that any Boolean combination of AP, TP and RP can be forced to hold at κ^{++} when κ is regular and there is a weak compact cardinal above κ. Moreover, if κ is measurable, we can simultaneously make κ singular of cofinality ω.

There is more work to be done. Note that the consistency strength of TP is a weak compact, but of RP and $\sim$ AP only a Mahlo.

Question 1. In the $\sim$ TP cases, can one get by with just a Mahlo instead of a weak compact above κ?

For $\sim$ TP, AP and $\sim$ RP we can simply force GCH at κ and then add a nonreflecting stationary subset of κ^{++}.

For $\sim$ TP, $\sim$ AP and $\sim$ RP, we can force with the product $\mathbb{T} \times \mathbb{Q}^* \times \mathbb{S}$ where $\mathbb{T}$ adds a λ-Suslin tree, $\mathbb{Q}^*$ is our version of Mitchell which kills AP and $\mathbb{S}$ adds a nonreflecting stationary subset of λ. We can view this as a three-step iteration in any order, so we easily get $\sim$ TP and $\sim$ RP. We also get $\sim$ AP as λ is still Mahlo after forcing with $\mathbb{T} \times \mathbb{S}$.

For $\sim$ TP, AP and RP we force GCH at κ and then apply the Harrington-Shelah forcing to get RP. As we still have the GCH at κ we have $\sim$ TP and AP.

I'm not sure how to do the remaining case, $\sim$ TP, $\sim$ AP and RP from just a Mahlo. We can first force RP at λ keeping λ Mahlo and then, if λ is not weak compact, continue by forcing with $\mathbb{Q}^*$. Then we have $\sim$ TP and $\sim$ AP. What is missing is a proof that we still have RP.

Question 2. If κ is hypermeasurable can we also get the cofinality of κ to be ω_1 in the Eightfold Way?

This might not be easy, as we have to generalise our analysis of Cohen followed by Prikry quotients to Cohen followed by Magidor quotients, where Magidor forcing is the analogue of Prikry forcing for getting cofinality ω_1.

Question 3. What can be said about the Eightfold Way at successors of weakly inaccessibles and at successors of singulars?

There is evidence that not all eight Boolean combinations of AP, TP and RP are consistent at the successor of a singular cardinal.

Question 4. Can one achieve the Eightfold Way for adjacent double-successors of regulars simultaneously and independently (Sixtyfourfold Way)?

So ends our discussion of the Eightfold Way. We'll now turn to something completely different. (The only thing in common with the Eightfold Way is that the next topic was also explored by the same AIM group: Cummings, Magidor, Rinot, Sinapova and myself.)

Ordinal definability at singulars

We prove the following result of Shelah.

Theorem 27 *Suppose that κ is singular of uncountable cofinality. Then for some subset x of κ, all subsets of κ belong to HOD_x, the class of sets which are hereditarily definable from ordinals and x.*

Proof. Let κ be a strong limit cardinal of uncountable cofinality μ. Let x be a subset of κ such that in $L[x]$ one finds an enumeration $(t_\eta \mid \eta < \kappa)$ of all elements of $H(\kappa)$ as well as a sequence $(\alpha_i \mid i < \mu)$ cofinal in κ.

To each set $X \subseteq \kappa$ we associate the function $f_X : \mu \to \kappa$, where $f_X(i) = \eta$ for the unique η such that $X \cap \alpha_i = t_\eta$. Note that if X, Y are distinct then f_X, f_Y differ on a final segment of μ. Define an ordering $\prec$ of the subsets of κ by $X \prec Y$ iff $f_X(i)$ is less than $f_Y(i)$ on a final segment of μ. As μ is greater than ω, this order is wellfounded.

For any ordinal α let R_α denote the subsets of κ of $\prec$-rank α. If X, Y are distinct elements of R_α then $f_X(i) > f_Y(i)$ for cofinally-many $i < \mu$, else $X \prec Y$. We claim that R_α has size at most 2^μ and hence size less than κ. If not, then let $(X_j \mid j < (2^\mu)^+)$ be distinct elements of R_α and define a colouring c of pairs from $(2^\mu)^+$ by $c(j_0, j_1) = i$ for the least i such that $f_{X_{j_0}}(i) > f_{X_{j_1}}(i)$. By the Erdős-Rado Theorem, c has a homogeneous set of ordertype μ^+, which is impossible as even a homogeneous set of ordertype ω yields an infinite descending sequence of ordinals.

Now define a wellorder $\prec_\alpha$ of R_α: Let A_α be the union of the Range(f_X) for $X \in R_\alpha$. Then A_α has size at most $2^\mu \times \mu < \kappa$ and let g_α be the order-preserving bijection between A_α and its ordertype. Then for X in R_α, $g_\alpha \circ f_X$ is an element of $H(\kappa)$ and we order R_α by $X \prec_\alpha Y$ iff $g_\alpha \circ f_X$ appears before $g_\alpha \circ f_Y$ in the enumeration $(t_\eta \mid \eta < \kappa)$ of $H(\kappa)$.

Putting together the wellorders $\prec_\alpha$ yields a wellorder of the subsets of κ which is definable from x. And each subset of κ appears as the β-th element of this wellorder for some ordinal β and therefore is definable from x together with an ordinal, i.e. belongs to HOD_x. $\Box$