Abstract

Vopěnka [2] proved long ago that every set of ordinals is set-generic over HOD, Gödel's inner model of hereditarily ordinal-definable sets. Here we show that the entire universe \( V \) is class-generic over \((HOD, S),\) and indeed over the even smaller inner model \( S = (L[S], S), \) where \( S \) is the Stability predicate. We refer to the inner model \( S \) as the Stable Core of \( V. \) The predicate \( S \) has a simple definition which is more absolute than any definition of HOD; in particular, it is possible to add reals which are not set-generic but preserve the Stable Core (this is not possible for HOD by Vopěnka’s theorem).

For an infinite cardinal \( \alpha, \) \( H(\alpha) \) consists of those sets whose transitive closure has size less than \( \alpha. \) Let \( C \) denote the closed unbounded class of all infinite cardinals \( \beta \) such that \( H(\alpha) \) has cardinality less than \( \beta \) whenever \( \alpha \) is an infinite cardinal less than \( \beta. \)

Definition 1 For a finite \( n > 0, \) we say that \( \alpha \) is \( n \)-Stable in \( \beta \) iff \( \alpha < \beta, \) \( \alpha \) and \( \beta \) are limit points of \( C \) and \( (H(\alpha), C \cap \alpha) \) is \( \Sigma_n \) elementary in \( (H(\beta), C \cap \beta). \)

The Stability predicate \( S \) places the Stability notion into a single predicate. \( S \) consists of all triples \((\alpha, \beta, n)\) such that \( \alpha \) is \( n \)-Stable in \( \beta. \) The \( \Delta_2 \) definable predicate \( S \) describes the “core” of \( V, \) in the following sense.

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Theorem 2  $V$ is generic over $(L[S], S)$ for an $(L[S], S)$-definable forcing. The same is true with $(L[S], S)$ replaced by $(M[S], S)$ for any definable inner model $M$.

Note that since $S$ is definable, $\text{HOD}[S] = \text{HOD}$. So we get:

Corollary 3  $V$ is generic over $\text{HOD}$ via a forcing which is definable in $V$.

In general the inner model $L[S]$ may be strictly smaller than HOD, as illustrated by the next result. For any model $N$, let $S^N$ denote $N$’s interpretation of the predicate $S$.

Theorem 4  (a) Suppose that $V$ is a set-generic extension of $M$. Then $S^M$ and $S^V$ agree above $\alpha$ for some ordinal $\alpha$. If $V$ is a $P$-generic extension of $M$ for a forcing $P$ of size less than the least $\beth$ fixed point of $M$, then $S^M$ equals $S^V$.

(b) Assuming GCH, $V$ has a generic extension of the form $L[R]$, where $R$ is a real not set-generic over $V$ and $S^V$ equals $S^{L[R]}$.

Corollary 5  It is consistent that $L[S]$ is properly contained in HOD.

The corollary follows from Theorem 4 by taking $V$ to be $L$ in part (b) of the theorem and observing that in the resulting model $L[R]$, $L[S]$ equals $L$, $R$ is not set-generic over $L$ but by Vopěnka’s theorem, $R$ is set-generic over $\text{HOD}$.

The proof of Theorem 2 comes in two parts. First we show that $V$ can be written as $L[F]$ where $F$ is a function from the ordinals to $2$ which “preserves” the Stability predicate $S$, in the sense that for $(\alpha, \beta, n)$ in $S$, $\alpha$ is $n$-Stable in $\beta$ relative to $F$. Then we use this function to prove the genericity of $V$ over $M[S]$ for any definable inner model $M$. The proof of Theorem 4 is via a refinement of the method of Jensen coding.

Forcing a Stability-preserving predicate

Our aim is to force a function $F$ from the ordinals to $2$ which codes $V$ (i.e., $V = L[F]$) and which obeys the following.
Suppose that $0 < n < \omega$ and $\alpha$ is $n$-Stable in $\beta$. Then $\alpha$ is $n$-Stable in $\beta$ relative to $F$: $(H(\alpha), C \cap \alpha, F \cap \alpha)$ is $\Sigma_n$ elementary in $(H(\beta), C \cap \beta, F \cap \beta)$.

To this end we define by induction on $\beta \in C$ a collection $P(\beta)$ of functions from $\beta$ to 2. For $0 < n < \omega$, we say that $\beta$ in $C$ is $n$-Admissible iff $\beta$ is a limit point of $C$ and $(H(\beta), C \cap \beta)$ satisfies $\Sigma_n$ replacement (with $C \cap \beta$ as an additional unary predicate). If $\alpha$ is $n$-Stable in some $\beta$ then $\alpha$ is $n$-Admissible.

If $\beta$ is not a limit point of $C$ then $P(\beta)$ consists of all functions $p : \beta \to 2$ such that $p \upharpoonright \alpha$ belongs to $P(\alpha)$ for all $\alpha \in C \cap \beta$. (Such functions exist, assuming that $P(\alpha)$ is nonempty for all $\alpha \in C \cap \beta$, a fact that we will verify.)

Suppose now that $\beta$ is a limit point of $C$. Let $P(< \beta)$ denote the union of the $P(\alpha)$, $\alpha \in C \cap \beta$, ordered by extension. Assuming extendibility for $P(< \beta)$, i.e. the statement that for $\alpha_0 < \alpha_1 < \beta$ in $C$, each $q_0$ in $P(\alpha_0)$ can be extended to some $q_1$ in $P(\alpha_1)$, this forcing adds a generic function which we denote by $\dot{f} : \beta \to 2$. We say that $p : \beta \to 2$ is $n$-generic for $P(< \beta)$ iff $G(p) = \{ p \upharpoonright \alpha \mid \alpha \in C \cap \beta \}$ meets every dense subset of $P(< \beta)$ of the form \{ $q \in P(< \beta) \mid q \vdash \varphi$ or $q \vdash \neg \varphi$ \}, where $\varphi$ is a $\Pi_n(H(\beta), C \cap \beta, \dot{f})$ sentence with parameters from $H(\beta)$. We define $P(\beta)$ to consist of all $p : \beta \to 2$ which are $n$-generic for $P(< \beta)$ for all $n$ such that $\beta$ is $n$-Admissible.

Let $P$ be the union of all of the $P(\beta)$’s, ordered by extension.

**Lemma 6** Assume Extendibility for $P$. Suppose that $G$ is $P$-generic over $V$ and let $F$ be the union of the functions in $G$. Then $V = L[F]$ and $(*)$ holds for $F$. Moreover, $V$ satisfies replacement with $F$ as an additional predicate.

**Proof.** Extendibility implies that it is dense to code any set of ordinals into the $P$-generic function $F$, from which it follows that $V$ is contained in $L[F]$. As $F \upharpoonright \alpha$ belongs to $V$ for each $\alpha \in C$ it also follows that $L[F]$ is contained in $V$ and therefore $L[F]$ equals $V$.

Suppose that $0 < n < \omega$ and $\alpha$ is $n$-Stable in $\beta$. The relation $q \vdash \varphi$ for $q$ in $P(< \beta)$ and $\Pi_1(H(\beta), C \cap \beta, \dot{f})$ sentences $\varphi$ with parameters from $H(\beta)$ is $\Pi_1$ over $(H(\beta), C \cap \beta)$: $q \vdash \varphi$ iff for all $r \leq q$ and transitive $T$ with $\text{Ord}(T) = \gamma \leq \text{Dom}(r)$, $(T, C \cap \gamma, r) \models \varphi$. It then follows by induction
on $n \geq 1$ that the relation $q \vDash \varphi$ for $q$ in $P(< \beta)$ and $\Pi_n(H(\beta), C \cap \beta, \dot{f})$ sentences $\varphi$ with parameters from $H(\beta)$ is $\Pi_n$ over $(H(\beta), C \cap \beta)$ (and the same for $\alpha$). As $F \upharpoonright \alpha$ is $n$-generic for $P(< \alpha)$, it follows that any true $\Pi_n(H(\alpha), C \cap \alpha, F \upharpoonright \alpha)$ sentence $\varphi$ with parameters from $H(\alpha)$ is $\Pi_n$ over $(H(\alpha), C \cap \alpha)$ (and the same for $\alpha$). As $F \upharpoonright \alpha$ is $n$-generic for $P(< \alpha)$, it follows that any true $\Pi_n(H(\alpha), C \cap \alpha, F \upharpoonright \alpha)$ sentence $\varphi$ with parameters from $H(\alpha)$ is forced by some condition $F \upharpoonright \alpha_0$, $\alpha_0 \in C \cap \alpha$. But then as $\alpha$ is $n$-Stable in $\beta$, $F \upharpoonright \alpha_0$ also forces “$\varphi$ holds in $(H(\beta), C \cap \beta, \dot{f} \upharpoonright \beta)$”; by the $n$-genericity of $F \upharpoonright \beta$, it follows that $\varphi$ holds in $(H(\beta), C \cap \beta, \dot{f} \upharpoonright \beta)$ when $\dot{f} \upharpoonright \beta$ is interpreted as the real $F \upharpoonright \beta$. Thus we have proved that $\alpha$ is $n$-Stable in $\beta$ relative to $F$. 

To verify replacement relative to $F$, we need only observe that the above implies that for each $n$, if $\alpha$ is $n$-Stable in $\text{Ord}$ (i.e., $(H(\alpha), C \cap \alpha)$ is $\Sigma_n$ elementary in $(V, C)$) then it remains so relative to $F$. □

We now turn to extendibility for $P$.

**Lemma 7** Suppose that $\alpha < \beta$ belong to $C$ and $p$ belongs to $P(\alpha)$. Then $p$ has an extension $q$ in $P(\beta)$.

**Proof.** By induction on $\beta$. The statement is immediate by induction if $\beta$ is not a limit point of $C$.

Suppose that $\beta$ is a limit point of $C$ but is not $1$-Admissible. Then there is a closed unbounded subset $D$ of $C \cap \beta$ of ordertype less than $\beta$ whose intersection with each of its limit points $\gamma < \beta$ is $\Delta_1$ definable over $(H(\gamma), C \cap \gamma)$. We can assume that both $\alpha$ and the ordertype of $D$ are less than the minimum of $D$. Now enumerate $D$ as $\beta_0 < \beta_1 < \cdots$ and using the induction hypothesis, successively extend $p$ to $q_0 \subseteq q_1 \subseteq \cdots$ with $q_i$ in $P(\beta_i)$, taking unions at limits. Note that for limit $i$, $q_i$ is indeed a condition because $\beta_i$ is not $1$-Admissible. The union of the $q_i$’s is the desired extension of $p$ in $P(\beta)$.

Next suppose that $\beta$ is $n$-Admissible but not $n + 1$-Admissible for some finite $n > 0$:

If $\beta$ is a limit of $n$-Stables (i.e., the set of $\alpha < \beta$ which are $n$-Stable in $\beta$ is cofinal in $\beta$), then proceed as in the previous paragraph: Choose a closed unbounded subset $D$ of $C \cap \beta$ of ordertype less than $\beta$ consisting of $n$-Stables in $\beta$, whose intersection with each of its limit points $\gamma < \beta$ is $\Delta_{n+1}$.
definable over \((H(\gamma), C \cap \gamma)\). Assume that both \(\alpha\) and the ordertype of \(D\) are less than the minimum of \(D\), enumerate \(D\) as \(\beta_0 < \beta_1 < \cdots\) and using the induction hypothesis, successively extend \(p\) to \(q_0 \subseteq q_1 \subseteq \cdots\) with \(q_i\) in \(P(\beta_i)\), taking unions at limits. For limit \(i\), \(q_i\) is indeed a condition because \(\beta_i\) is not \(n+1\)-Admissible and as it is a limit of \(n\)-Stables, \(q_i\) is \(n\)-generic for \(P(< \beta_i)\). The union of the \(q_i\)'s is the desired extension of \(p\) in \(P(\beta)\).

If \(\beta\) is not a limit of \(n\)-Stables then \(\beta\) must have cofinality \(\omega\) (else by \(n\)-Admissibility, we could find cofinally many \(n\)-Stables in \(\beta\) using the fact that \(\beta\) has uncountable cofinality). It suffices to show that any condition \(q\) in \(P(< \beta)\) can be extended to decide (i.e. force or force the negation of) each of fewer than \(\beta\)-many \(\Pi_n\) sentences with parameters from \(H(\beta)\) (given this, we can extend \(p\) in \(\omega\) steps to a condition in \(P(\beta)\) which is \(n\)-generic). To show this, let \((\phi_i | i < \delta)\) enumerate the given collection of \(\Pi_n\) sentences and if \(n > 1\), let \(D\) consist of all \(\gamma\) which are limits of \((n-1)\)-Stables in \(\beta\) and large enough so that \(H(\gamma)\) contains both \(q\) and this enumeration. (If \(n = 1\) then let \(D\) consist of all \(\gamma\) which are limit points of \(C\) and large enough so that \(H(\gamma)\) contains both \(q\) and this enumeration.) Now extend \(q\) successively to elements \(q_i\) of \(P(\gamma_i)\), where \(\gamma_{i+1} \geq \gamma_i\) is the least element of \(D\) so that either \(q_i\) forces \(\phi_i\) or \(q_{i+1}\) forces \(\psi_i = \text{the negation of } \phi_i\) (with corresponding witness to the \(\Sigma_n\) sentence \(\psi_i\)), taking unions at limits. For limit \(i\), \(q_i\) is a condition as \(\gamma_i\) is not \(n\)-Admissible but (in case \(n > 1\)) is a limit of \((n-1)\)-Stables. (The failure of \(\gamma_i\) to be \(n\)-Admissible uses the fact that the set of \(j < i\) such that \(q_{j+1}\) forces the negation of \(\phi_j\) can be treated as a parameter in \(H(\gamma_i)\).) As \(\beta\) is \(n\)-Admissible, this construction results in a sequence of \(q_i\)'s of length \(\delta\), whose union it the desired extension of \(q\) deciding all of the given \(\Pi_n\) sentences.

Finally, suppose that \(\beta\) is \(n\)-Admissible for every finite \(n\). Choose \(C\) to be closed unbounded in \(\beta\) so that any \(\gamma < \beta\) which is a limit point of \(C\) is a limit of \(n\)-Stables for every \(n\). (Note that we may choose \(C\) to be any cofinal \(\omega\)-sequence if \(\beta\) has cofinality \(\omega\).) Assume that \(\alpha\) is less than the least element of \(C\) and enumerate \(C\) as \(\beta_0 < \beta_1 < \cdots\). Then successively extend \(p\) to \(q_0 \subseteq q_1 \subseteq \cdots\) with \(q_i\) in \(P(\beta_i)\), taking unions at limits, and note that for limit \(i\), \(q_i\) is a condition because its \(n\)-genericity follows from the fact that \(\beta_i\) is a limit of \(n\)-Stables. The union of the \(q_i\)'s is the desired \(q\). \(\square\)

\(V\) is generic over the Stability predicate
Now fix a function $F : \text{Ord} \to 2$ as in the last section, i.e. with the following properties:

2. If $0 < n < \omega$ and $\alpha$ is $n$-Stable in $\beta$, then $\alpha$ is $n$-Stable in $\beta$ relative to $F$.

We devise a forcing $Q$ definable over $(L[S], S)$ such that for some $Q$-generic $G$, $V = L[S, G] = L[G]$ and $G$ is definable over $(V, F)$.

The language $\mathcal{L}$ is defined inductively as follows, where $\bar{F}$ is a unary function symbol.

1. For each ordinal $\alpha$, “$\bar{F}(\alpha) = 0$” and “$\bar{F}(\alpha) = 1$” are sentences of $\mathcal{L}$.
2. If $\Phi$ is a set of sentences of $\mathcal{L}$ and $\Phi$ belongs to $L[S]$, then $\bigwedge \Phi$ and $\bigvee \Phi$ are sentences of $\mathcal{L}$.

A sentence $\varphi$ of $\mathcal{L}$ is valid iff it is true when the symbol $\bar{F}$ is replaced by any function that belongs to a set-generic extension of $L[S]$. This notion is $L[S]$-definable and moreover if $\varphi$ is a sentence of $L[S]$ and $M$ is any outer model of $L[S]$, then $\varphi$ is valid in $L[S]$ iff it is valid in $M$.

Now let $T$ consist of all sentences of $\mathcal{L}$ of the form

$$\bigwedge (\Phi \cap H(\alpha)) \rightarrow \bigwedge (\Phi \cap H(\beta)),$$

where for some $\alpha < \beta$ and $1 < n < \omega$ we have:

(a) $\Phi$ is $\Sigma_n$ definable over $H(\beta) \cap L[S]$ using parameters from $H(\alpha) \cap L[S]$.
(b) $\alpha$ is $n$-Stable in $\beta$ (in $V$).

Note that (a) implies that $\Phi$ is $\Sigma_n$ definable over $(H(\beta), C \cap \beta)$ (using parameters from the $H(\alpha)$ of $V$). It follows that the sentences in $T$ are true

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1Indeed, if there is a function witnessing the non-validity of $\varphi$ in a set-generic extension of $M$ then we may assume that this generic extension is $M[G]$ where $G$ is generic for a Lévy collapse making $\varphi$ countable; then $L[S][G]$ also has a witness to the non-validity of $\varphi$, by Lévy absoluteness. Conversely, if the non-validity of $\varphi$ is witnessed in a set-generic extension of $L[S]$ then this will happen in $L[S][G]$ where $G$ is Lévy collapse generic over $L[S]$. Choose a condition in the Lévy collapse forcing this and $H$ containing this condition which is Lévy collapse generic over $M$; then the non-validity of $\varphi$ is witnessed in $M[H]$, a set-generic extension of $M$. 

6
when \( \hat{F} \) is interpreted as \( F \). Also note that \( T \) is \((L[S], S)\) definable, as (b) is expressed by the Stability predicate \( S \).

The desired forcing \( Q \) consists of all sentences \( \varphi \) of \( L \) which are consistent with \( T \), in the sense that for no subset \( T_0 \) of \( T \) is the sentence \( \bigwedge T_0 \to \sim \varphi \) valid. The sentences in \( Q \) are ordered by: \( \varphi \leq \psi \) iff \( T \) implies \( \varphi \to \psi \).

**Lemma 8** \( Q \) has the Ord-chain condition, i.e., any \((L[S], S)\)-definable maximal antichain in \( Q \) is a set.

**Proof.** Suppose that \( A \) is an \((L[S], S)\)-definable maximal antichain and consider \( \Phi = \{ \sim \varphi \mid \varphi \in A \} \). Then \( \Phi \) is also \((L[S], S)\)-definable. Choose \( n \) so that \( \Phi \) is \( \Sigma_n \)-definable over \((L[S], S)\) and choose \( \alpha \) to be \( n \)-Stable in \( \text{Ord} \) and large enough so that \( H(\alpha) \cap L[S] \) contains the parameters in the \( \Sigma_n \) definition of \( \Phi \). Then \( T \) together with \( \Phi \cap H(\alpha) \) implies \( \Phi \cap H(\beta) \) for all \( \beta \) greater than \( \alpha \) which are \( n \)-Stable in \( \text{Ord} \) and since there are arbitrarily large such \( \beta \), \( T \) together with \( \Phi \cap H(\alpha) \) implies all of \( \Phi \). It follows that \( A \) equals \( A \cap H(\alpha) \): Otherwise let \( \varphi \) belong to \( A \cap H(\alpha) \). As \( \sim \varphi \) belongs to \( \Phi \) it is implied by \( T \) together with \( \Phi \cap H(\alpha) \). But as \( A \) is an antichain, \( T \) together with \( \varphi \) implies \( \Phi \cap H(\alpha) \) and therefore \( T \) together with \( \varphi \) implies \( \sim \varphi \), contradicting the fact that \( \varphi \) belongs to \( Q \). \( \square \)

Now it is easy to see that \( V = L[F] = L[G] \) where \( G \) is \( Q \)-generic over \((L[S], S)\): Let \( G \) consist of all sentences in \( Q \) which are true when \( \hat{F} \) is interpreted as \( F \). It is obvious that \( G \) intersects all maximal antichains of \( Q \) which are sets in \( L[S] \), as if the set \( A \) is an antichain missed by \( G \) then \( \bigwedge \{ \sim \varphi \mid \varphi \in A \} \) is consistent with \( T \) and witnesses the failure of \( A \) to be maximal. By Lemma 8 this gives full genericity over \((L[S], S)\).

The above argument was carried out for the ground model \( L[S] \). But the same argument can be used for any ground model \( M[S] \) provided \( M \) is a definable inner model; simply replace \( n \) by \( n - k - 1 \) in (a) above, where \( M \) is \( \Sigma_k \)-definable. This completes the proof of Theorem 2.

**Preserving \( S \) when coding**

We sketch the proof of Theorem 4. Part (a) of the theorem is clear, because when applying a set-forcing \( P \), the Stability predicate is not affected above the size of \( P \).
(b) is proved as follows: Again write $V$ as $L[F]$ where $F$ preserves the Stability predicate. Now we describe a version of Jensen coding that produces a real $R$ such that:

i. $R$ is class-generic but not set-generic over $(V,F)$.
ii. $V$ is contained in $L[R]$ and $F$ is definable in $L[R]$ with parameter $R$.

Note that as we have assumed GCH, the class $C$ is just the class of all infinite cardinals.

Let $P_0$ be the version of Jensen coding defined in [1], Section 4.3, but with the following modification: We require that for limit cardinals $\alpha$ which are $n$-Admissible, conditions in $P_0^\mathcal{F} \setminus P_0(< \alpha)$ are $n$-generic for $P_0(< \alpha)$, i.e., decide all $\Pi_n(H(\alpha), C \cap \alpha, F \upharpoonright \alpha, \check{G}(< \alpha))$ sentences, where $\check{G}(< \alpha)$ denotes the $P_0(< \alpha)$-generic. This thinning of the forcing does not affect the proofs of extendibility and distributivity and has the consequence that if $G_0$ is $P_0$-generic and $\alpha$ is $n$-Stable in $\beta$ relative to $F$ then $\alpha$ is also $n$-Stable in $\beta$ relative to $F, G_0$. As $F$ preserves the Stability predicate, it follows that the $P_0$-generic real $R$ does as well. □

Some final remarks

Is the Stable Core $S$ the long sought-after “ultimate core model” of $V$? To answer this it is necessary to first answer the following questions:

Question 1. Does the existence of large cardinals in $V$ imply their existence in the Stable Core? Is the Stable Core rigid in the sense that there is no nontrivial elementary embedding of it to itself?

As $V$ is generic over the Stable Core there is reason to hope for a positive answer to Question 1.

Question 2. Does the Stable Core satisfy GCH and $\Box$ principles?

Unfortunately the Stable Core exhibits no condensation properties which would suggest a positive answer to Question 2. One may however hope to enrich the Stability predicate to obtain condensation and a positive answer to Question 2 for a modified version of the Stable Core.
Regardless of the answers to the above questions, the Stable Core does at least reveal the following: The notion of Stability is fundamental to our understanding of the structure of the set-theoretic universe.

References
