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STRONG CODING

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We present here a refinement of the method of Jensen coding [7] and apply it to the study of admissible ordinals. An ordinal α is recursively inaccessible if it is both admissible and the limit of admissible ordinals. Solovay asked if it is consistent to have a real R such that the R-admissible ordinals equal the recursively inaccessible ordinals. This is a problem in class forcing as any real in a set generic extension of L must preserve the admissibility of a final segment of the admissible ordinals.

Our main theorem provides an affirmative solution to Solovay's problem.

Theorem. $\mathrm{Con(ZF)} \to \mathrm{Con}(\mathrm{ZF} + \exists R \subseteq \omega$ (*R-admissibles = Recursively Inaccessibles*)).

Our proof strategy is described by:

Main Lemma. *Let M be a transitive model of $\mathrm{ZF} + V = L$. Then there is a $\Delta_1(M)$-class forcing notion $\mathcal{P}$ for producing a generic real R and $A \subseteq \mathrm{ORD}(M)$ such that*:

(a) *α is A-admissible iff α is recursively inaccessible, for $\alpha \in \mathrm{ORD}(M)$.*
(b) *$M[R]$ is a model of* ZF.
(c) *$A \cap \alpha$ is $\Delta_1(L_\alpha(R))$ for all admissible $\alpha \in M$.*
(d) *If $\alpha \in M$ is A-admissible, then α is R-admissible.*

Thus our solution to Solovay's problem is based on a 'strong coding' theorem, in which a certain predicate $A \subseteq \mathrm{ORD}$ is coded by a real R in such a way that the decoding of $A \cap \alpha$ from R can be carried out in $L_\alpha[R]$ for every admissible α. Note that it is possible to define $A \subseteq \mathrm{ORD}(M)$ so as to obey (a) above, A Δ_1 over M. However we find it necessary to build A simultaneously with the generic real R which strongly codes it.

Jensen's coding methods do not suffice as the recovery of $A \cap \alpha$ from R when α is admissible cannot necessarily be carried out in $L_\alpha[R]$. In fact, Jensen coding is designed so as to guarantee this recovery only when every L_α-cardinal is a cardinal of L.

Our coding method is an amalgamation of Jensen-style codings $\mathcal{P}^\beta$, one for each ordinal β which is either admissible or the limit of admissible ordinals. $\mathcal{P}^\beta$ is designed to produce a real R generic over L_β such that $A \cap \beta$ is $\Delta_1(L_\beta(R))$. This alone could be accomplished as in Jensen [7] by a 'reverse iteration' of almost disjoint set forcings, based on the cardinals in the sense of L_β. However the different $\mathcal{P}^\beta$ forcings must fit together in a special way, so that reals exist which are generic simultaneously for each of them.

It is this last coherence condition that is the main source of complexity in our construction. It requires that the conditions used in $\mathcal{P}^\beta$ be built out of sets which are partially generic for earlier forcings $\mathcal{P}^{\beta'}$, $\beta' < \beta$. Obtaining these partial generics is one of the main lemmas in our proof and draws on fine structure techniques from Friedman [5]. By defining $\mathcal{P}^\beta$ in this way as a 'generic Jensen coding' we can guarantee that any $\mathcal{P}$-generic real is $\mathcal{P}^\beta$-generic for each β, where $\mathcal{P} = \bigcup_\beta \mathcal{P}^\beta$.

The key lemma in Jensen [7] is the distributivity lemma, which is needed to show that his forcing is cardinal-preserving. An easier lemma is established first, the extendibility lemma, which states that a forcing condition can be extended 'arbitrarily far' in order to code more of the ground model. Similar lemmas occur here, however the built-in genericity of our conditions requires that both extendibility and a strengthened form of distributivity be established together, by a simultaneous induction.

Variants of our Main Lemma can be used to realize other 'admissibility spectra' by a class-generic real. The key hypothesis needed at this point is a strong definability assumption on the spectrum.

Corollary to Proof. *Suppose* $B \subseteq \mathrm{ORD}$ *and for all p.r. closed* β, $B \cap \beta$ *is* $\Delta_1(L_\beta)$, *uniformly. Let* $\Lambda =$ *admissible limits of* B. *Then* $\mathrm{Con(ZF)} \rightarrow \mathrm{Con}(\mathrm{ZF} + \exists R \subseteq \omega$ $(R\text{-}Admissibles = \Lambda))$.

Some final remarks before we begin the proof: (a) The (original) Jensen coding method does suffice to prove a weak form of our theorem: $\mathrm{Con(ZF)} \rightarrow \mathrm{Con}(\mathrm{ZF} + \exists R$ (Every R-admissible is recursively inaccessible)). This was established independently in David [2], where it is shown that 'is recursively inaccessible' can be replaced by 'belongs to X', X any Σ_1-class of admissibles containing all L-cardinals. Strong coding is not required for this result as the primary goal is to destroy admissibility, not to preserve it. (b) Further techniques of Jensen [7] will be used in Section 2, Part E to show: If $0^\#$ exists, then there is a real R such that R-admissibles = Recursively Inaccessibles.

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CONTENTS

SECTION ONE: THE CONDITIONS

A. Introduction

We will inductively define forcings $\mathcal{P}^\beta$ for $\beta \in \overline{\text{Adm}} = \{\beta \mid \beta$ is either admissible or the limit of admissibles$\}$. A $\mathcal{P}^\beta$-generic set over L_β determines a real R and $A \subseteq \beta$ such that A is $\Delta_1(L_\beta[R])$ and for $\alpha < \beta$: α is A-admissible iff α is recursively inaccessible. The definition of $\mathcal{P}^\beta$ depends not only on the definitions of $\mathcal{P}^{\beta'}$, $\beta' < \beta$ but is only comprehensible given that certain fundamental properties of these 'earlier forcings' have been established. We therefore begin by listing these properties in the form of lemmas, to be proved later by a simultaneous induction on β.

The definition of the $\mathcal{P}^\beta$ forcing requires us to define auxiliary forcings $\mathcal{P}^\beta_\kappa$ where $\beta \in \overline{\text{Adm}}$ and $\kappa \in \beta\text{-Card} = \{\gamma \mid \gamma$ is an infinite β-cardinal or $\gamma = 0\}$. If $\kappa \in \beta$-Card let $(\kappa^+)^{L_\beta}$ denote the least infinite β-cardinal greater than κ if there is such, β otherwise (thus $0^+ = \omega$ and we think of ω as a successor cardinal). A set $\mathcal{P}^\beta_\kappa$-generic over L_β determines a subset X of $(\kappa^+)^{L_\beta}$ and $A \subseteq \beta$ such that A is Δ_1 over $L_\beta[X]$ and for $\alpha \in [\kappa, \beta) = \{\gamma \mid \kappa \leqslant \gamma < \beta\}: \alpha$ is A-admissible iff α is recursively inaccessible. We shall have: $\mathcal{P}^\beta = \mathcal{P}^\beta_0$.

Lemma 1A.1. *Suppose β belongs to $\overline{\text{Adm}}$, $\kappa \in \beta$-Card. Then*:

(a) *$\mathcal{P}^\beta_\kappa \subseteq L_\beta$ and is uniformly Δ_1 over $\langle L_\beta, \beta\text{-Card}\rangle$.*

(b) *If $\beta > \beta' \in \overline{\text{Adm}}$, $\kappa < \beta'$, then $\mathcal{P}^{\beta'}_\kappa \subseteq \mathcal{P}^\beta_\kappa$. Whenever $p_1, p_2 \in \mathcal{P}^\beta_\kappa$ are compatible in $\mathcal{P}^\beta_\kappa$, say $p \in \mathcal{P}^\beta_\kappa$ and $p \leqslant p_1, p_2$, then $p' \leqslant p_1, p_2$ for some $p' \in \mathcal{P}^{\beta'}_\kappa$, $p \leqslant p'$ and hence p_1, p_2 are compatible in $\mathcal{P}^{\beta'}_\kappa$. If $p_1, p_2 \in \mathcal{P}^\beta_\kappa$ are incompatible in $\mathcal{P}^\beta_\kappa$, $p_1 \in \mathcal{P}^{\beta'}_\kappa$, then p_1, p_2' are incompatible in $\mathcal{P}^\beta_\kappa$ for some $p_2' \in \mathcal{P}^{\beta'}_\kappa$, $p_2 \leqslant p_2'$.*

Proof. Deferred.

Suppose $\mathcal{P}$ is a partial ordering of a subset of L_β. We say that $G \subseteq \mathcal{P}$ is *$\mathcal{P}$-generic* over L_β if:

(i) $p \in G$, $p \leqslant q \to q \in G$. $p_1, p_2 \in G \to p_1, p_2$ are compatible in $\mathcal{P}$.

(ii) $p_1, p_2 \in \mathcal{P} \to \exists q \in G$ ($q \leqslant p_1, p_2$ or q is incompatible with p_1 or q is incompatible with p_2).

(iii) $\mathcal{D} \in L_\beta$, $\mathcal{D}$ predense on $\mathcal{P} \to G \cap \mathcal{D} \neq \emptyset$, where $\mathcal{D}$ is *predense* on $\mathcal{P}$ if $\mathcal{D}^* = \{p \mid p \leqslant \text{some } q \in \mathcal{D}\}$ is dense on $\mathcal{P}$.

Lemma 1A.2. *There is a function $f(\kappa, \beta, X)$ such that*:

(a) *If $\kappa \in \beta$-Card, $\beta \in \overline{\text{Adm}}$, $X \subseteq [\kappa, \beta)$, then $f(\kappa, \beta, X) \subseteq L_\beta$ is Δ_1 over $\langle L_\beta[X \cap (\kappa^+)^{L_\beta}], \beta\text{-Card}\rangle$.*

(b) *If* $G \subseteq L_\beta$ *is* $\mathcal{P}^\beta_\kappa$*-generic over* L_β*, then* $G = f(\kappa, \beta, X)$ *for some unique* $X \subseteq [\kappa, \beta)$*. Moreover,* $X \cap \beta'$ *is uniformly* Δ_1 *over* $L_{\beta'}[G]$ *for admissible ordinals* $\beta' \leq \beta$ *(and this definition is independent of* X, β', G*).*

Proof. Deferred.

The function f describes the 'decoding' process. Intuitively, Lemma 1A.2 says that a set $\mathcal{P}^\beta_\kappa$-generic over L_β determines and is uniquely determined by a subset of $(\kappa^+)^{L_\beta}$. Note that it follows from Lemma 1A.2 that if $f(\kappa, \beta, X)$ is $\mathcal{P}^\beta_\kappa$-generic over L_β, then X is uniquely determined by $X \cap (\kappa^+)^{L_\beta}$.

Preserving the admissibility of an A-admissible ordinal β requires that we consider a stronger form of genericity.

Definition. Suppose $\mathcal{P}$ is a partial ordering of a subset of L_β and $T \subseteq \mathcal{P} \times \gamma$ for some $\gamma < \beta$. Let $\mathcal{D}(T)$ consist of all $p \in \mathcal{P}$ such that either:

(i) for some $\delta < \gamma$, $q \leq p \rightarrow (q, \delta) \notin T$, or

(ii) for all $\delta < \gamma$, $T_\delta = \{p \in \mathcal{P} \mid (p, \delta) \in T\}$ is dense below p (i.e., $q \leq p \rightarrow \exists r \leq q, r \in T_\delta$).

We say that $G \subseteq \mathcal{P}$ is *$\mathcal{P}$-Σ-generic* over L_β if G is $\mathcal{P}$-generic over L_β and $G \cap \mathcal{D}(T) \neq \emptyset$ for all $T \subseteq \mathcal{P} \times \gamma$, $\gamma < \beta$ which are Σ_1 over L_β and *persistent* ($(p, \delta) \in T$, $q \leq p \rightarrow (q, \delta) \in T$).

Remark. In most cases the condition 'Σ_1 over L_β' can be replaced by the stricter 'Δ_1 over L_β' by considering $T^* = \{(p, \delta) \mid \text{for some } q \geq p,\ L_\gamma \models q \in T_\delta$, where $\gamma = L\text{-rank}(p)\}$; under reasonable hypotheses $\mathcal{D}(T^*) \subseteq \mathcal{D}(T)$, T^* is Δ_1 over L_β; T^* is persistent. This is useful in the proof of Lemma 1D.2.

The point of Σ-genericity is that if $\mathcal{P} \Vdash KP$ (=admissibility theory) and the forcing relation of $\mathcal{P}$ restricted to ranked sentences is Σ_1, then $\langle L_\beta[G], G\rangle$ is admissible whenever $G \subseteq \mathcal{P}$ is $\mathcal{P}$-Σ-generic over L_β. For, the Truth Lemma holds for ranked sentences (using Σ-genericity) and so if $f: \gamma \rightarrow \beta$ is $\Sigma_1\langle L_\beta[G], G\rangle$ then Σ-genericity implies that $p \Vdash f$ is total, for some $p \in G$. Then $p \Vdash f$ is bounded as $\mathcal{P} \Vdash KP$.

Lemma 1A.3 (Genericity Lemma). *Suppose* $\beta_1 < \beta_2$ *belong to* $\overline{\text{Adm}}$, $\kappa \in \beta_1$-$\text{Card} \cap \beta_1$*. If* $\mathcal{D} \in L_{\beta_1}$ *is predense on* $\mathcal{P}^{\beta_1}_\kappa$*, then* $\mathcal{D}$ *is predense on* $\mathcal{P}^{\beta_2}_\kappa$*. If* β_1 *is recursively inaccessible and* $T \subseteq \mathcal{P}^{\beta_1}_\kappa \times \gamma$, $\gamma < \beta_1$ *is persistent and* $\Sigma_1(L_{\beta_1})$*, then* $\mathcal{D}(T)$ *is predense on* $\mathcal{P}^{\beta_2}_\kappa$.

Proof. Deferred.

Corollary 1A.4. *Suppose* $\beta \in \overline{\text{Adm}}$, $\kappa \in \beta$-Card, G *is* $\mathcal{P}^\beta_\kappa$*-generic over* L_β *and* $f(\kappa, \beta, X) = G$*. Then*:

(a) *If* $\beta > \beta' \in \overline{\text{Adm}}$, $\kappa < \beta'$, *then* $G \cap L_{\beta'}$ *is also* $\mathcal{P}_\kappa^{\beta'}$*-generic over* $L_{\beta'}$ *and* $f(\kappa, \beta', X \cap \beta') = G \cap L_{\beta'}$. *If in addition* β' *is recursively inaccessible, then* $G \cap L_{\beta'}$ *is* $\mathcal{P}_\kappa^{\beta'}$*-*$\Sigma$*-generic over* $L_{\beta'}$.

(b) G *is* Δ_1 *over* $L_\beta[X \cap (\kappa^+)^{L_\beta}]$.

Proof. (a) The genericity of $G \cap L_{\beta'}$ follows immediately from the Genericity Lemma and Lemma 1A.1(b). Suppose $f(\kappa, \beta', X') = G \cap L_{\beta'}$. By the uniformity in Lemma 1A.2(b), $X \cap \beta'$ has the same Δ_1-definition over $L_{\beta'}[G]$ as X' does over $L_{\beta'}[G \cap L_{\beta'}]$. Of course $L_{\beta'}[G] = L_{\beta'}[G \cap L_{\beta'}]$ so $X' = X \cap \beta'$.

(b) There are two cases (so the Δ_1-definition is not uniform): If there is a greatest β-cardinal, then this follows from Lemma 1A.2(a) since in this case β-Card is $\Delta_1(L_\beta)$. Otherwise it follows from (a) above that $G \cap L_{\beta'} = f(\kappa, \beta', X \cap \beta')$ whenever $(\kappa^+)^{L_\beta} < \beta' < \beta$, $\beta' \in \overline{\text{Adm}}$. But then G is Δ_1 over $L_\beta[X \cap (\kappa^+)^{L_\beta}]$ as $G \cap L_{\beta'}$ is uniformly Δ_2 over $L_{\beta'}[X \cap (\kappa^+)^{L_\beta}]$ for such β'. □

Corollary 1A.4(b) is important as we will later use it to conclude that $A \cap \beta$ is Δ_1 over $L_\beta[X \cap (\kappa^+)^{L_\beta}]$ whenever $f(\kappa, \beta, X)$ is $\mathcal{P}_\kappa^\beta$-generic over L_β. Thus it follows from Corollary 1A.4 that if G is $\mathcal{P}$-generic over L where $\mathcal{P} = \bigcup \{\mathcal{P}^\beta \mid \beta \in \overline{\text{Adm}}\}$, then for some $X \subset \text{ORD}$, $G \cap L_\beta = f(0, \beta, X \cap \beta)$ for all $\beta \in \overline{\text{Adm}}$ and hence $A \cap \beta$ is $\Delta_1(L_\beta[X \cap \omega])$ for all $\beta \in \overline{\text{Adm}}$. So $X \cap \omega$ 'strongly codes' A.

Lemma 1A.5 (The Generic Existence Lemma). *Suppose* $\beta \in \overline{\text{Adm}}$, $\kappa \in \beta$*-Card and* $\mu > \beta$ *is p.r. closed,* $L_\mu \vDash \text{card}(\beta) \leq \kappa$. *Then* $p \in \mathcal{P}_\kappa^\beta \to \exists G \in L_\mu$ *(*G *is* $\mathcal{P}_\kappa^\beta$*-generic over* L_β, $p \in G$*). If in addition* β *is recursively inaccessible, then* $p \in \mathcal{P}_\kappa^\beta \to \exists G \in L_\mu$ *(*G *is* $\mathcal{P}_\kappa^\beta$*-*Σ*-generic over* L_β, $p \in G$*).*

Proof. Deferred.

As was suggested earlier the forcing conditions in $\mathcal{P}_\kappa^\beta$ are built out of sets which are generic for forcings $\mathcal{P}_{\kappa'}^{\beta'}$, $\beta' < \beta$. The Generic Existence Lemma says that these sets exist in abundance.

Lemma 1A.6 (The Distributivity Lemma). *If* β *is recursively inaccessible,* $\kappa \in \beta$-Card, *then* $\mathcal{P}_\kappa^\beta$ *is* Σ*-distributive over* L_β*; that is, if* $\{T_i \mid i < \kappa\}$ *is a collection of predense subsets of* $\mathcal{P}_\kappa^\beta$ *and* $\{(i, p) \mid p \in T_i\} \in \Sigma_1(L_\beta)$*, then for any* $p \in \mathcal{P}_\kappa^\beta$ *there exists* $q \leq p$, $q \in \bigcap \{T_i^* \mid i < \kappa\}$.

Proof. Deferred.

The Distributivity Lemma is used for cardinal preservation, in the proof of the Generic Existence Lemma and in establishing that the forcing $\mathcal{P}$ preserves recursively inaccessibles. A much stronger form of distributivity will in fact be established later which is concerned with certain collections $\{\mathcal{D}_i \mid i<\kappa\}$ of predense sets on $\mathcal{P}^\beta_\kappa$ which are not necessarily definable over L_β.

Lemma 1A.7 (Factoring, Chain Condition). *If $\beta \in \overline{\text{Adm}}$, $\kappa \in \beta$-Card, then for all $\gamma \in \beta$-Card $\cap\, \kappa$, $\mathcal{P}^\beta_\gamma$ is equivalent to an iteration $\mathcal{P}^\beta_\kappa * \mathcal{P}^{\mathbf{G}_\kappa}_\gamma$ (the two forcings have dense subsets which are isomorphic via a $\Delta_1(L_\beta)$ isomorphism). Moreover, if β is recursively inaccessible, then $\mathcal{P}^\beta_\kappa \Vdash \mathcal{P}^{\mathbf{G}_\kappa}_\gamma$ has the Σ-κ^+-c.c. (any $\Sigma_1\langle L_\beta[\mathbf{G}_\kappa], \mathbf{G}_\kappa\rangle$-predense $D \subseteq \mathcal{P}^{\mathbf{G}_\kappa}_\gamma$ can be effectively thinned to a predense $D' \subseteq D$ which is an element of L_β of β-cardinality $\leqslant\kappa$).*

Proof. Deferred.

Lemma 1A.8 (Δ_1-Definability of Forcing). *If β is recursively inaccessible, $\kappa \in \beta$-Card, then the forcing relation for $\mathcal{P}^\beta_\kappa$ is $\Delta_1(L_\beta)$ when restricted to ranked sentences.*

Proof. Deferred.

The above lemmas imply that if R is a $\mathcal{P}$-generic real, then every recursively inaccessible ordinal is R-admissible. Indeed choose $X \subseteq \text{ORD}$ so that $X \cap \omega = R$ and $\bigcup \{f(0, \beta, X\cap\beta) \mid \beta \in \overline{\text{Adm}}\} = G$ is $\mathcal{P}$-generic (this is what we mean by the phrase 'R is $\mathcal{P}$-generic'). Then $G \cap L_\beta$ is $\mathcal{P}^\beta$-generic over L_β for $\beta \in \overline{\text{Adm}}$ by the Genericity Lemma and thus by the Δ_1-definability of Forcing, Distributivity, Factoring and Chain Condition we have that β recursively inaccessible, $L_\beta \models \kappa$ a cardinal$\rightarrow L_\beta[R] \models \kappa$ a cardinal. So it suffices to consider recursively inaccessible β such that L_β has a largest cardinal κ. Now $G \cap \mathcal{P}^\beta$ is $\mathcal{P}^\beta$-Σ-generic over L_β and so by Factoring, the Δ_1-definability of Forcing, the Chain Condition and Σ-distributivity if $f:\kappa \rightarrow \beta$ is $\Sigma_1(L_\beta[R])$ there exists $p \in G \cap \mathcal{P}^\beta$ so that $p = (p_\kappa, (p)^\kappa) \Vdash f$ is bounded (where $\Vdash$ refers to $\mathcal{P}^\beta \approx \mathcal{P}^\beta_\kappa * \mathcal{P}^{\mathbf{G}_\kappa}$). The fact that $\mathcal{P}$-generic reals destroy the admissibility of successor admissibles will follow easily from the definition and Extendibility properties of $\mathcal{P}$.

Having stated some of the basic properties of the forcings $\mathcal{P}^\beta_\kappa$ we now begin to describe the conditions used to code at successor cardinals.

B. Successor cardinal coding I: Generic codes

The forcings $\mathcal{P}^\beta_\kappa$ are in fact defined in terms of the more basic forcings $\mathcal{P}^s_\kappa$, where s is a certain type of partial characteristic function. We now describe the functions s that we wish to consider.

For $\beta \in \overline{\text{Adm}}$, $\kappa \in \beta$-Card we define a collection S^β_κ of characteristic functions on proper initial segments of $[\kappa, (\kappa^+)^{L_\beta}) = \{\gamma \mid \kappa \leqslant \gamma < (\kappa^+)^{L_\beta}\}$. If β is a limit of admissibles, we simply define $S^\beta_\kappa = \bigcup \{S^{\beta'}_\kappa \mid \kappa < \beta' < \beta, \beta' \in \overline{\text{Adm}}\}$. So we now focus on the case where β is a successor admissible. For any ordinal γ let $\tilde{\gamma}$ denote the least admissible greater than γ and $\gamma^- = \sup(\gamma \cap \mathrm{Adm})$.

Now fix a successor admissible $\tilde{\beta}$ where $\beta \in \overline{\mathrm{Adm}}$ and let $\kappa \in \tilde{\beta}$-Card. For notational simplicity we use κ^+ to denote $(\kappa^+)^{L_{\tilde{\beta}}}$. We let $\mathcal{O}(\kappa)$ denote $\{\xi \in [\kappa, \kappa^+) \mid L_\xi \models \mathrm{card}(\xi) = \kappa\}$. For $\xi \in \mathcal{O}(\kappa)$ define:

$$\mu_\xi^0 = \sup\{\mu_{\xi'} \mid \xi' < \dot{\xi}\} \quad (= \kappa \text{ if } \xi = \kappa),$$
$$\mu_\xi^{i+1} = \text{least p.r. closed } \mu > \mu_\xi^i \text{ s.t. } L_\mu \models \mathrm{card}(\xi) = \kappa,$$
$$\mu_\xi = \sup\{\mu_\xi^i \mid i \in \omega\}.$$

Note that either $L_{\mu_\xi} \models \mu_\xi^0 = \kappa^+$ or $\mu_\xi^- \leqslant \mu_\xi^0 < \mu_\xi$. Also extend the definition of μ_ξ^i to all $\xi \in [\kappa, \kappa^+)$ by setting $\mu_\xi^i = \mu_{\xi'}^i$ where $\xi' = \inf(\mathcal{O}(\kappa) - \xi)$. Now $S_\kappa^{\tilde{\beta}}$ consists of all $s : [\kappa, |s|) \to 2$, $\kappa \leqslant |s| < \kappa^+$ such that: Either $s \in S_\kappa^\beta$ or $\beta \leqslant |s| \in \mathcal{O}(\kappa)$ and:

(a) Let $X_s = \{\delta \in [\kappa, \beta) \mid s(\delta) = 1\}$. Then $f(\kappa, \beta, X_s)$ is $\mathcal{P}_\kappa^\beta$-generic over L_β (if β is recursively inaccessible it is $\mathcal{P}_\kappa^\beta$-Σ-generic over L_β) and $\langle L_\beta, s \upharpoonright \beta \rangle$ is inadmissible if β is a successor admissible.

(b) For $\kappa < \xi \leqslant |s|$, $s \upharpoonright \xi$ is $\Delta_1^*(\mathcal{A}(\mu_\xi^0))$.

Remarks. (1) Condition (a) has the intuitive meaning that $s \upharpoonright \beta$ is 'generic'. (b) exerts some control on the initial segments of s. Both the structure $\mathcal{A}(\nu)$ and the notion of Δ_1^*-definability are described later in this part. The condition "$s \upharpoonright \xi$ is $\Delta_1^*(\mathcal{A}(\mu_\xi^0))$" is much stronger than "$s \upharpoonright \xi \in L_{\mu_\xi^1}$".

(2) The function $\xi \to \mu_\xi$ is strictly increasing on $\mathcal{O}(\kappa)$ and discontinuous at each limit ordinal in $\mathcal{O}(\kappa)$. If $\xi \in [(\kappa^+)^{L_\beta}, \beta)$, then $\mu_\xi^- = \beta$ and $\mu_\xi^0 = (\kappa^+)^{L_\beta}$.

(3) In the above, we include the case $\kappa = \beta$, in which case clause (a) says nothing.

Now define $S_\kappa^\beta = \bigcup\{S_\kappa^{\beta'} \mid \beta' \in \mathrm{Adm} \cap \beta\}$ whenever $\kappa \in \beta$-Card, β a limit of admissibles. Thus we have defined S_κ^β whenever $\beta \in \overline{\mathrm{Adm}}$, $\kappa \in \beta$-Card. Also extend the definitions of μ_ξ^i to all $\xi \in [\kappa, (\kappa^+)^{L_\beta})$ for such β, κ. Note that if $\beta < \beta'$ belong to $\overline{\mathrm{Adm}}$, $\kappa \in \beta'$-Card, $\xi \in [\kappa, (\kappa^+)^{L_\beta})$, then the μ_ξ^i have the same definitions in L_β as they do in $L_{\beta'}$. We next prove some basic facts about the S_κ^β's.

Lemma 1B.1. *Suppose $\beta \in \overline{\mathrm{Adm}}$, $\kappa \in \tilde{\beta}$-Card. Then $S_\kappa^\beta \neq S_\kappa^{\tilde{\beta}}$ only if $\kappa = \mathrm{gc}\,\tilde{\beta}$ = largest $\tilde{\beta}$-cardinal.*

Proof. If $\kappa < \mathrm{gc}\,\tilde{\beta}$, then μ_β is not defined (in $L_{\tilde{\beta}}$) so (b) in the definition of $S_\kappa^{\tilde{\beta}}$ implies that $|s| < \beta$ whenever $s \in S_\kappa^{\tilde{\beta}}$. So $S_\kappa^{\tilde{\beta}} = S_\kappa^\beta$. □

The 'only if' in this lemma can be strengthened to 'if and only if'. This follows from Lemma 1.B4 and (a strong form of) the Generic Existence Lemma.

Lemma 1B.2. $s \in S_\kappa^\beta \to \mu_{|s|}^- \leqslant |s| < \mu_{|s|}$.

Proof. The inequality $|s| < \mu_{|s|}$ is easily established using the definition of μ_ξ. Now given $s \in S^\beta_\kappa$ let $\gamma \in \overline{\text{Adm}}$ be least so that $s \in S^{\bar\gamma}_\kappa$. Then $\gamma \leq |s|$ (else $s \in S^\gamma_\kappa$ and hence $s \in S^{\bar\delta}_\kappa$ for some $\delta \in \overline{\text{Adm}} \cap \gamma$, contradicting choice of γ). But then $\mu_{|s|} < \bar\gamma$ and hence $\gamma = \mu_{\overline{|s|}} \leq |s|$. □

Note that $s \in S^\beta_\kappa$, $\xi < |s|$ does not in general imply that $s \restriction [\kappa, \xi) \in S^\beta_\kappa$. For, suppose $L_{|s|} \models \kappa^+$ exists where $|s|$ is admissible, $\xi = (\kappa^+)^{L_{|s|}} < |s|$. Thus $\xi \notin \mathcal{O}(\kappa)$ so $s \restriction [\kappa, \xi) \notin S^\beta_\kappa$. The purpose of the next lemma is to show that this describes the only obstacle to the above implication.

Lemma 1B.3. *Suppose β belongs to $\overline{\text{Adm}}$, $s \in S^\beta_\kappa$ and $\xi \in \mathcal{O}(\kappa) \cap |s|$. Then $s \restriction [\kappa, \xi) \in S^\beta_\kappa$.*

Proof. It's enough to show that $s \restriction [\kappa, \xi) \in S^{\bar\xi}_\kappa$. Let $\beta' = \mu_{\bar\xi} = (\xi)^-$. By Corollary 1A.4(a), $f(\kappa, \beta', X_s \cap \beta')$ is $\mathcal{P}^{\beta'}_\kappa$-generic over $L_{\beta'}$ (and $\mathcal{P}^{\beta'}_\kappa$-$\Sigma$-generic over $L_{\beta'}$ if β' is recursively inaccessible). But $X_s \cap \beta' = X_{s \restriction [\kappa, \beta')}$ so (a) is satisfied. (b) is satisfied as it holds for s and can be verified inside any L_ν such that $s \restriction \xi \in L_\nu$, $L_\nu \models \kappa =$ largest cardinal. □

The next lemma is useful in establishing that a partial function s belongs to S^β_κ. It also points out a redundancy in our definition of $S^{\bar\beta}_\kappa$.

Lemma 1B.4. *Suppose $\beta \in \overline{\text{Adm}}$, $\kappa = \text{gc}\,\bar\beta$, $s : [\kappa, \beta) \to 2$, s is $\Delta^*_1(\mathcal{A}(\mu^0_\beta))$. Define X_s as in* (a) *above. If $f(\kappa, \beta, X_s)$ is $\mathcal{P}^\beta_\kappa$-generic over L_β ($\mathcal{P}^\beta_\kappa$-Σ-generic over L_β if β is recursively inaccessible), then $s \in S^{\bar\beta}_\kappa$.*

The proof of Lemma 1B.4 depends upon a technical fact concerning the forcings $\mathcal{P}^\beta_\kappa$. This fact intuitively states that conditions in $\mathcal{P}^\beta_\kappa$ are constructed out of element of the various $S^{\beta'}_{\kappa'}$.

Lemma 1B.5. *Suppose $\beta \in \overline{\text{Adm}}$, $\kappa \in \beta$-Card, $s : [\kappa, \beta) \to 2$ and $f(\kappa, \beta, X_s)$ is $\mathcal{P}^\beta_\kappa$-generic over L_β. Let $\kappa \leq \kappa' \in \beta$-Card. Then $s \restriction [\kappa', \xi) \in S^\beta_{\kappa'}$ for $\xi \in ((\kappa')^+)^{L_\beta} \cap \mathcal{O}(\kappa')$.*

Proof. Deferred.

Now we can provide:

Proof of Lemma 1B.4. We must verify that (b) holds for s. Note that $|s| = \beta \in \mathcal{O}(\kappa)$ as κ is the greatest $\bar\beta$-cardinal. We must show that $\xi \in \mathcal{O}(\kappa) \cap \beta \to s \restriction \xi$ is $\Delta^*_1(\mathcal{A}(\mu^0_\xi))$. But $\xi \in \mathcal{O}(\kappa) \cap \beta \to \xi < (\kappa^+)^{L_\beta}$ so by Lemma 1B.5 we have that $s \restriction \xi \in S^\beta_\kappa$. Thus by definition of S^β_κ we have that $s \restriction \xi$ is $\Delta^*_1(\mathcal{A}(\mu^0_\xi))$. □

Lemma 1B.4 implies that the definition of $S_\kappa^{\bar\beta}$ would remain unchanged were condition (b) replaced by "$s \restriction \xi$ is $\Delta_1^*(\mathscr{A}(\mu_\xi^0))$ for all $\xi \in \mathcal{O}(\kappa) \cap [\beta, |s|]$." This is the redundancy referred to earlier.

We end this discussion of the S_κ^β's by stating a lemma concerning extendibility. It is closely related to the Generic Existence Lemma.

Lemma 1B.6. *Suppose $\beta \in \overline{\text{Adm}}$, $\kappa \in \beta$-Card, $s \in S_\kappa^\beta$. Then for all $\xi < (\kappa^+)^{L_\beta}$ there exists $t \supseteq s$, $t \in S_\kappa^\beta$, $|t| \geq \xi$.*

Proof. Deferred.

Generic codes

A condition in $\mathscr{P}_\kappa^\beta$ is obtained by fitting together elements of S_γ^β for $\kappa \leq \gamma \in \beta$-Card. The part of the condition in S_γ^β helps to code the part of the condition in $S_{\gamma^+}^\beta$ into a subset of γ^+ (where γ^+ denotes $(\gamma^+)^{L_\beta}$).

Fix $\beta \in \overline{\text{Adm}}$ and let γ be a β-cardinal less than gc β. To simplify notation we use γ^+ throughout to denote $(\gamma^+)^{L_\beta}$. Also fix $s \in S_{\gamma^+}^\beta$. We wish to describe the forcing R^s for coding s into a subset of γ^+. This is a variant of almost disjoint coding (with generic codes). The following lemma is proved in the appendix to Jensen [7]:

Lemma (Jensen). *There exists a sequence $\langle b_\xi \mid \gamma^+ \leq \xi < \gamma^{++} \rangle$ of subsets of γ^+ such that b_ξ is (uniformly) definable over L_{μ_ξ} and whenever $g : \gamma \to [\xi, \gamma^{++})$, $g \in L_\beta$, the sequence $\langle b_{g(i)} \mid i < \gamma \rangle$ is Cohen generic (as a sequence) over L_{μ_ξ}.*

Jensen establishes this lemma using $\diamondsuit$ and a gap-1 morass at γ^+.

The above lemma can be used as in [7] to code s into a subset D of γ^+ by requiring: $s(\xi) = 1$ iff $D \cap S(b_\xi)$ is bounded in γ^+ (where $S(b) = \{\text{Code}(b \restriction i) \mid i < \gamma^+\} \subseteq \gamma^+$). This coding is not good enough for our purposes. The reason is that we must have that initial segments of (the characteristic function) of D belong to S_γ^β, and are thus generic codings for forcings associated with elements of $\overline{\text{Adm}} \cap \gamma^+$. Thus we want that not only are the b_ξ's mutually generic but the same is also true of the restrictions $b_\xi \cap \alpha$ for many $\alpha \in \overline{\text{Adm}} \cap \gamma^+$ (with respect to some appropriate forcing). Cohen forcing can no longer be used as Cohen genericity for b_ξ implies that for large intervals (α_1, α_2) below γ^+, $b_\xi \cap (\alpha_1, \alpha_2) = \emptyset$. In fact the appropriate forcings $\mathscr{C}^\alpha$ must be defined by induction on α.

Now it is too much to ask that $b_\xi \cap \alpha$ be $\mathscr{C}^\alpha$-generic for every α as by $\diamondsuit$ there are stationary many $\alpha < \gamma^+$ such that $b_\xi \cap \alpha$ is constructed in L 'quickly' after α. Instead we require that $b_\xi \cap \alpha$ is either generic or coincides with some b_η^α, where the b_η^α's relate to α much as the b_ξ's relate to γ^+.

Fitting the above requirements together requires the use of a gap-1 morass at

γ^+. In fact we shall make use of the particular morass constructed in Stanley [8]. Our construction of the b_ξ's is similar in spirit to Jensen's proof of the above lemma, but is complicated by the fact that we require so much genericity.

A word of explanation: the generic codes construction in this part is not sufficient for the correct definition of R^s. We include it, however, both as motivation for the supergeneric codes of Part C and as a model for later extendibility arguments.

We now describe the construction of a morass in [8]. First we review the S-hierarchy for L (see Devlin [3]). The S_β's, $\beta \in \mathrm{ORD}$ are increasing transitive sets such that $\bigcup\{S_\beta \mid \beta \in \mathrm{ORD}\} = L$ and $S_\beta \cap \mathrm{ORD} = \beta$ for limit ordinals β. Moreover $S_{\beta+\omega}$ = Rudimentary Closure of $S_\beta \cup \{S_\beta\}$ and S_β carries a Σ_1-definable well-ordering, uniformly in β. The usual Skolem hull and transitive collapse arguments from L work inside each S_β, β a limit ordinal. Fix a limit ordinal β. The Σ_n-projectum of β, ρ_n^β, is the least ordinal γ such that there is a $\Sigma_n(S_\beta)$-injection from β into γ. Jensen showed that if X is a bounded subset of ρ_n^β, X is $\Sigma_n(S_\beta)$, then $X \in S_\beta$. He also proved the existence of a *Σ_n-master code* for β; a set $A \subseteq \rho_n^\beta$ which is $\Sigma_n(S_\beta)$ such that $B \subseteq \rho_n^\beta$ is $\Sigma_1\langle L_{\rho_n^\beta}, A\rangle$ iff B is $\Sigma_{n+1}(S_\beta)$. We let A_n^β denote a canonical choice for a Σ_n-master code as in [3].

Now for all α let $T_\alpha = \{\nu \mid \nu$ is p.r. closed, $L_\nu \models \alpha$ is the largest cardinal$\}$. For any $\nu \in T_\alpha$ we set $\beta(\nu)$ = least β s.t. ν is singular in $S_{\beta+\omega}$, $n(\nu)$ = least n s.t. ν is $\Sigma_n(S_{\beta(\nu)})$-injectible into α, $\rho(\nu) = \rho_{n(\nu)-1}^{\beta_\nu}$ and $A(\nu) = A_{n(\nu)-1}^{\beta_\nu}$. Also $p(\nu)$ = least p s.t. there is a $\Sigma_1\langle S_{\rho(\nu)}, A(\nu)\rangle$-injection of $S_{\rho(\nu)}$ into α with parameter p. Finally $\mathscr{A}(\nu) = \langle S_{\rho(\nu)}, A(\nu)\rangle$. We write $\alpha(\nu) = \alpha$ when $\nu \in T_\alpha$.

If $\bar\nu$, ν are p.r. closed we write $f : \bar\nu \Rightarrow \nu$ if $f : \mathscr{A}(\bar\nu) \xrightarrow{\Sigma_1} \mathscr{A}(\nu)$, Range$(f) \supseteq \{\alpha(\nu), p(\nu)\}$ (and $\nu \in$ Range(f) if $\nu < \rho(\nu)$). Also $\lambda(f) = \sup f[\bar\nu]$ and $C_\nu = \{\lambda(f) \mid f : \bar\nu \Rightarrow \nu$ for some $\bar\nu$, $\lambda(f) < \nu\}$.

It is convenient to modify the definition of C_ν in the case where C_ν is bounded in ν. In that case there is a least parameter $q(\nu)$ s.t. the $\Sigma_1(\mathscr{A}(\nu))$-Skolem hull of $\{p(\nu), q(\nu)\}$ is unbounded in ν (when intersected with ν).

Let h_ν be the canonical Σ_1-Skolem function for $\mathscr{A}(\nu)$ and let $\bar\xi_n^\nu$ be the canonical ordinal code for $L_\nu \cap h_\nu[n \times \{p(\nu), q(\nu)\}]$ and define $\xi_n^\nu = \langle \bar\xi_n^\nu, \hat\xi_n^\nu\rangle$ where $\langle \hat\xi_n^\nu \mid n \in \omega\rangle$ is an ω-sequence cofinal in ν which codes a $\Delta_1(\nu)$-Master Code for $\mathscr{A}(\nu)$; i.e., a subset of L_ν is Δ_1 over $\mathscr{A}(\nu)$ iff it is Δ_1 over $\langle L_\nu, \langle \hat\xi_n^\nu \mid n \in \omega\rangle\rangle$. This is easily accomplished using the fact that Σ_1-cof$(\mathscr{A}(\nu)) = \Sigma_1(\mathscr{A}(\nu))$-cof$(\nu)$. Then set $C'_\nu = \{\xi_n^\nu \mid n \in \omega\}$. Also choose the $\hat\xi_n^\nu$ so that part (e) of the Jensen theorem below remains true. See Lemma 6.41 of Beller–Jensen–Welch [1]. Define $C'_\nu = \{\xi_n^\nu \mid n \in \omega\}$ if C_ν is bounded in ν; $C'_\nu = C_\nu$ otherwise. Jensen (essentially) proves:

Theorem (Jensen). (a) *C'_ν is uniformly p.r. in ν, $\beta(\nu)$ and is closed unbounded in ν.*

(b) *If ν' is a limit point of C'_ν, then $C'_{\nu'} = C'_\nu \cap \nu'$.*

(c) *Ordertype$(C'_\nu) \leqslant \alpha(\nu)$.*

(d) *If* $f:\bar{\nu}\Rightarrow\nu$, *then* $f\restriction L_{\bar{\nu}}:\langle L_{\bar{\nu}}, C'_{\bar{\nu}}\rangle \xrightarrow{\Sigma_0} \langle L_\nu, C'_\nu\rangle$

(e) *If* $g:\langle L_{\bar{\nu}}, \bar{C}\rangle \xrightarrow{\Sigma_1} \langle L_\nu, C'_\nu\rangle$, *then* $\bar{C}=C'_{\bar{\nu}}$ *and g extends uniquely to* $f:\bar{\nu}\Rightarrow\nu$, $f(p(\bar{\nu}))=p(\nu)$.

We are almost ready to define a gap-1 morass at γ^+. A *Q-formula* is one of the form $\forall\sigma\,\exists\sigma'>\sigma\,\phi$ where ϕ is Σ_1 (and σ, σ' are variables for ordinals). Suppose $f:\bar{\mathscr{A}}\to\mathscr{A}$ is a monomorphism of amenable structures. We write $f:\bar{\mathscr{A}}\xrightarrow{Q}\mathscr{A}$ if whenever $\psi(a_1,\ldots,a_n)$ is a Q-formula with parameters $\bar{a}_1,\ldots,\bar{a}_n$ from $\bar{\mathscr{A}}$, $\bar{\mathscr{A}}\models\psi(\bar{a}_1,\ldots,\bar{a}_n)$ iff $\mathscr{A}\models\psi(f(\bar{a}_1),\ldots,f(\bar{a}_n))$. In this case we say that f is a *Q-embedding*. A Σ_0-embedding which is cofinal is automatically a Q-embedding. A Q-embedding need not be a Σ_2-embedding and a Σ_1-embedding need not be a Q-embedding.

We also define here the notion Δ_1^* which is used in (b) of the definition of $S_\kappa^{\bar{\beta}}$. Let $\nu\in T_\alpha$ and let $\mathscr{A}=\mathscr{A}(\nu)$. Then $X\subseteq L_\nu$ is $\Sigma_1^*(\mathscr{A})$ if X can be defined over $\langle L_\nu, C'_\nu\rangle$ by a Σ_1-formula. If both X, $L_\nu - X$ are $\Sigma_1^*(\mathscr{A})$, then we say that X is $\Delta_1^*(\mathscr{A})$. It can be shown that $\Delta_1(\mathscr{A})\subseteq\Delta_1^*(\mathscr{A})\subseteq\Delta_2(\mathscr{A})$. An important property of $\Delta_1^*(\mathscr{A})$-sets is that if $f:\bar{\mathscr{A}}\to\mathscr{A}$ is a cofinal Σ_1-embedding, $p(\nu)\in\text{Range}(f)$ and $X\subseteq L_\nu$ is $\Delta_1^*(\mathscr{A})$, then $f^{-1}[X]$ is $\Delta_1^*(\bar{\mathscr{A}})$. This will be useful in our discussion of the forcing R^s in Part D.

Work now in L_β. Let $\mathscr{U}$ consist of all $\alpha\in\overline{\text{Adm}}$, $\alpha>\gamma$ such that $L_\alpha\models\gamma$ is the largest cardinal. (Recall that γ is a β-cardinal $<\text{gc}\,\beta$.) $T=\bigcup\{T_\alpha \mid \alpha\in\mathscr{U}\}$. If $\bar{\nu}$, $\nu\in T$ we write $\bar{\nu}\leqslant\nu$ if there exists $g:\bar{\nu}\Rightarrow\nu$ such that $g\restriction\alpha(\bar{\nu})=\text{id}\restriction\alpha(\bar{\nu})$, $g(\alpha(\bar{\nu}))=\alpha(\nu)$ and $g\restriction L_{\bar{\nu}}:L_{\bar{\nu}}\xrightarrow{Q}L_\nu$. Moreover, g is unique and we write $g\restriction L_{\bar{\nu}}=\pi_{\bar{\nu}\nu}$. Also define $\bar{\nu}<\nu$ iff ($\bar{\nu}\leqslant\nu$ and $\bar{\nu}\neq\nu$). It is shown in [8] that we have defined a gap-1 morass at γ^+ in this way. Thus the following properties hold:

(M1) If $\bar{\nu}\leqslant\nu$, then $\pi_{\bar{\nu}\nu}$ maps $T_{\alpha(\bar{\nu})}\cap\bar{\nu}$ into $T_{\alpha(\nu)}\cap\nu$ so that (denoting $\pi_{\bar{\nu}\nu}(\bar{\gamma})$ by γ):

(a) $\bar{\gamma}$ initial in $T_{\alpha(\bar{\nu})}\to\gamma$ initial in $T_{\alpha(\nu)}$,
(b) $\bar{\gamma}$ the $T_{\alpha(\bar{\nu})}$-successor of $\bar{\beta}\to\gamma$ the $T_{\alpha(\nu)}$-successor of $\beta=\pi_{\bar{\nu}\nu}(\bar{\beta})$,
(c) $\bar{\gamma}$ a limit point of $T_{\alpha(\bar{\nu})}\to\gamma$ a limit point of $T_{\alpha(\nu)}$.

(M2) $\bar{\nu}\leqslant\nu$, $\bar{\gamma}\in T_{\alpha(\bar{\nu})}\cap\bar{\nu}\to\bar{\gamma}\leqslant\pi_{\bar{\nu}\nu}(\bar{\gamma})=\gamma$ and $\pi_{\bar{\gamma}\gamma}=\pi_{\bar{\nu}\nu}\mid L_{\bar{\gamma}}$.

(M3) $\{\alpha(\bar{\nu})\mid\bar{\nu}<\nu\}$ is closed in $\alpha(\nu)$.

(M4) ν not maximal in $T_{\alpha(\nu)}\to\{\alpha(\bar{\nu})\mid\bar{\nu}<\nu\}$ is unbounded in $\alpha(\nu)$.

(M5) $\{\alpha(\bar{\nu})\mid\bar{\nu}<\nu\}$ unbounded in $\alpha(\nu)\to L_\nu=\bigcup\{\text{Range}(\pi_{\bar{\nu}\nu})\mid\bar{\nu}<\nu\}$.

(M6) $\bar{\nu}$ a limit of $T_{\alpha(\bar{\nu})}$, $\bar{\nu}\leqslant\nu$, $\lambda=\bigcup(\text{Range}(\pi_{\bar{\nu}\nu})\cap\nu)\to\bar{\nu}\leqslant\lambda$ and $\pi_{\bar{\nu}\lambda}\restriction\bar{\nu}=\pi_{\bar{\nu}\nu}\restriction\bar{\nu}$.

(M7) Suppose $\bar{\nu}$ is a limit point of $T_{\alpha(\bar{\nu})}$, $\bar{\nu} \leqslant \nu$ and $\pi_{\bar{\nu}\nu}$ is cofinal. If $\alpha \in \{\alpha(\gamma) \mid \bar{\gamma} \leqslant \gamma \leqslant \pi_{\bar{\nu}\nu}(\bar{\gamma})\}$ for all $\bar{\gamma} \in T_{\alpha(\bar{\nu})} \cap \bar{\nu}$, then $\alpha \in \{\alpha(\gamma) \mid \bar{\nu} \leqslant \gamma \leqslant \nu\}$.

The verification of (M1–7) can be found in [8]. The need for Q-embeddings (as opposed to Σ_1-embeddings) is in the verification of M1(c).

We now define the morass relation $\dashv$. Set $\bar{\nu} <_* \nu$ if $\bar{\nu}$ immediately precedes ν in $\leqslant$. Then $\tau \dashv \nu$ if there are $\bar{\nu} <_* \nu$, $\bar{\tau} \in T_{\alpha(\bar{\nu})} \cap \bar{\nu}$, $\bar{\tau} <_* \tau < \pi_{\bar{\nu}\nu}(\bar{\tau})$.

Lemma 1B.7. (a) *$\dashv$ is a tree.*

(b) $\tau \dashv \nu \rightarrow \alpha(\tau) < \alpha(\nu)$

(c) *If $\bar{\nu} <_* \nu$ and $\pi_{\bar{\nu}\nu}$ is cofinal, then ν is a limit in $\dashv$ and $\alpha(\nu) = \bigcup \{\alpha(\tau) \mid \tau \dashv \nu\}$.*

Proof. (a) This is clear, using M2.

(b) If $\tau \dashv \nu$, then $\alpha(\tau) < \alpha(\tau')$ where $\tau' = \pi_{\bar{\nu}\nu}(\bar{\tau})$ and $\bar{\tau} <_* \tau$. But $\alpha(\tau') = \alpha(\nu)$.

(c) For $\bar{\tau} \in T_{\alpha(\bar{\nu})} \cap \bar{\nu}$ let $\eta(\bar{\tau})$ be defined by $\bar{\tau} <_* \eta(\bar{\tau}) < \pi_{\bar{\nu}\nu}(\bar{\tau})$. Then ν is the $\dashv$-limit of the $\eta(\bar{\tau})$. By M7, $\alpha' = \bigcup \{\alpha(\eta(\bar{\tau})) \mid \bar{\tau} \in T_{\alpha(\bar{\nu})} \cap \bar{\nu}\}$ must equal $\alpha(\nu')$ for some ν' such that $\bar{\nu} < \nu' \leqslant \nu$. As $\bar{\nu} <_* \nu$ we must have $\nu' = \nu$. □

We now proceed to discuss the generic codes b_ξ for $\xi \in T_{\gamma^+} = \{\xi \mid \xi$ is p.r. closed, $\xi < \beta$, $L_\xi \models \gamma^+$ is the largest cardinal$\}$. To do so we must define a number of auxiliary notions. For $\nu \in T$ let $W(\nu) = T_{\alpha(\bar{\nu})} - \bar{\nu}$ where $\bar{\nu} <_* \nu$ (if such a $\bar{\nu}$ exists). To define the b_ξ's, $\xi \in T_{\gamma^+}$ we must in fact define sets b_ν for all $\nu \in T$. The definition of b_ν for $\alpha(\nu) < \alpha \in \mathcal{U}$ is by an induction on α in which we simultaneously define $b_{\nu\tau}$, $\tau \in W(\nu)$, $\alpha(\nu) < \alpha$ and forcings $\mathscr{C}_X^\alpha$ where $X \in I_\alpha$ is described below. For $\alpha \in \mathcal{U}$, $\nu(\alpha)$ denotes $\max(T_\alpha)$ if $T_\alpha \neq \emptyset$, α otherwise. Also set $T_\alpha^+ = T_\alpha \cup \{\nu(\alpha) + 1\}$.

Definition. Suppose $\alpha \in \mathcal{U}$, $\sigma < \tau$ belong to T_α^+. Then $T(\sigma, \tau) = \{\nu \in T_\alpha \mid \sigma \leqslant \nu < \tau\}$. For $\beta \in \mathcal{U}$ set $I_\beta = \{T(\sigma, \tau) \mid \sigma < \tau$ belong to T_α^+, $\sigma = \min T_\alpha$ and $\alpha \in \mathcal{U} \cap \beta\}$.

We attempt to offer some explanation for the sets I_β. We wish to obtain genericity for an arbitrary γ-sequence $\langle b_{g(i)} \mid j < \gamma \rangle$ as in Jensen's lemma. It is convenient to only consider sequences of a special form (and argue that this is sufficient). Using I_β we can describe which sequences from $\{b_\nu \mid \nu \in T_\beta\}$ that we allow: We typically consider $\langle b_{\pi(\nu)} \mid \nu \in T(\sigma, \tau) \rangle$ where $T(\sigma, \tau) \in I_\beta$, $\tau < \tau' \in T_\beta$, $\pi = \pi_{\tau\tau'}$. One type of condition in $\mathscr{C}_X^\alpha$ is such a sequence where $X = T(\sigma, \tau)$, $\beta < \alpha$. Note that we can have $\tau =$ the $T_{\alpha(\sigma)}$-successor of σ, in which case $T(\sigma, \tau)$ is a singleton.

We now define the b_ν's, $b_{\nu\tau}$'s, $\mathscr{C}_X^\alpha$'s. We want to maintain the following properties:

(a) $b_\nu : [\gamma, |b_\nu|) \rightarrow 2$, $b_{\nu\tau} : [\gamma, |b_{\nu\tau}|) \rightarrow 2$ where $|b_\nu| = \alpha(\nu)$, $|b_{\nu\tau}| = \alpha(\nu)$ if $\tau \in W(\nu)$.

(b) $\bar{\nu} \leqslant \nu \rightarrow b_{\bar{\nu}} \subseteq b_\nu$.

(c) $\bar{\nu} <_* \nu \rightarrow b_{\nu\bar{\nu}} = b_\nu$.

(d) $\tau \dashv \nu$, $\sigma \in W(\nu) \rightarrow b_{\tau\sigma} \subseteq b_{\nu\sigma}$.

(e) If $p \in \mathscr{C}^\alpha_X$, then $\operatorname{dom}(p) = X \in I_\alpha$ and for some $|p| < \alpha$, all $i \in X$, $p(i) : [\gamma, |p|) \rightarrow 2$, $p(i) \supseteq b_i$. If $p, q \in \mathscr{C}^\alpha_X$, then $p \leqslant q$ iff $p(i) \supseteq q(i)$ for all $i \in X$. If $\alpha_1 < \alpha_2$ belong to $\mathscr{U}$, then $\mathscr{C}^{\alpha_1}_X \subseteq \mathscr{C}^{\alpha_2}_X$ for all $X \in I_{\alpha_1}$. If $p \in \mathscr{C}^\alpha_X$, $\eta < \alpha$, then there exists $q \leqslant p$, $|q| \geqslant \eta$.

Let $\mathscr{C}_X = \bigcup \{\mathscr{C}^\alpha_X \mid \alpha \in \mathscr{U}$ and $X \in I_\alpha\}$. For $X = T(\sigma, \tau)$ we use $|X|$ to denote $\alpha(\sigma)$.

(f) Suppose $p \in \mathscr{C}_X$ and $\eta > |X|$. Then $p^\eta \in \mathscr{C}_X$ where $p^\eta(i) = p(i) \upharpoonright \eta$.

(g) Suppose $X = T(\sigma, \tau)$, $\min(X) = \min(T_{|X|}) = \sigma$, $\tau < \tau' \in T_\alpha$. Define p by $p(i) = b_{\pi(i)}$ for $i \in X$, where $\pi = \pi_{\tau\tau'}$. Then $p \in \mathscr{C}_X$. Also for each $i \in X$ let $\mathscr{C}(p, i) = \{q \upharpoonright (X - i) \mid q \in \mathscr{C}^\alpha_X$ and $q(j) \subseteq p(j)$ for all $j \in X \cap i\}$ and $G(p, i) = \{r \in \mathscr{C}^\alpha_{X-i} \mid r(j) \subseteq p(j)$ for all $j \in X - i\}$. Then $G(p, i)$ is $\mathscr{C}(p, i)$-generic over $L_{\pi(i)}(p \upharpoonright i)$.

A special case of the last statement in (g) is when $i = \sigma$: Then we are asserting that $\{q \in \mathscr{C}^\alpha_X \mid p \leqslant q\}$ is $\mathscr{C}^\alpha_X$-generic over $L_{\pi(\sigma)}$.

By induction on $\alpha \in \mathscr{U}$ we define $\{b_\nu \mid \alpha(\nu) < \alpha\}$, $\{b_{\nu\tau} \mid \alpha(\nu) < \alpha\}$, $\mathscr{C}^\alpha_X$ for $X \in I_\alpha$. If $\alpha = \min(\mathscr{U})$, then $I_\alpha = \emptyset$ so there is nothing to define. If α is the limit of elements of $\mathscr{U}$, then $\mathscr{C}^\alpha_X = \bigcup \{\mathscr{C}^{\alpha'}_X \mid \alpha' \in \mathscr{U} \cap \alpha, X \in I_{\alpha'}\}$ for all $X \in I_\alpha$. Also note that $b_{\nu\tau}$, b_ν for $\alpha(\nu) < \alpha$ are already defined by induction. Now suppose that α' is the least element of $\mathscr{U}$ greater than $\alpha \in \mathscr{U}$ and we want to define $\{b_\nu \mid \alpha(\nu) = \alpha\}$, $\{b_{\nu\tau} \mid \alpha(\nu) = \alpha\}$, $\mathscr{C}^{\alpha'}_X$ for $X \in I_{\alpha'}$.

The definition of $b_{\nu\tau}$, b_ν for $\alpha(\nu) = \alpha$, $\tau \in W(\nu)$ breaks into cases. Fix ν such that $\alpha(\nu) = \alpha$.

Case 1: *ν is initial in* $<$. Set $b_\nu(\eta) = 0$ for all $\eta \in [\gamma, \alpha(\nu))$.

Case 2: *ν a limit in* $<$. Then $b_\nu = \bigcup \{b_\tau \mid \tau < \nu\}$.

Case 3: *ν a successor in* $<$. Let $\bar{\nu} <_* \nu$ and $\bar{\alpha} = \alpha(\bar{\nu})$.

(3a) *ν is initial in T_α.* Let $X = T(\bar{\nu}, \nu(\bar{\alpha}) + 1) = T_{\bar{\alpha}}$. Then $X \in I_\alpha$. Choose the L-least G such that G is $\mathscr{C}^\alpha_X$-generic over L_ν (the existence of G is justified by Lemma 1B.9 below). For $\tau \in T_{\bar{\alpha}} = W(\nu)$ let $b_{\nu\tau} = \bigcup \{p(\tau) \mid p \in G\}$ and $b_\nu = b_{\nu\bar{\nu}}$.

(3b) *ν a successor element of T_α.* Then ν is a successor in $\dashv$; let ν' immediately precede ν in $\dashv$. Define $X = T(\bar{\nu}, \nu(\bar{\alpha}) + 1) = W(\nu)$. Then $X \in I_\alpha$. Also let $Y = T_{\bar{\alpha}}$.

We define the forcing $\mathscr{C}^\nu_X$ by: p belongs to $\mathscr{C}^\nu_X$ iff $p = q \upharpoonright X$ where $q \in \mathscr{C}^\alpha_Y$ and $q(i) \subseteq b_{\pi_{\bar{\nu}\nu}(i)}$ for all $i \in Y - X = T_{\bar{\alpha}} \cap \bar{\nu}$. Define $p(i) = b_{\nu' i}$ for $i \in X$. Then $p \in \mathscr{C}^\nu_X$ (to be justified later). Now let G be the L-least $\mathscr{C}^\nu_X$-generic over L_ν such that $p \in G$ (the existence of G will be justified later). For $\tau \in X$ set $b_{\nu\tau} = \bigcup \{p'(\tau) \mid p' \in G\}$ and $b_\nu = b_{\nu\bar{\nu}}$.

(3c) *ν is a limit in T_α.* Let $\lambda = \bigcup \pi_{\bar{\nu}\nu}[\bar{\nu}]$. If $\lambda = \nu$, then set $b_{\nu\tau} = \bigcup \{b_{\nu'\tau} \mid \nu' \dashv \nu\}$ and $b_\nu = b_{\nu\bar{\nu}}$. If $\lambda < \nu$ set $\bar{\nu} <_* \nu' < \lambda$, $X = T(\bar{\nu}, \nu(\bar{\alpha}) + 1) = W(\nu)$ and $Y = T_{\bar{\alpha}}$.

We define the forcing $\mathscr{C}^\nu_X$ by: p belongs to $\mathscr{C}^\nu_X$ iff $p = q \upharpoonright X$ where $q \in \mathscr{C}^\alpha_Y$ and $q(i) \subseteq b_{\pi_{\bar{\nu}\nu}(i)}$ for all $i \in Y - X = T_{\bar{\alpha}} \cap \bar{\nu}$. Define $p(i) = b_{\nu' i}$ for $i \in X$. Then $p \in \mathscr{C}^\nu_X$ (to be justified later). Now let G be the L-least G such that $p \in G$ and G is

$\mathscr{C}_X^\nu$-generic over L_ν. (The existence of G will be justified later). For $\tau \in X$ set $b_{\nu\tau} = \bigcup \{p'(\tau) \mid p' \in G\}$ and $b_\nu = b_{\nu\bar{\nu}}$.

This completes our construction of the b_ν's, $b_{\nu\tau}$'s.

We now consider $\mathscr{C}_X^{\alpha'}$ for $X \in I_{\alpha'}$. First we define $\{p \in \mathscr{C}_X^{\alpha'} \mid |p| = \alpha\}$ when $X \in I_\alpha$. This consists of all p such that $p \in L_{\mu_\alpha}$, $\mathrm{Dom}(p) = X$, $p(i):[\gamma, \alpha) \to 2$, $p(i) \supseteq b_i$ for all $i \in X$ and:

(i) Let $X_0(p) = \{i \in X \mid p(i)$ is equal to b_ν for some $\nu \in T_\alpha\}$. Then $X_0(p)$ is an initial segment of X. If i belongs to $X_0(p)$, then $p(i) = b_{\nu(i)}$ where $i < \nu(i) \in T_\alpha$. If $i < j$ belong to $X_0(p)$, then $\nu(i) = \pi_{j\nu(j)}(i)$.

(ii) Suppose $\sigma = \sup(X_0(p))$ and $\sigma \in X_0(p)$, $\sigma <_* \nu(\sigma)$. Then $p(i) \supseteq b_{\nu(\sigma)i}$ for all $i \in X - \sigma$.

(iii) For each $i \in X$ let $G(p, i) = \{q \in \mathscr{C}_{X-i}^\alpha \mid q(j) \subseteq p(j)$ for all $j \in X - i\}$ and $\mathscr{C}(p, i) = \{r \restriction (X - i) \mid r \in \mathscr{C}_X^\alpha$ and $r(j) \subseteq p(j)$ for all $j \in X \cap i\}$. If $i \in X_0(p)$, then $G(p, i)$ is $\mathscr{C}(p, i)$-generic over $L_{\nu(i)}(p \restriction i)$. Suppose $X_1(p) = X - X_0(p)$ is non-empty and let $\nu(p) = \min(X_1(p))$. Then $G(p, \nu(p))$ is $\mathscr{C}(p, \nu(p))$-generic over $L_{\nu(\alpha)}(p \restriction \nu(p))$.

Continue to assume that $X \in I_\alpha$. We now want to define $\mathscr{C}_X^{\alpha'}$ (including those p such that $|p| > \alpha$). Basically we put little restriction on $p(i) \restriction [\alpha, |p|)$ but we do want a key property: $p(i) \restriction [\alpha, |p|)$ uniquely determines $\langle p(j) \restriction [\alpha, |p|) \mid j < i \rangle$, at least for p.r. closed ordinals $|p|$. This is captured by the following definition: $p \in \mathscr{C}_X^{\alpha'}$ iff $p \in \mathscr{C}_X^\alpha$ or:

(i′) $\mathrm{Dom}(p) = X$, $p(i):[\gamma, |p|) \to 2$ for all $i \in X$, $\alpha \leq |p| < \alpha'$.

(ii′) Suppose $i < j$ belong to X and $\langle \alpha + i, \xi \rangle < |p|$. Then $p(j)(\langle \alpha + i, \xi \rangle) = p(i)(\xi)$. ($\langle \cdot, \cdot \rangle$ is a canonical pairing on ORD × ORD).

(iii′) Define p^η by $p^\eta(i) = p(i) \restriction \eta$. Then $p^\alpha \in \mathscr{C}_X^{\alpha'}$ and $p^\eta \in L_{\mu_\eta}$ for all η.

If $X \in I_{\alpha'} - I_\alpha$, then $\mathscr{C}_X^{\alpha'}$ consists of all p such that $p(i) \supseteq b_i$ for all $i \in X$ and p obeys (i′), (ii′) and (iii′) with "$p^\alpha \in \mathscr{C}_X^{\alpha'}$" deleted. If $p, q \in \mathscr{C}_X^{\alpha'}$, then $p \leq q$ iff $q = p^\eta$ for some η. This completes the definition of the forcings $\mathscr{C}_X^\alpha$.

We now must verify properties (a)–(g) and justify the various steps in the construction of the b_ν's, $b_{\nu\tau}$'s. This verification is dependent upon a number of lemmas, the key one being the Extendibility Lemma for $\mathscr{C}_X^\alpha$ (Lemma 1B.9).

Lemma 1B.8. (a) *If $p \in \mathscr{C}_X^\alpha$ and $\eta > |X|$, then $p^\eta \in \mathscr{C}_X^\alpha$.*

(b) *If $p, q \in \mathscr{C}_X^\alpha$, $|p| \in \mathscr{U}$ and $p(i) = q(i)$, then $p(j) = q(j)$ for all $j \in X \cap i$.*

Proof. (a) We can assume that $|p| \in \mathscr{U}$ and $\min(X) = \min(T_{|X|}) = i$. By definition, $G(p, i)$ is $\mathscr{C}(p, i)$-generic over $L_{|p|}$ so it follows that $p^\eta \restriction (X - i) \in \mathscr{C}(p, i)$ for each $\eta \in (|X|, |p|)$. But $X - i = X$ and $\mathscr{C}(p, i) = \mathscr{C}_X^{|p|}$ so $p^\eta \in \mathscr{C}_X^\alpha$ for such η. It is clear that for $\eta \geq |p|$, $p^\eta = p \in \mathscr{C}_X^\alpha$.

(b) By induction on $|p| = |q|$. If $|p|$ is a limit of elements of $\mathscr{U}$, then the result follows from (a) and induction. If $|p|$ is a successor element of $\mathscr{U}$, then by definition of $\mathscr{C}_X^{\alpha'}$ we have that $p(j)(\xi) = p(i)(\langle \alpha + j, \xi \rangle) = q(i)(\langle \alpha + j, \xi \rangle) = q(j)(\xi)$ for all $\xi < |p|$, where $\alpha = \mathscr{U}$-predecessor to $|p|$. □

Our most important lemma in this part is the following Extendibility Lemma for $\mathscr{C}_X^\alpha$:

Lemma 1B.9. *Suppose* $\alpha \in \mathscr{U} \cap \gamma^+$ *and* $X \in I_\alpha$.

(a) *If* $p \in \mathscr{C}_X^\alpha$, *then there exists* $G \in L_{\mu_\alpha}$ *such that* $p \in G$ *and* G *is* $\mathscr{C}_X^\alpha$*-generic over* $L_{\nu(\alpha)}$.

(b) *If* $p \in \mathscr{C}_X^\alpha$, $\eta < \alpha$, *then there exists* $q \leqslant p$, $|q| \geqslant \eta$.

Proof. By induction on $\alpha \in \mathscr{U} \cap \gamma^+$. If α is a successor element of $\mathscr{U}$, argue as follows: Let β be the $\mathscr{U}$-predecessor of α and first suppose that $X \in I_\beta$. By (b) applied to β we have that $\{q \in \mathscr{C}_X^\beta \mid |q| \geqslant \eta\}$ is dense on $\mathscr{C}_X^\beta$ for all $\eta < \beta$, so by (a) applied to β we have $\forall p \in \mathscr{C}_X^\alpha \exists q \leqslant p\ (|q| \geqslant \beta)$. But it is obvious that $p \in \mathscr{C}_X^\alpha$, $|p| \geqslant \beta \rightarrow \exists q \leqslant p\ (|q| \geqslant \eta)$ for any $\eta < \alpha$. So (b) holds for α. If $X \in I_\alpha - I_\beta$, then (b) clearly holds. To get (a) for α note that $\nu(\alpha) = \alpha$, so we are only concerned with predense sets which belong to L_α. This type of genericity is very weak: Any $G \subseteq \mathscr{C}_X^\alpha$ which is compatible, closed upward and contains conditions q of arbitrarily large length $|q| < \alpha$, is automatically $\mathscr{C}_X^\alpha$-generic over L_α. To satisfy (a) for α pick $q_0 \leqslant p$, $|q_0| \geqslant \beta$ and let $G = \{q \in \mathscr{C}_X^\alpha \mid q_0,\ q$ are compatible, $q(i)(\eta) = 1$ iff either $q_0(i)(\eta) = 1$ or $\eta = \langle \beta + j, \xi \rangle$ where $j \in X \cap i$ and $q(j)(\xi) = 1\}$. Elements q of G of large length can be easily constructed by defining $q(i)$ inductively on $i \in X$.

Now suppose that α is a limit of elements of $\mathscr{U}$, $\alpha < \gamma^+$. Then (b) follows trivially for α by induction. To prove (a) for α fix $p \in \mathscr{C}_X^\alpha$ and let $\beta = \beta(\alpha)$, $n = n(\alpha)$, $\rho = \rho_{n-1}^\beta$, $\mathscr{A} = \mathscr{A}(\alpha) = \langle S_\rho, A(\alpha) \rangle$ where $A(\alpha)$ is a Σ_{n-1}-master code for β. Note that $\nu(\alpha) \leqslant \beta$ as $L_{\nu(\alpha)} \vDash \alpha$ is a cardinal.

First suppose that C_α is unbounded in α. Let $\hat{\mathscr{C}}_X^\alpha$ consists of all $q \in \mathscr{C}_X^\alpha$ such that $X_0(q) = \emptyset$ (equivalently: $X = X_1(q)$). Now define a sequence $p_0 \geqslant p_1 \geqslant p_2 \geqslant \cdots$ of conditions in $\hat{\mathscr{C}}_X^\alpha$ inductively as follows: $p_0 = p$; $p_{i+1} = L$-least $q \leqslant p_i$ in $\hat{\mathscr{C}}_X^\alpha$ such that $|p_i| < |q| \in C_\alpha$; $p_\lambda = \bigcup \{p_i \mid i < \lambda\}$ for limit $\lambda <$ ordertype(C_α). We claim that p_i is a condition in $\hat{\mathscr{C}}_X^\alpha$ for each i and that $G = \{q \in \mathscr{C}_X^\alpha \mid p_i \leqslant q$ for some $i\}$ is $\mathscr{C}_X^\alpha$-generic over $L_{\nu(\alpha)}$, $G \in L_{\mu_\alpha}$.

The proof that $p_i \in \hat{\mathscr{C}}_X^\alpha$ for each i goes by induction on i. The case of i a successor ordinal follows by applying (a) inductively. Suppose now that $i = \lambda$, a limit ordinal. The fact that $C_\alpha \cap |p_\lambda| = C_{|p_\lambda|}$ implies that $p_\lambda \in L_{\mu_{|p_\lambda|}}$ as $C_{|p_\lambda|}$ is definable over $L_{\beta(|p_\lambda|)}$, $\beta(|p_\lambda|) < \mu_{|p_\lambda|}$. So we need only verify that $G_\lambda = \{q \in \mathscr{C}_X^{|p_\lambda|} \mid p_i \leqslant q$ for some $i < \lambda\}$ is $\mathscr{C}_X^{|p_\lambda|}$-generic over $L_{\nu(|p_\lambda|)}$. For $i > 0$ let $\alpha_i = |p_i|$ and choose $f_i : \gamma_i \Rightarrow \alpha$, $\lambda(f_i) = \alpha_i$. Define $\sigma_i = \bigcup \mathrm{Range}(f_i)$. Let h be the canonical Σ_1^p-Skolem function for $\mathscr{A}$ where $p = p(\alpha)$ and for each $\sigma < \rho$ let h_σ be the canonical Σ_1^p-Skolem function for $\mathscr{A}_\sigma = \langle S_\sigma, A(\alpha) \cap \sigma \rangle$ (when this structure is amenable). $\mathscr{A}_{\sigma_i}$ is amenable as $\sigma \in \mathrm{Range}(f_i) \rightarrow A(\alpha) \cap \sigma \in \mathrm{Range}(f_i)$. And, $\sigma_i < \rho$ since $\sigma \in \mathrm{Range}(f_i) \rightarrow \bigcup (h_\sigma[\omega \times \gamma] \cap \alpha)$ is less than α (as α is a cardinal in $S_\rho) \rightarrow \bigcup (h_\sigma[\omega \times \gamma] \cap \alpha) \in \mathrm{Range}(f_i)$; so $\mathrm{Range}(f_i)$ unbounded in $\rho \rightarrow \mathrm{Range}(f_i) \cap \alpha$ unbounded in $\bigcup (h[\omega \times \gamma] \cap \alpha) = \alpha$, contradicting $\lambda(f_i) = \alpha_i < \alpha$.

Similarly $h_{\sigma_i}[\omega\times\alpha_i]\cap\alpha=\alpha_i$ as for each $\alpha'\in\text{Range}(f_i)\cap\alpha$, $\sigma\in\text{Range}(f_i)$ we have $\bigcup(h_\sigma[\omega\times\alpha']\cap\alpha)\in\text{Range}(f_i)$, so $(h_{\sigma_i}[\omega\times\alpha_i])\cap\alpha\subseteq\bigcup(\text{Range}(f_i)\cup\alpha)=\alpha_i$. Let $\pi_i:T'_i\simeq h_{\sigma_i}[\omega\times\alpha_i]$ be the inverse of the transitive collapse of $h_{\sigma_i}[\omega\times\alpha_i]$. Then by Jensen's extension of embeddings lemma (Devlin [3, p. 100]) there is $\hat{\alpha}_i$ such that $T'_i=\langle S^{\hat{\alpha}_i}_{\rho_{n-1}},A^{\hat{\alpha}_i}_{n-1}\rangle$. Thus for $i<j$ we have $\pi_j^{-1}\circ\pi_i=\pi_{ij}:\langle S^{\hat{\alpha}_i}_{\rho_{n-1}},A^{\hat{\alpha}_i}_{n-1}\rangle\to\langle S^{\hat{\alpha}_j}_{\rho_{n-1}},A^{\hat{\alpha}_j}_{n-1}\rangle$. But note that $T'_i\vDash\alpha_i$ is a cardinal (or $\text{ORD}(T'_i)=\alpha_i$) and yet there is a partial cofinal $\Sigma_1(T'_i)$-function from γ_i into α_i. As $L_{\alpha_i}\vDash\gamma$ is the largest cardinal, it follows that $\hat{\alpha}_i=\beta(\alpha_i)$, $T'_i=\mathscr{A}(\alpha_i)$. Thus we have Σ_0-embeddings $\pi_{ij}:\mathscr{A}(\alpha_i)\to\mathscr{A}(\alpha_j)$ for $i<j$ and hence Σ_{n-1}-embeddings $\hat{\pi}_{ij}:S_{\beta(\alpha_i)}\to S_{\beta(\alpha_j)}$. For limit j, $S_{\beta(\alpha_j)}=\text{Direct Lim}(\langle S_{\beta(\alpha_i)}\mid i<j\rangle,\hat{\pi}_{ij})$.

In particular, $S_{\beta(\alpha_\lambda)}=\text{Direct Lim}(\langle S_{\beta(\alpha_i)}\mid i<\lambda\rangle,\hat{\pi}_{i\lambda})$. Suppose $\mathscr{D}\in L_{\nu(\alpha_\lambda)}$ is predense on $\mathscr{C}^{\alpha_\lambda}_X$. Then $\mathscr{D}\in\text{Range}(\hat{\pi}_{i\lambda})$ for some $i<\lambda$, $X\in L_{\alpha_i}$. Then $\mathscr{D}_i=\hat{\pi}^{-1}_{i\lambda}(\mathscr{D})=\mathscr{D}\cap L_{\alpha_i}$ is predense on $\mathscr{C}^{\alpha_i}_X$ as either "$\mathscr{D}$ is predense on $\mathscr{C}^{\alpha_\lambda}_X$" is a Σ_0-statement over $S_{\beta(\alpha_\lambda)}$ (when $\beta(\alpha_\lambda)>\alpha_\lambda$) or $\mathscr{D}\in L_{\alpha_i}$ (when $\beta(\alpha_\lambda)=\alpha_\lambda$). (In the latter case, $\mathscr{D}$ is predense on $\mathscr{C}^{\alpha_i}_X$ as if $p,q\in\mathscr{C}^{\alpha_i}_X$ are compatible in $\mathscr{C}^{\alpha_\lambda}_X$, they must be compatible in $\mathscr{C}^{\alpha_i}_X$.) If $\nu(\alpha_\lambda)<\beta(\alpha_\lambda)$, we can assume that $\nu(\alpha_\lambda)\in\text{Range}(\hat{\pi}_{i\lambda})$. In this case it follows that $\mathscr{D}_i\in L_{\nu(\alpha_i)}$, as $\nu(\alpha_i)=\hat{\pi}^{-1}_{i\lambda}(\nu(\alpha_\lambda))$. If $\nu(\alpha_\lambda)=\beta(\alpha_\lambda)$, then either $n>1$, in which case $\nu(\alpha_i)=\beta(\alpha_i)$ so again $\mathscr{D}_i\in L_{\nu(\alpha_i)}$, or $n=1$. In this final case ($\nu(\alpha_\lambda)=\beta(\alpha_\lambda)$, $n=1$) we know that $\beta(\alpha_\lambda)$ is a limit point of $T_{\alpha\lambda}$ as the σ_i's, $i<\lambda$ are p.r. closed and cofinal in σ_λ and $L_{\beta(\alpha_\lambda)}$ is isomorphic to a Σ_1-elementary substructure of L_{σ_λ}. Thus we can assume that $\mathscr{D}\in L_\tau$ for some $\tau\in T_{\alpha_\lambda}-\{\nu(\alpha_\lambda)\}$ and hence for sufficiently large $i<\lambda$, $\mathscr{D}_i\in L_{\nu(\alpha_i)}$. Thus in all cases we can assume that $\mathscr{D}_i\in L_{\nu(\alpha_i)}$. As $p_i\in\hat{\mathscr{C}}^{\alpha_i}_X$ we have that $p_i\leqslant$ some $q\in\mathscr{D}_i$ and hence G_λ meets $\mathscr{D}$.

Next we consider what happens if C_α is bounded in α. In this case there is a cofinal increasing $\Sigma_2(\mathscr{A}(\alpha))$-function $g':\omega\to\rho(\alpha)$. As before let $A(\alpha)=A^{\beta(\alpha)}_{n(\alpha)-1}$; if $\langle S_{g'(i)},A(\alpha)\cap g'(i)\rangle$ were amenable for each $i\in\omega$ we could proceed much as before. Instead we take a different approach. First suppose that $\rho(\alpha)$ is a limit of p.r. closed ordinals (this is automatic if $n(\alpha)>1$). Define $g:\omega\to\rho(\alpha)$ to be cofinal and $\Sigma_2(\mathscr{A}(\alpha))$ so that $A(\alpha)\cap g(i)\in S_{g(i+1)}$ for each $i\in\omega$. We also assume that $p(\alpha)\in S_{g(0)}$ and $g(i)$ is a limit ordinal for each i. There are two cases: $\rho(\alpha)>\alpha$ and $\rho(\alpha)=\alpha$. First assume the former. Let $H_i=h_{g(i+1)}[\omega\times(\gamma\cup\{A(\alpha)\cap g(i)\})]$ where $h_{g(i+1)}$ is a $\Sigma_1^{p(\alpha)}$-Skolem function for $S_{g(i+1)}$. Let $\pi_i:T'_i\simeq H_i$ be the inverse of the transitive collapse of H_i and set $\alpha_i=T'_i\cap\alpha$. Then $T'_i\vDash\alpha_i$ is a cardinal, yet there is a $\Sigma_1(T'_i)$-injection of α_i into γ. So $T'_i=S_{\beta(\alpha_i)}$ and $n(\alpha_i)=1$. As before let $\hat{\mathscr{C}}^\alpha_X$ denote $\{q\in\mathscr{C}^\alpha_X\mid X_1(q)=X\}$. Define $p_0=p$ and $p_{i+1}=L$-least $q\leqslant p_i$ such that $q\in\hat{\mathscr{C}}^\alpha_X$, $X\in L_{|q|}$, $|q|=\alpha_j$ for some $j>i$. Let $G=\{q\in\mathscr{C}^\alpha_X\mid p_i\leqslant q$ for some $i\}$. We claim that G is $\mathscr{C}^\alpha_X$-generic. Indeed, suppose $\mathscr{D}\in L_{\nu(\alpha)}$ is predense on $\mathscr{C}^\alpha_X$. Then for some i, $\mathscr{D}\in H_i$ as $\bigcup\{H_i\mid i\in\omega\}=L_{\rho(\alpha)}$ and $\rho(\alpha)>\alpha$. We can assume that $\mathscr{D}\in L_\sigma$ where $\sigma\in T_\alpha\cap H_i$ (since $\rho(\alpha)$ is a limit of p.r. closed ordinals). Thus $\mathscr{D}_i=\mathscr{D}\cap L_{\alpha_i}$ belongs to $L_{\nu(\alpha_i)}$ and hence $p_i\leqslant$ some $q\in\mathscr{D}_i$. So G meets $\mathscr{D}$.

We must consider the possibility that $\rho(\alpha)=\alpha$. If $n(\alpha)=1$, then $\beta(\alpha)=\alpha$ is inadmissible with Σ_2-cofinality ω (as otherwise C_α is unbounded in α). Choose a $\Sigma_2(L_\alpha)$-cofinal sequence $\langle \alpha_i \mid i \in \omega \rangle$ and given $p_0 \in \mathscr{C}_X^\alpha$ define $\hat{\mathscr{C}}_X^\alpha$ and $\langle p_i \mid i \in \omega \rangle$ as before. Then $\mathscr{D} \in L_\alpha$, $\mathscr{D}$ predense on $\mathscr{C}_X^\alpha$ implies $\mathscr{D} \in L_{\alpha_i}$ for some i, so p_i meets $\mathscr{D}$. Thus $G = \{q \in \mathscr{C}_X^\alpha \mid p_i \leqslant q$ for some $i\}$ is $\mathscr{C}_X^\alpha$-generic over L_α. Similarly if $\beta(\alpha) = \alpha$, $n(\alpha) > 1$, then choose a $\Sigma_2(\mathscr{A}(\alpha))$-cofinal sequence $\langle \alpha_i \mid i < \omega \rangle$ in α and given $p_0 \in \mathscr{C}_X^\alpha$ define $\hat{\mathscr{C}}_X^\alpha$ and $\langle p_i \mid i \langle \omega \rangle$ as before. Then $G = \{q \in \mathscr{C}_X^\alpha \mid p_i \leqslant q$ for some $i\}$ is again $\mathscr{C}_X^\alpha$-generic over L_α.

So we can assume that $n(\alpha) > 1$, $\beta(\alpha) > \alpha$. Let $j < n(\alpha) - 1$ be the largest j such that $\rho_j^{\beta(\alpha)} > \alpha$ ($0 \leqslant j$ exists since $\beta(\alpha) > \alpha$). Let $\rho'(\alpha) = \rho_j^{\beta(\alpha)}$, $\mathscr{A}'(\alpha) = \langle S_{\rho'(\alpha)}, A_j^{\beta(\alpha)} \rangle$ and $p'(\alpha) =$ least $p \in S_{\rho'(\alpha)}$ such that there is a $\Sigma_1(\mathscr{A}'(\alpha))$-injection of $S_{\rho'(\alpha)}$ into α with parameter p. Now choose $g : \omega \to \alpha$ to be $\Sigma_2(\mathscr{A}(\alpha))$, cofinal and define $\alpha_0 < \alpha_1 < \cdots$, $H_0 \subseteq H_1 \subseteq \cdots$ inductively by: $H_0 = \Sigma_1^{p'(\alpha)}$-Skolem hull of $g(0)$ inside $\mathscr{A}'(\alpha)$, $\alpha_0 = H_0 \cap \alpha$; $H_{i+1} = \Sigma_1^{p'(\alpha)}$-Skolem hull of $g(i+1) \cup \{\alpha_i\}$ inside $\mathscr{A}'(\alpha)$, $\alpha_{i+1} = H_{i+1} \cap \alpha$. Then $\bigcup \{\alpha_i \mid i \in \omega\} = \alpha$. Given $p_0 \in \mathscr{C}_X^\alpha$ define $\hat{\mathscr{C}}_X^\alpha$, p_i as before. If $\mathscr{D} \in L_{\nu(\alpha)}$, then $\mathscr{D} \in H_i$ for some i as $\nu(\alpha) \leqslant \rho'(\alpha)$, $\bigcup \{H_i \mid i \in \omega\} = \mathscr{A}'(\alpha)$. If $\nu(\alpha) < \rho'(\alpha)$, then $\mathscr{D} \cap L_{\alpha_i} \in L_{\nu(\alpha_i)}$, provided $\nu(\alpha) \in H_i$. If $\nu(\alpha) = \rho'(\alpha)$ then $\nu(\alpha_i) = \mathrm{ORD}(T_i')$ where $\pi_i : T_i' \simeq H_i$ is the transitive collapse of H_i, since π_i is a Σ_1-embedding into $\mathscr{A}'(\alpha)$. So again $\mathscr{D} \cup L_{\alpha_i} \in L_{\nu(\alpha_i)}$. Thus, if $\mathscr{D}$ is predense on $\mathscr{C}_X^\alpha$, it follows that p_i meets $\mathscr{D} \cap L_{\alpha_i}$ and hence $G = \{q \in \mathscr{C}_X^\alpha \mid p_i \leqslant q$ for some $i\}$ meets $\mathscr{D}$. Thus, G is $\mathscr{C}_X^\alpha$-generic over $L_{\nu(\alpha)}$.

We consider the case: C_α bounded, $\rho(\alpha) > \alpha$ but where $\rho(\alpha)$ is not the limit of p.r. closed ordinals. Thus we have $\rho(\alpha) = \beta(\alpha)$, $n(\alpha) = 1$. The argument that we used earlier succeeds if $\nu(\alpha) < \beta(\alpha)$ for then $\mathscr{D} \in L_{\nu(\alpha)}$ implies $\mathscr{D} \cap L_{\alpha_i} \in L_{\nu(\alpha_i)}$ for some i such that $\nu(\alpha) \in H_i$. We are left with the case $\nu(\alpha) = \beta(\alpha)$ is a successor element of T_α. Choose a cofinal $\Sigma_2(L_{\beta(\alpha)})$-function $g : \omega \to \beta(\alpha)$. Our main claim is that if $p_0 \in \mathscr{C}_X^\alpha$, $i \in \omega$, then there exists $p \leqslant p_0$ in $\hat{\mathscr{C}}_X^\alpha$ such that $p \in \mathscr{D}^*$ for all predense $\mathscr{D} \subseteq \mathscr{C}_X^\alpha$, $\mathscr{D} \in H_i = \Sigma_1^{p(\alpha)}$-Skolem hull of γ inside $L_{g(i)}$. To prove this define $p_0 \geqslant p_1 \geqslant \cdots$ by: p_0 is defined. $\beta_0 = 0$. $p_{j+1} =$ least $p \leqslant p_j$ in $\hat{\mathscr{C}}_X^\alpha$ such that for some $\mathscr{D} \in H_i$ as above, $p \in \mathscr{D}^*$, $p_j \notin \mathscr{D}^*$. $\beta_{j+1} = H'_{j+1} \cap \alpha$ where $H'_{j+1} = \Sigma_1^{p(\alpha)}$-Skolem hull of $|p_j| + 1$ inside $L_{g(i)}$. $p_\lambda = \bigcup \{p_i \mid i < \lambda\}$, $\beta_\lambda = \bigcup \{\beta_i \mid i < \lambda\}$ for limit λ. Then p_λ is a condition for limit λ. The desired $p \leqslant p_0$ is p_δ where δ is least so that $p_{\delta+1}$ is undefined. Now given this claim, for any $p_0 \in \mathscr{C}_X^\alpha$ we can choose $p_0 = q_0 \geqslant q_1 \geqslant \cdots$ successively in $\hat{\mathscr{C}}_X^\alpha$ so that $\bigcup \{|q_i| \mid i < \omega\} = \alpha$ and $q_{i+1} \in \mathscr{D}^*$ for all predense $\mathscr{D} \subseteq \mathscr{C}_X^\alpha$, $\mathscr{D} \in H_i$. Thus $G = \{q \in \mathscr{C}_X^\alpha \mid q_i \leqslant q$ for some $i\}$ is $\mathscr{C}_X^\alpha$-generic over $L_{\nu(\alpha)}$, as desired. □

We next want to establish a version of the preceding lemma for the forcings $\mathscr{C}(p, i)$. To do so we need a somewhat more general class of forcings to consider.

Definition. Suppose $\min T_\beta = \sigma_0 < \tau < \nu$ belong to T_β^+ and let $X = T(\sigma_0, \tau)$, $Y = T(\tau, \nu)$. Suppose $p \in C_X$. Then $\mathscr{C}_Y^p$ consists of all $q \in \mathscr{C}_Y^{|p|}$ such that $p^{|q|} \cup q \in \mathscr{C}_{X \cup Y}$.

Lemma 1B.10. *Let X, Y be as in the preceding definition.*

(a) *Suppose $p \in \mathscr{C}_X$, $|p| = \alpha \in \mathscr{U}$ and $q \in \mathscr{C}^{\varrho}_Y$. Then there exists $G \in L_{\mu_\alpha}$, $q \in G$ such that G is $\mathscr{C}^{\varrho}_Y$-generic over $L_{\nu(\alpha)}(p)$.*

Moreover, if $q(G)$ is defined by $q(G)(i) = \bigcup \{q(i) \mid q \in G\}$ for each $i \in Y$, then $p \cup q(G) \in \mathscr{C}_{X \cup Y}$.

(b) *Suppose $p \in \mathscr{C}_X$, $q \in \mathscr{C}^{\varrho}_Y$, $\eta < |p| = \alpha \in \mathscr{U}$. Then there exists $q' \leqslant q$ in $\mathscr{C}^{\varrho}_Y$, $|q'| \geqslant \eta$.*

(c) *Let $r \in \mathscr{C}_X$, $|r| = \alpha \in \mathscr{U}$ and suppose that $G = G(r \restriction (X - \sigma)) = \{p \in \mathscr{C}(r, \sigma) \mid p(i) \subseteq r(i)$ for all $i \in X - \sigma\}$ is $\mathscr{C}(r, \sigma)$-generic over $L_\nu(r \restriction \sigma)$ where $b_\nu = r(\sigma)$ if $\sigma \in X_0(r)$; $\nu = \nu(\alpha)$ otherwise. Also suppose that H is $\mathscr{C}^r_Y$-generic over $L_{\nu(\alpha)}(r)$. Then $\{p \cup q \mid p \in G, q \in H, |p| = |q|\}$ is $\mathscr{C}^{r \restriction \sigma}_{(X-\sigma) \cup Y}$-generic over $L_\nu(r \restriction \sigma)$.*

(d) *Suppose $p \in \mathscr{C}_X$, $p' \leqslant p$ in $\mathscr{C}_X$, $|p'| = \alpha \in \mathscr{U}$ and $r \in \mathscr{C}_{X \cup Y}$, $r \restriction X = p$. Then there exists $r' \leqslant r$ in $\mathscr{C}_{X \cup Y}$, $r' \restriction X = p'$. Conversely, if $r \in \mathscr{C}_{X \cup Y}$, then $r \restriction X \in \mathscr{C}_X$.*

Proof. By a simultaneous induction on $\alpha \in \mathscr{U}$. We prove (c) using (d) for $\alpha' < \alpha$, (a) using (c) and (b) for $\alpha' < \alpha$, (d) using (a) and finally (b) using (d) and Lemma 1B.9.

(c) Suppose $\mathscr{D} \in L_\nu(r \restriction \sigma)$ is predense on $\mathscr{C}^{r \restriction \sigma}_{X-\sigma \cup Y}$. We wish to find $p \in G$, $q \in H$ so that $p \cup q$ meets $\mathscr{D}$. Let $\mathscr{D}_0 = \{p \in \mathscr{C}(r, \sigma) \mid p = p' \restriction X$ for some $p' \in \mathscr{D}\}$. We claim that $\mathscr{D}_0$ is predense on $\mathscr{C}(r, \sigma)$. Indeed, if $p_0 \in \mathscr{C}(r, \sigma)$, then by (d) inductively we can find q_0 so that $p_0 \cup q_0 \in \mathscr{C}^{r \restriction \sigma}_{X-\sigma \cup Y}$ and then choose $p' \leqslant p_0 \cup q_0$ meeting $\mathscr{D}$. Then $p'_0 = p' \restriction X - \sigma$ meets $\mathscr{D}_0$ and $p'_0 \in \mathscr{C}^{r \restriction \sigma}_{X-\sigma}$ by the second clause of (d), inductively. Now, as $\mathscr{D}_0$ is predense on $\mathscr{C}(r, \sigma)$ and $\mathscr{D}_0 \in L_\nu(r \restriction \sigma)$, the genericity of G implies that $\mathscr{D}_1 = \{q \in \mathscr{C}^r_Y \mid p \cup q \in \mathscr{D}$ for some $p \in G\}$ is predense on $\mathscr{C}^r_Y$. The genericity of H over $L_{\nu(\alpha)}(r)$ and the fact that $\mathscr{D}_1 \in L_{\nu(\alpha)}(r) = L_{\nu(\alpha)}(G)$ imply that H meets $\mathscr{D}_1$. So $\{p \cup q \mid p \in G, q \in H, |p| = |q|\}$ meets $\mathscr{D}$.

(a) To prove the first statement the idea is to imitate the proof of Lemma 1B.9(a), using (b) inductively in replace of 1B.9(b). To verify that the argument goes through one need only note two facts: First, the fine structure theory that we applied to the structures $\mathscr{A}(\alpha)$, $\alpha \in \mathscr{U}$ (and $\mathscr{A}(\nu)$, $\nu \in T$) still works when relativized to a predicate $p \subseteq L_\alpha (p \subseteq L_{\alpha(\nu)}$, respectively). This is because $L_\alpha \models \gamma$ is the largest cardinal so when Skolem hulls inside $\mathscr{A}^p(\alpha)$ (the relativized version of $\mathscr{A}(\alpha)$) are collapsed, p collapses to $p \cap L_\beta$ for some $\beta < \alpha$ and Jensen's extension of embeddings lemma therefore still applies (similarly for $\mathscr{A}^p(\nu)$). The predicate p that we must consider is of course the p in the statement of (a), or more properly $\{q \in \mathscr{C}_X \mid p \leqslant q, p \neq q\} \subseteq L_\alpha$. Second, we claim that p preserves the property "$L_{\nu(\alpha)} \models \alpha$ is a cardinal" or equivalently "$L_{\nu(\alpha)} \models \gamma^+$ exists". In other words if $p \in \mathscr{C}_X$, $|p| = \alpha$, then $L_{\nu(\alpha)}[p] \models \alpha$ is a cardinal.

To prove this last assertion we first make the following observations: The proof of Lemma 1B.9 showed that $\mathscr{C}^\alpha_X$ is γ-distributive in $L_{\nu(\alpha)}$. Indeed, note that in the course of building a $\mathscr{C}^\alpha_X$-generic set over $L_{\nu(\alpha)}$ the components of any given γ-sequence from $L_{\nu(\alpha)}$ of predense sets are all met by some stage of the construction. Also one has the following general forcing fact: If $\mathscr{P} \in L_\nu$ is γ-distributive in L_ν, $L_\nu \models \gamma^+$ exists, then $L_\nu[G] \models \gamma^+$ exists for any G which is $\mathscr{P}$-generic over L_ν.

Now we can establish our earlier assertion. First note that for all $\eta < \nu(\alpha)$, $q_\eta = p \restriction \{i \in X \mid p(i) = b_\nu$ for some $\nu \leq \eta\}$ belongs to $L_{\nu(\alpha)}$, for there exists $\tau \in T_\alpha$, $\tau \leq \eta$ and $\sigma < \tau$ such that $q_\eta(i) = b_{\pi_{\sigma\tau}}(i)$ for all i (namely set $\tau = \sup\{\nu \mid b_\nu \in \text{Range}(p), \nu \leq \eta\}$); thus q_η is definable over $\mathscr{A}(\tau)$ and so belongs to $L_{\nu(\alpha)}$. Now suppose that $L_{\nu(\alpha)}(p) \models \alpha$ is not a cardinal and choose $\eta < \nu(\alpha)$ so that $L_\eta(p) \models \alpha$ is not a cardinal. But $p = q_\eta \cup p'$ where $p' = p \restriction X'$, $X' = X - \text{Dom}(q_\eta)$ and $G(p')$ is $\mathscr{C}^{q_\eta}_{X'}$-generic over L_τ, $\tau =$ least element of T_α greater than η. But then (by induction) $G(p')$ is generic over L_τ for a forcing $\mathscr{P} \in L_\tau$ which is γ-distributive in L_τ. So $L_\tau(G(p')) = L_\tau(p') = L_\tau(q_\eta, p') = L_\tau(p) \models \alpha$ is a cardinal, contradicting the choice of $\eta < \tau$.

We now consider the second statement in (a). To verify that $p \cup q(G) \in \mathscr{C}_{X \cup Y}$ it suffices to show, letting $r = p \cup q(G)$, that $G(r, i)$ is (sufficiently) $\mathscr{C}(r, i)$-generic for all $i \in X_0(p) \cup \{\nu(p)\}$. (Condition (ii) in the definition of $\mathscr{C}_{X \cup Y}$ is met as $q(G)$ is the union of conditions in $\mathscr{C}^p_Y$; if $i \in X_0(r) \cup \{\nu(r)\}$, then either $i \in X_0(p) \cup \{\nu(p)\}$ or $i = \min Y$, in which case the genericity of $G(r, i)$ follows from the choice of G.) But this follows immediately from (c), replacing $X - \sigma$ in (c) by $X - i$, r by p and noting that the genericity of $G(p \restriction X - i) = G(p, i)$ follows from the fact that $p \in \mathscr{C}_X$.

(d) The first statement follows immediately from (a) by replacing p by p' and q by $r \restriction Y$. For the second statement we need only check that $G(r \restriction X, i)$ is sufficiently $\mathscr{C}(r \restriction X, i)$-generic for $i \in X_0(r \restriction X) \cup \{\nu(r \restriction X)\}$. We know that $G(r, i)$ is sufficiently $\mathscr{C}(r, i)$-generic for $i \in X_0(r) \cup \{\nu(r)\} \supseteq X_0(r \restriction X) \cup \{\nu(r \restriction X)\}$. Now let $\mathscr{D}$ be an appropriate predense set on $\mathscr{C}(r \restriction X, i)$ which we wish to show is met by $G(r \restriction X, i)$. Let $\mathscr{D}' = \{q \in \mathscr{C}(r, i) \mid q \restriction (X - i) \in \mathscr{D}\}$. It suffices to show that $\mathscr{D}'$ is predense on $\mathscr{C}(r, i)$ for then the genericity of $G(r, i)$ implies that $G(r \restriction X, i)$ meets $\mathscr{D}$. So let $r' \in \mathscr{C}(r, i)$ and choose $p' \leq r' \restriction (X - i)$, p' meets $\mathscr{D}$. But now by the first statement in (d), there exists $r'' \leq r'$ such that $r'' \restriction (X - i) = p'$ and therefore r'' meets $\mathscr{D}'$.

(b) This follows from (d), replacing p by $p^{|q|}$, p' by p, r by $(p^{|q|}) \cup q$ and then the desired q' is $r'^{\eta} \restriction Y$. □

We need one more lemma in order to complete our study of the generic codes. As with the preceding lemma, though we have only one property in mind to establish (part (b) in Lemma 1B.10) we must 'carry along' a number of other statements in order to provide an inductive argument. The property that we are now after is the claim "Then $p \in \mathscr{C}_X$" in property (g). Conditions of the form $p(i) = b_{\pi(i)}$ where π is a morass map ($\pi = \pi_{\sigma\tau}$ for some $\sigma < \tau$) are called *standard* conditions. Thus we are trying to show that p defined in this way is in fact a condition (in $\mathscr{C}_X$, $X = \text{Dom}(p)$).

Establishing this depends upon other properties which are based on the idea of 'thinning'.

Definition. Suppose $p \in \mathscr{C}_X$ where $X = T(\sigma, \tau)$, $\sigma = \min T_{|X|}$. Suppose $\bar{X} = T(\bar{\sigma}, \bar{\tau})$, $\bar{\sigma} = \min T_{|\bar{X}|}$ where $\bar{\tau} \preccurlyeq \tau' \in X \cup \{\tau\}$ and $\pi = \pi_{\bar{\tau}\tau'}$. A *thinning* of p is a function q of the form $q(i) = p(\pi(i))$ where $\mathrm{Dom}(q) = \bar{X}$, π are as above.

Lemma 1B.11 (Thinning Lemma). (a) *Suppose $X \in I_\alpha$, $\sup(X) = \bar{\tau} < \tau \in T_\alpha$ and p is defined on X by $p(i) = b_{\pi_{\bar{\tau}\tau}(i)}$. Then $p \in \mathscr{C}_X$.*

(b) *Suppose $X = T(\sigma, \tau)$, $Y = T(\tau, \eta)$ where $\min T_\beta = \sigma < \tau < \eta \in T_\beta^+$ and $p \in \mathscr{C}_X$, $|p| = \alpha \in \mathscr{U}$. Suppose G is $\mathscr{C}_Y^p$-generic over $L_{\nu(\alpha)}(p)$. Define q by $q(i) = \bigcup \{q'(i) \mid q' \in G\}$ for $i \in Y$ and suppose that $\bar{p} \cup \bar{q}$ is a thinning of $p \cup q$, $\bar{p}$ a thinning of p, $\bar{q}$ a thinning of q. Then $G(\bar{q}) = \{\bar{q}' \in \mathscr{C}_{\bar{Y}}^{\bar{p}} \mid \bar{q}'(i) \subseteq \bar{q}(i)$ for all $i \in \bar{Y}\}$ is $\mathscr{C}_{\bar{Y}}^{\bar{p}}$-generic over $L_{\nu(\alpha)}(\bar{p})$ where $\bar{Y} = \mathrm{Dom}(\bar{q})$.*

(c) *If $p \in \mathscr{C}_X$, $|p| = \alpha$ and q is a thinnning of p, then $q \in \mathscr{C}_X$.*

(d) *Suppose $X = T(\sigma, \tau)$, $Y = T(\tau, \nu)$ where $\min T_\beta = \sigma < \tau < \nu \in T_\beta^+$ and $p \in \mathscr{C}_X$, $|p| = \alpha \in \mathscr{U}$. Suppose $\bar{\nu} < \nu' \in X \cup Y \cup \{\nu\}$ and let $\bar{X} = \pi^{-1}[X]$, $\bar{Y} = \pi^{-1}[Y]$ where $\pi = \pi_{\bar{\nu}\nu'}$. Suppose $q \in \mathscr{C}_Y^p$ and $\bar{q}(i) \supseteq q(\pi(i))$ for each $i \in \bar{Y}$ where $\bar{p} = p \circ \pi \restriction \bar{X}$, $\bar{p} \cup \bar{q} \in \mathscr{C}_{\bar{X} \cup \bar{Y}}$ and $G(\bar{q}) = \{\bar{q}' \in \mathscr{C}_{\bar{Y}}^{\bar{p}} \mid \bar{q} \leqslant \bar{q}'\}$ is $\mathscr{C}_{\bar{Y}}^{\bar{p}}$-generic over $L_{\nu(\alpha)}(\bar{p})$. Then there exists $q' \leqslant q$ in $\mathscr{C}_Y$ such that $p \cup q' \in \mathscr{C}_{X \cup Y}$ and $q' \circ \pi \restriction \bar{Y} = \bar{q}$.*

(e) *Suppose $\bar{p} \in \mathscr{C}_{\bar{X}}$ is a thinning of $p \in \mathscr{C}_X$ and $\bar{q} \leqslant \bar{p}$ in $\mathscr{C}_{\bar{X}}$, $|\bar{q}| = \alpha \in \mathscr{U}$. Then there exists $q \leqslant p$ in $\mathscr{C}_X$ such that $\bar{q}$ is a thinning of q.*

Proof. By induction on $\alpha \in \mathscr{U}$. We prove (a) using (c) and induction, (b) using (e) for $\beta < \alpha$, (c) using (b) and (a) for $\beta < \alpha$, (d) using (e) for $\beta < \alpha$ and (e) using Lemma 1B.8, (a), (d). (Lemma 1B.10 is also used.)

(a) We can assume that τ is a T_α-successor or τ is a T_α-limit and $\pi_{\bar{\tau}\tau}[\bar{\tau}]$ is cofinal in τ. For, otherwise replace τ by $\tau' = \sup \pi_{\bar{\tau}\tau}[\bar{\tau}]$ and then (M6) implies that $\bar{\tau} < \tau'$. We must show that for $i \in X$, $G(p, i)$ is sufficiently $\mathscr{C}(p, i)$-generic.

Let $\mathscr{D} \in L_{\pi(i)}(p \restriction i)$ be predense on $\mathscr{C}(p, i)$. If τ is a $<$-limit, then we can choose $\tau' < \tau$ so that $\mathscr{D} \in \mathrm{Range}(\pi_{\tau'\tau})$. Choose $i' \in T_{\alpha(\tau')}$ so that $i' < \pi(i)$; this is possible as it is easily checked that (in general) $\sigma < \tau$, $\tau' \in \mathrm{Range}(\pi_{\sigma\tau})$, $\sigma < \nu < \tau \to \exists \nu' < \tau' \, (\alpha(\nu') = \alpha(\nu))$. Note that $\mathscr{D} \in \mathrm{Range}(\pi_{i'\pi(i)})$ and $\pi_{i'\pi(i)}(\mathscr{D} \cap L_{\alpha(\tau')}) = \mathscr{D}$. By induction $p' \in \mathscr{C}_{X-i}$ where $p'(j) = b_{\pi_{\bar{\tau}\tau'}(j)}$ and therefore $G(p')$ meets $\mathscr{D} \cap L_{\alpha(\tau')}$, since $\mathscr{D} \cap L_{\alpha(\tau')} \in L_{i'}(p^{\alpha(\tau')} \restriction i)$. So $G(p, i)$ meets $\mathscr{D}$.

Now suppose that $\tau' <_* \tau$. By (c) it suffices to consider the case $\tau' = \bar{\tau}$. First suppose that τ is a T_α-successor. Let $X' = (X - i) - \{\bar{\tau}\}$ ($\bar{\tau} \in X$ since $\bar{\tau}$ is a $T_{\alpha(\bar{\tau})}$-successor and $\bar{\tau} = \sup X$). Let $p' = p \restriction X'$ and $\bar{\alpha} = \alpha(\bar{\tau})$. Then by construction $b_\tau = \bigcup \{q(\bar{\tau}) \mid q \in G\}$ where G is sufficiently $\mathscr{C}_{T_\alpha - \bar{\tau}}^{p'}$-generic. Now pick any $\nu' < \nu = T_\alpha$-predecessor of τ, $\bar{\nu} = (T_{\bar{\alpha}}$-predecessor of $\bar{\tau}) < \nu'$, $\mathscr{D} \in \mathrm{Range}(\pi_{\nu'\nu})$. Then by induction $q' \in \mathscr{C}_{X'}$ where $q'(j) = b_{\pi_{\bar{\nu}\nu'}(j)}$ and also $G(q)$ is sufficiently $\mathscr{C}_{T_\alpha - \bar{\tau}}^{q'}$-generic where $q \in G$, $|q| = \alpha(\nu')$. It follows from Lemma 1B.10 that

$q' \cup q \in \mathscr{C}_{T_{\bar\alpha}-i}$ and hence $q'' = q' \cup \{\langle \bar\tau, b_{v'} \rangle\} \in \mathscr{C}_{X-i}$. Now as before $G(q'')$ meets $\mathscr{D} \cap L_{\alpha(v')}$ so $G(p, i)$ meets $\mathscr{D}$.

If τ is a T_α-limit, then pick $v' \dashv \tau$ so that $\mathscr{D} \in \text{Range}(\pi_{v'\tau'})$ where $\bar v <_* \bar v' < \tau' = \pi_{\bar\tau\tau}(\bar v)$. Let $p' = p \restriction (X - i) \cap \bar v$ and note that for each $j \in T_{\bar\alpha} \cap [\bar v, \bar\tau]$, $b_{v'j} = \bigcup \{q(j) \mid q \in G\}$ where G is a (fixed) sufficiently $\mathscr{C}^{p'}_{T_{\bar\alpha}-\bar v}$-generic set. By Lemma 1B.10, $p' \cup q' \in \mathscr{C}_{T_{\bar\alpha} \cap [i, \bar\tau]}$ where $q'(j) = b_{v'j}$ for $j \in T_{\bar\alpha} \cap [\bar v, \bar\tau]$. Thus $G(p' \cup q')$ meets $\mathscr{D} \cap L_{\alpha(v')}$ and hence $G(p, i)$ meets $\mathscr{D}$.

(b) Let $\bar{\mathscr{D}} \in L_{v(\alpha)}(\bar p)$ be predense on $\mathscr{C}^{\bar p}_{\bar Y}$; we want to show that $G(\bar q)$ meets $\bar{\mathscr{D}}$. Let $\mathscr{D} = \{q' \in \mathscr{C}^p_Y \mid q' \circ \pi \restriction \bar Y \in \bar{\mathscr{D}}\}$ where π 'witnesses' the fact that $\bar q$ is a thinning of q. It suffices to show that $\mathscr{D}$ is predense on $\mathscr{C}^p_Y$ for then the genericity of G implies that $G(\bar q)$ meets $\bar{\mathscr{D}}$. Given $q' \in \mathscr{C}^p_Y$ let $\bar q'$ extend $q' \circ \pi \restriction \bar Y$, $\bar q'$ meets $\bar{\mathscr{D}}$. By (e) inductively there exists $q'' \leq q$ so that $\bar q'$ is a thinning of q''. Of course the witness to this last fact must be $\pi \restriction \bar Y$ and so q'' meets $\mathscr{D}$.

(c) This is clear from (a) if $X_0(p) = X$. So we can assume that $v(p)$ is defined. Again by (a), $q \restriction v(q) \in \mathscr{C}_{X_0(q)}$. By Lemma 1B.10 it suffices to show that $G(q, v(q))$ is sufficiently $\mathscr{C}(q, v(q))$-generic. By definition, $q \restriction X_0(q)$ is a thinning of $p \restriction X_0(p)$ so (b) applies to show the genericity of $G(q, v(q))$.

(d) This is proved much like Lemma 1B.10(a), whose proof is patterned in turn on the proof of Lemma 1B.9(a). In this case we want to successively extend q to meet predense sets on $\mathscr{C}^p_Y$ which belong to $L_{v(\alpha)}(p)$ but with the restraint imposed on all conditions q'' that $q'' \circ (\pi \restriction \bar Y) \in G(\bar q)$. (If we succeed in meeting all of those predense sets in this way, then we have constructed a sufficiently $\mathscr{C}^p_Y$-generic G and hence by Lemma 1B.10 we have the desired q' defined by $q'(i) = \bigcup \{q''(i) \mid q'' \in G\}$.) The key question is whether the restraint imposed interferes with meeting predense sets. It does not provided that given an appropriate $\mathscr{D}$ which is predense on $\mathscr{C}^p_Y$ and $q'' \in \mathscr{C}^p_Y$ such that $q'' \circ (\pi \restriction \bar Y) \in G(\bar q)$, we can extend q'' to q''' meeting $\mathscr{D}$ so that $q''' \circ (\pi \restriction \bar Y) \in G(\bar q)$. $\bar{\mathscr{D}} = \{q''' \circ (\pi \restriction \bar Y) \mid q''' \in \mathscr{D}\}$ is predense on $\mathscr{C}^{\bar p}_{\bar Y}$ by (e) inductively. Thus the proof reduces to the following:

Lemma 1B.12. *Suppose $\bar p \in \mathscr{C}_X$, $\bar p \cup \bar q \in \mathscr{C}_{\bar X \cup \bar Y}$ where $\bar X, \bar X \cup \bar Y \in I_\alpha$, $|\bar p| = |\bar q| = |\bar p \cup \bar q| = \alpha \in U$ and $X_0(\bar p \cup \bar q) \subseteq \bar X$. Also suppose that $\bar p$ is a thinning of $p \in \mathscr{C}_X$ and* Range$(\bar q) \cap$ Range$(p) = \emptyset$. *Then $G(\bar q) = \{\bar q' \in \mathscr{C}^{\bar p}_{\bar Y} \mid \bar q \leq \bar q'\}$ is $\mathscr{C}^{\bar p}_{\bar Y}$-generic over $L_{v(\alpha)}(p)$.*

Proof. Deferred to Part C.

(e) Choose $\tau' \in T_\alpha$ to be $\sup\{v \in T_\alpha \mid b_v \in \text{Range}(\bar q)\}$. Pick $\tau < \tau'$, $\tau \in T_{|X|}$ (this is possible by (M7), the 'second continuity principle' for morasses). Then $p' \in \mathscr{C}_{X \cap \tau}$ where $p'(i) = b_{\pi_{\tau\tau'}(i)} \restriction |p|$, by (a) inductively. But $p \restriction \tau$ agrees with p' on cofinally many $i \in X \cap \tau$ so $p \restriction \tau = p'$ by Lemma 1B.8(b). So we define $q(i) = b_{\pi_{\tau\tau'}(i)}$ for $i \in X \cap \tau$. Also $G(\bar q \restriction (\bar X - \bar\tau))$ is $\mathscr{C}(\bar q, \bar\tau)$-generic over $L_{v(\alpha)}(\bar q \restriction \bar\tau)$ where $\bar\tau \in T_{|\bar X|}$, $\bar\tau < \tau$. So by (d) there exists $q' \in \mathscr{C}_{X-\tau}$, $(q \restriction (X \cap$

$\tau)) \cup q' \in \mathscr{C}_X$, $q' \circ \pi \restriction (\bar{X} - \bar{\tau}) = \bar{q}$ (where π 'witnesses' that $\bar{p}$ is a thinning of p). Finally let $q = (q \restriction (X \cap \tau)) \cup q'$. Then q is as desired. □

At last we can verify properties (a)–(g) and show that the construction of the b_ν's, $b_{\nu\tau}$'s is well-defined. The assertions "$|b_\nu| = \alpha(\nu)$", "$|b_{\nu\tau}| = \alpha(\nu)$" in (a) follows from 1B.11(a) and 1B.10(b), using the definition of $\mathscr{C}_X^p$. (b), (c), (d) are clear by construction. (e) is clear, its last statement following from Lemma 1B.9(b). (f) follows from Lemma 1B.8(a). (g) follows from 1B.11(a) and from the definition of $\mathscr{C}_X$.

Now for the construction of the b_ν's, $b_{\nu\tau}$'s: Case (3a) is handled by 1B.9(a). For (3b) use Lemma 1B.10(a). Case (3c) uses 1B.10(a), including its second statement to justify "$p \in \mathscr{C}_X^\nu$".

C. Successor cardinal coding II: Supergeneric codes

In this part we refine the construction of the generic codes given above. The need for this is to establish Lemma 1B.12, which is not true for the generic codes as built in Part B. This 'mutual-genericity', or 'amalgamation' property is also needed in the proof of extendibility for the forcing R^s to be defined in Part D. We also deal with Σ-genericity in this part.

We must now define $\{p \in \mathscr{C}_X \mid |p| = \alpha\}$ for all $X \in I_\alpha$ simultaneously to guarantee the desired mutual genericity. In order to also establish forms of Lemmas 1B.10, 1B.11 in this new context we are led to the definitions below.

As before we define $\{b_\nu \mid \nu \in T_{<\alpha}\}$, $\{b_{\nu\tau} \mid \nu \in T_{<\alpha},\ \tau \in W(\nu)\}$ and $\mathscr{C}_X^\alpha$, $X \in I_\alpha$ by induction on $\alpha \in U = U(\gamma)$ (where $U = \mathscr{U}$ of Part B and $T_{<\alpha}$ denotes $\bigcup\{T_{\alpha'} \mid \alpha' \in U \cap \alpha\}$). The heart of the matter is to define $\{p \in \mathscr{C}_X^{\alpha'} \mid |p| = \alpha\}$ for $X \in I_\alpha$ where α' = U-successor of α. For $X \cup Y \in I_\alpha$, X and Y disjoint and $p \in \mathscr{C}_X$, $|p| = \alpha$ we let $\mathscr{C}_Y^p$ denote (as before) $\{q \in \mathscr{C}_Y^\alpha \mid p^{|q|} \cup q \in \mathscr{C}_{X \cup Y}\}$. Also for $p \in \mathscr{C}_X$, $|p| = \alpha$, $X_0(p)$ denotes $\{\sigma \in X \mid p(\sigma) = b_\nu$ for some $\nu \in T_\alpha\}$. Set $\hat{\mathscr{C}}_Y^p = \{q \in \mathscr{C}_Y^p \mid X_0((p \cup q)^\eta) \subseteq X$ for all $\eta \in U \cup (|X|, |q|]\} \cup \{q_Y\}$ where $\mathrm{Dom}(q_Y) = Y$, $q_Y(i) = b_i$ for all $i \in Y$. (This definition of $\hat{\mathscr{C}}_Y^p$ is not entirely unrelated to the $\hat{\mathscr{C}}_X$ defined in the proof of Lemma 1B.9.)

To each 'α-condition' c is associated a canonical condition $p(c)$ in $\mathscr{C}_{X(c)}$, $|p(c)| = \alpha$. The notion of 'α-condition' is explained by:

Definition. An *α-condition* is a sequence $c = (p_0, (p_1', p_1), \ldots, (p_n', p_n))$ where, setting $Y_i = \mathrm{Dom}(p_i)$:

(a) Either $p_0 = \emptyset$ or: $p_0 \in \mathscr{C}_{Y_0}$, $|p_0| = \alpha$ is *standard*; i.e., $p_0(\sigma) \in \{b_\nu \mid \nu \in T_\alpha\}$ for $\sigma \in Y_0$.

(b) p_{i+1}' is a thinning of $p(p_0, (p_1', p_1), \ldots, (p_i', p_i))$.

(c) $p_{i+1} \in \hat{\mathscr{C}}_{Y_{i+1}}^{p_{i+1}'}$.

(d) $\nu(\alpha_{i+1}) \in Y_{i+1}$ where $\alpha_{i+1} = |Y_{i+1}|$.

If c is an α-condition as above, then $X(c)$ denotes $\mathrm{Dom}(p'_n \cup p_n)$ and $\mathrm{length}(c)=n$. Also set $\sigma(c)=\sup\{\nu \mid b_\nu \in \mathrm{Range}(p_0)\}$ if $p_0 \neq \emptyset$; $\sigma(c)=0$ otherwise.

We note that the trivial thinnings $p=q\circ\pi$, $\pi=\mathrm{id}$ are allowed to occur in (b) above.

Once we have defined $p(c)$ for the α-conditions c, $\{p \in \mathscr{C}_X^{\alpha'} \mid |p|=\alpha\}$ can be defined to consist of all thinnings q of conditions of the form $p(c)$, c an α-condition, which have domain X, together with all q' obtainable from such q as follows: Pick $i_0<\cdots<i_n$ from $X\cup\{\bigcup X+1\}$, $\beta_0,\ldots,\beta_n \in U$. Then define $q'(i)=q(i)$ for $i\in X$ so that the least k, $i<i_k$, is even. Define $q'(i)\upharpoonright\beta_k = q(i)\upharpoonright\beta_k$, $q'(i)(\eta)=0$ for all $\eta\in[\beta_k,\alpha)$ if the least $k, i<i_k$, is odd. (The introduction of the conditions q' is necessary for the proof of the Genericity Lemma for R^s. See Sublemma 1D.3.)

The conditions $\{p(c) \mid c$ an α-condition of length $n\}$ are defined by induction on n. We wish to maintain the following properties:

(a)–(f) as in Part B.

(g) As in Part B but with the final added clause "and $G(p,i)$ is $\mathscr{C}(p,i)$-Σ-generic over $L_{\pi(i)}(p\upharpoonright i)$ if $\pi(i)$ is $p\upharpoonright i$-admissible".

The next property is a slightly modified version of Lemma 1B.12. As suggested above, we must introduce certain 'dummy' conditions into $\mathscr{C}_X$; $\mathscr{C}_X^*$ is obtained by discarding them from $\mathscr{C}_X$ and is defined just before Fact 2 below.

(h) Suppose $\bar p \in \mathscr{C}_{\bar X}^*$, $\bar p\cup\bar q \in \mathscr{C}_{\bar X\cup\bar Y}^*$, $X_0(\bar p\cup\bar q)\subseteq\bar X$ and $|\bar p|=|\bar q|=|\bar p\cup\bar q|=\alpha\in U(\gamma)$. If $\bar p$ is a thinning of p, $\mathrm{Range}(\bar q)\cap\mathrm{Range}(p)=\emptyset$, then $G(\bar q)$ is $\mathscr{C}_{\bar Y}^{\bar p}$-generic over $L_{\nu(\alpha)}(p)$ (is $\mathscr{C}_{\bar Y}^{\bar p}$-$\Sigma$-generic over $L_{\nu(\alpha)}(p)$ if $\nu(\alpha)$ is p-admissible). ($\approx$Lemma 1B.12).

(i) c an α-condition, $\beta\in C_\alpha$, c^β a β-condition$\rightarrow p(c)\leqslant p(c^\beta)$ (where if $c=(p_0,(p'_1,p_1),\ldots,(p'_n,p_n))$ then $c^\beta=(p_0^\beta,(p_1'^\beta,p_1^\beta),\ldots,(p_n'^\beta,p_n^\beta)))$.

(j) Suppose $\bar\nu <_* \nu$ and let $X_\nu = T_{\alpha(\bar\nu)}\cap\bar\nu$, $Y_\nu=T_{\alpha(\bar\nu)}-X_\nu$. Let $c(\nu)$ be the $\alpha(\nu)$-condition $(\pi_\nu,(\pi_\nu,q_{Y_\nu}))$ where $\mathrm{Dom}(\pi_\nu)=X_\nu$, $\pi_\nu(i)=b_{\pi_{\bar\nu\nu}}(i)$ for all $i\in X_\nu$. Then for $j\in Y_\nu$, $b_{\nu j}=p(c(\nu))(j)$.

(k) Suppose $\bar\nu<_*\nu$, c an $\alpha(\nu)$-condition, $\sigma(c)=\nu$. Also suppose that $c=(p_0,(p'_1,p_1),\ldots,(p'_n,p_n))$. Then $p(c)=p(c')$ where $c'=(\pi_\nu,(\pi_\nu,q_{Y_\nu}),(p'_1,p_1),\ldots,(p'_n,p_n))$.

(l) Suppose $\nu'\dashv\nu$, c an $\alpha(\nu)$-condition and $c^{\alpha(\nu')}$ an $\alpha(\nu')$-condition. Then $p(c)\leqslant p(c^{\alpha(\nu')})$.

To express the remaining of our properties we introduce the morass relation $\dashv\!\vdash'$. This relation applies in each of three situations:

(1) Suppose $\alpha\in U(\gamma)$, $\alpha=\nu(\alpha)$ and $\alpha^*=(\Sigma_1\text{-projectum of }\alpha)=\gamma$. Let $B_\alpha=\{\alpha'<\alpha \mid (\alpha')^*=\gamma,\ p(\alpha')=p(\alpha),\ C'_{\alpha'}=C'_\alpha\cap\alpha'\}$. Then $\beta\dashv\!\vdash'\alpha$ iff $\beta\in B_\alpha$.

(2) Suppose $\nu=\nu(\alpha)>\alpha$, $\nu^*=\alpha$ and ν is $<$-minimal. For any $\alpha'<\alpha$ define $H_{\alpha'}^\nu=\Sigma_1$-Skolem hull of $\alpha'\cup\{p(\nu)\}$ inside L_ν. Then $\sigma\dashv\!\vdash'\nu$ if σ is $<$-minimal, $\sigma^*=\alpha(\sigma)$ and $L_\sigma\simeq H_\alpha^\nu(\sigma)$.

(3) Suppose $\nu = \nu(\alpha) >_* \bar{\nu}$ and $\nu^* = \alpha$. Define $H^\nu_{\alpha'}$ as in (2). Then $\sigma \dashv\!\!\|' \nu$ if $\bar{\nu} <_* \sigma$, $\sigma^* = \alpha(\sigma)$ and $L_\sigma \simeq H^\nu_{\alpha(\sigma)}$.

Remarks. (a) Note that $C_\alpha \subseteq B_\alpha$ in (1) above. If α is admissible in (1), then B_α is $\pi_1(L_\alpha)$ and unbounded in α. Also in (2), (3) we have that Σ_1-cof$(\nu) \geq \alpha$ and so $\{\alpha' < \alpha \mid \alpha' = H^\nu_{\alpha'} \cap \alpha\}$ is unbounded in α.

(b) Suppose α' is a successor point of C_α in (1). Then $B_\alpha \cap \alpha'$ is $\Delta_1(L_{\alpha'})$. This is important for our treatment of Σ-genericity when α is admissible as in (1). If ν is admissible as in (2), (3), then we will use the $\pi_1(L_\nu)$-regularity of $\nu^* = \alpha$. See the Fact in the proof of Lemma 1C.3. Note that if ν is admissible as in (1), (2) or (3), then $\{\sigma \mid \sigma \dashv\!\!\|' \nu\}$ is $\pi_1(L_\nu)$.

In the course of our construction we shall define the property "ν is active" for ν as in Cases (1), (2), (3) above. We set $\sigma \dashv\!\!\| \nu$ iff ($\sigma \dashv\!\!\|' \nu$ is active) for ν as in (2), (3) and $\beta \dashv\!\!\| \alpha$ iff ($\beta \dashv\!\!\|' \alpha$ and either β is active or $\beta \in C_\alpha$) in (1). We cannot define "ν is active" at present as its definition is influenced by the outcome of our construction.

We can now introduce:

(m) Suppose $\nu' \dashv\!\!\| \nu$, c an $\alpha(\nu)$-condition and $c^{\alpha(\nu')}$ an $\alpha(\nu')$-condition. Then $p(c) \leq p(c^{\alpha(\nu')})$.

In property (m) we mean to include the possibility $\nu' = \alpha(\nu')$, $\nu = \alpha(\nu)$ in which case $\nu' \dashv\!\!\|' \nu$ is defined via (1) above. We will prove a number of lemmas below which imply that the relation $\dashv\!\!\|'$ is well-behaved and interacts well with $\dashv$ and the C_α's.

We now discuss the definitions of $b_{\nu(\alpha)}$, $\{b_{\nu(\alpha)\bar{\sigma}} \mid \bar{\sigma} \in W(\nu(\alpha))\}$ (if $\nu(\alpha)$ is a $<$-successor) and $p(c)$ for α-conditions c. Our major concern is to arrange condition (h), which necessitates consideration of the following forcings.

Motivation. We want to define $p(p_0, (p'_1, p_1), \ldots, (p'_{i+1}, p_{i+1}))$ so that its restriction to $Y_{i+1} = \mathrm{Dom}(p_{i+1})$ is $\mathscr{C}^{p'_{i+1}}_{Y_{i+1}}$-generic over $L_{\nu(\alpha)}(p'_{i+1})$. Of course the forcing $\mathscr{C}^{p'_{i+1}}_{Y_{i+1}}$ is defined in terms of $p(p_0, (p'_1, p_1), \ldots, (p'_i, p_i))$ which is itself generic; thus we are dealing here with a finite iteration. What we actually need to consider is many such finite iterations with the following coherence property: If c, d are α-conditions with a largest common initial segment e, then $p(c) \restriction X$, $p(d) \restriction Y$ should be mutually-generic over $L_{\nu(\alpha)}(p(e))$ and $p(c) \restriction X(c) - X$, $p(d) \restriction X(d) - Y$ should be thinnings of $p(e)$, where $X = \{i \in X(c) \mid p(c)(i) \notin \mathrm{Range}(p(e))\}$ and $Y = \{i \in X(d) \mid p(d)(i) \notin \mathrm{Range}(p(e))\}$. Thus we must deal with a 'tree' of iterations. The nodes of this tree are 'α-names', defined below.

Definition. An *α-name* is a sequence $\bar{c} = (p_0, (\bar{p}_1, p_1), \ldots, (\bar{p}_n, p_n))$ where setting $Y_i = \mathrm{Dom}(p_i)$:

(a) Either $p_0 = \emptyset$ or: $p_0 \in \mathscr{C}_{Y_0}$, $|p_0| = \alpha$ is standard; i.e., $p_0(\sigma) \in \{b_\nu \mid \nu \in T_\alpha\}$ for $\sigma \in Y_0$.

(b) $\bar{p}_1$ is a thinning of p_0 and $p_1 \in \mathscr{C}^{\bar{p}_1}_{Y_1}$. Also let $\mathscr{C}(p_0, (\bar{p}_1, p_1)) = \{q \in \mathscr{C}^{\bar{p}_1}_{Y_1} \mid q \leq p_1\}$ and let $\mathbf{G}^1$ denote the generic for forcing with $\mathscr{C}(p_0, (\bar{p}_1, p_1))$ over $L_{\nu(\alpha)}(p_0)$. In addition set $\mathbf{G}(p_0, (\bar{p}_1, p_1)) = \bar{p}_1 \cup \mathbf{G}^1$ (where $\mathbf{G}^1$ is naturally identified with a function on Y_1).

(c) For some $(p'_1, \ldots, p'_i) \leq (p_1, \ldots, p_i)$, $(p'_1, \ldots, p'_i) \Vdash \bar{p}_{i+1}$ is a thinning of $\mathbf{G}(p_0, (\bar{p}_1, p_1), \ldots, (\bar{p}_i, p_i))$ and $p_{i+1} \in \mathscr{C}^{\bar{p}_{i+1}}_{Y_{i+1}}$, where $\Vdash$ refers to the p.o. $\mathscr{C}(p_0, (\bar{p}_1, p_1)) * \cdots * \mathscr{C}(p_0, (\bar{p}_1, p_1), \ldots, (\bar{p}_i, p_i))$.

(d) $\mathscr{C}(p_0, (\bar{p}_1, p_1), \ldots, (\bar{p}_{i+1}, p_{i+1})) = \{q \leq p_{i+1} \mid$ for some $(p'_1, \ldots, p'_i) \leq (p_1, \ldots, p_i)$, $(p'_1, \ldots, p'_i) \Vdash q \in \mathscr{C}^{\bar{p}_{i+1}}_{Y_{i+1}}\}$ and $\mathbf{G}(p_0, (\bar{p}_1, p_1), \ldots, (\bar{p}_{i+1}, p_{i+1})) = \bar{p}_{i+1} \cup \mathbf{G}^{i+1}$. ($\mathbf{G}^{i+1}$ denotes the generic for forcing with $\mathscr{C}(p_0, (\bar{p}_1, p_1), \ldots, (\bar{p}_{i+1}, p_{i+1}))$ over $L_{\nu(\alpha)}(p_0, \mathbf{G}^1, \ldots, \mathbf{G}^i)$.)

(e) $\nu(\alpha_{i+1}) \in Y_{i+1}$ where $\alpha_{i+1} = |Y_{i+1}|$.

Note that we have chosen 'canonical names' in the sense that the p_i's are not terms but restrictions to Y_i of actual elements of $\mathscr{C}^\alpha$. We shall also assume that $\bar{p}_{i+1}$ is a term of the form $\mathbf{G}(p_0, (\bar{p}_1, p_1), \ldots, (\bar{p}_i, p_i)) \circ \pi_{i+1}$, where π_{i+1} is as in the definition of thinning.

To each α-name $\bar{c} = (p_0, (\bar{p}_1, p_1), \ldots, (\bar{p}_n, p_n))$ we have the associated iteration $\mathscr{C}^*(\bar{c}) = \mathscr{C}(p_0, (\bar{p}_1, p_1)) * \cdots * \mathscr{C}(p_0, (\bar{p}_1, p_1), \ldots, (\bar{p}_n, p_n))$. Now let $J_0(\alpha)$ denote all α-conditions c such that $\sigma(c) < \nu(\alpha)$. Similarly define $\bar{J}_0(\alpha) =$ all α-names $\bar{c}$ such that $\sigma(\bar{c})$ ($=\sup\{\nu \in T_\alpha \mid b_\nu \in \text{Range}(p_0), p_0 = \text{1st component of } \bar{c}\}$) is less than $\nu(\alpha)$. If $\bar{c}$ is an α-name, $i \leq \text{length}(\bar{c})$, then $\bar{c}(\leq i)$ denotes the α-name $(p_0, (\bar{p}_1, p_1), \ldots, (\bar{p}_i, p_i))$, where $\bar{c} = (p_0, (\bar{p}_1, p_1), \ldots, (\bar{p}_n, p_n))$.

Let $\mathscr{D}_0(\alpha) = \{\text{finite } D \subseteq \bar{J}_0(\alpha) \mid \bar{c} \in D \rightarrow \bar{c}(\leq i) \in D \text{ for all } i\}$. Then we define the forcing $\mathscr{P}(\alpha, D)$ to consist of all functions f with domain D such that for all $\bar{c} \in D \in \mathscr{D}_0(\alpha)$: $f(\bar{c}) = (q_1, \ldots, q_k) \in \mathscr{C}^*(\bar{c})$ and $\bar{c}_1 < \bar{c}_2 \rightarrow f(\bar{c}_1) < f(\bar{c}_2)$ ($s < t$ denotes "s is an initial segment of t"). Thus $\mathscr{P}(\alpha, D)$ is a 'tree-iteration' of the forcings $\mathscr{C}^*(\bar{c})$, $\bar{c} \in D$.

We think of a $\mathscr{P}(\alpha, D)$-generic set G as an assignment of a sequence $(g_0, g_1, \ldots, g_k)$, $g_i \in \mathscr{C}_{Y_i} = \{h \restriction Y_i \mid h \in \mathscr{C}_{T_{\alpha_i}}\}$ where $\alpha_i = |Y_i|$ to each α-name $\bar{c} = (p_0, (\bar{p}_1, p_1), \ldots, (\bar{p}_k, p_k)) \in D$, $Y_i = \text{Dom}(p_i)$. (Namely: $g_0 = p_0$ and for $i > 0$, $g_i = \bigcup\{q_i \mid (q_1, \ldots, q_k) \in f(\bar{c}) \text{ for some } f \in G\}$.) We want to associate $p(\bar{c}) \in \mathscr{C}_{X(\bar{c})}$, $X(\bar{c}) = \text{Dom}(\bar{p}_k \cup p_k)$ to each 'proper' α-name $\bar{c} \in \bar{J}_0(\alpha)$ so that the assignment $\bar{c} = (p_0, (\bar{p}_1, p_1), \ldots, (\bar{p}_k, p_k)) \mapsto (p(p_0), p(p_0, (\bar{p}_1, p_1)) \restriction \text{Dom}(p_1), \ldots, p(p_0, \ldots, (\bar{p}_k, p_k)) \restriction \text{Dom}(p_k))$ is $\mathscr{P}(\alpha, D)$-generic over $L_{\nu(\alpha)}$ (is $\mathscr{P}(\alpha, D)$-Σ-generic over $L_{\nu(\alpha)}$ if $\nu(\alpha)$ is admissible), for each $D \in \mathscr{D}_0(\alpha)$ consisting solely of 'proper' α-names. (The notion of 'proper' is defined inductively, simultaneously with the definition of the $p(\bar{c})$'s.) If this is done, then we can define $p(c)$ for any $c \in J_0(\alpha)$ by canonically assigning a 'proper' α-name $\bar{c} \in \bar{J}_0(\alpha)$ to each $c \in J_0(\alpha)$ and setting $p(c) = p(\bar{c})$. As a result properties analogous to (a)–(m) for α-names will be maintained. We shall also define $p(c)$, $p(\bar{c})$ when c is an α-condition, $\bar{c}$ a 'proper' α-name but $\sigma(c) = \sigma(\bar{c}) = \nu(\alpha)$.

Our next two lemmas deal with $\diamondsuit$-sequences. (We will use the relation $\dashv\vert$ to deal with Σ-genericity but use $\diamondsuit$ instead to obtain regular genericity.) Let $E = \{\alpha \in U(\gamma) \mid C_\alpha$ is bounded in α and either $\beta(\nu(\alpha)) < \beta(\alpha)$ or $\alpha = \nu(\alpha) < \beta(\alpha)\}$.

Lemma 1C.1. *Suppose $\alpha \in U(\gamma)$, $\alpha < \nu(\alpha)$. Then $E \cap \alpha$ is an $L_{\nu(\alpha)}$-stationary subset of α.*

Proof. Suppose $C \in L_{\nu(\alpha)}$ where the $<_L$-least closed unbounded subset of α disjoint from $E \cap \alpha$ and let ν = greatest p.r. closed ν' such that $C \notin L_{\nu'}$. Then $\nu < \nu(\alpha)$. Let $\eta = \hat{\eta} \cdot \omega$ where $\hat{\eta}$ is the least $\hat{\eta} \geqslant \beta(\nu)$ such that $C \in L_{\hat{\eta}}$ and $p(\eta)$ = (least p such that $L_\eta = \Sigma_1$-Skolem hull of $\alpha \cup \{p\}$ inside L_η).

Now consider $H = \Sigma_1$-Skolem hull of $\gamma \cup \{p(\eta)\}$ inside L_η and let $\bar{\alpha} = H \cap \alpha$, $L_{\bar{\eta}}$ = collapse(H). We claim that $\bar{\alpha} \in E \cap C$, contradicting the choice of C.

Note that $L_{\bar{\eta}} \vDash \bar{\alpha}$ is a cardinal, yet the Σ_1-projectum of $\bar{\eta}$ equals γ. So $\mathscr{A}(\bar{\alpha}) = L_{\bar{\eta}}$, $\beta(\bar{\alpha}) = \bar{\eta}$ and $\bar{\alpha} < \nu(\bar{\alpha}) \rightarrow \beta(\nu(\bar{\alpha})) < \beta(\bar{\alpha}) = \bar{\eta}$ ($\nu(\bar{\alpha}) < \bar{\eta}$ as $\bar{\eta}$ is not p.r. closed). Also $C \cap \bar{\alpha} \in L_{\bar{\eta}}$ and $C \cap \bar{\alpha}$ is unbounded in $\bar{\alpha}$. As C is closed, $\bar{\alpha} \in C$.

Let $J = \Sigma_1$-Skolem hull of $\{\gamma, p(\bar{\alpha})\}$ inside $L_{\bar{\eta}}$. Then J contains the image of C under $\pi : H \simeq L_{\bar{\eta}}$ $(= C \cap \bar{\alpha})$ and hence that of ν, $\beta(\nu)$, $\beta(\nu) \cdot n$ for each n. Thus J is unbounded in $L_{\bar{\eta}}$. But then $J \cap \bar{\alpha}$ is unbounded in $\bar{\alpha}$ since J contains $\bar{\alpha} \cap (\Sigma_1$-Skolem hull of $\gamma \cup \{p(\bar{\alpha})\}$ inside $L_{\beta(\bar{\nu}) \cdot n})$ for each n, where $\bar{\nu}$ = collapse of ν. We have shown that $C_{\bar{\alpha}} = \emptyset$. So $\bar{\alpha} \in E$. □

Lemma 1C.2. *There exists $\langle D_\delta \mid \delta \in E \rangle \in L_\beta$ such that for each $\alpha < \nu(\alpha)$, $\langle D_\delta \mid \delta \in E \cap \alpha \rangle \in L_{\nu(\alpha)}$ is a $\diamondsuit(E \cap \alpha)$-sequence for $L_{\nu(\alpha)}$; i.e., $\{\delta \in E \cap \alpha \mid X \cap \delta = D_\delta\}$ is $L_{\nu(\alpha)}$-stationary for each $X \subseteq \alpha$, $X \in L_{\nu(\alpha)}$.*

Proof. Define D_δ exactly as in Jensen [6]: Pick the least $\langle D_\delta, C_\delta \rangle$ such that C_δ is a closed unbounded subset of δ, $\{\delta' \in E \cap \delta \mid D_\delta \cap \delta' = D_{\delta'}\} \cap C_\delta = \emptyset$. The proof that this works is exactly as in [6]. □

Our next group of lemmas is concerned with the relations $\dashv\!\!\!|'$, $\dashv$, $<$ and their interaction with the sets C_α.

Lemma 1C.3. (a) *$\dashv\!\!\!|'$ is a tree. If $\nu^* = \alpha(\nu) < \nu$ is not a $<$-limit, then $\{\alpha(\sigma) \mid \sigma \dashv\!\!\!|' \nu\}$ is a closed subset of $\alpha(\nu)$. If $\alpha^* = \gamma$, $\alpha \in U(\gamma)$, then $\{\beta \mid \beta \dashv\!\!\!|' \alpha\}$ is a closed subset of α.*

(b) *If ν is admissible, $\nu^* < \nu$ and $\nu^* \in U(\gamma) \cup \{\gamma\}$, then ν is either a $<$-limit or a $\dashv\!\!\!|'$-limit.*

Proof. (a) The first statement is clear as if $L_\sigma \simeq H^\nu_{\alpha(\sigma)}$ and $\tau < \sigma$, then $L_\tau \simeq H^\sigma_{\alpha(\tau)}$ iff $L_\tau \simeq H^\nu_{\alpha(\tau)}$. For, if $\pi : L_\alpha \simeq H^\nu_{\alpha(\sigma)}$, then $H^\nu_{\alpha(\tau)} = \pi[H^\sigma_{\alpha(\tau)}]$. To verify the second statement we need the following.

Sublemma 1C.4. *Suppose* $\nu \in T_\alpha$, $\nu^* < \nu$ *and* $\pi : L_{\bar\nu} \to L_\nu$ *is a Q-embedding,* $p(\nu) \in \mathrm{Range}(\pi)$. *If* $\Sigma_1\text{-cof}(\nu) \geq \alpha$, *then* π *is* Σ_2*-elementary.*

Given the Sublemma we make the following observations. Suppose $\nu^* = \alpha(\nu) < \nu$ is not a $<$-limit. Then $\Sigma_1\text{-cof}(\nu) \geq \alpha(\nu)$ for otherwise for all $\alpha' < \alpha(\nu)$ there exists $\alpha'' < \alpha(\nu)$ so that $\alpha'' \geq \alpha'$ and $H^\nu_{\alpha''} \cap \mathrm{ORD}$ is cofinal in ν; but then $H^\nu_{\alpha''} \simeq L_\sigma \neq L_\nu$ since $\nu^* \geq \alpha(\nu)$ and thus $\sigma < \nu$. This contradicts the assumption that ν is not a $<$-limit. Thus Sublemma 1C.4 applies to ν and in fact to any $\sigma \dashv\!\vdash' \nu$ as in that case $L_\sigma \simeq H^\nu_{\alpha(\sigma)} <_{\Sigma_1} L_\nu$ and thus $\Sigma_1\text{-cof}(\sigma) \geq \alpha(\sigma)$. So the assertion that σ is $<$-minimal (that $\bar\nu <_* \sigma$) is equivalent to the assertion that Σ_2-Skolem hull $(\gamma \cup \{p(\sigma)\})$ in L_σ equals L_σ (that Σ_2-Skolem hull $(\alpha(\bar\nu) + 1 \cup \{p(\sigma)\})$ in L_σ equals L_σ). (Note that the canonical Σ_2-Skolem function for L_σ needs only $p(\sigma)$ as parameter.)

We now prove the second statement in (a). If $\langle \sigma_i \mid i < \lambda \rangle$ is an increasing sequence so that $\sigma_i \dashv\!\vdash' \nu$ for all i, then let $\alpha' = \bigcup \{\alpha(\sigma_i) \mid i < \lambda\}$ and $L_{\sigma'} \simeq H^\nu_{\alpha'}$. Clearly $(\sigma')^* = \alpha'$ as for each $i < \lambda$, $H^{\sigma'}_{\alpha(\sigma_i)}$ does not contain $\alpha(\sigma_{i+1})$ (using the fact that $\sigma^*_{i+1} = \alpha(\sigma_{i+1})$). To see that σ' is $<$-minimal (that $\nu <_* \sigma'$) if ν is $<$-minimal (if $\bar\nu <_* \nu$) it is enough to show that Σ_2-Skolem hull of $\gamma \cup \{p(\sigma')\}$ in $L_{\sigma'}$ equals $L_{\sigma'}$ (that Σ_2-Skolem hull of $\alpha(\bar\nu) + 1 \cup \{p(\sigma')\}$ in $L_{\sigma'}$ equals $L_{\sigma'}$). But this is clear, using the fact that this is true for the σ_i's and $L_{\sigma'} = \bigcup \{H^{\sigma'}_{\alpha(\sigma_i)} \mid i < \lambda\}$.

The third statement in (a) is easily verified. The Sublemma will be proved after we verify (b).

(b) If $\nu^* = \gamma$, then $\nu = \alpha \in U(\gamma)$ and we need only observe that $C_\alpha \subseteq B_\alpha$ is unbounded in α. So assume that $\nu \in T_\alpha$, $\nu^* = \alpha$ and ν is not a $<$-limit. Given $\alpha' < \alpha$ we can choose $\alpha'_0 \geq \alpha'$ so that $\{\alpha'_0\}$ is Σ_2-definable in L_ν with parameters from $\gamma \cup \{p(\nu)\}$ (from $\alpha(\bar\nu) \cup \{\alpha(\bar\nu), p(\nu)\}$) if ν is $<$-minimal (if $\bar\nu <_* \nu$). Now inductively define $H_0 = H^\nu_{\alpha'_0+1}$, $\alpha'_1 = H_0 \cap \alpha, \ldots, H_i = H^\nu_{\alpha'_i+1}$, $\alpha'_{i+1} = H_i \cap \alpha, \ldots$ and let $\alpha'' = \bigcup \{\alpha'_i \mid i < \omega\}$, $L_\sigma \simeq H^\nu_{\alpha''}$. Then $\sigma^* = \alpha'' = \alpha(\sigma)$ as $H^\sigma_{\alpha'_i}$ does not contain α'_i as an element. Also the Σ_2-Skolem hull of $\gamma \cup \{p(\sigma)\}$ in L_σ (the Σ_2-Skolem hull of $\alpha(\bar\nu) + 1 \cup \{p(\sigma)\}$ in L_σ) contains α'_0, hence α'_i for each $i < \omega$, hence all of L_σ. We have proved that σ is $<$-minimal (that $\bar\nu <_* \sigma$) and so $\sigma \dashv\!\vdash' \nu$, provided we argue that $\sigma \neq \nu$. Note that $\Pi_1\text{-cof}(\sigma) = \Pi_1(L_\sigma)\text{-cof}(\alpha(\sigma)) = \omega$.

Fact. *Suppose* $\Sigma_1\text{-cof}(\beta) > \kappa$ *and* $L_\beta \vDash \kappa$ *is regular. Then* κ *is* $\Pi_1(L_\beta)$*-regular.*

Proof of Fact. This is really Lemma 2.3 of Sacks–Simpson [10]. Suppose $f : \gamma \to \kappa$ is $\Pi_1(L_\beta)$, $\gamma < \kappa$ and let $A_i = \{\delta \mid \delta < f(i)\} = \{\delta \mid \forall \delta' \leq \delta \ \sim\phi(i, \delta')\}$ where $f(i) = \delta$ iff $\phi(i, \delta)$, ϕ $\Pi_1(L_\beta)$. As $\Sigma_1\text{-cof}(\beta) \geq \kappa$ we know that A_i is $\Sigma_1(L_\beta)$, uniformly in i. But $\langle A_i \mid i < \gamma \rangle$ must belong to L_β (and hence Range(f) is bounded since $L_\beta \vDash \kappa$ is regular) as otherwise the first κ stages of a 1–1 $\Sigma_1(L_\beta)$-enumeration of $\bigcup \{A_i \mid i < \gamma\}$ gives a $\Sigma_1(L_\beta)$-injection of κ onto the γ-union of sets of size $<\kappa$; this injection belongs to L_β as $\Sigma_1\text{-cof}(\beta) > \kappa$, contradicting the hypothesis that $L_\beta \models \kappa$ is regular. This proves the Fact, and hence the lemma is reduced to Sublemma 1C.4.

Proof of Sublemma 1C.4. It is enough to show that any Σ_2-predicate on L_ν with parameter p is $(\sim Q)$-definable over L_ν with parameter $\langle p, p(\nu)\rangle$. Work in L_ν. Suppose $A = \{x \mid \exists y\, \forall z\, \phi(x, y, z)\}$ where ϕ is Δ_0 with parameter p. Then $x \in A$ iff $\exists \beta < \alpha\, [\beta \in \mathrm{Dom}(h)$ and $\forall z\, \phi(x, h(\beta), z)]$ where h is a partial Σ_1-function from α onto the universe $(=L_\nu)$ with parameter $p(\nu)$. But then $x \in A$ iff $\{\sigma \mid \forall \sigma' < \sigma$ (least $\beta < \alpha$ s.t. $\beta \in \mathrm{Dom}(h^{\sigma'})$ and $\forall z \in L_{\sigma'}\, \phi(x, h(\beta), z)$ if exists $\neq$ least $\beta < \alpha$ s.t. $\beta \in \mathrm{Dom}(h^\sigma)$ and $\forall z \in L_\sigma\, \phi(x, h(\beta), z))\}$ is bounded, using the fact that Σ_1-cofinality $\geq \alpha$. This proves the Sublemma. □

Lemma 1C.5. *Suppose $\nu \in T_\alpha$ is a $<$-limit. Then for sufficiently large $\beta < \alpha$, $\beta \in C_\alpha \to$ there exists $\bar\nu < \nu$ such that $\alpha(\bar\nu) = \beta$.*

Proof. First assume that $\beta(\nu(\alpha)) = \beta(\alpha)$, $n(\nu(\alpha)) = n(\alpha)$. Then $\nu(\alpha)$ is not a $<$-limit: For some $\beta < \alpha$, $p(\alpha) \in \Sigma_1$-Skolem hull of $\beta \cup \{p(\nu(\alpha))\}$ inside $\mathcal{A}(\alpha) = \mathcal{A}(\nu(\alpha))$ since the Σ_1-Skolem hull of $\alpha \cup \{p(\nu(\alpha))\}$ inside $\mathcal{A}(\alpha)$ equals $\mathcal{A}(\alpha)$. But then $\bar\nu < \nu(\alpha)$ implies $\alpha(\bar\nu) < \beta$ as $\mathcal{A}(\alpha) = \Sigma_1$-Skolem hull of $\gamma \cup \{p(\alpha)\}$ inside $\mathcal{A}(\alpha)$. So we have that $\nu \neq \nu(\alpha)$ and we can assume that C_α is unbounded in α.

Now choose $g : \bar\alpha \Rightarrow \alpha$ such that $\mathrm{Range}(g) \cap \nu(\alpha)$ contains an ordinal $> \nu$. This is possible, as otherwise for some ρ, $\mathrm{Range}(g) \subseteq \rho < \rho(\alpha)$ for all such g ($\sigma \in \mathrm{Range}(g) \to$ (Σ_1-Skolem hull of α inside $\mathcal{A}_\sigma = \langle L_\sigma, A(\alpha) \cap \sigma\rangle) \cap \nu(\alpha)$ has supremum in $\mathrm{Range}(g)$); this implies that C_α is definable over $\mathcal{A}_\rho$, contradicting the fact that α is regular in $\mathcal{A}(\alpha)$. Let $\nu' = \sup(\mathrm{Range}(g) \cap \nu(\alpha)) < \nu(\alpha)$, $\rho' = \sup(\mathrm{Range}(g)) < \rho(\alpha)$. Choose $\beta < \alpha$ so that $h : \bar\alpha \Rightarrow \alpha$, $\lambda(h) \geq \beta \to \mathrm{Range}(h)$ contains an ordinal $\geq \rho'$; for example, let β be greater than the supremum of $\alpha \cap$ (Σ_1-Skolem hull of $\gamma \cup \{p(\alpha)\}$ inside $\mathcal{A}'_\rho$).

We claim that $h : \bar\alpha \Rightarrow \alpha$, $\lambda(h) \geq \beta$ implies that $\lambda(h) = \alpha(\bar\nu)$ for some $\bar\nu < \nu'$ (if $\nu(\alpha)$ is a limit of elements of T_α) or for some $\bar\nu < \nu_0 = T_\alpha$-predecessor to $\nu(\alpha)$. The latter case follows easily as if $\nu(\alpha)$ is a T_α-successor, then we can assume that $\nu_0 \in \mathrm{Range}(h)$ and hence $\mathcal{A}(\nu_0) \in \mathrm{Range}(h)$, ($\Sigma_2$-Skolem hull $\lambda(h) \cup \{p(\nu_0)\}$ inside $\mathcal{A}(\nu_0)) \cap \alpha = \lambda(h)$ so $\lambda(h) = \alpha(\bar\nu)$ where $\bar\nu$ is the height of the transitive collapse of this latter Skolem hull. Now suppose that $\nu(\alpha)$ is a T_α-limit so $\nu' \in T_\alpha$. We next determine $\mathcal{A}(\nu')$: Let $H = \Sigma_1$-Skolem hull of $\alpha \cup \{p(\nu(\alpha))\}$ inside $\mathcal{A}_{\rho'}$ and let $\mathcal{A}$ be the transitive collapse of H, with collapsing map $\pi : H \simeq \mathcal{A}$. Then by extension of embeddings lemma, $\mathcal{A} = \mathcal{A}(\pi(\nu(\alpha)))$ and $p(\pi(\nu(\alpha))) = \pi(p(\nu(\alpha)))$. But $H \cap \nu(\alpha) = \nu'$ so $\mathcal{A} = \mathcal{A}(\nu')$, $\pi(p(\nu(\alpha))) = p(\nu')$. We can now show that $\lambda(h) = \alpha(\bar\nu)$ for some $\bar\nu < \nu'$: Indeed, (Σ_1-Skolem hull of $\lambda(h) \cup \{p(\alpha)\}$ inside $\mathcal{A}_\rho) \cap \alpha = \lambda(h)$ where $\rho = \bigcup \mathrm{Range}(h)$. But as $p(\nu(\alpha)) =$ least p such that $p(\alpha) \in \Sigma_1$-Skolem hull of $\alpha \cup \{p\}$ inside $\mathcal{A}(\alpha)$, $p(\nu(\alpha)) \in$ the preceding Skolem hull. It follows that (Σ_1-Skolem hull of $\lambda(h) \cup \{p(\nu(\alpha))\}$ inside $\mathcal{A}_{\rho'}) \cap \alpha = \lambda(h)$

so applying π, (Σ_1-Skolem hull of $\lambda(h) \cup \{p(\nu')\}$ inside $\mathscr{A}(\nu')) \cap \alpha = \lambda(h)$. But now let $\bar{\nu}$ = the height of the transitive collapse of this last Skolem hull.

To conclude this case ($\beta(\alpha) = \beta(\nu(\alpha))$, $n(\alpha) = n(\nu(\alpha))$) note that, if $\nu(\alpha)$ is a T_α-successor, then if ν_0 = the T_α-predecessor to $\nu(\alpha)$ we must have $\{\alpha(\bar{\nu}) \mid \bar{\nu} < \nu_0\} - \beta \subseteq \{\alpha(\bar{\nu}) \mid \bar{\nu} < \nu'\}$ for some $\beta < \alpha$ (choose $\beta = \alpha(\bar{\nu})$ where $\pi_{\bar{\nu}\nu_0}$ has ν' in its range, if $\nu' \neq \nu_0$).

Now suppose $\beta(\alpha) \neq \beta(\nu(\alpha))$ or $n(\alpha) \neq n(\nu(\alpha))$. The preceding argument can be modified as follows: First assume that $n(\alpha) = 1$, so $\beta(\alpha) = \rho(\alpha) > \nu(\alpha)$. Then we show that $\beta \in \{\alpha(\bar{\nu}) \mid \bar{\nu} < \nu(\alpha)\}$ for sufficiently large $\beta \in C_\alpha$: For sufficiently large $\beta \in C_\alpha$ we can choose $g : \bar{\alpha} \Rightarrow \alpha$ such that $\lambda(g) = \beta$ and Range$(g) \nsubseteq \nu(\alpha)$. Then $\nu(\alpha) \in$ Range(g) and so $\mathscr{A}(\nu(\alpha)) \in$ Range(g) and (Σ_2-Skolem hull $\beta \cup \{p(\nu(\alpha))\}$ inside $\mathscr{A}(\nu(\alpha))) \cap \alpha = \beta$. Thus $\beta = \alpha(\bar{\nu})$ where $\bar{\nu}$ = height of the collapse of this Skolem hull and $\bar{\nu} < \nu(\alpha)$. Now assume that $n(\alpha) > 1$ so $\rho(\alpha) = \alpha$. Suppose $g : \bar{\alpha} \Rightarrow \alpha$ and let $\bar{g} : L_{\bar{\beta}} \to L_{\beta(\alpha)}$ be the canonical extension of g to a $\Sigma_{n(\alpha)}$-embedding (via the extension of embeddings lemma). If $\beta(\alpha) > \nu(\alpha)$, assume that $\nu(\alpha) \in$ Range$(\bar{g})$. Otherwise we know that $p(\nu(\alpha)) \in$ Range$(\bar{g})$ since $\bar{g}$ is $\Sigma_{n(\alpha)}$-elementary and $n(\nu(\alpha)) < n(\alpha)$. In either case we can find such a g with $\lambda(g) = \beta$, for sufficiently large $\beta \in C_\alpha$, and then we know that (Σ_1-Skolem hull of $\lambda(g) \cup \{p(\nu(\alpha))\}$ inside $\mathscr{A}(\nu(\alpha))) \cap \alpha = \lambda(g)$. Thus if $(\beta(\nu(\alpha)), n(\nu(\alpha))) \neq (\nu(\alpha), 1)$, then $\beta \in \{\alpha(\bar{\nu}) \mid \bar{\nu} < \nu(\alpha)\}$ for sufficiently large $\beta \in C_\alpha$. Similarly, if $\beta(\alpha) > \beta(\nu(\alpha))$ or $n(\alpha) \geq 3$, then we can replace Σ_1 by Σ_2 in the above Skolem hull and obtain the same result.

There remains the case: $(\beta(\nu(\alpha)), n(\nu(\alpha))) = (\nu(\alpha), 1)$, $(\beta(\alpha), n(\alpha)) = (\nu(\alpha)), 2)$. For this case we need Sublemma 1C.4. If Σ_1-cof$(\nu(\alpha)) \geq \alpha$, then $\nu(\alpha)$ is not a $<$-limit for otherwise by Sublemma 1C.4 and the definition of $\bar{\nu} < \nu(\alpha)$ we cannot have Σ_2-projectum$(\nu(\alpha)) = \gamma$. So we can assume that ν as given in the hypothesis of the lemma is less than $\nu(\alpha)$. But we have already established that $\{\alpha(\bar{\nu}) \mid$ there exists $g : L_{\bar{\nu}} \xrightarrow{\Sigma_1} L_{\nu(\alpha)}$, $p(\nu(\alpha)) \in$ Range$(g)\}$ contains a final segment of C_α, which easily gives the same assertion for $\{\alpha(\bar{\nu}) \mid \bar{\nu} < \nu\}$. Finally, if Σ_1-cof$(\nu(\alpha)) < \alpha$, then $g : L_{\bar{\nu}} \xrightarrow{\Sigma_1} L_{\nu(\alpha)}$, $p(\nu(\alpha)) \in$ Range(g) implies $\bar{\nu} < \nu(\alpha)$ for sufficiently large such $\bar{\nu}$, $\alpha(\bar{\nu}) < \alpha$ as, if $a(\bar{\nu})$ is large enough, the corresponding $g : L_{\bar{\nu}} \xrightarrow{\Sigma_1} L_{\nu(\alpha)}$, $p(\nu(\alpha)) \in$ Range(g) must be cofinal. □

Lemma 1C.6. *Suppose $\bar{\nu} <_* \nu \in T_\alpha$ and either $\pi_{\bar{\nu}\nu}$ is cofinal into ν or ν is not a T_α-limit. Then for all $\beta \in C_\alpha$ there exists $\nu' \dashv \nu$ such that $\alpha(\nu') = \beta$.*

Proof. The argument of the preceding lemma showed that if $(\beta(\nu(\alpha)), n(\nu(\alpha))) \neq (\nu(\alpha), 1)$ and $(\beta(\alpha), n(\alpha)) \neq (\beta(\nu(\alpha)), n(\nu(\alpha)))$, then $\nu(\alpha)$ is a $<$-limit. Also note that by Sublemma 1C.4 we also have that $\nu(\alpha)$ is a $<$-limit in the case: $(\beta(\nu(\alpha)), n(\nu(\alpha))) = (\nu(\alpha), 1) \neq (\beta(\alpha), n(\alpha))$, Σ_1-cof$(\nu(\alpha)) < \alpha$. The hypothesis of this lemma implies that if $(\beta(\nu(\alpha)), n(\nu(\alpha))) = (\nu(\alpha), 1)$, then Σ_1-cof$(\nu) < \alpha$. (Also note $\nu = \nu(\alpha)$.) Thus we can conclude that in the present situation we must have: $(\beta(\alpha), n(\alpha)) = (\beta(\nu(\alpha)), n(\nu(\alpha)))$ and so $\mathscr{A}(\alpha) = \mathscr{A}(\nu(\alpha))$.

Our next claim is that if $g : \bar{\alpha} \Rightarrow \alpha$ then $\alpha(\bar{\nu}) \in \text{Range}(g)$. First note that $p(\nu(\alpha)) = p(\nu) \in \text{Range}(g)$: Indeed $p(\nu) =$ least p such that $p(\alpha) \in \Sigma_1$-Skolem hull of $\alpha \cup \{p\}$ inside $\mathscr{A}(\alpha)$, and in general, if A is Σ_1-definable with parameter p, then the least element of A belongs to the Σ_1-Skolem hull of $\{p\}$. This proves that $p(\nu) \in \Sigma_1$-Skolem hull of $\{p(\alpha), \alpha\}$ inside $\mathscr{A}(\alpha)$ so $p(\nu) \in \text{Range}(g)$. Now let $\beta < \alpha$ be least so that $p(\alpha) \in \Sigma_1$-Skolem hull of $\gamma \cup \{\beta, p(\nu(\alpha))\}$ inside $\mathscr{A}(\alpha)$. Then $\beta \geqslant \alpha(\bar{\nu})$ as $(\Sigma_1\text{-Skolem hull } \alpha(\bar{\nu}) \cup \{p(\nu(\alpha))\}) \cap \nu(\alpha) = \text{Range}(\pi_{\bar{\nu}\nu}) \neq \nu(\alpha)$. But since $\bar{\nu} <_* \nu$ we have that the Σ_1-Skolem hull of $\alpha(\bar{\nu}) \cup \{\alpha(\bar{\nu}), p(\nu(\alpha))\}$ inside $\mathscr{A}(\alpha)$ must equal $\mathscr{A}(\alpha)$ so $\beta = \alpha(\bar{\nu})$. Thus $\alpha(\bar{\nu}) \in \Sigma_1$-Skolem hull of $\{p(\alpha), \gamma, p(\nu(\alpha))\}$ inside $\mathscr{A}(\alpha)$ and so $\alpha(\bar{\nu}) \in \text{Range}(g)$.

Now to conclude note that unboundly many $\nu' \in \text{Range}(g) \cap \nu(\alpha)$ must belong to the Σ_1-Skolem hull of $\alpha(\bar{\nu}) \cup \{p(\nu(\alpha))\}$ inside $\mathscr{A}(\alpha)$, since this Skolem hull has unbounded intersection with $\nu(\alpha)$ (it contains $\text{Range}(\pi_{\bar{\nu}\nu})$). Thus $\hat{\nu} = \bigcup(\text{Range}(g) \cap \nu(\alpha))$ is a limit of elements of $\text{Range}(\pi_{\bar{\nu}\nu})$ and $\bar{\nu} < \nu$, else $\bigcup(\text{Range}(g) \cap \nu(\alpha)) = \nu(\alpha)$ which would imply that $\lambda(g) = \alpha$. Pick $\nu' < \hat{\nu}$, $\nu' \dashv \nu(\alpha)$.

As in the preceding lemma we determine $\mathscr{A}(\hat{\nu})$: Let $H = \Sigma_1$-Skolem hull of $\alpha \cup \{p(\nu(\alpha))\}$ inside $\mathscr{A}_\sigma = \langle L_\sigma, A(\alpha) \cap \sigma \rangle$ where $\sigma = \bigcup \text{Range}(g)$. Let $\pi : H \simeq \mathscr{A}$ be the transitive collapse. By the extension of embeddings lemma $\mathscr{A} = \mathscr{A}(\pi(\nu(\alpha)))$ and $p(\pi(\nu(\alpha))) = \pi(p(\nu(\alpha)))$. But $H \cap \nu(\alpha) = \hat{\nu}$ so $\mathscr{A} = \mathscr{A}(\hat{\nu})$, $\pi(p(\nu(\alpha))) = p(\hat{\nu})$. Now by the definition of ν' we have that $\alpha(\nu') = (\Sigma_1$-Skolem hull of $\alpha(\bar{\nu}) \cup \{\alpha(\bar{\nu}), p(\hat{\nu})\}$ inside $\mathscr{A}(\hat{\nu})) \cap \alpha$. But this equals ($\Sigma_1$-Skolem hull of $\alpha(\bar{\nu}) \cup \{\alpha(\bar{\nu}), p(\nu(\alpha))\}$ inside $\mathscr{A}_\sigma) \cap \alpha$, by applying π^{-1}. This latter intersection equals $\bigcup(\text{Range}(g) \cap \alpha)$ and so $\lambda(g) = \alpha(\nu')$. □

Lemma 1C.7. *Suppose* $\bar{\nu} <_* \nu = \nu(\alpha)$, $\lambda = \bigcup \pi_{\bar{\nu}\nu}[\bar{\nu}] < \nu$. *Let* $\bar{\nu} <_* \bar{\lambda} < \lambda$. *If* $\beta \in C_\alpha$, $\beta < \alpha(\bar{\lambda})$, *then* $\beta = \alpha(\sigma)$ *for some* $\sigma \dashv \nu$. *If* $\nu^* = \alpha$, $\beta \in C_\alpha$, *then* $\nu(\beta) \dashv\!\mid' \nu$.

Proof. Suppose $\beta \in C_\alpha$ and $g : \delta \Rightarrow \alpha$, $\lambda(g) = \beta$. Let $\hat{g}$ be the canonical extension of g to a Σ_1-elementary map from $\mathscr{A}(\nu(\delta))$ to $\mathscr{A}(\nu)$ if $\mathscr{A}(\nu) \neq \mathscr{A}(\alpha)$ (and therefore $\nu^* = \alpha$). As in the proof of Lemma 1C.6, $\alpha(\bar{\nu}) \in \text{Range}(g)$. Note that $\alpha(\bar{\lambda}) = \alpha \cap (\Sigma_1$-Skolem hull of $\alpha(\bar{\nu}) \cup \{\alpha(\bar{\nu}), p(\nu)\}$ inside $\mathscr{A}(\lambda))$ and therefore $\beta = \lambda(g) < \alpha(\bar{\lambda})$ iff $\bigcup(\text{Range}(\hat{g}) \cap \nu) < \lambda$. Thus as in Lemma 1C.6, $\beta < \alpha(\bar{\lambda})$ implies that $\nu(\beta) \dashv \bar{\lambda}$ since $\text{Range}(\hat{g}) \cap \text{Range}(\pi_{\bar{\nu}\lambda})$ must be cofinal in $\text{Range}(\hat{g}) \cap \nu$ and thus $\bar{\sigma} <_* \nu(\beta) < \pi_{\bar{\nu}\bar{\lambda}}(\bar{\sigma})$ where $\bar{\sigma} = \sup\{\sigma \mid \pi_{\bar{\nu}\lambda}(\sigma) \in \text{Range}(\hat{g})\}$.

Now suppose that $\nu^* = \alpha$ and let $\sigma = \bigcup(\text{Range}(\hat{g}) \cap \nu) < \nu$. Then let $\pi : L_{\bar{\sigma}} \simeq H^\sigma_\beta$. We know that $\Sigma_1\text{-cof}(\nu) \geqslant \alpha$ as otherwise ν is a $<$-limit, so the function $\alpha' \mapsto \bigcup(H^\nu_{\alpha'} \cap \text{ORD})$ is Σ_2-definable over L_ν with parameter $p(\nu)$ and maps α cofinally into ν. As $\hat{g}$ is Σ_2-elementary, we have that $H^\sigma_\beta = H^\nu_\beta$ and therefore $L_{\bar{\sigma}} \simeq H^\nu_{\alpha(\bar{\sigma})} = H^\nu_\beta$, $\bar{\sigma}^* = \alpha(\bar{\sigma})$. Also $\bar{\nu} < \bar{\sigma}$ as the Σ_2-elementariness of $\pi_{\bar{\nu}\nu}$ (see Sublemma 1C.4) implies that of $\pi^{-1} \circ \pi_{\bar{\nu}\nu} = \pi_{\bar{\nu}\bar{\sigma}}$. To see that $\bar{\nu} <_* \bar{\sigma}$ note that

Σ_2-Skolem hull $(\alpha(\bar{\nu}) \cup \{\alpha(\bar{\nu}), p(\sigma)\})$ in $L_{\bar{\sigma}}$ must contain Range(g) and hence all of $L_{\bar{\sigma}}$. But Σ_1-cof$(\bar{\sigma}) \geq \alpha(\bar{\sigma})$ so $\sigma < \bar{\sigma}$ implies that $\pi_{\bar{\sigma}\sigma}$ is Σ_2-elementary. □

Note. The proof of Lemma 1C.7 shows that $\beta \in C_\alpha$, $\beta \geq \alpha(\bar{\lambda})$ implies $\sigma \dashv \nu(\beta)$ whenever $\sigma \dashv \bar{\lambda}$. This fact will be of use later.

Lemma 1C.8. *Suppose ν is $<$-minimal, $\nu^* = \alpha(\nu) = \alpha$. If $\beta \in C_\alpha$, then $\nu(\beta) \dashv\!\!\|' \nu$.*

Proof. The argument of Lemma 1C.7 shows that if $\beta \in C_\alpha$, then $\beta = \alpha(\bar{\sigma})$ where $L_{\bar{\sigma}} \simeq H^\nu_{\alpha(\bar{\sigma})}$, Σ_1-cof$(\bar{\sigma}) \geq \beta$, $p(\beta) = p(\alpha)$ and $\bar{\sigma}^* = \beta$. Note that $p(\alpha) = 0$ as otherwise there is $\tau < \nu$, $\alpha(\tau) = \bigcup(\Sigma_1$-Skolem hull of $\{\gamma\}$ inside $\mathscr{A}(\alpha))$. But $\bar{\tau} < \bar{\sigma}$ implies $\pi_{\bar{\tau}\bar{\sigma}}$ is Σ_2-elementary, so $\pi_{\bar{\tau}\bar{\sigma}} =$ identity. □

Lemma 1C.9. *Suppose $\alpha < \nu = \nu(\alpha)$ is neither a $<$-limit nor a T_α-limit. Then C_α is bounded in α.*

Proof. As $\nu(\alpha)$ is not a T_α-limit, Σ_1-cof$(\nu(\alpha)) = \omega$. Thus $\mathscr{A}(\alpha) = \mathscr{A}(\nu(\alpha))$ for otherwise for sufficiently large $\alpha' < \alpha$, (Σ_1-Skolem hull of $\alpha' \cup \{p(\nu(\alpha))\}$ inside $L_{\nu(\alpha)}$) is cofinal in but unequal to $L_{\nu(\alpha)}$; so $\nu(\alpha)$ is a $<$-limit. Now let $\nu^- = T_\alpha$-predecessor to ν if $\nu \neq \min(T_\alpha)$, $= \alpha$ if $\nu = \min(T_\alpha)$. If $p(\nu) \notin L_{\nu^-}$, then $g : \delta \Rightarrow \alpha$ implies $\bigcup \text{Range}(\hat{g}) = \nu$ and hence $\lambda(g) = \alpha$. Otherwise let $\beta = (\Sigma_1$-Skolem hull of $\bar{\alpha} \cup \{p(\nu)\}$ inside $L_{\nu^-}) \cap \alpha$, where $\bar{\alpha} = \alpha(\bar{\nu})$ if $\bar{\nu} <_* \nu$, $= \gamma$ if ν is $<$-minimal. Then $\lambda(g) > \beta \rightarrow \bigcup \text{Range}(g) = \nu \rightarrow \lambda(g) = \alpha$. So C_α is bounded by β. □

Lemma 1C.10. *Suppose $\bar{\sigma} < \sigma \in T_\alpha \cap \nu(\alpha)$.*

(a) *If C_α is unbounded in α, then for sufficiently large $\beta \in C_\alpha$, $\bar{\sigma} < \tau < \sigma$ for some $\tau \in T_\beta$.*

(b) *If $\nu(\alpha)$ is a $\dashv$-limit (i.e., $\alpha = \bigcup\{\alpha(\bar{\nu}) \mid \bar{\nu} \dashv \nu(\alpha)\}$), then for sufficiently large $\bar{\nu} \dashv \nu(\alpha)$, $\bar{\sigma} < \tau < \sigma$ for some $\tau \in T_{\alpha(\bar{\nu})}$.*

(c) *If $\nu(\alpha)$ is a $\dashv\!\!\|'$-limit, then for sufficiently large $\bar{\nu} \dashv\!\!\|' \nu(\alpha)$, $\bar{\sigma} < \tau < \sigma$ for some $\tau \in T_{\alpha(\bar{\nu})}$.*

Proof. (a) follows from Lemma 1C.5 as clearly σ is a $<$-limit ($\sigma \in T_\alpha \cap \nu(\alpha)$). For (b), first choose $\sigma' > \sigma$, $\sigma' \in T_\alpha \cap \text{Range}(\pi_{\tau\nu(\alpha)})$ where $\tau <_* \nu(\alpha)$. Then if $\tau' = \pi^{-1}_{\tau\nu(\alpha)}(\sigma')$, $\alpha(\bar{\nu}) \in \{\alpha(\eta) \mid \tau' < \eta < \sigma'\}$ for all $\bar{\nu} \dashv \nu(\alpha)$, $\pi_{\tau\nu(\alpha)}(\bar{\nu}) > \sigma'$. But for sufficiently large $\eta' < \sigma'$, there exists $\eta < \sigma$ such that $\alpha(\eta) = \alpha(\eta')$. This proves (b). For (c), choose $\alpha' < \alpha$ so that $\sigma \in \Sigma_1$-Skolem hull of $\alpha' \cup \{p(\nu(\alpha))\}$ inside $L_{\nu(\alpha)}$. Then for sufficiently large $\bar{\nu} \dashv\!\!\|' \nu(\alpha)$, $\alpha(\bar{\nu}) > \alpha'$ and $\bar{\nu} < \nu \in T_\alpha$ where $\alpha \in \Sigma_1$-Skolem hull of $\alpha' \cup \{p(\nu(\alpha))\}$ inside L_ν. Then $\alpha(\bar{\nu}) = \alpha(\tau)$ for some $\tau < \sigma$ as $\pi_{\bar{\nu}\nu} \restriction L_\tau : L_\tau \rightarrow L_\sigma$ is elementary, where $\tau = \pi^{-1}_{\bar{\nu}\nu}(\sigma)$. □

Remarks. (1) The reason for introducing $\dashv\!\!\|$ is to deal with Σ-genericity. Lemma

1C.3(b), which also holds for $\dashv\!\!\mid$, is the key property. It allows us to divide the task of meeting predense sets $\mathscr{D}(W_e)$, $e \in L_{\nu(\alpha)}$ into a sequence of steps by following the relation $\dashv\!\!\mid$ or by reflection, using the relation $<$.

(2) Lemmas 1C.5–1C.8 are needed to see that coherence restrictions (in the inductive construction of the b_ν's) imposed by the relations $\dashv$, $\dashv\!\!\mid$ do not conflict with those imposed by the C_α's. These lemmas essentially show that we have in fact a 'morass with linear limits' (see Velleman [9] and Donder [4]).

(3) Lemmas 1C.9, 1C.10 are needed to justify certain steps in the construction of the b_ν's.

(4) Note that $\sigma \dashv \tau \dashv\!\!\mid \nu \to \sigma \dashv \nu$. This is useful in verifying that there is no conflict between the coherence conditions imposed by $\dashv$, $\dashv\!\!\mid$.

Now for the construction of $\{b_\nu \mid \nu \in T_\alpha\}$, $\{b_{\nu\sigma} \mid \sigma \in W(\nu),\ \nu = \nu(\alpha)\}$ (if $\nu(\alpha)$ is a $<$-successor) and $p(\bar{c})$ when $\bar{c}$ is a 'proper' α-name. The collection of proper α-names is defined by induction on α: the idea is to arrange that if $\bar{c} = (p_0, (\bar{p}_1, p_1), \ldots, (\bar{p}_n, p_n))$ is proper, $g_i = p(p_0, (\bar{p}_1, p_1), \ldots, (\bar{p}_i, p_i))$ and $g_i \restriction \mathrm{Dom}(p_i) \leq q_i \leq p_i$ for each $i \leq i_0 < n$, then $\bar{d} = (p_0, (\bar{p}_1, q_1), \ldots, (\bar{p}_{i_0}, q_{i_0}), (\bar{p}_{i_0+1}, p_{i_0+1}), \ldots, (\bar{p}_n, p_n))$ is also a proper α-name. This closure property does not hold for the collection of all α-names as the choice of the g_i's 'rules out' certain α-names.

Fix a $\diamondsuit(E)$-sequence $\langle D_\delta \mid \delta \in E\rangle$ for L_β as in Lemma 1C.2, where $E = \{\sigma \in U(\gamma) \mid \beta(\nu(\delta)) < \beta(\delta)$ or $\delta = \nu(\delta) < \beta(\delta)$, and C_δ is bounded in $\delta\}$ (as before). The construction breaks up into a number of cases.

Case 1: *C_α is unbounded in α.*

Case 1A: *$\nu(\alpha)$ is $<$-minimal.*

Set $b_\nu = \bigcup\{b_\sigma \mid \sigma < \nu\}$ for $\nu \in T_\alpha \cap \nu(\alpha)$ and define $b_{\nu(\alpha)}$: $[\gamma, \alpha) \to 2$ by $b_{\nu(\alpha)}(\eta) = 0$ for all η. We must define $p(\bar{c})$ for each proper α-name $\bar{c}$. Note that $\sigma(\bar{c}) < \nu(\alpha)$ since $\nu(\alpha)$ is $<$-minimal.

We know (by Lemma 1C.10(a)) that for sufficiently large $\beta \in C_\alpha$ there exists $\bar{\nu} \in T_\beta$ such that $\bar{\nu} < \sigma(\bar{c})$ (if $\sigma(\bar{c}) \neq 0$). Thus for sufficiently large $\beta \in C_\alpha$, $\bar{c}^\beta = (p_0^\beta, (\bar{p}_1^\beta, p_1), \ldots, (\bar{p}_n^\beta, p_n))$ if $\bar{c} = (p_0, (\bar{p}_1, p_1), \ldots, (\bar{p}_n, p_n)))$ is a β-name. We say that $\bar{c}$ is proper if $\bar{c}^\beta$ is proper for such $\beta \in C_\alpha$. And, as suggested by condition (i), $p(\bar{c}) = \bigcup\{p(\bar{c}^\beta) \mid \beta \in C_\alpha,\ \bar{c}^\beta$ a β-name$\}$. (It will follow from our definition of properness that $\bar{c}^\beta$ proper, $\beta' \in C_\beta$, $\bar{c}^{\beta'}$ a β'-name $\to \bar{c}^{\beta'}$ proper.)

Case 1B: *$\nu(\alpha)$ is a $<$-limit.*

Set $b_\nu = \bigcup\{b_\sigma \mid \sigma < \nu\}$ for all $\nu \in T_\alpha$. As in Case 1A, if $\bar{c}$ is an α-name then for sufficiently large $\beta \in C_\alpha$, $\bar{c}^\beta$ is a β-name; we say that $\bar{c}$ is proper if $\bar{c}^\beta$ is proper for such β and in this case define $p(\bar{c}) = \bigcup\{p(\bar{c}^\beta) \mid \beta \in C_\alpha,\ \bar{c}^\beta$ a β-name$\}$.

Case 1C: *$\nu(\alpha)$ is a $<$-successor.* Let $\bar{\nu} <_* \nu(\alpha)$.

Case 1C(i): *$\nu(\alpha)$ is T_α-minimal.*

Case 1C(ii): *$\nu(\alpha)$ is a T_α-successor.*

Due to our hypothesis that C_α is unbounded in α, these last two cases cannot occur. See Lemma 1C.9.

Case 1C(iii): *$\nu(\alpha)$ is a T_α-limit,* Range($\pi_{\bar\nu\nu(\alpha)}$) *bounded in* $\nu(\alpha)$. Let $X = T_{\alpha(\bar\nu)} \cap \bar\nu$, $Y = T_{\alpha(\bar\nu)} - X$ and define the α-name $c(\nu(\alpha)) = (\pi_{\nu(\alpha)}, (\pi_{\nu(\alpha)}, q_Y))$ by: $\mathrm{Dom}(\pi_{\nu(\alpha)}) = X$, $\pi_{\nu(\alpha)}(i) = b_{\pi_{\bar\nu\nu(\alpha)}}(i)$. For sufficiently large $\beta \in C_\alpha$, $c(\nu(\alpha))^\beta$ is a proper β-name (as $\sigma(c(\nu(\alpha))) < \nu(\alpha)$) and we define $p(c(\nu(\alpha))) = \bigcup \{p(c(\nu(\alpha))^\beta) \mid \beta \in C_\alpha,\ c(\nu(\alpha))^\beta$ a β-name$\}$. Now for $j \in W(\nu(\alpha))$, $b_{\nu(\alpha)j} = p(c(\nu(\alpha)))(j)$. And, $b_{\nu(\alpha)} = b_{\nu(\alpha)\bar\nu}$.

If $\bar c \in \bar J_0(\alpha)$, then for sufficiently large $\beta \in C_\alpha$, $\bar c^\beta$ is a β-name; in this case $\bar c$ is proper iff $\bar c^\beta$ is proper for such β and then $p(\bar c) = \bigcup \{p(\bar c^\beta) \mid \beta \in C_\alpha,\ \bar c^\beta$ a β-name$\}$. If $\bar c = (p_0, (\bar p_1, p_1), \ldots, (\bar p_n, p_n))$ is an α-name of length >0, $\sigma(\bar c) = \nu(\alpha)$, then p_0 is a thinning of $p(c(\nu(\alpha)))$ and we identify $\bar c$ with the α-name $\bar d = (\pi_{\nu(\alpha)}, (\pi_{\nu(\alpha)}, q_Y), (\bar p_1, p_1), \ldots, (\bar p_n, p_n))$ and define: $\bar c$ is proper iff $\bar d$ is proper, in this case $p(\bar c) = p(\bar d)$. Note that $\sigma(\bar d) < \nu(\alpha)$ so $p(\bar d)$ is already defined. If length$(\bar c) = 0$, then $\bar c = (p_0)$ is proper and $p(\bar c) = p_0$.

Case 1C(iv): *$\nu(\alpha)$ is a T_α-limit,* Range($\pi_{\bar\nu\nu(\alpha)}$) *unbounded in* $\nu(\alpha)$. For each $\sigma \dashv \nu(\alpha)$ and for $\sigma = \nu(\alpha)$ let $X_\sigma = T_{\alpha(\bar\sigma)} \cap \bar\sigma$, $Y_\sigma = T_{\alpha(\bar\sigma)} - X_\sigma$ where $\bar\sigma <_* \sigma$ and define π_σ by: $\mathrm{Dom}(\pi_\sigma) = X_\sigma$, $\pi_\sigma(i) = b_{\pi_{\bar\sigma\sigma}(i)}$ for each i. Also consider the $\alpha(\sigma)$-name $c(\sigma) = (\pi_\sigma, (\pi_\sigma, q_{Y_\sigma}))$. Then $\sigma_1 \dashv \sigma_2 \dashv \nu(\alpha)$ implies $p(c(\sigma_2)) \leq p(c(\sigma_1))$. Define $p(c(\nu(\alpha))) = \bigcup \{p(c(\sigma)) \mid \sigma \dashv \nu(\alpha)\}(c(\nu(\alpha))$ is proper). Now for $j \in W(\nu(\alpha))$ set $b_{\nu(\alpha)j} = p(c(\nu(\alpha)))(j)$ and $b_{\nu(\alpha)} = b_{\nu(\alpha)\bar\nu}$.

If $\bar c \in \bar J_0(\alpha)$, then $\bar c^\beta$ is a β-name for sufficiently large $\beta \in C_\alpha$; $\bar c$ is proper if $\bar c^\beta$ is proper for such β and in this case $p(\bar c) = \bigcup \{p(\bar c^\beta) \mid \beta \in C_\alpha,\ \bar c^\beta$ a β-name$\}$. If $\bar c$ is an α-name of positive length, $\bar c = (p_0, (\bar p_1, p_1), \ldots, (\bar p_n, p_n))$ and $\sigma(\bar c) = \nu(\alpha)$, then for sufficiently large $\sigma \dashv \nu(\alpha)$, $p_0^{\alpha(\sigma)}$ is a thinning of $p(c(\sigma))$ and we form the $\alpha(\sigma)$-name $\bar d_\sigma = (\pi_\sigma, (\pi_\sigma, q_{Y_\sigma}), (\bar p_1^{\alpha(\sigma)}, p_1), \ldots, (\bar p_n^{\alpha(\sigma)}, p_n))$; we say that $\bar c$ is proper if $\bar d_\sigma$ is proper for such σ. Then $p(\bar c) = \bigcup \{p(\bar d_\sigma) \mid \sigma \dashv \nu(\alpha),\ \bar d_\sigma$ an $\alpha(\sigma)$-name$\}$. (As suggested by condition (1), we have that $\sigma_1 \dashv \sigma_2$, $\bar d_{\sigma_1}$ a proper $\alpha(\sigma_1)$-name$\rightarrow p(\bar d_{\sigma_2}) \leq p(\bar d_{\sigma_1})$.) If length$(\bar c) = 0$, then $\bar c = (p_0)$ is proper and $p(\bar c) = p_0$.

Case 1D: $\nu(\alpha) = \alpha$.

In this case $\sigma(\bar c) = 0$ for all α-names $\bar c$. For sufficiently large $\beta \in C_\alpha$, $\bar c^\beta = \bar c$; we say that $\bar c$ is proper if $\bar c^\beta$ is proper for such β. Finally let $p(\bar c) = \bigcup \{p(\bar c^\beta) \mid \beta \in C_\alpha,\ \bar c = \bar c^\beta\}$, for proper $\bar c$.

Case 2: *C_α is bounded in α.*

Case 2A: *$\nu(\alpha)$ is $<$-minimal.*

Case 2A(i): $\nu(\alpha)^* \neq \alpha$. Set $b_\nu = \bigcup \{b_\sigma \mid \sigma < \nu\}$ for $\nu \in T_\alpha \cap \nu(\alpha)$ and define $b_{\nu(\alpha)} : [\gamma, \alpha) \rightarrow 2$ by $b_{\nu(\alpha)}(\eta) = 0$ for all $\eta < \alpha$. If $\bar c$ is an α-name, then $\sigma(\bar c) < \nu(\alpha)$ since $\nu(\alpha)$ is $<$-minimal.

As C_α is bouned in α it follows that there exists a $\Pi_1(\mathscr{A}(\alpha))$ ω-sequence $\sup(C_\alpha) = \alpha_0 < \alpha_1 < \cdots$ cofinal in α such that for any $\sigma \in T_\alpha \cap \nu(\alpha)$, $\alpha_i \in \{\alpha(\bar\sigma) \mid \bar\sigma < \sigma\}$ for sufficiently large i.

Now fix a sequence $\alpha_0 < \alpha_1 < \cdots$ as above and suppose $\bar{c} = (p_0, (\bar{p}_1, p_1), \ldots, (\bar{p}_n, p_n))$ is an α-name, $Y_i = \mathrm{Dom}(p_i)$. Then $\bar{c}^{\alpha_j}$ is an α_j-name for sufficiently large j. We now describe the process of "extending $\bar{c}$ along the α_j's," which will be used repeatedly in the remaining cases. The definition is by induction on n. Let $j'_0, j'_1, \ldots$ be those j such that $\bar{c}^{\alpha_j}$ is an α_j-name and for all $i < n : j = j^i_m$ for some m. Let $j_0 = j^n_0 = j'_0$, $j_m = j^n_m = j'_{m+1}$ for $m > 0$ (there is a reason for omitting j'_1: See the proof of Lemma 1C.11). Let $\bar{c}_{j_0} = \bar{c}(\leq n-1)_{j_0} * (\bar{q}_n, p_n)$ where $\bar{q}_n$ thins $\mathbf{G}(\bar{c}(\leq n-1)_{j_0})$ as $\bar{p}_n$ thins $\mathbf{G}(\bar{c}(\leq n-1))$, if $n > 1$; $\bar{c}_{j_0} = \bar{c}^{\alpha_{j_0}}$ if $n = 1$. We say that $\bar{c}$ is proper iff $\bar{c}_{j_0}$ is a proper α_{j_0}-name. Now $\bar{c}_{j_{m+1}} = \bar{c}(\leq n-1)_{j_{m+1}} * (\bar{q}_n, q_n)$ where $\mathrm{Dom}(q_n) = Y_n$, $q_n(k) = p(\bar{c}_{j_m})(k)$ for $k \in Y_n$, $\bar{q}_n$ thins $\mathbf{G}(\bar{c}_{j_{m+1}}(\leq n-1))$ exactly as $\bar{p}_n$ thins $\mathbf{G}(\bar{c}(\leq n-1))$, if $n > 1$; $\bar{c}_{j_{m+1}} = (p_0^{\alpha_{j_{m+1}}}, (\bar{q}_1, q_1))$ where $\mathrm{Dom}(q_1) = Y_1$, $q_1(k) = p(\bar{c}_{j_m})(k)$ for $k \in Y_1$, $\bar{q}_1$ thins $p_0^{\alpha_{j_{m+1}}}$ as $\bar{p}_1$ thins p_0, if $n = 1$. Note that $p(\bar{c}_{j_{m_2}}) \leq p(\bar{c}_{j_{m_1}})$ if $m_1 \leq m_2$. Set $p(\bar{c}) = \bigcup \{p(\bar{c}_{j_m}) \mid m \geq 0\}$.

In the next two cases define $\sigma \dashv\!| \, v(\alpha)$ iff $\sigma \dashv\!| \, ' v(\alpha)$, σ is active.

Case 2A(ii): $v(\alpha)^* = \alpha$, $v(\alpha)$ *a* $\dashv\!|$*-limit*. Then $v(\alpha)$ is active. Set $b_v = \bigcup \{b_\sigma \mid \sigma < v\}$ for $v \in T_\alpha \cap v(\alpha)$ and define $b_{v(\alpha)} : [\gamma, \alpha) \to 2$ by $b_{v(\alpha)}(\eta) = 0$ for all η. If $\bar{c}$ is an α-name, then $\sigma(\bar{c}) < v(\alpha)$ so by Lemma 1C.10(c), $\bar{c}^{\alpha(\sigma)}$ is an $\alpha(\sigma)$-name for sufficiently large $\sigma \dashv\!| \, v(\alpha)$. We say that $\bar{c}$ is proper if $\bar{c}^{\alpha(\sigma)}$ is proper for such σ and, as suggested by condition (m), $p(\bar{c}) = \bigcup \{p(\bar{c}^{\alpha(\sigma)}) \mid \sigma \dashv\!| \, v(\alpha), \bar{c}^{\alpha(\sigma)}$ an $\alpha(\sigma)$-name$\}$.

Case 2A(iii): $v(\alpha)^* = \alpha$, $v(\alpha)$ *not a* $\dashv\!|$*-limit*. Let $\sigma =$ the $\dashv\!|$-predecessor to $v(\alpha)$, if it exists. As in Case 2A(i) we can choose a $\Pi_1(\mathcal{A}(\alpha))$ ω-sequence $\alpha_0 < \alpha_1 < \cdots$ cofinal in α such that for any $v \in T_\alpha \cap v(\alpha)$, $\alpha_i \in \{\alpha(\bar{v}) \mid \bar{v} < v\}$ for sufficiently large i; also choose $\alpha_0 = \alpha(\sigma)$ if $v(\alpha)$ is a $\dashv\!|$-successor, $\alpha_0 = \sup(C_\alpha)$ otherwise.

Set $b_v = \bigcup \{b_\sigma \mid \sigma < v\}$ for $v \in T_\alpha \cap v(\alpha)$ and define $b_{v(\alpha)} : [\gamma, \alpha) \to 2$ by $b_{v(\alpha)}(\eta) = 0$ for all η. If $\bar{c}$ is an α-name, then $\bar{c}^{\alpha_j}$ is an α_j-name for sufficiently large j. If $\bar{c}^{\alpha_0}$ is an α_0-name, then we say that $\bar{c}$ is proper iff $\bar{c}^{\alpha_0}$ is proper.

Choose a (canonical) partial $\Sigma_1(L_{v(\alpha)})$-function h_α from α onto $L_{v(\alpha)}$. We say that $\bar{e} < \alpha$ *α-codes* the pair (D, e) if $D \in \bar{\mathcal{D}}_0(\alpha)$ consists solely of proper α-names $\bar{c}$ such that $\bar{c}^{\alpha_0}$ is an α_0-name and $h_\alpha(\bar{e}) = (D, e)$. For such a pair (D, e) consider $f \in \mathcal{P}(\alpha, D)$ defined by: if $\bar{c} \in D$ is of the form $(p_0, (\bar{p}_1, p_1), \ldots, (\bar{p}_n, p_n))$, then $f(\bar{c}) = (q_1, \ldots, q_n)$ where $q_j = p(\bar{c}^{\alpha_0}(\leq j)) \restriction \mathrm{Dom}(p_j)$. Then (D, e) is *alive* if $f \notin \mathcal{D}(W_e^*)$, where $W_e = e$th $\Sigma_1(L_{v(\alpha)})$-set and $W_e^* = \{(p, \delta) \mid p \leq q$ for some $(q, \delta) \in W_e \cap \mathcal{P}(\alpha, D) \times \alpha\}$. (Recall that $\mathcal{D}(T) = \{p \mid T_\delta$ is dense below p for all $\delta < \alpha$ or for some $\delta < \alpha$, $q \leq p \to q \notin T_\delta\}$, for persistent $T \subseteq \mathcal{P} \times \alpha$.)

Now let $\bar{e} < \alpha$ be the least α-code of an alive pair (D, e) (if there is one). We define $\mathcal{D}'(W_e^*)$ to consist of all $f \in \mathcal{D}(W_e^*)$ so that either for some $\delta < \alpha$, $f' \leq f \to f' \notin (W_e^*)_\delta$ or for some $W \subseteq W_e^*$, $W \in L_{v(\alpha)}$, W_δ is dense below f for all $\delta < \alpha$. Then let $f' \leq f$ be the $L_{v(\alpha)}$-least f' in $\mathcal{D}'(W_e^*)$ (if such an f' exists) so that f' is 1–1 and for some fixed η, $(r_1, \ldots, r_l) \in \mathrm{Range}(f') \to |r_i| = \eta$ for all i. If $\bar{e}$ and f' as defined above do exist, then $v(\sigma)$ is active; otherwise $v(\alpha)$ is not active and set $D = \emptyset$. Choose i_0 so that $\mathrm{Range}(f') \subseteq L_{\alpha_{i_0}}$, should f' exist.

If $\bar{c} = (p_0, (\bar{p}_1, p_1), \ldots, (\bar{p}_n, p_n)) \in \bar{J}_0(\alpha)$ we define properness for $\bar{c}, p(\bar{c})$ as follows: First, if $\bar{c}^{\alpha_0}$ is an α_0-name, then let $\bar{c}_1 = (p_0, (\bar{q}_1, q_1), \ldots, (\bar{q}_n, q_n))$ where $q_i = p(\bar{c}^{\alpha_0}(\leqslant i)) \restriction \mathrm{Dom}(p_i)$ and $\bar{q}_{i+1}$ thins $\mathbf{G}(\bar{c}_1(\leqslant i))$ as $\bar{p}_{i+1}$ thins $\mathbf{G}(\bar{c}(\leqslant i))$. Otherwise let $\bar{c}_1 = \bar{c}$, $\bar{q}_i = \bar{p}_i$, $q_i = p_i$. Second, suppose that $\bar{c}(\leqslant i) \in D$ for some $i \geqslant 1$. Then let $\bar{d} = (p_0, (\bar{r}_1, r_1), \ldots, (\bar{r}_m, r_m), (\bar{q}_{m+1}, q_{m+1}), \ldots, (\bar{q}_n, q_n))$ where m is largest so that $\bar{c}(\leqslant m) \in D$, $f'(\bar{c}(\leqslant m)) = (r_1, \ldots, r_m)$ and $\bar{r}_{i+1}$ thins $\mathbf{G}(\bar{d}(\leqslant i))$ as $\bar{p}_{i+1}$ thins $\mathbf{G}(\bar{c}(\leqslant i))$. Define $\bar{c}$ is proper, $p(\bar{c})$ to coincide with $\bar{d}$ is proper, $p(\bar{d})$ where the latter are defined by extending $\bar{d}$ along the α_j's. Lastly suppose that $\bar{c}(\leqslant 1) \notin D$. Then we extend $\bar{c}$, so as to 'code' $\bar{c}$ at level α_i, some $i > i_0$: Namely, we can canonically identify $((\bar{p}_1, p_1), \ldots, (\bar{p}_n, p_n))$ with an ordinal $\beta < \alpha$, using the canonical well-ordering $<_L$. Then choose $i > i_0$ to be least so that $\beta < \alpha_i$ and $\bar{c}_1^{\alpha_i}$ is an α_i-name. Now pick $(r_1, \ldots, r_n) \in \mathscr{C}^*(\bar{c}_1)$ so that $|r_j| = \alpha_i + \beta$ for each j, $1 \leqslant j \leqslant n$. Then let $\bar{d} = (p_0, (\bar{r}_1, r_1), \ldots, (\bar{r}_n, r_n))$ where $\bar{r}_{i+1}$ thins $\mathbf{G}(\bar{d}(\leqslant i))$ as $\bar{p}_{i+1}$ thins $\mathbf{G}(\bar{c}(\leqslant i))$ and define $\bar{c}$ is proper, $p(\bar{c})$ to coincide with $\bar{d}$ is proper, $p(\bar{d})$, the latter being defined by extending $\bar{d}$ along the α_j's.

The fact that we have 'coded' $\bar{c}$ into $\bar{d}$ in the last case above is important for the proof of Lemma 1C.11.

Case 2B: *$v(\alpha)$ is a $<$-limit.*

Case 2B(i): $\alpha \notin E$. Set $b_v = \bigcup \{b_\sigma \mid \sigma < v\}$ for $v \in T_\alpha$. Also choose a cofinal $\Pi_1(\mathscr{A}(\alpha))$ ω-sequence $\sup C_\alpha = \alpha_0 < \alpha_1 < \cdots$ below α so that $\alpha_i \in \{\alpha(\sigma) \mid \sigma < v(\alpha)\}$ for each $i > 0$. Then, if $\bar{c}$ is an α-name, it follows that $\bar{c}^{\alpha_j}$ is an α_j-name for sufficiently large j. Define properness for $\bar{c}$ and $p(\bar{c})$ by extending $\bar{c}$ along the α_j's.

Case 2B(ii): $\alpha \in E$. For $v \in T_\alpha$ define $b_v = \bigcup \{b_\sigma \mid \sigma < v\}$. Let $\alpha_0 = \sup C_\alpha$. If $\bar{c}^{\alpha_0}$ is an α_0-name, then we say that $\bar{c}$ is proper exactly if $\bar{c}^{\alpha_0}$ is proper.

In this case we make use of the canonical $\diamondsuit(E)$-sequence $\langle D_\delta \mid \delta \in E \rangle$. First we need to define a generalization of the forcing $\mathscr{P}(\alpha, D)$, defined earlier for $D \in \bar{\mathscr{D}}_0(\alpha)$.

We say that $D \in \bar{\mathscr{D}}(\alpha)$ if D is a finite subset of $\bar{J}(\alpha)$, $\bar{c} \in D \to \bar{c}(\leqslant i) \in D$ for all i. (Note that $\bar{J}(\alpha) \in L_{\beta(\alpha)}$ as $\alpha \in E$.) Now for $D \in \bar{\mathscr{D}}(\alpha)$, $\mathscr{P}(\alpha, D)$ consists of all functions f with domain D such that for $\bar{c} \in D$, $f(\bar{c}) \in \mathscr{C}^*(\bar{c})$, $\bar{c}_1 < \bar{c}_2$ implies $f(\bar{c}_1) < f(\bar{c}_2)$.

Say that α is *active* if $D_\alpha \subseteq \alpha$ codes a triple $(L_{\beta(\alpha)}, D, \mathscr{S})$ where $D \in \bar{\mathscr{D}}(\alpha)$ consists of proper α-names $\bar{c}$ such that $\bar{c}^{\alpha_0}$ is an α_0-name, ω^d divides $\beta(\alpha)$ where $d = \mathrm{card}(D)$ and $\mathscr{S} \in L_{\beta(\alpha)}$ is predense on $\mathscr{P}(\alpha, \mathscr{D})$. (This means that $D_\alpha = \{\langle 0, \alpha_1, \alpha_2 \rangle \mid (\alpha_1, \alpha_2) \in R\} \cup \{\langle 1, \eta_0 \rangle, \langle 2, \eta_1 \rangle\}$ where for some h, $h: \langle \alpha, R \rangle \cong \langle L_{\beta(\alpha)}, \varepsilon \rangle$, $h(\eta_0) = D$, $h(\eta_1) = \mathscr{S}$.) If α is not active, then proceed exactly as in Case 2B(i). Otherwise choose $(L_{\beta(\alpha)}, D, \mathscr{S})$ as above. Also insist that $\alpha_0 < \alpha_1 < \cdots$ be $\Pi_1(\mathscr{A}(\alpha))$ and have the following 'stability property': Pick a limit ordinal $\beta < \beta(\alpha)$, $p \in L_\beta$ and $n \in \omega$ so that $\bar{\mathscr{D}}(\alpha)$, D, $\mathscr{S}$, $p(\alpha)$, α_0 are all $\Sigma_n(S_\beta)$ with parameter p. Then we require that for $i > 0$, (Σ_{n+3}-Skolem hull of $\alpha_i \cup \{p\}$ inside $S_\beta) \cap \alpha = \alpha_i$ (this is possible as α is regular in $L_{\beta(\alpha)}$). Then $\bar{c}$ an α_i-name $\to \bar{c}^{\alpha_i}$ an α_i-name for sufficiently large i.

Now define D' to consist of all α-names $\bar{c} = (p_0, (\bar{p}_1, p_1), \ldots, (\bar{p}_n, p_n))$ such that for some $\bar{d} \in D$, $p_0 =$ first component of $\bar{d}$ and $(p_1, \ldots, p_n) \in \mathscr{C}^*(\bar{d})$. Note that $\bar{c}\in D'\to\bar{c}(\leq i)\in D'$ for all i. We first define properness for $\bar{c}$, $p(\bar{c})$ for $\bar{c}\in D'$, by induction on length$(\bar{c})=n$. We will also need to 'code' $\bar{c}$ at various points in the construction, much as in Case 2A(iii). Let β be the ordinal code for $((\bar{p}_1,p_1),\ldots,(\bar{p}_n,p_n))$ using $<_L$. Note that each α_i is a limit of p.r. closed ordinals and $\bar{c}^{\alpha_i}$ an α_i-name$\to\beta<\alpha_i$.

First suppose that length$(\bar{c})=1$, so $\bar{c}=(p_0,(\bar{p}_1,p_1))$. Then $\bar{c}$ is proper. We shall more or less "extend $\bar{c}$ along the α_i's", but with some changes. Let $\bar{d}_0,\ldots,\bar{d}_s$ be those elements $\bar{d}$ of D such that $p_0=$ first component of $\bar{d}$, $p_1\in\mathscr{C}^*(\bar{d})$ and let $D_t=D-\{\bar{d}\neq\bar{d}_t\mid\bar{d}(\leq 1)=\bar{d}_t\}$, for $0\leq t\leq s$. Then $D_t\in\mathscr{D}(\alpha)$ for each t. Also set $\mathscr{S}_t=\{f\restriction D_t\mid f\in\mathscr{S}\}$. Then $\mathscr{S}_t$ is predense on $\mathscr{P}(\alpha,D_t)$. By Lemma 1C.13, $\mathscr{P}(\alpha,D^*)$ is γ-distributive and obeys the $(<\alpha)$-chain condition in $L_{\beta(\alpha)}$ for any $D^*\in\mathscr{D}(\alpha)$ (consisting of proper α-names $\bar{c}$ such that $\bar{c}^{\alpha_0}$ is an α_0-name) of cardinality $<$card(D). Thus for any $f_t\in\mathscr{P}(\alpha,\{\bar{d}_t\})$ and any $g_t\in\mathscr{P}(\alpha,D_t-\{\bar{d}_t\})$ there exists a maximal antichain M_{g_t} in $\{g\in\mathscr{P}(\alpha,D_t-\{\bar{d}_t\})\mid g\leq g_t\}$ and $f'_t\leq f_t$ in $\mathscr{P}(\alpha,\{\bar{d}_t\})$ such that for all $g\in M_{g_t}$, $f'_t\cup g\leq$ some $h\in\mathscr{S}_t$. Then in fact it follows that for any $f_t\in\mathscr{P}(\alpha,\{\bar{d}_t\})$ there exists $f'_t\leq f_t$ in $\mathscr{P}(\alpha,\{\bar{d}_t\})$ such that for all $g_t\in\mathscr{P}(\alpha,D_t-\{\bar{d}_t\})$ there exists a maximal antichain M_{g_t} in $\{g\in\mathscr{P}(\alpha,D_t-\{\bar{d}_t\})\mid g\leq g_t\}$ such that $g\in M_{g_t}\to f'_t\cup g\leq$ some $h\in\mathscr{S}_t$. Lastly note tha if $f\leq\{(\bar{d}_t,p_1)\}$ in $\mathscr{P}(\alpha,\{\bar{d}_t\})$, then $f'_t\in\mathscr{P}(\alpha,\{\bar{d}_{t'}\})$ for all t', $0\leq t'\leq s$, where $f'_t(\bar{d}_{t'})=f(\bar{d}_t)$. Thus we get: if $f\in\bigcup\{\mathscr{P}(\alpha,\{\bar{d}_t\})\mid 0\leq t\leq s\}$, then there exists $f'\leq f$ (in $\mathscr{P}(\alpha,\{\bar{d}_t\})$ where $f\in\mathscr{P}(\alpha,\{\bar{d}_t\})$) such that for all $g\in\mathscr{P}(\alpha,D_{t'}-\{\bar{d}_{t'}\})(0\leq t'\leq s)$, there exists a maximal anti-chain M_g in $\{g'\in\mathscr{P}(\alpha,D_{t'}-\{\bar{d}_{t'}\})\mid g'\leq g\}$ such that $g'\in M_g\to f'_{t'}\cup g'\leq$ some $h\in\mathscr{S}_{t'}$. This property is exactly what we need to provide the proper definition of $p(\bar{c})$.

Let $\bar{d}_t=(p_0,(\bar{p}_1,p_1^t))$ and $\bar{d}=(p_0,(\bar{p}_1,r_1))$ where $r_1(k)=\bigcup\{p_1^t(k)\mid 0\leq t\leq s$ for each $k\in\mathrm{Dom}(p_1)$. (Then $\bar{d}=\bar{d}_{t_0}$ for some t_0.) Let $j'_0,j'_1,\ldots$ be those j such that $\bar{c}^{\alpha_j}$ is an α_j-name and let $j_0=j'_0$, $j_m=j'_{m+1}$ for $m>0$. We define a sequence $\bar{c}_{j_0},\bar{c}_{j_1},\ldots$ of names as follows: Let $\bar{c}_{j_0}=\bar{c}^{\alpha_{j_0}}$, $\bar{c}_{j_1}=(p_0^{\alpha_{j_1}},(\bar{q}_1,q_1))$ where $q_1=p(\bar{c}_{j_0})\restriction Y_1$ ($Y_1=\mathrm{Dom}(p_1)$) and $\bar{q}_1$ thins $p_0^{\alpha_{j_1}}$ exactly as $\bar{p}_1$ thins p_0 (this is exactly as before). Now let $\{(\bar{d},q_2)\}\leq\{(\bar{d},q_1)\}$ in $\mathscr{P}(\alpha,\{\bar{d}\})$ be the least obeying the property expressed in the preceding paragraph, where $f=\{(\bar{d},q_1)\}$ and $f'=\{(\bar{d},q_2)\}$. Also choose $|q_2|=\gamma'+\beta$ where γ' is p.r. closed (this is to 'code' $\bar{c}$). Then $q_2\in L_{\alpha_{j_2}}$ by the way the α_i's were defined. Set $\bar{c}_{j_2}=(p_0^{\alpha_{j_2}},(\bar{q}_2,q_2))$ where $\bar{q}_2$ thins $p_0^{\alpha_{j_2}}$ exactly as $\bar{p}_1$ thins p_0. Then $\bar{c}_{j_3}=(p_0^{\alpha_{j_3}},(\bar{q}_3,q_3))$ where $q_3=p(\bar{c}_{j_2})\restriction Y_1$ and $\bar{q}_3$ thins $p_0^{\alpha_{j_3}}$ as $\bar{p}_1$ thins p_0. More generally, for $m>1$ let $\{(\bar{d},q_{2m})\}\leq\{(\bar{d},q_{2m-1})\}$ in $\mathscr{P}(\alpha,\{\bar{d}\})$ be the least so that $|q_{2m}|=\gamma'+\beta$ where γ' is p.r. closed.

Set $\bar{c}_{j_{2m}}=(p_0^{\alpha_{j_{2m}}},(\bar{q}_{2m},q_{2m}))$ where $\bar{q}_{2m}$ thins $p_0^{\alpha_{j_{2m}}}$ exactly as $\bar{p}_1$ thins p_0. Then $\bar{c}_{j_{2m+1}}=(p_0^{\alpha_{j_{2m+1}}},(\bar{q}_{2m+1},q_{2m+1}))$ where $q_{2m+1}=p(\bar{c}_{j_{2m}})\restriction Y_1$ and $\bar{q}_{2m+1}$ thins $p_0^{\alpha_{j_{2m+1}}}$ as $\bar{p}_1$ thins p_0. Then set $p(\bar{c})=\bigcup\{p(\bar{c}_{j_m})\mid m\geq 0\}$. Thus we have defined $p(\bar{c})$ by "extending along the α_j's", but we have also taken care to arrange that $p(\bar{c})$ meets certain dense sets.

Now we consider the general case, length$(\bar{c})=k+1>1$, so $\bar{c}$

$(p_0, (\bar{p}_1, p_1), \ldots, (\bar{p}_k, p_k), (\bar{p}_{k+1}, p_{k+1}))$. Let $\bar{d}_0, \ldots, \bar{d}_s$ be those elements $\bar{d}$ of D such that $p_0 =$ first component of $\bar{d}$ and $(p_1, \ldots, p_{k+1}) \in \mathscr{C}^*(\bar{d})$ and let $D_t = D - \{\bar{d} \neq \bar{d}_t \mid \bar{d}(\leq k+1) = \bar{d}_t\}$, for $0 \leq t \leq s$. Also let $D_t^i = \{\bar{d} \in D_t \mid \bar{d} < \bar{d}_t(\leq i)$ or $\bar{d}(\leq i) = \bar{d}_t(\leq i)\}$, for $0 \leq t \leq s$, $1 \leq i \leq k+1$. By Lemma 1C.13, for any $f_t \in \mathscr{P}(\alpha, D_t^1)$ there exists $f_t' \leq f_t$ in $\mathscr{P}(\alpha, D_t^1)$ such that for all $g_t \in \mathscr{P}(\alpha, D_t - D_t^1)$, there exists a maximal antichain M_{g_t} in $\{g \in \mathscr{P}(\alpha, D_t - D_t^1) \mid g \leq g_t\}$ such that $g \in M_{g_t} \rightarrow f_t' \cup g \leq$ some $h \in \mathscr{S}_t$ $(= \{f \upharpoonright D_t \mid f \in \mathscr{S}\})$. Thus the property satisfied by f_t' in the preceding statement is dense in $\mathscr{P}(\alpha, D_t^1)$. Let $\mathscr{S}_t^1$ be a maximal antichain of conditions in $\mathscr{P}(\alpha, D_t^1)$ satisfying the above property. Now by applying the same reasoning to $\mathscr{S}_t^1$ and $\mathscr{P}(\alpha, D_t^2)$ we get: For any $f_t^1 \in \mathscr{P}(\alpha, D_t^2)$ there exists $(f_t^1)' \leq f_t^1$ in $\mathscr{P}(\alpha, D_t^2)$ such that for all $g_t^1 \in \mathscr{P}(\alpha, D_t^1 - \{\bar{c} \in D_t^2 \mid \text{length}(\bar{c}) \geq 2\})$, there exists a maximal antichain $M_{g_t^1}$ below g_t^1 such that $g \in M_{g_t^1} \rightarrow (f_t^1)' \cup g \leq$ some $h \in \mathscr{S}_t^1$, or $g(\bar{d}_t(\leq 1))$, $(f_t^1)'(\bar{d}_t(\leq 1))$ are incompatible. Let $\mathscr{S}_t^2$ be a maximal antichain of conditions in $\mathscr{P}(\alpha, D_t^2)$ obeying the above property satisfied by $(f_t^1)'$. Continuing inductively, we finally obtain a maximal antichain $\mathscr{S}_t^k$ in $\mathscr{P}(\alpha, D_t^k)$ and the following property: $(*)_t$ for any $f_t^k \in \mathscr{P}(\alpha, D_t^{k+1})$ there exists $(f_t^k)' \leq f_t^k$ in $\mathscr{P}(\alpha, D_t^{k+1})$ such that for all $g_t^k \in \mathscr{P}(\alpha, D_t^k - \{\bar{d}_t\})$, there exists a maximal antichain $M_{g_t^k}$ below g_t^k such that $g \in M_{g_t^k} \rightarrow (f_t^k)' \cup g \leq$ some $h \in \mathscr{S}_t^k$, or $g(\bar{d}_t(\leq k))$, $(f_t^k)'(\bar{d}_t(\leq k))$ are incompatible. Note that $D_t^{k+1} = \{\bar{d} \mid \bar{d} < \bar{d}_t\}$. Finally, as before we can allow t to vary: For any $f \in \bigcup \{\mathscr{P}(\alpha, D_t^{k+1}) \mid 0 \leq t \leq s\}$ there exists $f' \leq f$ (in $\mathscr{P}(\alpha, D_t^{k+1})$ if $f \in \mathscr{P}(\alpha, D_t^{k+1})$) such that for all $g \in \mathscr{P}(\alpha, D_{t'}^k - \{\bar{d}_{t'}\})$ $(0 \leq t' \leq s)$ there exists a maximal antichain M_g below g in $\mathscr{P}(\alpha, D_{t'}^k - \{\bar{d}_{t'}\})$ such that $g' \in M_g \rightarrow f_{t'}' \cup g' \in \mathscr{S}_{t'}^k$, or $g'(\bar{d}_{t'}(\leq k))$, $f_t'(\bar{d}_{t'}(\leq k))$ are incompatible (where $f_{t'}'(\bar{d}_{t'}(\leq i)) = f'(\bar{d}_t(\leq i))$.

Now we can define $p(\bar{c})$. We assume inductively that for each $i \leq k$ we have assigned sequences $j_0^i, j_1^i, j_2^i, \ldots$ and $\bar{c}(\leq i)_{j_0^i}, \bar{c}(\leq i)_{j_1^i}, \ldots$ to the α-name $\bar{c}(\leq i)$. Let $j_0', j_1', \ldots$ be those j such that $\bar{c}^{\alpha_j}$ is an α_j-name and for all $i \leq k$: $j = j_m^i$ for some m. Let $j_0 = j_0^{k+1} = j_0'$ and $j_m = j_m^{k+1} = j_{m+1}'$ for $m > 0$. Let $\bar{c}_{j_0} = \bar{c}(\leq k)_{j_0} * (\bar{q}_0, p_{k+1})$ where $\bar{q}_0$ thins $\mathbf{G}(\bar{c}(\leq k)_{j_0})$ as $\bar{p}_{k+1}$ thins $\mathbf{G}(\bar{c}(\leq k))$. Then $\bar{c}$ is proper iff $\bar{c}_{j_0}$ is a proper α_{j_0}-name. Let $\bar{c}_{j_1} = \bar{c}(\leq k)_{j_1} * (\bar{q}_1, q_1)$ where $q_1 = p(\bar{c}_{j_0}) \upharpoonright Y_{k+1}$ $(Y_{k+1} = \text{Dom}(p_{k+1}))$, $\bar{q}_1$ thins $\mathbf{G}(\bar{c}(\leq k)_{j_1})$ as $\bar{p}_{k+1}$ thins $\mathbf{G}(\bar{c}(\leq k))$. (So far we are just extending $\bar{c}$ along the α_j's.) Write $\bar{c}_{j_1} = (p_0^{\alpha_{j_1}}, (\bar{r}_1, r_1), \ldots, (\bar{r}_k, r_k), (\bar{q}_1, q_1))$. Now let $f_t \in \mathscr{P}(\alpha, D_t^{k+1})$ for all $t \in [0, s]$ be defined by $f_t(\bar{d}_t(\leq i)) = (r_1, \ldots, r_i)$ if $i \leq k$, $f_t(\bar{d}_t) = (r_1, \ldots, r_k, q_1)$. By induction we assume that $(p(\bar{c}(\leq 1)) \upharpoonright Y_1, \ldots, p(\bar{c}(\leq k)) \upharpoonright Y_k)$ is $\mathscr{C}^*(\bar{c}(\leq k))$-generic over $L_{\beta(\alpha)}$; thus there exists $(r_1', \ldots, r_k', q_2) \leq (r_1, \ldots, r_k, q_1)$ such that $(p(\bar{c}(\leq i)) \upharpoonright Y_i) \leq r_i'$ for all i and for all t, f_t' obeys the property of the preceding paragraph, where $f = f_t$, $f' = f_t'$ and $f_t' \in \mathscr{P}(\alpha, D_t^{k+1})$ is defined by: $f_t'(\bar{d}_t(\leq i)) = (r_1', \ldots, r_i')$ if $i \leq k$, $f_t'(\bar{d}_t) = (r_1', \ldots, r_k', q_2)$. Also choose $|q_2| = \gamma' + \beta$ where γ' is p.r. closed. We can also have $|r_1'|, \ldots, |r_k'|, |q_2| < \alpha_{j_2}$ by the choice of the α_i's. Set $\bar{c}_{j_2} = \bar{c}(\leq k)_{j_2} * (\bar{q}_2, q_2)$ where $\bar{q}_2$ thins $\mathbf{G}(\bar{c}_{j_2}(\leq k))$ exactly as $\bar{p}_{k+1}$ thins $\mathbf{G}(\bar{c}(\leq k))$. Then $j_3 = j_{i_1+1}'$ and $\bar{c}_{j_3} = \bar{c}(\leq k)_{j_3} * (\bar{q}_3, q_3)$ where $\bar{q}_3$ thins $\mathbf{G}(\bar{c}_{j_3}(\leq k))$ as $\bar{p}_{k+1}$ thins $\mathbf{G}(\bar{c}(\leq k))$ and $q_3 = p(\bar{c}_{j_2}) \upharpoonright Y_{k+1}$.

More generally, for $m>0$ let $(r'_1, \ldots, r'_k, q_{2m}) \leq (r_1, \ldots, r_k, q_{2m-1})$ (where $\bar{c}_{j_{2m-1}} = (p_0^{\alpha_{j_{2m-1}}}, (\bar{r}_1, r_1), \ldots, (\bar{r}_k, r_k), (\bar{q}_{2m-1}, q_{2m-1})))$ be such that $p(\bar{c}(\leq i)) \upharpoonright Y_i \leq r'_i$ for all i and $|q_{2m}| = \gamma' + \beta$ where γ' is p.r. closed. We can have $|r'_1|, \ldots, |r'_k|, |q_{2m}| < \alpha_{j_{2m}}$ by choice of the α_i's. Set $\bar{c}_{j_{2m}} = \bar{c}(\leq k)_{j_{2m}} * (\bar{q}_{2m}, q_{2m})$ where $\bar{q}_{2m}$ thins $\mathbf{G}(\bar{c}_{j_{2m}}(\leq k))$ as $\bar{p}_{k+1}$ thins $\mathbf{G}(\bar{c}(\leq k))$. Then $\bar{c}_{j_{2m+1}} = \bar{c}(\leq k)_{j_{2m+1}} * (\bar{q}_{2m+1}, q_{2m+1})$ where $\bar{q}_{2m+1}$ thins $\mathbf{G}(\bar{c}_{j_{2m+1}}(\leq k))$ as $\bar{p}_{k+1}$ thins $\mathbf{G}(\bar{c}(\leq k))$ and $q_{2m+1} = p(\bar{c}_{j_{2m}}) \upharpoonright Y_{k+1}$.

Finally, set $p(\bar{c}) = \bigcup \{p(\bar{c}_{j_m}) \mid m \geq 0\}$.

If $\bar{c} \notin D'$, then proceed as follows. Let $\bar{c}(\leq m) \in D'$ for some largest $m \geq 0$. Thus $p(\bar{c}(\leq m))$ is already defined. Then define $p(\bar{c})$ by starting with $p(\bar{c}(\leq m))$ and then extending along the α_j's (beginning with the sequence $j_0^m, j_1^m, \ldots$ resulting from the construction of $p(\bar{c}(\leq m)))$ exactly as in the case $\bar{c} \in D'$, with the exception that all reference to f'_t is omitted. In this way we have still coded $\bar{c}$ into $p(\bar{c})$.

Case 2C: *$\nu(\alpha)$ is a $<$-successor.* Let $\bar{\nu} <_* \nu(\alpha)$.

Case 2C(i): *$\nu(\alpha)$ is T_α-minimal.* Choose a $\Pi_1(\mathscr{A}(\alpha))$-sequence $\alpha(\bar{\nu}) = \alpha_0 < \alpha_1 < \cdots$ cofinal in α so that $\alpha_i \in U(\gamma)$ for each i. If $\bar{c} = (p_0, (\bar{p}_1, p_1), \ldots, (\bar{p}_n, p_n))$ is an α-name, $\sigma(\bar{c}) < \nu(\alpha)$ (hence $\sigma(\bar{c}) = 0$) then $\bar{c}^{\alpha_j}$ is an α_j-name for sufficiently large j. Then define properness for $\bar{c}$, $p(\bar{c})$ by extending $\bar{c}$ along the α_j's.

In particular we have defined $p((\emptyset, (\emptyset, q_Y)))$ where $Y = T_{\alpha(\bar{\nu})}$. Let $b_{\nu(\alpha)\sigma} = p((\emptyset, (\emptyset, q_Y)))(\sigma)$ for $\sigma \in T_{\alpha(\bar{\nu})} = W(\nu(\alpha))$. Then $b_{\nu(\alpha)} = b_{\nu(\alpha)\bar{\nu}}$.

If $\bar{c} \in \bar{J}(\alpha)$, $\sigma(\bar{c}) = \nu(\alpha)$ and length$(\bar{c}) > 0$, then we identify $\bar{c} = (p_0, (\bar{p}_1, p_1), \ldots, (\bar{p}_n, p_n))$ with the α-name $\bar{d} = (\emptyset, (\emptyset, q_Y), (\bar{p}_1, p_1), \ldots, (\bar{p}_n, p_n))$ and define: $\bar{c}$ is proper iff $\bar{d}$ is proper; in this case $p(\bar{c}) = p(\bar{d})$. If length$(\bar{c}) = 0$, then $\bar{c}$ is proper and if $\bar{c} = (p_0)$, then $p(\bar{c}) = p_0$.

Case 2C(ii): *$\nu(\alpha)$ is a T_α-successor.* Let σ be the T_α-predecessor to $\nu(\alpha)$ and σ' the $\dashv$-predecessor to $\nu(\alpha)$. Choose a $\Pi_1(\mathscr{A}(\alpha))$ ω-sequence $\alpha(\sigma') = \alpha_0 < \alpha_1 < \cdots$ cofinal in α so that $\alpha_i \in \{\alpha(\bar{\sigma}) \mid \bar{\sigma} < \sigma\}$ for each i. If $\bar{c}$ is an α-name such that $\sigma(\bar{c}) < \nu(\alpha)$, then $\bar{c}^{\alpha_j}$ is an α_j-name for sufficiently large j. Define properness for $\bar{c}$, $p(\bar{c})$ by extending $\bar{c}$ along the α_j's.

Finally define the proper α-name $c(\nu(\alpha))$ to be $(\pi_{\nu(\alpha)}, (\pi_{\nu(\alpha)}, q_Y))$ where $Y = T_{\alpha(\bar{\nu})} - \alpha(\bar{\nu})$ and $\mathrm{Dom}(\pi_{\nu(\alpha)}) = T_{\alpha(\bar{\nu})} \cap \bar{\nu}$ is defined by $\pi_{\nu(\alpha)}(i) = b_{\pi_{\bar{\nu}\nu(\alpha)}}(i)$. As $\nu(\alpha)$ is a T_α-successor we have that $\sigma(c(\nu(\alpha))) < \nu(\alpha)$ so $p(c(\nu(\alpha)))$ has been defined. For $j \in W(\nu(\alpha))$ let $b_{\nu(\alpha)j} = p(c(\nu(\alpha)))(j)$, $b_{\nu(\alpha)} = b_{\nu(\alpha)\bar{\nu}}$. If $\bar{c} = (p_0, (\bar{p}_1, p_1), \ldots, (\bar{p}_n, p_n))$ is an α-name of positive length, $\sigma(\bar{c}) = \nu(\alpha)$, then we identify $\bar{c}$ with the α-name $\bar{d} = (\pi_{\nu(\alpha)}, (\pi_{\nu(\alpha)}, q_Y), (\bar{p}_1, p_1), \ldots, (\bar{p}_n, p_n))$ and define: $\bar{c}$ is proper iff $\bar{d}$ is proper; in this case $p(\bar{c}) = p(\bar{d})$. If $\bar{c} = (p_0)$, then $\bar{c}$ is proper and $p(\bar{c}) = p_0$.

Case 2C(iii): *$\nu(\alpha)$ is a T_α-limit,* Range$(\pi_{\bar{\nu}\nu(\alpha)})$ *is bounded in* $\nu(\alpha)$. Let $\lambda = \bigcup \mathrm{Range}(\pi_{\bar{\nu}\nu(\alpha)})$.

Case 2C(iii)(a): *$\nu(\alpha)^* \neq \alpha$.* Let $\bar{\nu} <_* \sigma < \lambda$ and choose a $\Pi_1(\mathscr{A}(\alpha))$ ω-sequence $\alpha_0 < \alpha_1 < \cdots$ cofinal in α so that for any $\nu \in T_\alpha \cap \nu(\alpha)$, $\alpha_i \in \{\alpha(\tau) \mid \tau <$

$\nu\}$ for sufficiently large i. Also, define $\alpha_0 = \max(\alpha(\sigma), \sup C_\alpha)$. If $\bar{c} \in \bar{J}_0(\alpha)$, then define properness for $\bar{c}$, $p(\bar{c})$ by extending $\bar{c}$ along the α_j's.

Finally the proper α-name $c(\nu(\alpha)) = (\pi_{\nu(\alpha)}, (\pi_{\nu(\alpha)}, q_Y)) \in \bar{J}_0(\alpha)$ is defined exactly as in Case 2C(ii) as are $b_{\nu(\alpha)j}$, $j \in W(\nu(\alpha))$ and $p(\bar{c})$, $\sigma(\bar{c}) = \nu(\alpha)$.

Case 2C(iii)(b): $\nu(\alpha)^* = \alpha$, $\nu(\alpha)$ *a* $\dashv\!|$*-limit.* Then $\nu(\alpha)$ is active. Proceed exactly as in Case 2A(ii) to define b_ν, $\nu \in T_\alpha \cap \nu(\alpha)$ and $p(\bar{c})$ when $\sigma(\bar{c}) < \nu(\alpha)$. Then proceed exactly as in Case 2C(ii) to define $b_{\nu(\alpha)j}$, $j \in W(\nu(\alpha))$ and $p(\bar{c})$, $\sigma(\bar{c}) = \nu(\alpha)$.

Case 2C(iii)(c): $\nu(\alpha)^* = \alpha$, $\nu(\alpha)$ *not a* $\dashv\!|$*-limit.* Define α_0 to be the largest of $\alpha(\sigma)$, $\sup C_\alpha$ and $\alpha(\sigma')$ where $\bar{\nu} <_* \sigma < \lambda$ and $\sigma' = \dashv\!|$-predecessor to $\nu(\alpha)$, if exists. Then proceed exactly as in Case 2A(iii) to define b_ν, $\nu \in T_\alpha \cap \nu(\alpha)$ and $p(\bar{c})$ when $\sigma(\bar{c}) < \nu(\alpha)$. Proceed as in Case 2C(ii) to define $b_{\nu(\alpha)j}$, $j \in W(\nu(\alpha))$ and $p(\bar{c})$, $\sigma(\bar{c}) = \nu(\alpha)$.

Case 2C(iv): $\nu(\alpha)$ *is a* T_α*-limit,* Range$(\pi_{\bar{\nu}\nu(\alpha)})$ *is unbounded in* $\nu(\alpha)$. For $\sigma \dashv \nu(\alpha)$ and for $\sigma = \nu(\alpha)$ define X_σ, Y_σ, π_σ as in Case 1C(iv). Also define $b_{\nu(\alpha)j}$ for $j \in W(\nu(\alpha))$ as in that case. If $\bar{c} \in \bar{J}_0(\alpha)$, then $\bar{c}^{\alpha(\sigma)}$ is an $\alpha(\sigma)$-name for sufficiently large $\sigma \dashv \nu(\alpha)$; $\bar{c}$ is proper if $\bar{c}^{\alpha(\sigma)}$ is proper for such σ and in this case $p(\bar{c}) = \bigcup\{p(\bar{c}^{\alpha(\sigma)}) \mid \sigma \dashv \nu(\alpha),\ \bar{c}^{\alpha(\sigma)}$ an $\alpha(\sigma)$-name$\}$. If $\bar{c}$ is an α-name of positive length, $\bar{c} = (p_0, (\bar{p}_1, p_1), \ldots, (\bar{p}_n, p_n))$ and $\sigma(\bar{c}) = \nu(\alpha)$, then for sufficiently large $\sigma \dashv \nu(\alpha)$ we form the $\alpha(\sigma)$-name $\bar{d}_\sigma = (\pi_\sigma, (\pi_\sigma, q_{Y_\sigma}), (\bar{p}_1^{\alpha(\sigma)}, p_1), \ldots, (\bar{p}_n^{\alpha(\sigma)}, p_n))$ and say that $\bar{c}$ is proper if $\bar{d}_\sigma$ is proper for such σ. Then $p(\bar{c}) = \bigcup\{p(\bar{d}_\sigma) \mid \sigma \dashv \nu(\alpha),\ \bar{d}_\sigma$ an $\alpha(\sigma)$-name$\}$. If length$(\bar{c}) = 0$, $\bar{c} = (p_0)$, then $\bar{c}$ is proper and $p(\bar{c}) = p_0$.

Case 2D: $\nu(\alpha) = \alpha$.

Case 2D(i): $\alpha \in E$. Choose a $\Pi_1(\mathscr{A}(\alpha))$ ω-sequence $\alpha_0 < \alpha_1 < \cdots$ cofinal in α so that $\alpha_i \in U(\gamma)$ for all i, $\alpha_0 = \sup C_\alpha$. Then proceed exactly as in Case 2B(ii) to define properness, $\bar{\mathscr{D}}(\alpha)$, $\mathscr{P}(\alpha, D)$ for $D \in \mathscr{D}(\alpha)$ and $p(\bar{c})$ for proper α-names $\bar{c}$.

Case 2D(ii): $\alpha^* = \alpha$, $\alpha \notin E$. Choose $\alpha_0 < \alpha_1 < \cdots$ as in Case 2D(i) and then proceed exactly as in Case 2B(i).

In the next two cases we define $\beta \dashv\!| \alpha$ iff $\beta \dashv\!|' \alpha$ and either β is active or $\beta \in C_\alpha$.

Case 2D(iii): $\alpha^* = \gamma$, α *a* $\dashv\!|$*-limit.* Then α is active. If $\bar{c}$ is an α-name, then $\bar{c}^\beta$ is also a β-name for sufficiently large $\beta \dashv\!| \alpha$. Then $\bar{c}$ is proper iff $\bar{c}^\beta$ is proper for such β and in that case $p(\bar{c}) = \bigcup\{p(\bar{c}^\beta) \mid \beta \dashv\!| \alpha,\ \bar{c}^\beta$ a β-name$\}$.

Case 2D(iv): $\alpha^* = \gamma$, α *not a* $\dashv\!|$*-limit.* Let β be the $U(\gamma)$-predecessor to α, should it exist. If β exists, we must define $\mathscr{C}_X^\alpha$ when $X \in I_\alpha$, in order to complete the definition of 'α-name' in this case. By induction we have defined $p(\bar{c})$ for all proper β-names $\bar{c}$; if $X \in I_\beta$, then $\{p \in \mathscr{C}_X^\alpha \mid |p| = \beta\}$ consists of all p, Dom$(p) = X$, which are thinnings of conditions of the form $p(\bar{c})$, $\bar{c}$ a proper β-name or which can be obtained from such q as follows: Pick $i_0 < \cdots < i_n$ from $X \cup \{\bigcup X + 1\}$, $\alpha_0, \ldots, \alpha_n \in U$. Then define $p(i) = q(i)$ for $i \in X$ so that the least k, $i < i_k$ is even. Define $p(i) \restriction \alpha_k = q(i) \restriction \alpha_k$, $p(i)(\eta) = 0$ for all $\eta \in [\alpha_k, \beta)$ if the least k, $i < i_k$, is odd.

Then we define $\bar{\mathscr{C}}_X^\alpha$ (for $X \in I_\beta$) much as in Part B. Let $\hat{U}(\gamma) = \{\eta \mid L_\eta \models \gamma$ is the largest cardinal$\} - U(\gamma)$. Then $p \in \bar{\mathscr{C}}_X^\alpha$ iff $p \in \mathscr{C}_X^\beta$ or:

(i) $\mathrm{Dom}(p) = X$, $p(i):[\gamma, |p|) \to 2$ for $i \in X$, $\beta \leq |p| < \alpha$ and $\widetilde{p(i)} \subseteq \hat{U}(\gamma)$ for each $i \in X$. (This last requirement is needed for the proof of Lemma 1D.2.)

(ii) Suppose $i < j$ belong to X and $\langle \beta + i, \xi \rangle < |p|$. Then $p(j)(\langle \beta + i, \xi \rangle) = p(i)(\xi)$.

(iii) Define p^η by $p^\eta(i) = p(i) \restriction \eta$. Then $p^\beta \in \mathscr{C}_X^\alpha$ and p^η is $\Delta_1^*(\mathscr{A}(\eta))$ for $\eta \in T_\gamma = \{\nu \mid \nu$ is p.r. closed and $L_\nu \models \gamma$ is the largest cardinal$\}$.

If $X \in I_\alpha - I_\beta$, then $\bar{\mathscr{C}}_X^\alpha$ consists of all p obeying (i), (ii), (iii) with "$p^\beta \in \mathscr{C}_X^\alpha$" deleted, such that $p(i) \supseteq b_i$ for all $i \in X$. Finally, $\mathscr{C}_X^\alpha$ is defined by closing $\bar{\mathscr{C}}_X^\alpha$ under the earlier operation: Pick $i_0 < \cdots < i_n$ from $X \cup \{\bigcup X + 1\}$, $\beta_0, \ldots, \beta_n \in U$. Then define $p(i) = q(i)$ for $i \in X$ so that the least k, $i < i_k$, is even. Define $p(i) \restriction \beta_k = q(i) \restriction \beta_k$, $p(i)(\eta) = 0$ for all $\eta \in [\beta_k, \alpha)$ if the least k, $i < i_k$, is odd.

This completely defines $\mathscr{C}_X^\alpha$ for $X \in I_\alpha$ and therefore $\mathscr{C}_Y^p$, $\bar{\mathscr{C}}_Y^p$ ($p \in \mathscr{C}_X$, $X \cup Y \in I_\alpha$, X and Y disjoint). Then the collection of α-names can be fully defined.

Now let $\beta' =$ the $\dashv\!\vdash$-predecessor of α if it exists; $\beta' = 0$ otherwise. We no longer assume that β exists; if it does not, then set $\beta = \beta'$. We define "$\bar{c}$ is proper" and $p(\bar{c})$ for α-names $\bar{c} = (\emptyset, (\emptyset, p_1), \ldots, (\bar{p}_n, p_n))$ by induction on n. We want to do this in such a way that each $b \in \mathrm{Range}(p(\bar{c}) \restriction Y_n)$ 'codes' the α-name $\bar{c}$ (where $Y_n = \mathrm{Dom}(p_n)$). Now using the canonical well-ordering $<_L$ we can identify $\bar{c}$ with an ordinal $\delta = \mathrm{ord}(\bar{c}) =$ rank of $\bar{c}$ in $<_L$. We say that $b:[\gamma, \alpha) \to 2$ *codes* $\bar{c}$ if for $\delta < \alpha : \tilde{b} = \{\eta \mid b(\eta) = 0\}$ is almost disjoint from $x_\delta = \{\langle 0, \langle \delta, \eta \rangle \rangle \mid \eta \in \mathcal{O}(\gamma) \cap \alpha\}$ (i.e., $\tilde{b} \cap x_\delta$ is bounded in α) iff $\delta = \mathrm{ord}(\bar{c})$. Given any $s:[\gamma, \alpha') \to 2$, $\alpha' < \alpha$ and $\bar{c}$, it is easy to construct $b \supseteq s$ which codes $\bar{c}$.

We must also deal with Σ-genericity. As in Case 2A(iii) choose a (canonical) partial $\Sigma_1(L_\alpha)$-function h_α from γ onto L_α. We say that $\bar{e} < \gamma$ *α-codes* the pair (D, e) if $D \in \bar{\mathscr{D}}_0(\alpha)$ consists solely of α-names $\bar{c}$ such that $\bar{c}^{\beta'}$ is a proper β'-name and $h_\alpha(\bar{e}) = (D, e)$. For such a pair consider $f \in \mathscr{P}(\alpha, D)$ defined by: if $\bar{c} \in D$ is of the form $(\emptyset, (\emptyset, p_1), \ldots, (\bar{p}_n, p_n))$, then $f(\bar{c}) = (q_1, \ldots, q_n)$ where $q_j = p(\bar{c}^{\beta'}(\leq j)) \restriction \mathrm{Dom}(p_j)$. Then we define "$(D, e)$ is alive" as follows. Let $\beta'' = \max C_\alpha$. Then (D, e) is alive if $\beta'' = 0$ or if $h_\delta(\bar{e}) = (D_\delta, e_\delta)$ is alive at unboundedly many active stages $\delta \dashv\!\vdash \beta''$, or if (D'_δ, e'_δ) is alive at unboundedly many stages $\delta \dashv\!\vdash \beta''$, where $h_\delta(\bar{e}') = (D'_\delta, e'_\delta)$, $\bar{e}' < \bar{e}$. (By induction we have defined "(D'_δ, e'_δ) is alive" at $\dashv\!\vdash$-successor stages $\delta < \alpha$).)

First suppose that γ is L_α-regular. Choose $\bar{e} < \gamma$ to be the least α-code of an alive pair, if there is one. Then choose the L_α-least $f' \leq f$ so that $\{f'\} \times \gamma \subseteq W_e^*$ if possible; otherwise choose the L_α-least $f' \leq f$ so that for some $\delta < \gamma$ we have $\{f'\} \times \delta \subseteq W_e^*$ but $\{f\} \times \delta \nsubseteq W_e^*$, where $W_e = e$th $\Sigma_1(L_\alpha)$-set. Also require that f' is 1–1 and for some $\eta \geq \beta : (r_1, \ldots, r_l) \in \mathrm{Range}(f') \to |r_i| = \eta$ for all i. We say that α is active iff $\bar{e}$, f' both exist. If $\bar{e}$ or f' as above does not exist, then set $D = \emptyset$.

If γ is L_α-singular, fix the L_α-least sequence $\langle \gamma_i \mid i < L_\alpha\text{-cof}(\gamma) = \kappa \rangle$ so as to be continuous, increasing and cofinal. Then choose $i < \kappa$ to be least so that there an α-code $\bar{e}$ of an alive pair such that $\bar{e} < \gamma_i$. Also let $f' \leqslant f$ be least so that for some such $\bar{e} < \gamma_i$ and some $\delta < \bar{e}$ we have that $\{f'\} \times \delta \subseteq W_e^*$ but $\{f\} \times \delta \nsubseteq W_e^*$, where $h_\alpha(\bar{e}) = (D, e)$, f' and f belong to $\mathscr{P}(\alpha, D)$ and $W_e = r$th $\Sigma_1(L_\alpha)$-set. Also require that f' is 1–1 and for some $\eta \geqslant \beta : (r_1, \ldots, r_l) \in \text{Range}(f') \rightarrow |r_i| = \eta$ for all i. We say that α is active iff both $\bar{e}$, f' as above exist. If $\bar{e}$ or f' as above do not exist, then set $D = \emptyset$.

Now suppose $n = 1$. Then $\bar{c} = (\emptyset, (\emptyset, p_1))$ is proper. If $|p_1| < \beta'$, then let $\bar{c}_0 = (\emptyset, (\emptyset, q_1))$ where $q_1 = p(\bar{c}^{\beta'})$. If $\bar{c} \in D$, then let $\bar{d} = (\emptyset, (\emptyset, r_1))$ where $r_1 = f'(\bar{c})$. Then as in the proof of Lemma 1B.9 we can easily extend r_1 to s_1 so that $\text{Dom}(s_1) = \text{Dom}(r_1)$, $s_1(i) : [\gamma, \alpha) \rightarrow 2$, $s_1^{\sim}(i) \subseteq \mathcal{O}(\gamma)$ and $s_1(i) \supseteq r_1(i)$ for each i, s_1 obeys (ii), (iii) above and $s_1(i)$ 'codes' $\bar{c}$ for all i. Set $p(\bar{c}) = s_1$. If $\bar{c} \notin D$, then let $\bar{d} = (\emptyset, (\emptyset, r_1))$ where $r_1 = p(\bar{c}_0^\beta)$ if $\beta' < \beta$, $= q_1$ otherwise, and then using this definition of $\bar{d}$ proceed as above. If $|p_1| \in [\beta', \beta)$, then let $\bar{d} = (\emptyset, (\emptyset, r_1))$ where $r_1 = p(\bar{c}^\beta)$ and proceed as above. If $|p_1| \geqslant \beta$, then let $\bar{d} = \bar{c}$ and proceed as above.

Now suppose that we have defined "$\bar{c}$ is proper" and $p(\bar{c})$ for α-names of length k and $\bar{c} = (\emptyset, (\emptyset, p_1), \ldots, (\bar{p}_{k+1}, p_{k+1}))$ is an α-name of length $k+1$. Let p'_{k+1} thin $p(\bar{c}(\leqslant k))$ as $\bar{p}_{k+1}$ thins $\mathbf{G}(\bar{c}(\leqslant k))$. Then $\bar{c}$ is proper exactly if $\bar{c}(\leqslant k)$ is proper and $p_{k+1} \in \mathscr{C}_Y^{p'_{k+1}}$, $Y = \text{Dom}(p_{k+1})$. If this is the case we define $p(\bar{c})$ as follows: If $\bar{c}^{\beta'}$ is a β'-name, then let $\bar{c}_0 = (\emptyset, (\emptyset, q_1), \ldots, (\bar{q}_{k+1}, q_{k+1}))$ where $q_i = p(\bar{c}^{\beta'}(\leqslant i)) \upharpoonright \text{Dom}(p_i)$, $\bar{q}_{i+1}$ thins $\mathbf{G}(\bar{c}_0(\leqslant i))$ as $\bar{p}_{i+1}$ thins $\mathbf{G}(\bar{c}(\leqslant i))$. If $\bar{c}(\leqslant i) \in D$ for some $i \geqslant 1$, then let $\bar{d} = (\emptyset, (\emptyset, r_1), \ldots, (\bar{r}_m, r_m), (\bar{q}_{m+1}, q_{m+1}), \ldots, (\bar{q}_{k+1}, q_{k+1}))$ where m is largest so that $\bar{c}(\leqslant m) \in D$, $f'(\bar{c}(\leqslant m)) = (r_1, \ldots, r_m)$ and $\bar{r}_{i+1}$ thins $\mathbf{G}(\bar{d}(\leqslant i))$ as $\bar{p}_{i+1}$ thins $\mathbf{G}(\bar{c}(\leqslant i))$. Now if $|q_{k+1}| \geqslant \beta$, then extend q_{k+1} to s_{k+1} so that $\text{Dom}(s_{k+1}) = \text{Dom}(q_{k+1})$, $s_{k+1}(i) : [\gamma, \alpha) \rightarrow 2$, $s_{k+1}^{\sim}(i) \subseteq \mathcal{O}(\gamma)$ and $s_{k+1}(i) \supseteq q_{k+1}(i)$ for each i, $s'_{k+1} \cup s_{k+1}$ obeys (ii), (iii) above and $s_{k+1}(i)$ 'codes' $\bar{c}$ for each i, where s'_{k+1} thins $p(\bar{d}(\leqslant k))$ as $\bar{p}_{k+1}$ thins $\mathbf{G}(\bar{c}(\leqslant k))$. Set $p(\bar{c}) = s'_{k+1} \cup s_{k+1}$. If $|q_{k+1}| < \beta$, then by Fact 4 below we can choose a canonical β-name $\bar{d}_0$ so that $p(\bar{d}(\leqslant k))^\beta$ is obtained from $p(\bar{d}_0)$ as in the definition of $\mathscr{C}_X^\alpha$; then $\bar{d}_0 * (\bar{r}_{k+1}, q_{k+1})$ is a proper β-name where $\bar{r}_{k+1}$ thins $\mathbf{G}(\bar{d}_0)$ as $\bar{p}_{k+1}$ thins $\mathbf{G}(\bar{c}(\leqslant k))$. Let $r_{k+1} = p(\bar{d}_0 * (\bar{r}_{k+1}, q_{k+1})) \upharpoonright \text{Dom}(p_{k+1})$ and then define s'_{k+1}, s_{k+1}, $p(\bar{c})$ as above, using r_{k+1} in place of q_{k+1}. If $\bar{c}(\leqslant 1) \notin D$, then let $\bar{d} = (\emptyset, (\emptyset, r_1), \ldots, (\bar{r}_{k+1}, r_{k+1}))$ where $r_i = p(\bar{c}_0^\beta(\leqslant i)) \upharpoonright \text{Dom}(p_i)$ and $\bar{r}_{i+1}$ thins $\mathbf{G}(\bar{d}(\leqslant i))$ as $\bar{p}_{i+1}$ thins $\mathbf{G}(\bar{c}(\leqslant i))$, if $\beta' < \beta$; $\bar{d} = \bar{c}_0$ otherwise. Then proceed as above with this definition of $\bar{d}$. If $\bar{c}^{\beta'}$ is not a β'-name but $\bar{c}^\beta$ is a β-name, then let $\bar{d} = (\emptyset, (\emptyset, r_1), \ldots, (\bar{r}_{k+1}, r_{k+1})$ where $r_i = p(\bar{c}^\beta(\leqslant i)) \upharpoonright \text{Dom}(p_i)$, $\bar{r}_{i+1}$ thins $\mathbf{G}(\bar{d}(\leqslant i))$ as $\bar{p}_{i+1}$ thins $\mathbf{G}(\bar{c}(\leqslant i))$. Then proceed as above.

If $\bar{c}^\beta$ is not a β-name and $|p_{k+1}| \geqslant \beta$, then let $\bar{d} = \bar{c}$ and proceed as above. If $|p_{k+1}| < \beta$, then by Fact 4 below we can choose a canonical β-name $\bar{d}_0$ so that $p(\bar{c}(\leqslant k))^\beta$ is obtained from $p(\bar{d}_0)$ as in the definition of $\mathscr{C}_X^\alpha$; then $\bar{d}_0 * (\bar{q}_{k+1}, p_{k+1})$ is a β-name, where $\bar{q}_{k+1}$ thins $\mathbf{G}(\bar{d}_0)$ as $\bar{p}_{k+1}$ thins $\mathbf{G}(\bar{c}(\leqslant k))$. Let $q_{k+1} = p(\bar{d}_0 * (\bar{q}_{k+1}, p_{k+1})) \upharpoonright Y_{k+1}$, $\bar{d} = (\emptyset, (\emptyset, p_1), \ldots, (\bar{p}_k, p_k), (\bar{p}_{k+1}, q_{k+1}))$ and proceed as above. This completes Case 2D(iv).

Finally we set $\mathscr{C}_X^\alpha = \bigcup \{\mathscr{C}_X^\beta \mid \beta \in U(\gamma) \cap \alpha, X \in I_\beta\}$ when $X \in I_\alpha$ and α is a

$U(\gamma)$-limit. This completes the definition of $\{b_\nu \mid \nu \in T\}$, $\{b_{\nu\sigma} \mid \nu$ a $<$-successor, $\sigma \in W(\nu)\}$, $p(\bar{c})$ for proper α-names $\bar{c}$ and $\mathscr{C}_X^\alpha$ for $X \in I_\alpha$ $(\alpha \in U(\gamma))$.

Remarks. (5) The 'essential' steps in the above induction are the cases: $\dashv\!\vdash$-successor, $\alpha \in E$. The former deals with Σ-genericity and the latter with regular genericity. Note that Lemmas 1C.3, 1C.7, 1C.8 also hold for the relation $\dashv\!\vdash$: Lemma 1C.3(a) is clear as $\sigma \dashv\!\vdash \nu$ iff ($\sigma \dashv\!\vdash' \nu$, σ is active) and the $\dashv\!\vdash'$-limit of active ordinals is active (for ν as in (2), (3). The argument for ν as in (1) is similar.) To argue for Lemma 1C.3(b) see the proof of the Genericity Lemma 1C.13. For Lemmas 1C.7, 1C.8 note that $\beta \in C_\alpha$, $g: \delta \Rightarrow \alpha$, $\lambda(g) = \beta$ implies that $\hat{g}: \mathscr{A}(\nu(\delta)) \to \mathscr{A}(\nu(\alpha))$ is Σ_2-elementary and hence $\nu(\beta)$ is the $\dashv\!\vdash'$-limit of active ordinals. So $\nu(\beta) \dashv\!\vdash \nu(\alpha)$ as $\nu(\beta) \dashv\!\vdash' \nu(\alpha)$ and $\nu(\beta)$ is active.

(6) The need to work with α-names rather than α-conditions is that defining the collection of α-conditions of length $k > 1$ requires that we have already defined $p(c)$ for α-conditions c of length $<k$; but we want $p(c)$ to depend on the definition of $p(d)$ for α-conditions d of length $\geq k$, for the sake of the mutual genericity property (h). The collection of α-names can be defined at the start.

(7) 'Properness' for α-names reveals its meaning only in the last case of the above induction. In all other cases the properness of an α-name $\bar{c}$ is reduced to that of a β-name, $\beta < \alpha$.

Now as promised we associate a proper α-name $\bar{c}$ to each α-condition $c = (p_0, (p_1', p_1), \ldots, (p_n', p_n))$. (We do this while simultaneously defining $p(c)$ to be $p(\bar{c})$ where $\bar{c}$ is the α-name associated to c.) Namely, to c we associate $\bar{c} = (p_0, (\bar{p}_1, p_1), \ldots, (\bar{p}_n, p_n))$ where $\bar{p}_{k+1} = \mathbf{G}(\bar{c}(\leq k)) \circ \pi_k$, if $p_{k+1}' = p(c(\leq k)) \circ \pi_k$ (thus $\bar{p}_{k+1}$ thins $\mathbf{G}(\bar{c}(\leq k))$ exactly as p_{k+1}' thins $p(c(\leq k))$). We must verify that $\bar{c}$ is proper. But the following is easily verified.

Fact 1. *Suppose $\bar{c} = (p_0, (\bar{p}_1, p_1), \ldots, (\bar{p}_n, p_n))$ is an α-name. Then $\bar{c}$ is proper iff for all $k < n$, $p_{k+1} \in \mathscr{C}_{k+1}^{p(\bar{c}(\leq k))}$, where $Y_{k+1} = \mathrm{Dom}(p_{k+1})$.*

Proof. By induction on α. All cases except α a $U(\gamma)$-successor are easy as we have defined $p(\bar{c})$ to extend $p(\bar{d})$ for a β-name $\bar{d}$, $\beta < \alpha$, and have defined $\bar{c}$ proper $\leftrightarrow \bar{d}$ is proper. If α is a $U(\gamma)$-successor, then this is immediate from the definition of proper in Case 2D(iv). $\square$

Now the properness of $\bar{c}$ follows from Fact 1 and the definition of α-condition. Similarly, Fact 1 can be used in conjunction with properties (i)–(m) to justify the properness of names considered in the construction.

Let $\mathscr{C}_X^* = \{p \in \mathscr{C}_X \mid$ for all $i \in \mathrm{Dom}(p)$, $\widetilde{p(i)} \cap \alpha$ is unbounded in α where $\alpha = |p(i)| \in U(\gamma)\}$.

Fact 2. *Suppose $p \in \mathscr{C}_X^*$ where $X \in I_\alpha$, $|p| = \alpha \in U(\gamma)$. Then p is a thinning of $p(\bar{c})$ for some proper α-name $\bar{c}$.*

Proof. Let $\alpha' = U(\gamma)$-successor to α. Then $p \in \mathscr{C}_X^{\alpha'}$ as $\alpha' < \beta \in U(\gamma)$, $p \in \mathscr{C}_X^{\beta}$, $|p| < \alpha' \rightarrow p \in \mathscr{C}_X^{\alpha'}$ can be checked by induction on β. But then the conclusion is clear by the definition of $\{p \in \mathscr{C}_X^{\alpha'} \mid |p| = \alpha\}$ given in Case 2D(iv). □

Fact 3. *Suppose $p \in \mathscr{C}_X$, $|p| < \eta < \gamma^+$. Then there exists $q \leq p$ in $\mathscr{C}_X$, $|q| = \eta$.*

Proof. We can assume that $p \in \mathscr{C}_X^*$.

It is easily checked by induction that $|p(\bar{c})(i)| = \alpha$ for any proper α-name $\bar{c}$, $i \in \mathrm{Dom}(p(\bar{c}))$. Note that Fact 1 implies that if $\bar{c} = (p_0, (\bar{p}_1, p_1), \ldots, (\bar{p}_n, p_n))$ is an α-name, $|p_i| \geq |p_{i+1}|$ for all $i < n$, then $\bar{c}$ is proper. Now we prove Fact 3 by induction on η. If η is not a limit of elements of $U(\gamma)$, then the result is easy, using induction and the definition of $p(\bar{c})$, $\bar{c}$ a proper α-name, α a $U(\gamma)$-successor. Thus we can assume that $|p| = \beta \in U(\gamma)$. Now let $\alpha =$ least element of $U(\gamma)$ greater than β and if p is a thinning of $\bar{c} = (p_0, (\bar{p}_1, p_1), \ldots, (\bar{p}_n, p_n))$, define $\bar{d} = (\emptyset, (\emptyset, q_0), (\bar{q}_1, q_1), \ldots, (\bar{q}_n, q_n))$ by: $q_0 = q_{Y_0}$ where $Y_0 = T_\beta$, $q_i = p(\bar{c}(\leq i)) \restriction Y_i$ ($Y_i = \mathrm{Dom}(p_i)$) and $\bar{q}_{i+1}$ thins $\mathbf{G}(\bar{d}(\leq i))$ as $\bar{p}_{i+1}$ thins $\mathbf{G}(\bar{c}(\leq i))$, for $i > 0$. To define $\bar{q}_1$ choose σ, π so that $p_0(i) = b_{\pi(i)}$ and $\bar{p}_1 = \mathbf{G}((p_0)) \circ \sigma$. Then $\bar{q}_1 = \mathbf{G}((\emptyset, (\emptyset, q_0))) \circ \pi \circ \sigma$. Then $|q_i| = \beta$ for all $i > 0$ so $\bar{d}^\eta$ is a proper η-name. But then $q \leq p$ where $q = p(\bar{d}^\eta)$, $|q| = \eta$ and we are done. □

Fact 4. *Suppose $p \in \mathscr{C}_X$, $|X| < \eta < |p|$. Then $p^\eta \in \mathscr{C}_X$.*

Proof. This is clear if $|p|$ is not a limit of elements of $U(\gamma)$, using the definition of $p(\bar{c})$ given in Case 2D(iv). Otherwise let p be obtained from a thinning of $p(\bar{c})$ as in the definition of $\mathscr{C}_X^\alpha$, where $\bar{c}$ is a proper $|p|$-name. An inspection of the construction shows that $p(\bar{c})$ is the union of conditions of the form $p(\bar{d})$, $\bar{d}$ a proper α-name for some $\alpha < |p|$. Thus p^η is obtained from a thinning of $p(\bar{d})^\eta$ for some proper α-name $\bar{d}$, $\alpha < |p|$ and so by induction $p^\eta \in \mathscr{C}_X$. □

Fact 5. *Suppose $p \in \mathscr{C}_X$, $|p| < \alpha \in U(\gamma)$. Then there exist $q_0, q_1 \leq p$ in $\mathscr{C}_X^\alpha$ such that q_0, q_1 are incompatible.*

Proof. This is clear from the definition of $\mathscr{C}_X^\alpha$ in Case 2D(iv). □

Fact 6. *Suppose $\bar{c} = (p_0, (\bar{p}_1, p_1), \ldots, (\bar{p}_n, p_n))$ is a proper α-name of positive length, $\beta \in U(\gamma) \cap \alpha$ and $\bar{c}^\beta$ is a β-name. Then $p(\bar{c})^\beta = p(p_0^\beta, (\bar{p}_1^\beta, q_1), \ldots, (\bar{p}_n^\beta, q_n))$ for some $(q_1, \ldots, q_n) \in \mathscr{C}^*(\bar{c}) \cap L_\beta$.*

Proof. By induction on α. If C_α is unbounded in α and $\sigma(\bar{c})$ is a $<$-limit, then the result follows by induction as $p(\bar{c}) = \bigcup \{p(\bar{c}^\beta) \mid \beta \in C_\alpha$, $\bar{c}^\beta$ a β-name$\}$ and $\beta \in C_\alpha$, $\bar{c}^\beta$ a β-name $\rightarrow \mathscr{C}^*(\bar{c}^\beta) = \mathscr{C}^*(\bar{c}) \cap L_\beta$. If $\sigma(\bar{c})$ is a $<$-successor, then $p(\bar{c}) \supseteq \bigcup \{p(\pi_\sigma, (\pi_\sigma, q_{Y_\sigma}), (\bar{p}_1^{\alpha(\sigma)}, p_1), \ldots, (\bar{p}_n^{\alpha(\sigma)}, p_n)) \mid \sigma \dashv \nu(\alpha)$, $\bar{c}^{\alpha(\sigma)}$ an $\alpha(\sigma)$-name$\}$; but $\bar{c}^\beta$ a β-name $\rightarrow \beta < \alpha(\sigma)$ for all $\sigma \dashv \nu(\alpha) \rightarrow \bar{c}^{\alpha(\sigma)}$ an $\alpha(\sigma)$-name for all

$\sigma \dashv \nu(\alpha)$. So we are again done by induction unless $\nu(\alpha)$ is $\dashv$-minimal. But then an inspection of Case 2C(i) reveals that $p(\bar{c}) \leqslant p(\bar{c}^{\alpha(\bar{\nu})})$ where $\bar{\nu} <_* \nu(\alpha)$ and we can then apply the induction hypothesis as $\beta \leqslant \alpha(\bar{\nu})$. If $\nu(\alpha)$ is a $\dashv$-limit (is a $\dashv\!\!\mid$-limit) and $\sigma(\bar{c})$ is a $<$-limit, then $p(\bar{c}) = \bigcup \{p(\bar{c}^{\alpha(\sigma)}) \mid \sigma \dashv \nu(\alpha),\ \bar{c}^{\alpha(\sigma)}$ an $\alpha(\sigma)$-name$\}$ $(=\bigcup \{p(\bar{c}^{\alpha(\sigma)}) \mid \sigma \dashv\!\!\mid \nu(\alpha),\ \bar{c}^{\alpha(\sigma)}$ an $\alpha(\sigma)$-name$\})$ and we are done by induction once again.

In all other cases, except Cases 2A(iii), 2B(ii), 2C(iii)(c), 2D(i), 2D(iv), $p(\bar{c})$ is defined by "extending $\bar{c}$ along the α_j's" for some appropriate sequence $\alpha_0 < \alpha_1 < \cdots$ cofinal in α (this process is described in Case 2A(i)). Then $p(\bar{c}) = \bigcup \{p(\bar{c}_{j_m}) \mid m \geqslant 0\}$ and an inspection of the definition of $\bar{c}_{j_m}$ shows that $p(\bar{c})^{\alpha_{j_m}} = p(p_0^{\alpha_{j_m}}, (\bar{p}_1^{\alpha_{j_m}}, q_1), \ldots, (\bar{p}_n^{\alpha_{j_m}}, q_n))$ for some $(q_1, \ldots, q_n) \in \mathscr{C}^*(\bar{c}) \cap L_{\alpha_{j_m}}$. Then we can apply the induction hypothesis by selecting $\alpha_{j_m} > \beta$. In Cases 2A(iii), 2C(iii)(c), $p(\bar{c})$ is defined by replacing $\bar{c}$ by $\bar{d} = (p_0, (\bar{r}_1, r_1), \ldots, (\bar{r}_n, r_n))$ where $(r_1, \ldots, r_n) \in \mathscr{C}^*(\bar{c})$ and $\bar{r}_{i+1}$ thins $\mathbf{G}(\bar{d}(\leqslant i))$ exactly as $\bar{p}_{i+1}$ thins $\mathbf{G}(\bar{c}(\leqslant i))$, and then $p(\bar{c}) = p(\bar{d})$ is defined by extending $\bar{d}$ along the α_j's. For all k, $p(\bar{c})^{\alpha_{j_k}} = p(p_0^{\alpha_{j_k}}, (\bar{p}_1^{\alpha_{j_k}}, r_1'), \ldots, (\bar{p}_n^{\alpha_{j_k}}, r_n'))$ for some $(r_1', \ldots, r_n') \in \mathscr{C}^*(\bar{d}) \cap L_{\alpha_{j_k}}$ (where $j_0, j_1, \ldots,$ come from extending $\bar{d}$ along the α_j's). Then $(r_1', \ldots, r_n') \in \mathscr{C}^*(\bar{c})$ so we can apply the induction hypothesis, selecting $\alpha_{j_k} > \beta$. Cases 2B(ii), 2D(i) constitute a variation on "extending along the α_j's" which does not alter the above argument.

Finally we consider Case 2D(iv). Then $\beta \leqslant (U(\gamma)$-predecessor to $\alpha) = \bar{\alpha}$. If $\beta = \bar{\alpha}$, then the Fact follows easily from the definition of $p(\bar{c})$. Otherwise we can apply the induction hypothesis to the $\bar{\alpha}$-name $\bar{d}$ where $p(\bar{d}) = p(\bar{c})^{\bar{\alpha}}$,

$$\bar{d} = (\emptyset^{\bar{\alpha}}, (\emptyset^{\bar{\alpha}}, q_1), \ldots, (\bar{p}_n^{\bar{\alpha}}, q_n)), \quad (q_1, \ldots, q_n) \in \mathscr{C}^*(\bar{c}) \cap L_{\bar{\alpha}}. \qquad \square$$

Fact 7. *Suppose $p \in \mathscr{C}_X^\alpha$ and $b = p(i)$ for some $i \in X$. Then $\tilde{b} \subseteq \hat{U}(\gamma)$.*

Proof. By induction on $|b|$. If $|b|$ is a $U(\gamma)$-limit, then the result is clear by induction. Otherwise the result follows from clause (i) of the definition of $\mathscr{C}_X^\alpha$ in Case 2D(iv). $\square$

Fact 7 is needed in the extendibility proof for R^s.

Definition. An α-name $\bar{c} = (p_0, (\bar{p}_1, p_1), \ldots, (\bar{p}_n, p_n))$ is *canonical* if $\nu(\alpha)$ a $<$-successor, $b_{\nu(\alpha)} \in \text{Range}(p(\bar{c})) \to b_{\nu(\alpha)} \in \text{Range}(p_0)$.

Recall that $p \in \mathscr{C}_X$ is *standard* if $p(i) = b_{\pi(i)}$ for some morass map π.

Lemma 1C.11 (Canonical α-Names). *Suppose $p \in \mathscr{C}_X^*$, $|p| = \alpha \in U(\gamma)$ and p is not standard. Then there is a unique canonical proper α-name $\bar{c}$ such that p is a thinning of $p(\bar{c})$, p is not a thinning of $p(\bar{c}(\leqslant i))$ for any $i <$ length$(\bar{c})$.*

For any α-name $\bar{c}$ define $\mathrm{Range}(\bar{c}) = \cup\{\mathrm{Range}(p(\bar{c}(\leqslant i))) \mid 0 \leqslant i \leqslant \mathrm{length}(\bar{c})\}$. Then if $p, \bar{c}$ are as in the lemma, we have $\mathrm{Range}(p) \subseteq \mathrm{Range}(\bar{c})$, $\mathrm{Range}(p) \nsubseteq \mathrm{Range}(\bar{c}(\leqslant i))$ for $i < \mathrm{length}(\bar{c})$.

If $p \in \mathscr{C}^*_X$, $|p| = \alpha \in U(\gamma)$, then the *canonical α-name associated to p* is (p) if p is standard, is the canonical proper α-name $\bar{c}$ of the lemma if p is not standard.

Proof of Lemma 1C.11. We actually show somewhat more, by induction on α: If $b:[\gamma, \alpha) \to 2$, $b \notin \{b_\nu \mid \nu \in T_\alpha\}$, then there is at most one canonical proper α-name $\bar{c}$ such that $b \in \mathrm{Range}(\bar{c})$, $b \notin \mathrm{Range}(\bar{c}(\leqslant i))$ for $i < \mathrm{length}(\bar{c})$. Note that by Fact 2 there does exist a proper α-name $\bar{c}$ such that p is a thinning of $p(\bar{c})$. Also note that if $\bar{c}$ is not canonical then there exists a canonical $\bar{d}$ such that $p(\bar{c}) = p(\bar{d})$. Then by replacing $\bar{d}$ by $\bar{d}(\leqslant m)$ where m is least so that p is a thinning of $p(\bar{d}(\leqslant m))$, we have established the existence part of the lemma.

Now we establish the above claim. Suppose that $\bar{c} = (p_0, (\bar{p}_1, p_1), \ldots, (\bar{p}_n, p_n))$ and $\bar{d} = (q_0, (\bar{q}_1, q_1), \ldots, (\bar{q}_m, q_m))$ are canonical proper α-names of positive length such that $b \in \mathrm{Range}(\bar{c}) - \mathrm{Range}(\bar{c}(\leqslant n-1))$, $b \in \mathrm{Range}(\bar{d}) - \mathrm{Range}(\bar{d}(\leqslant m-1))$. We will show that $\bar{c} = \bar{d}$.

If C_α is unbounded in α, then for sufficiently large $\beta \in C_\alpha$, $p(\bar{c})^\beta = p(\bar{c}^\beta)$, $p(\bar{d})^\beta = p(\bar{d}^\beta)$, $b \restriction \beta \notin \{b_\nu \mid \nu \in T_\beta\}$ and therefore by induction $p(\bar{c}^\beta) = p(\bar{d}^\beta)$. Again by induction either $\bar{c}^\beta = \bar{d}^\beta$ for such β, $\bar{d}^\beta =$ unique canonical β-name $\bar{e}$ such that $p(\bar{e}) = p(\bar{c}^\beta)$ for such β or $\bar{c}^\beta =$ unique canonical β-name $\bar{e}$ such that $p(\bar{e}) = p(\bar{d}^\beta)$ for such β. The first case implies that $\bar{c} = \bar{d}$ and the other two cases imply that $\bar{c} = \bar{d}$ or one of $\bar{c}, \bar{d}$ is not canonical.

If $\nu(\alpha)$ is a $\dashv$-limit (is a $\dashv\!\vdash$-limit, respectively) then $p(\bar{c}) = \bigcup\{p(\bar{c}^{\alpha(\sigma)}) \mid \sigma \dashv \nu(\alpha)$, $\bar{c}^{\alpha(\sigma)}$ an $\alpha(\sigma)$-name$\}$ $(= \bigcup\{p(\bar{c}^{\alpha(\sigma)}) \mid \sigma \dashv\!\vdash \nu(\alpha)$, $\bar{c}^{\alpha(\sigma)}$ an $\alpha(\sigma)$-name$\}$, respectively) and as the same is true for $p(\bar{d})$, we have $\bar{c} = \bar{d}$ by induction once again, using the fact that for sufficiently large $\sigma \dashv \nu(\alpha)(\sigma \dashv\!\vdash \nu(\alpha)$, respectively) we have $b \restriction \alpha(\sigma) \notin \{b_\nu \mid \nu \in T_{\alpha(\sigma)}\}$.

In all other cases, except Cases 2A(iii), 2B(ii), 2C(iii)(c), 2D(i), 2D(iv), $p(\bar{c})$ and $p(\bar{d})$ are defined by "extending along the α_j's" for some appropriate sequence $\alpha_0 < \alpha_1 \cdots$ cofinal in α. (It is here that we use the fact that "j_1' was omitted" in the definition of "extending along the α_j's.") Thus we have defined sequences $\langle \bar{c}_{j_k} \mid k \geqslant 0\rangle$, $\langle \bar{d}_{j_k^*} \mid k \geqslant 0\rangle$ where $\bar{c}_{j_k}$ is an α_{j_k}-name, $\bar{d}_{j_k^*}$ is an $\alpha_{j_k^*}$-name and we defined $p(\bar{c}) = \bigcup\{p(\bar{c}_{j_k}) \mid k \geqslant 0\}$, $p(\bar{d}) = \bigcup\{p(\bar{d}_{j_k^*}) \mid k \geqslant 0\}$. Note that $\{j_k \mid k \geqslant 0\}$ and $\{j_k^* \mid k \geqslant 0\}$ each contain a final segment of ω. Thus by induction for sufficiently large j, $p(\bar{c}_j) = p(\bar{d}_j)$ and $b \restriction \alpha_j \notin \{b_\nu \mid \nu \in T_{\alpha_j}\}$, hence either $\bar{c}_j = \bar{d}_j$ or $\bar{c}_j =$ unique canonical α_j-name $\bar{e}$ such that $p(\bar{e}) = p(\bar{d}_j)$ or $\bar{d}_j =$ unique canonical α_j-name $\bar{e}$ such that $p(\bar{e}) = p(\bar{c}_j)$. But the last two possibilities are ruled out if j is chosen large enough so that $j - 1 \in \{j_k \mid k \geqslant 0\} \cap \{j_k^* \mid k \geqslant 0\}$ and $\alpha_{j-1} > \max(\beta, \beta^*)$ where $\mathrm{Dom}(p_0) \subseteq T_\beta$, $\mathrm{Dom}(q_0) \subseteq T_{\beta^*}$. (For example, the second possibility implies that $\mathrm{Dom}(\bar{c}_j(\leqslant 0)) = \mathrm{Dom}(p_0)$ is contained in $T_{\alpha_{j-1}}$.) So we know that for sufficiently large j, $\bar{c}_j = \bar{d}_j$. Thus in particular, $p(\bar{c}(\leqslant i)) = p(\bar{d}(\leqslant i))$ for each i.

We now show that $\bar{c}(\leqslant i)=\bar{d}(\leqslant i)$ for $i\geqslant 0$, by induction on i. Suppose $i=1$. Let $\langle j_{k,1}\mid k\geqslant 0\rangle$ and $\langle j^*_{k,1}\mid k\geqslant 0\rangle$ be the sequences used to define $p(\bar{c}(\leqslant 1))$, $p(\bar{d}(\leqslant 1))$ by extending along the α_j's. If $j_{0,1}=j^*_{0,1}$, then we must have $\bar{c}(\leqslant 1)=\bar{d}(\leqslant 1)$ as otherwise by induction $p(\bar{c}(\leqslant 1)_{j_{0,1}})\neq p(\bar{d}(\leqslant 1)_{j_{0,1}})$ and hence $p(\bar{c}(\leqslant 1)_j)\neq p(\bar{d}(\leqslant 1)_j)$ for all sufficiently large j. So assume that $j_{0,1}<j^*_{0,1}$ (without loss of generality). Let $j=j^*_{1,1}=j_{k+1,1}$. Then $p(\bar{c}(\leqslant 1)_j)=p(p_0^{\alpha_j},(\bar{r}_1,r_1))$ where $|r_1|=\alpha_{j_{k,1}}$, and $p(\bar{d}(\leqslant 1)_j)=p(q_0^{\alpha_j},(\bar{s}_1,s_1))$ where $|s_1|=\alpha_{j^*_{0,1}}$. But $j_{k,1}\neq j^*_{0,1}$ as $j_{k,1}$ was 'omitted' from the sequence $(j^*_{0,1},j^*_{1,1},j^*_{2,1},\ldots)$. Thus $p(\bar{c}(\leqslant 1)_j)\neq p(\bar{d}(\leqslant 1)_j)$ and this contradicts $p(\bar{c}(\leqslant 1))=p(\bar{d}(\leqslant 1))$.

Now suppose $i=i'+1>1$. We know by induction that $\bar{c}(\leqslant i')=\bar{d}(\leqslant i')$ and this allows us to repeat the preceding argument: Let $\langle j_{k,i}\mid k\geqslant 0\rangle$ and $\langle j^*_{k,i}\mid k\geqslant 0\rangle$ be the sequences used to define $p(\bar{c}(\leqslant i))$, $p(\bar{d}(\leqslant i))$ by extending along the α_j's. If $j_{0,i}=j^*_{0,i}$, then we must have $\bar{c}(\leqslant i)=\bar{d}(\leqslant i)$ as otherwise by induction $p(\bar{c}(\leqslant i)_{j_{0,i}})\neq p(\bar{d}(\leqslant i)_{j^*_{0,i}})$ and hence $p(\bar{c}(\leqslant i)_j)\neq p(\bar{d}(\leqslant i)_j)$ for all sufficiently large j. So assume without loss of generality that $j_{0,i}<j^*_{0,i}$. Let $j=j^*_{1,i}=j_{k+1,i}$. Then $p(\bar{c}(\leqslant i)_j)=p(p_0^{\alpha_j},(\bar{r}_1,r_1),\ldots,(\bar{r}_i,r_i))$ where $|r_i|=\alpha_{j_{k,i}}$ and $p(\bar{d}(\leqslant i)_j)=p(q_0^{\alpha_j},(\bar{s}_1,s_1),\ldots,(\bar{s}_i,s_i))$ where $|s_i|=\alpha_{j^*_{0,i}}$. But $j_{k,i}\neq j^*_{0,i}$ as $j_{k,i}$ was 'omitted' from the sequence $(j^*_{0,i},j^*_{1,i},\ldots)$. Thus $p(\bar{c}(\leqslant i)_j)\neq p(\bar{d}(\leqslant i)_j)$ and this contradicts $p(\bar{c}(\leqslant i))=p(\bar{d}(\leqslant i))$.

In Cases 2A(iii), 2C(iii)(c), $p(\bar{c})$ and $p(\bar{d})$ are defined by replacing $\bar{c}$, $\bar{d}$ by some appropriate $\bar{c}'$, $\bar{d}'$ and then extending $\bar{c}'$, $\bar{d}'$ along the α_j's. Thus the preceding argument shows that $p(\bar{c})=p(\bar{d})\to\bar{c}'=\bar{d}'$. But an inspection of the definitions of $\bar{c}'$, $\bar{d}'$ reveals that $\bar{c}'=\bar{d}'\to\bar{c}=\bar{d}$ (as $\bar{c}$, $\bar{d}$ were 'coded' into $\bar{c}'$, $\bar{d}'$). Similarly, in Cases 2B(ii), 2D(i) we are dealing with a variant of "extending along the α_j's" in which: $\bar{c}\neq\bar{d}\to\bar{c}_j\neq\bar{d}_j$ for all j for which both $\bar{c}_j$, $\bar{d}_j$ are defined (we have 'coded' into each $\bar{c}_{j_{2m}}$ where $j_0,j_1,\ldots$ come from the defintion of $p(\bar{c})$; at j_{2m+1} we are just "extending along the α_j's"). Thus once again, $\bar{c}=\bar{d}$ follows as in the above argument.

Finally we consider Case 2D(iv). By the definitions given in that case, we must have that $p(\bar{c})\restriction i=p(\bar{d})\restriction j$ where $p(\bar{c})(i)=p(\bar{d})(j)=b$ (see clause (ii) in the definition of $\mathscr{C}^\beta_X$). But $\bar{c}$ is coded into $p(\bar{c})(i)$ for all $i\in\mathrm{Dom}(p_n)$ and $\bar{d}$ is coded into $p(\bar{d})(j)$ for all $j\in\mathrm{Dom}(q_m)$ (see the definition of *code* in Case 2D(iv)). Thus as $b=p(\bar{c})(i)=p(\bar{d})(j)$ for some $i\in\mathrm{Dom}(p_n)$, $j\in\mathrm{Dom}(q_m)$ we must have $\bar{c}=\bar{d}$. □

We can now begin the verification of properties (a)–(m). The key genericity properties (g), (h) will be dealt with in Lemmas 1C.13–1C.16.

Lemma 1C.12. *Properties* (a)–(f), (i)–(m) *hold.*

Proof. (a)–(d) follow just as they did in Part B. Fact 3 yields the last statement (e) (the rest of (e) is clear). Fact 4 handles property (f). It remains to verify (i)–(m), which we proceed to do by a simultaneous induction on α, discussi each case separately. Note that it suffices to verify these properties with 'condition' replaced by 'name' throughout.

Case 1A. Property (i) is easily verified by examining the construction. There are no new cases of (j), (k) or (l). Property (m) is verified using Lemma 1C.8, the fact that C_α is unbounded in α and induction.

Case 1B. Property (i) follows from the construction. There are no new cases of (j)–(m).

Case 1C(iii). Property (i) is clear. (j), (k), are clear from the construction. (l) holds by induction and the Note after Lemma 1C.7. (m) follows by induction and the last statement of Lemma 1C.7.

Case 1C(iv). Property (i) follows from Lemma 1C.6 and the construction (using induction). Properties (j), (k), (l) are clear from the construction and induction. There are no new cases of (m).

Case 1D. Property (i) is clear. There are no new cases of (j), (k), or (l). Property (m) follows from the Note after Lemma 1C.7.

Case 2A(i). Property (i) follows from the fact that $\alpha_0 = \sup(C_\alpha)$ and from the fact that $\bar{c}^{\alpha_0}$ a proper α_0-name$\rightarrow p(\bar{c}) \leqslant p(\bar{c}^{\alpha_0})$ (this is easily seen by checking the construction of "extending along the α_i's"). There are no new cases of (j)–(m).

Case 2A(ii). Property (i) follows from the construction, Lemma 1C.8 and induction. There are no new cases of (j), (k), (l) and property (m) follows from the construction.

Case 2A(iii). Property (i) follows from Lemma 1C.8, induction, the fact that $\alpha_0 = \alpha(\sigma)$ where σ immediately $\dashv\!\!|$-precedes $\nu(\alpha)$ (or $\alpha_0 = \sup(C_\alpha)$) and the fact that we defined $p(\bar{c})$ to extend $p(\bar{c}^{\alpha_0})$ when $\bar{c}$ is a proper α_0-name. There are no new cases of (j), (k), (l). Property (m) is checked as was property (i).

Case 2B(i). Property (i) holds as we chose $\alpha_0 = \sup(C_\alpha)$ and $p(\bar{c}) \leqslant p(\bar{c}^{\alpha_0})$ when $\bar{c}^{\alpha_0}$ is a proper α_0-name. There are no new cases of (j)–(m).

Case 2B(ii). Same as Case 2B(i).

Case 2C(i). By Lemma 1C.6, $C_\alpha = \emptyset$. Thus there are no new cases of property (i). Property (j) is clear from the construction. Properties (k)–(m) present no new cases.

Case 2C(ii). Property (i) follows from Lemma 1C.6, the construction and induction. Properties (j), (k), (l) are clear from the fact that we chose $\alpha_0 = \alpha(\sigma')$ where σ' immediately $\dashv$-precedes $\nu(\alpha)$. There are no new cases of (m).

Case 2C(iii)(a). Property (i) follows from induction, the Note after Lemma 1C.7 and the definition of α_0. Also (j), (k), (l) are clear from the construction. There are no new cases of (m).

Case 2C(iii)(b). Property (i) follows fromr Lemma 1C.7 and induction. Properties (j), (k), (l) follow from the construction as $\sigma \dashv \nu(\alpha)$, $\tau \dashv\!\!| \nu(\alpha) \rightarrow \sigma \dashv \tau$. Property (m) follows from the construction.

Case 2C(iii)(c). Argue as in Case 2A(iii) for (i), (m). Properties (j), (k), (l) follows as in Case 2C(ii).

Case 2C(iv). Just like Case 1C(iv), using Lemma 1C.6 to verify property (i).

Cases 2D(i), (ii), (iii). Property (i) follows from the construction and the choice of α_0. There are no new cases of (j)–(l). Property (m) is verified by examining the construction.

Case 2D(iv). Note that we have defined $p(\bar{c})$ to extend $p(\bar{c}^\beta)$ if $\bar{c}^\beta$ is a proper β-name, where β = ⫞-predecessor to α (if it exists). We can then verify property (i). Properties (j)–(l) present no new cases. Property (m) follows from the above remark and induction. □

We now come to the main lemma of this part.

Lemma 1C.13 (Genericity Lemma). *For any proper α-name $\bar{c}=(\bar{p}_0, (\bar{p}_1, p_1), \ldots, (\bar{p}_n, p_n))$ let $g(\bar{c})=(g_1, \ldots, g_n)$ where $g_i = p(\bar{c}(\leqslant i)) \restriction \mathrm{Dom}(p_i)$. Suppose $D \in \mathscr{D}_0(\alpha)$ consists solely of proper α-names. Then the assignment $\bar{c} \mapsto g(\bar{c})$ is $\mathscr{P}(\alpha, D)$-generic over $L_{\nu(\alpha)}$ (is $\mathscr{P}(\alpha, D)$-Σ-generic over $L_{\nu(\alpha)}$ if $\nu(\alpha)$ is admissible). Moreover, if $\beta(\nu(\alpha)) < \beta(\alpha)$ or $\alpha = \nu(\alpha) < \beta(\alpha)$, then for any $D \in \bar{\mathscr{D}}(\alpha)$ consisting solely of proper α-names: ω^{d+1} divides $\beta(\alpha)$ (where $d = \mathrm{Card}(D)) \to \mathscr{P}(\alpha, D)$ has the $(<\alpha)$-chain condition in $L_{\beta(\alpha)}$, and hence the assignment $(\bar{c} \mapsto g(\bar{c}))$ is $\mathscr{P}(\alpha, D)$-generic over $L_{\beta(\alpha)}$.*

Remark. Much as in the proof of (T a κ-tree, $T(<\kappa)$-c.c.$\to T(<\kappa)$-distributive) it can be easily shown that $\mathscr{P}(\alpha, D)$ has the $(<\alpha)$-c.c. in $L_{\beta(\alpha)} \to \mathscr{P}(\alpha, D)$ is $(<\alpha)$-distributive in $L_{\beta(\alpha)}$.

Proof. We establish the first assertion by induction on α. The different cases are examined, according to the nature of C_α, $\nu(\alpha)$.

C_α unbounded. Suppose $\nu(\alpha) > \alpha$. If $\mathscr{S} \in L_{\nu(\alpha)}$ is predense on $\mathscr{P}(\alpha, D)$ then, as $\nu(\alpha)$ is either a $<$-limit or a T_α-limit, there exists $\bar{\sigma} < \sigma \in T_\alpha$ such that $\mathscr{S} \in \mathrm{Range}(\pi_{\bar{\sigma}\sigma})$. Then by induction we can choose such a $\bar{\sigma}$ so that $\bar{c} \mapsto g(\bar{c})^{\alpha(\bar{\sigma})}$ is $\mathscr{P}(\alpha(\bar{\sigma}), \bar{D})$-generic over $L_{\bar{\sigma}}$ where if $g(\bar{c}) = (g_1, \ldots, g_n)$ then $g(\bar{c})^\eta = (g_1^\eta, \ldots, g_n^\eta)$, and where $\bar{D}$ is defined as follows: By Fact 6, for each $\bar{c} \in D$ we can choose $\bar{d}$ so that $p(\bar{c}^{\alpha(\bar{\sigma})}) = p(\bar{d})$ where for some $(q_1, \ldots, q_n) \in \mathscr{C}^*(\bar{c})$, $\mathscr{C}^*(\bar{d}) = \{\bar{q} \in \mathscr{C}^*(\bar{c}^{\alpha(\bar{\sigma})}) \mid \bar{q} \leqslant (q_1, \ldots, q_n)\}$. Let $\bar{D}$ consist of all such $\bar{d}$. Thus $\{\bar{q} \in \mathscr{P}(\alpha(\bar{\sigma}), \bar{D}) \mid \bar{q} \leqslant$ some element of $\pi_{\bar{\sigma}\sigma}^{-1}(\mathscr{S})\}$ is dense. So $(\bar{c} \mapsto g(\bar{c})^{\alpha(\bar{\sigma})})$ meets $\pi_{\bar{\sigma}\sigma}^{-1}(\mathscr{S})$ and $(\bar{c} \mapsto g(\bar{c}))$ meets $\mathscr{S}$. This proves genericity over $L_{\nu(\alpha)}$. If $\nu(\alpha) = \alpha$, then genericity over $L_\alpha = L_{\nu(\alpha)}$ is trivial.

Now to prove Σ-genericity over $L_{\nu(\alpha)}$ if $\nu(\alpha)$ is admissible: First note that we can assume that $\nu(\alpha)^* \leqslant \alpha$ as otherwise Σ-genericity reduces to genericity (since $\mathscr{P}(\alpha, D) \subseteq L_\alpha$). Moreover, if $\nu(\alpha) = \alpha$, then we can assume that $\alpha^* = \gamma$: Otherwise α is the limit of α-stable ordinals β and the Σ-genericity of $(\bar{c} \mapsto g(\bar{c}))$ follows from that of $(\bar{c} \mapsto g(\bar{c})^\beta)$ for the α-stable β (using Fact 6). If $\nu(\alpha)$ is a $<$-limit, then the Σ-genericity of $(\bar{c} \mapsto g(\bar{c}))$ reduces to that of $(\bar{c} \mapsto g(\bar{c})^{\alpha(\sigma)})$ for $\sigma < \nu(\alpha)$.

So we can assume that $\nu(\alpha)$ is not a $<$-limit and $\nu(\alpha)^* = \alpha < \nu(\alpha)$ or

$\nu(\alpha)^* = \gamma$. But then by Lemma 1C.3(b), $\nu(\alpha)$ is a $\dashv\!\vdash$-limit. We show now that our definitions from Cases 2A(iii), 2C(iii)(c), 2D(iv) imply the Σ-genericity of $(\bar{c} \mapsto g(\bar{c}))$. Let $G(\bar{c} \mapsto g(\bar{c})) \subseteq \mathcal{P}(\alpha, D)$ be the collection of conditions f extended by $(\bar{c} \mapsto g(\bar{c}))$.

First consider the case: $\nu(\alpha)$ is $\prec$-minimal. As α is the largest $\nu(\alpha)$-cardinal it suffices to show: For $e \in L_{\nu(\alpha)}$ there exists $f \in G(\bar{c} \mapsto g(\bar{c}))$ such that $f \in \mathcal{D}(W_e^*)$, where $W_e = W_e^{\nu(\alpha)} = e$th $\Sigma_1(L_{\nu(\alpha)})$-set. Pick $\bar{e} < \alpha$ so that $h_\alpha(\bar{e}) = e$, where h_α is the (canonical) $\Sigma_1(L_{\nu(\alpha)})$ partial function from α onto $L_{\nu(\alpha)}$ (with parameter $p(\nu(\alpha))$). Notice that if $\nu \dashv\!\vdash \nu(\alpha)$, $\alpha(\nu) > \bar{e}$, $f \in \mathcal{D}'(W^*_{h_{\alpha(\nu)}(\bar{e})})$, then $f \in \mathcal{D}'(W_e^*) \subseteq \mathcal{D}(W_e^*)$, using the fact that $H^{\nu(\alpha)}_{\alpha(\nu)} \simeq L_\nu$. Also by the **Fact** of Lemma 1C.3 we know that Π_1-cof$(\nu(\alpha))$ is equal to α. We can now show by induction on $\bar{e}$ that there exists $\nu \dashv\!\vdash \nu(\alpha)$ so that $f_\nu = (\bar{c} \mapsto g(\bar{c}^{\alpha(\nu)})) \in \mathcal{D}(W_e^*)$, where $e = h_\alpha(\bar{e})$. Indeed, as $\langle f_\nu \mid \nu \dashv\!\vdash \nu(\alpha) \rangle$ is $\Pi_1(L_{\nu(\alpha)})$ we can choose $\nu \dashv\!\vdash \nu(\alpha)$ sufficiently large so that either $f_\nu \leqslant$ some element of $\mathcal{D}(W^*_{h_{\alpha(\nu)}(\bar{e})})$ or $\bar{e}$ is the least $\alpha(\nu)$-code of an alive pair. In the latter case, either (a) $(W_e^*)_\delta$ is dense below f_ν for all $\delta < \alpha$, in which case by the admissibility of $\nu(\alpha)$, $(W)_\delta$ is dense below f_ν for all $\delta < \alpha$ for some $W \in L_{\nu(\alpha)}$, $W \subseteq W_e^*$. As $L_\nu \simeq H^{\nu(\alpha)}_{\alpha(\nu)}$, the same is true for $W^*_{h_{\alpha(\nu)}(\bar{e})}$ and thus by construction $f_\nu \leqslant$ some element of $\mathcal{D}'(W^*_{h_{\alpha(\nu)}(\bar{e})})$; or (b) there is a least $\nu' \dashv\!\vdash' \nu(\alpha)$, $\nu' \geqslant \nu$ so that for some $f' \leqslant f$ in $L_{\nu'}$, $\delta < \alpha(\nu')$ we have $g \leqslant f' \to g \notin (W^*_{h_{\alpha(\nu')}(\bar{e})})_\delta$. But then by construction $f_{\nu'} \in \mathcal{D}(W_e^*)$ and ν' is active. So $\nu' \dashv\!\vdash \nu(\alpha)$ and $G(\bar{c} \mapsto g(\bar{c}))$ meets $\mathcal{D}(W_e^*)$.

The case of $\nu(\alpha)$ a $\prec$-successor is exactly the same, using the construction defined in Case 2C(iii)(c).

Finally suppose $\nu(\alpha) = \alpha$. We must show that if $h_\alpha(\bar{e}) = (D, e)$, then there exists $\beta \dashv\!\vdash \alpha$ so that $f_\beta = (\bar{c} \mapsto g(\bar{c}^\beta)) \in \mathcal{D}(W_e^*)$, where $W_e = e$th $\Sigma_1(L_\alpha)$-set. First we show this when γ is L_α-regular. In this case Π_1-cof$(\alpha) = \gamma$ and hence the function $\bar{e}' \mapsto$ least $\beta \in C_\alpha$ so that $h_\beta(\bar{e}')$ is not alive (at stage β) is bounded on $\{\bar{e}' \mid \bar{e}' \leqslant \bar{e}\}$, provided we show that it is total. To see this, by induction choose $\beta_0 \in C_\alpha$ so that $h_\beta(\bar{e}')$ is not alive at stage β_0 for all $\bar{e}' < \bar{e}$ and argue as follows: If $h_\beta(\bar{e})$ is not of the form (D_β, e_β) where D_β is a member of $\bar{\mathcal{D}}_0(\beta)$, then of course we are done. Otherwise, (D_β, e_β) is alive at only boundedly many stages β as the function $(\delta \mapsto$ least $\beta \in C_\alpha$, $\beta > \beta_0$ so that $f_\beta \leqslant$ some g, $(g, \delta) \in W^*_{h_\beta(\bar{e})})$ is $\Pi_1(L_\alpha)$ and hence either bounded (if its Domain is not all of γ) or has constant value $\beta_1 =$ least element of C_α greater than β_0 (if f_{β_0} has an extension g so that $\{g\} \times \gamma \subseteq W^*_{h_\alpha(\bar{e})}$; this uses the fact that $\beta_1 = \sup(\Sigma_1$-Skolem hull of $\{\beta_0, p(\alpha)\}$ inside L_α)).

Now if $\beta \in C_\alpha$ is the least stage in C_α at which $h_\beta(\bar{e})$ is not alive, then either $\{f_\beta\} \times \gamma \subseteq W_e^*$ or $h_\beta(\bar{e})$ is active at only boundedly many stages $< \beta$. Let $\beta_0 = C_\alpha$-predecessor to β. As $\beta = \sup(\Sigma_1$-Skolem hull of $\{\beta_0, p(\alpha)\}$ inside L_α) it must be that $g \leqslant f_\beta \to (g, \delta) \notin W_e^*$ where δ is least so that $(f_\beta, \delta) \notin W_e^*$. So $f_\beta \in \mathcal{D}(W_e^*)$ and we are done.

When γ is L_α-singular and $\langle \gamma_i \mid i < \kappa \rangle$ is the $L_\alpha$least continuous, increasing cofinal sequence below γ, $\kappa = L_\alpha$-cof(γ), then we argue that $(i \mapsto$ least $\beta \in C_\alpha$ so that $h_\beta(\bar{e}')$ is not alive at stage β for all $\bar{e}' < \gamma_i$) is total and bounded on any $i_0 < \kappa$. To see this we use the fact that the partial function $(\bar{e}, \delta) \mapsto$ (least $\beta \in C_\alpha$ so that $(g, \delta) \in W^*_{h_\beta(\bar{e}')}$) restricted to $\{(\bar{e}', \delta) \mid \bar{e}' < \gamma_i, \delta < \bar{e}'\}$, is not cofinal in any $\beta \in C_\alpha$ of L_α-cofinality γ_i^+. (The condition $f_{\beta,\bar{e}'}$ is defined in $\mathcal{P}(\alpha, D_{\bar{e}'})$ as f_β is defined in $\mathcal{P}(\alpha, D)$, where $h_\alpha(\bar{e}') = (D_{\bar{e}'}, e)$.) Such β's exist (for example let $\beta = \bigcup(\Sigma_1$-Skolem hull of $\gamma_i^+ \cup \{p(\alpha)\}$ in L_α)) and the above function is L_β-definable when intersected with L_β. Now we argue for the desired result by induction on i_0. As $\Pi_1(L_\alpha)$-cof$(\kappa) = \kappa$ there must be a stage $\beta \in C_\alpha$ so that L_α-cof$(\beta) \geq \gamma_{i_0}^+$ and $i_0 \leq$ least i so that $h_\beta(\bar{e}')$ is alive at stage β for some $\bar{e}' < \gamma_i$. But then $i_0 <$ least i so that $h_\beta(\bar{e}')$ is alive at stage β for some $\bar{e}' < \gamma_i$, since L_α-cof$(\beta) > \gamma_{i_0}$.

Now for any $i < \kappa$ pick $\beta \in C_\alpha$ to be least so that $h_\beta(\bar{e}')$ is not active at stage β for all $\bar{e}' < \gamma_i$. If $\beta' = C_\alpha$-predecessor of β, then $\beta = \sup((\Sigma_1$-Skolem hull $\{\beta', p(\alpha)\}$ in $L_\alpha) \cap \mathrm{ORD})$ and this Skolem hull contains the parameter i. Thus for any $\bar{e}' < \gamma_i$ we must have that either $h_\beta(\bar{e}')$ is not of the form (D, e) or $\{f_\beta\} \times \bar{e}' \subseteq W^*_{h_\alpha(\bar{e}')}$ or for some $\delta < \bar{e}'$, $(g, \delta) \notin W^*_{h_\alpha(\bar{e}')}$ for all $g \leq f_\beta$. Thus $G(\bar{c} \mapsto g(\bar{c}))$ meets $\mathcal{D}(W^*_e)$ for all $e \in L_\alpha$.

C_α bounded. By the first argument in the 'C_α unbounded' case the genericity of $(\bar{c} \mapsto g(\bar{c}))$ over $L_{\nu(\alpha)}$ follows easily if $\nu(\alpha)$ is either a T_α-limit a $\prec$-limit or if $\nu(\alpha) = \alpha$. By Lemma 1C.3(b) if $\nu(\alpha)$ is admissible and projectible, then $\nu(\alpha)$ is a $\dashv\!\vdash$-limit and we can apply the argument in the 'C_α unbounded' case to establish the Σ-genericity of $(\bar{c} \mapsto g(\bar{c}))$. If $\nu(\alpha)$ is admissible and nonprojectible, then either Σ-genericity reduces to ordinary genericity (if $\nu(\alpha) > \alpha$) or Σ-genericity can be established using the fact that $\nu(\alpha) = \alpha$ is a limit of α-stable ordinals.

So it remains to establish genericity of $(\bar{c} \mapsto g(\bar{c}))$ over $L_{\nu(\alpha)}$ when $\nu(\alpha)$ is either T_α-minimal or a T_α-successor and either $\prec$-minimal or a $\prec$-successor (and, we must establish the last assertion of the lemma). We shall deal only with the case ($\nu(\alpha)$ a T_α-successor, $\prec$-successor) as the other cases are handled in almost exactly the same way.

Suppose $\mathcal{S} \in L_{\nu(\alpha)}$ is predense on $\mathcal{P}(\alpha, D)$. Let $\alpha_0 < \alpha_1 < \cdots$ be as defined in Case 2C(ii) and let $d = \mathrm{Card}(D)$. We can choose i so large that for $j \geq i : \omega^{d+1}$ divides $\beta(\alpha_j)$, $\mathcal{S} \cap L_{\alpha_j} \in L_{\beta(\alpha_j)}$ and for each $\bar{c} \in D$, $\bar{c}_j$ is defined where $\bar{c}_0, \bar{c}_1, \ldots$ is obtained by extending $\bar{c}$ along the α_j's. Let $\bar{D} = \{\bar{c}_i \mid \bar{c} \in D\}$. It suffices to show that $(\bar{c}_i \mapsto g(\bar{c}_i))$ is $\mathcal{P}(\alpha_{i+1}, \bar{D})$-generic over $L_{\beta(\alpha_{i+1})}$, as $g(\bar{c}) \leq g(\bar{c}_i)$. But this follows by induction from the last assertion of this Lemma.

This last assertion is proved by induction on d. Suppose α is as given in that assertion. We can assume that C_α is bounded in α, as otherwise the claim follows easily by induction. The proof of Lemma 1C.1 shows that $E \cap \alpha$ is stationary in $L_{\beta(\alpha)}$ and that $\langle D_\delta \mid \delta \in E \cap \alpha \rangle$ is a $\diamondsuit(E)$-sequence for $L_{\beta(\alpha)}$. Now suppose that $\mathcal{S} \in L_{\beta(\alpha)}$ is predense on $\mathcal{P}(\alpha, D)$. The hypothesis that ω^{d+1} divides $\beta(\alpha)$ implies that for unboundedly many $\delta \in E \cap \alpha$, $D_\delta \subseteq \delta$ codes $(L_{\beta(\delta)}, \bar{D}, \bar{\mathcal{S}})$ where ω^d divides $\beta(\delta)$, $\bar{D} = \{\bar{c}^\delta \mid \bar{c} \in D\}$, $\bar{\mathcal{S}} = \{\bar{f} \mid$ for some $f \in \mathcal{S} \bar{f}(\bar{c}^\delta) = f(\bar{c}) \in L_\delta$ for all $\bar{c} \in D\}$ is predense on $\mathcal{P}(\delta, \bar{D})$. We can also assume that $\sup C_\delta = \delta_0$ is large enough so that $\bar{c}^{\delta_0}$ is a δ_0-name for all $\bar{c} \in D$. Then by construction (see Case 2B(ii)) if for each $\bar{c} \in D$ we let $\bar{c}' \in \bar{J}(\delta)$ be defined (via Fact 6) by $p(\bar{c}') = p(\bar{c})^{\delta}$, then we have that $(\bar{c}^{\delta} \mapsto g(\bar{c}'))$ meets $\bar{\mathcal{S}}$ and hence $(\bar{c} \mapsto g(\bar{c}))$ meets $\mathcal{S}$. Note that the construction in Case 2B(ii) makes use of this lemma, inductively. We have established the desired genericity for $(\bar{c} \mapsto g(\bar{c}))$. But note that (using the definition of $\hat{\mathscr{C}}^{\bar{p}}_{Y}$) the above argument shows that $f \leqslant$ some element f' of $\mathcal{S}$, $f'(\bar{c}) \in L_{\delta}$ for all $\bar{c} \in D$, for *any* $f \in \mathcal{P}(\alpha, D)$ such that $|f(\bar{c})| \geqslant \delta$ for all $\bar{c} \in D$. Thus if $\mathcal{S}$ were an antichain in $\mathcal{P}(\alpha, D)$ we have shown that $f \in \mathcal{S} \rightarrow |f(\bar{c})| < \delta$ for all $\bar{c} \in D$ and thus $\mathcal{S}$ has cardinality $< \alpha$ in $L_{\beta(\alpha)}$. $\square$

Now using the Canonical α-Names Lemma 1C.11 and the Genericity Lemma 1C.13 we can establish property (h). First we need the following:

Fact. *Suppose that $\check{\mathscr{C}}(\bar{c})$ and $\check{P}(\alpha, D)$ are defined as were $\mathscr{C}(\bar{c})$, $P(\alpha, D)$ but with $\hat{\mathscr{C}}^{\bar{p}_{i+1}}_{Y_{i+1}}$ replaced by $\mathscr{C}^{\bar{p}_{i+1}}_{Y_{i+1}}$. Then the Genericity Lemma holds for $\check{P}(\alpha, D)$.*

Proof. It suffices to show that if $\bar{c} = (p_0, (\bar{p}_1, p_1), \ldots, (\bar{p}_n, p_n))$ is a proper α-name, then there exists $(q_1, \ldots, q_n) \in \mathscr{C}^*(\bar{c})$ so that $(r_1, \ldots, r_n) \in \check{\mathscr{C}}^*(\bar{c})$, $(r_1, \ldots, r_n) \leqslant (q_1, \ldots, q_n) \rightarrow (r_1, \ldots, r_n) \in \mathscr{C}^*(\bar{c})$. This is clear if $\sigma(\bar{c})$ is not a $<$-limit, and we can assume that $n = 1$. Let $\sigma' = \min(\mathrm{Dom}(p_1))$. Let $\tau_0 < \sigma(\bar{c})$, $\bar{c}^{\alpha(\tau_0)}$ an $\alpha(\tau_0)$-name, $\alpha_0 = \alpha(\tau_0)$ and choose $q_1 \leqslant p_1$ in $\hat{\mathscr{C}}^{\bar{p}_1}_{Y_1}$ so that $q_1 = p(\bar{d}^{\alpha_0}) \restriction Y_1$ where $\bar{d}^{\alpha_0} = (p_0^{\alpha_0}, (\bar{s}_1, s_1), (\bar{p}_1, p_1))$ and $\bar{s}_1 \neq p_0^{\alpha_0}$. Now suppose $r_1 \leqslant q_1$, $r_1 \notin \mathscr{C}^*(\bar{c}) = \mathscr{C}(\bar{c})$ and thus $r_1(\sigma') = b_{\tau_1'}$ for some $\tau_1' \in T_{\beta}$, $\beta = |r_1|$. Let $\tau_1 \in T_{\beta}$, $\bar{p}_1(\sigma) = b_{\pi_{\sigma\tau_1}}(\bar{\sigma})$ for $\bar{\sigma} \in \mathrm{Dom}(\bar{p}_1)$ and $\sigma = \bigcup \mathrm{Dom}(\bar{p}_1) < \tau_1$. Then by Fact 6 (assuming without loss of generality that $\sigma' <_* \tau_1'$) we have that $r_1(\sigma') \restriction \alpha_0 = b_{\tau_1'} \restriction \alpha_0 = p((\pi_{\tau_1'}, (\pi_{\tau_1'}, q_{Y_{\tau_1'}})))(\sigma')^{\alpha} = p((\pi^{\alpha_0}_{\tau_1'}, (\pi^{\alpha_0}_{\tau_1'}, q)))$ where $\pi_{\tau'}$ is as in Case 1C(iii) and $q \in \mathscr{C}((\pi_{\tau_1'}, (\pi_{\tau_1'}, q_{Y_{\tau_1'}})))$. But $r_1(\sigma') \restriction \alpha_0 = q_1(\sigma') = p(\bar{d}^{\alpha_0})(\sigma')$ and this contradicts the Canonical α-Names Lemma 1C.11.

Lemma 1C.14. *Property* (h) *holds.*

Proof. By induction on α, and for fixed α by induction on length $(\bar{d})$ where $\bar{d}$ is the canonical proper α-name associated to $\bar{p} \cup \bar{q}$. If length$(\bar{d}) = 0$, then $\bar{p} \cup \bar{q}$ is standard so $X_0(\bar{p} \cup \bar{q}) = \bar{X} \cup \bar{Y}$ and there is nothing to prove. We will assume that $v(\alpha)$ is not p-admissible, as otherwise the argument is the same with 'Σ-generic' replacing 'generic'.

First assume that $\bar{p} \cup \bar{q} = p(\bar{d})$ and $\sigma(p) < v(\alpha)$. Let $\bar{\sigma} = \min \bar{Y}$ and $n =$ length$(\bar{d})$. There are three possibilities. First, it may be that $\bar{\sigma} = \bar{\sigma}_0$, where $\bar{\sigma}_0 =$ least σ such that $(\bar{p} \cup \bar{q})(\sigma) \notin \mathrm{Range}(p(\bar{d}(\leqslant n-1))$. Then by the Genericity Lemma, $G(\bar{q})$ is $\mathscr{C}^{\bar{p}}_{Y}$-generic over $L_{v(\alpha)}(p(d) - p(\bar{c}), p(\bar{c}))$ where d is the canonical α-name associated to p and $\bar{c}$ is the longest common initial segment of $\bar{d}$ and d. (Indeed, apply the Genericity Lemma and the Fact above to

$D = \{e \mid e < d \text{ or } e < \bar{d}\}$. Note that $\bar{c} < \bar{d}(\leqslant n-1)$ as $\mathrm{Range}(\bar{q}) \cap \mathrm{Range}(p) = \emptyset$.) Thus we get that $G(\bar{q})$ is $\mathscr{C}^{\bar{p}}_{\bar{Y}}$-generic over $L_{\nu(\alpha)}(p(d)) \supseteq L_{\nu(\alpha)}(p)$.

Second, it may be that $\bar{\sigma} > \bar{\sigma}_0$. Then $\bar{p}$ must be trivial thinning of p (due to the definition of $\hat{\mathscr{C}}^{\bar{p}}_{\bar{Y}}$) and it suffices to show that $G(\bar{q})$ is $\mathscr{C}^{\bar{p}}_{\bar{Y}}$-generic over $L_{\nu(\alpha)}(\bar{p})$. But by the Genericity Lemma we have that $G((\bar{p} \cup \bar{q}) \upharpoonright (\bar{X} \cup \bar{Y} - \bar{\sigma}_0))$ is $\mathscr{C}^{\bar{p} \upharpoonright \bar{\sigma}_0}_{\bar{X} \cup \bar{Y} - \bar{\sigma}_0}$-generic over $L_{\nu(\alpha)}(\bar{p} \upharpoonright \bar{\sigma}_0)$ and this implies that $G(\bar{p} \upharpoonright (X - \bar{\sigma}))$ is $\mathscr{C}^{\bar{p} \upharpoonright \bar{\sigma}_0}_{\bar{X} - \bar{\sigma}_0}$-generic over $L_{\nu(\alpha)}(\bar{p} \upharpoonright \bar{\sigma}_0)$, $G(\bar{q})$ is $\mathscr{C}^{\bar{p}}_{\bar{Y}}$-generic over $L_{\nu(\alpha)}(\bar{p} \upharpoonright \bar{\sigma}_0,$ $\bar{p} \upharpoonright (\bar{X} - \bar{\sigma}_0))$ (we are writing $\mathscr{C}_{\bar{X} \cup \bar{Y}}$ as a two-step iteration). So we are done.

Third, it may be that $\bar{\sigma} < \bar{\sigma}_0$. By induction (on $\mathrm{length}(\bar{d}) = n$) we know that $G(\bar{q} \upharpoonright \bar{\sigma}_0)$ is $\mathscr{C}^{\bar{p}}_{\bar{Y} \cap \bar{\sigma}_0}$-generic over $L_{\nu(\alpha)}(p)$ and, as in the first case, by the Genericity Lemma we get that $G(\bar{q} \upharpoonright Y - \bar{\sigma}_0)$ is $\mathscr{C}^{\bar{p} \cup \bar{q} \upharpoonright \bar{\sigma}_0}_{\bar{Y} - \bar{\sigma}_0}$-generic over $L_{\nu(\alpha)}(p, \bar{q} \upharpoonright \bar{\sigma}_0)$. Putting this together we have that $G(\bar{q})$ is $\mathscr{C}^{\bar{p}}_{\bar{Y}}$-generic over $L_{\nu(\alpha)}(p)$.

Now if $\bar{p} \cup \bar{q} = p(\bar{d})$ but $\sigma(p) = \nu(\alpha)$ we must use induction on α. For $\eta < \nu(\alpha)$ let $\sigma_\eta =$ least $\sigma \in \mathrm{Dom}(p)$ such that $p(\sigma) \notin \{b_\tau \mid \tau \in T_\alpha \cap \eta\}$ and $p_\eta = p \upharpoonright (\mathrm{Dom}(p) - \sigma_\eta)$. It suffices to show that for unboundedly many $\eta < \nu(\alpha)$, $G(\bar{q})$ is $\mathscr{C}^{\bar{p}}_{\bar{Y}}$-generic over $L_\eta(p_\eta)$. Now also let $\sigma'_\eta = \sigma_\eta$ if $\mathrm{Range}(\bar{p}) \subseteq \mathrm{Range}(p \upharpoonright \sigma_\eta)$ and $\sigma'_\eta =$ least σ such that $\mathrm{Range}(\bar{p}) \subseteq \mathrm{Range}(p \upharpoonright \sigma)$, otherwise. If we set $p'_\eta = p \upharpoonright (\mathrm{Dom}(p) - \sigma'_\eta)$, then it suffices (by the product lemma) to show that $G(p'_\eta)$ is $\mathscr{C}^{p \upharpoonright \sigma'_\eta}_{\mathrm{Dom}(p) - \sigma'_\eta}$-generic over $L_\eta(\bar{p} \cup \bar{q})$, for unboundedly many $\eta < \nu(\alpha)$. First suppose that $\nu(\alpha)$ is a T_α-limit, so we need only establish the preceding for T_α-successors $\eta < \nu(\alpha)$. Choose a $<$-successor $\tau < \eta$; then by induction (on α) we have that $G((p'_\eta)^{\alpha(\tau)})$ is $\mathscr{C}^{p^{\alpha(\tau)} \upharpoonright \sigma'_\eta}_{\mathrm{Dom}(p'_\eta)}$-generic over $L_\tau((\bar{p} \cup \bar{q})^{\alpha(\tau)})$. But as η is a T_α-successor we have that $(\bar{p} \cup \bar{q})^{\alpha(\tau)} = (\bar{p}_0)^{\alpha(\tau)} \cup (\bar{p}_1 \cup \bar{q})^{\alpha(\tau)}$ where $(\bar{p}_0)^{\alpha(\tau)} = p^{\alpha(\tau)} \upharpoonright \{\sigma \mid p^{\alpha(\tau)}(\sigma) \in L_\tau\}$ belongs to L_τ. Note that $G((\bar{p}_1 \cup \bar{q})^{\alpha(\tau)})$ is $\mathscr{C}^{(\bar{p}_0)^{\alpha(\tau)}}_{\mathrm{Dom}(\bar{p}_1 \cup \bar{q})}$-generic over L_τ, so $G((p'_\eta)^{\alpha(\tau)}) \times G((\bar{p}_1 \cup \bar{q})^{\alpha(\tau)})$ is $\mathscr{C}^{p^{\alpha(\tau)} \upharpoonright \sigma'_\eta}_{\mathrm{Dom}(p'_\eta)} \times \mathscr{C}^{(\bar{p}_0)^{\alpha(\tau)}}_{\mathrm{Dom}(\bar{p}_1 \cup \bar{q})}$-generic over L_τ. As $\pi_{\tau\eta}$ is Σ_1-elementary for each $\tau < \eta$ we get that $G(p'_\eta) \times G(\bar{p}_1 \cup \bar{q})$ is $\mathscr{C}^{p \upharpoonright \sigma'_\eta}_{\mathrm{Dom}(p'_\eta)} - \mathscr{C}^{\bar{p}_0}_{\mathrm{Dom}(\bar{p}_1 \cup \bar{q})}$-generic over L_η and hence $G(p'_\eta)$ is $\mathscr{C}^{p \upharpoonright \sigma'_\eta}_{\mathrm{Dom}(p'_\eta)}$-generic over $L_\eta(G(\bar{p}_1 \cup \bar{q})) = L_\eta(\bar{p} \cup \bar{q})$, as desired. Now, if $\nu(\alpha)$ is a T_α-successor or T_α-minimal, use the last statement of the Genericity Lemma to establish (inductively) that for $\eta \in (\nu', \nu(\alpha))$, $\nu' = T_\alpha$-predecessor to $\nu(\alpha)(= \alpha$ if $\nu(\alpha)$ is T_α-minimal) we have: $G(p'_\eta)$ is $\mathscr{C}^{p \upharpoonright \sigma'_\eta}_{\mathrm{Dom}(p) - \sigma'_\eta}$-generic over $L_\eta(\bar{p} \cup \bar{q})$. For Σ-genericity we can reflect along $\{\tau \mid \tau < \nu(\alpha)\}$ if $\nu(\alpha)$ is a $<$-limit and otherwise along $\{\tau \mid \tau \dashv\!\!| \, \nu(\alpha)\}$ (as Σ-genericity reduces to genericity unless $\nu(\alpha)^* = \alpha$).

Finally suppose $\bar{p} \cup \bar{q} \neq p(\bar{d})$. We can assume that $\bar{p} \cup \bar{q} = p(\bar{d}) \circ \pi$ where π is a morass map different from the identity, as otherwise the genericity of $G(\bar{q})$ follows from that of $G(p(\bar{d}) - \bar{p})$. Let $\bar{\sigma} = \min(\mathrm{Dom}(\bar{q}))$ and $\sigma = \pi(\bar{\sigma})$. If $X = \mathrm{Dom}(p)$, $|X| \geqslant \alpha(\sigma)$, then by the preceding two paragraphs we know that $G(p(\bar{d}) \upharpoonright (\mathrm{Dom}(p(\bar{d})) - \sigma))$ is $\mathscr{C}^{p(\bar{d}) \upharpoonright \sigma}_{\mathrm{Dom}(p(\bar{d})) - \sigma}$-generic over $L_{\nu(\alpha)}(p)$. If $|X| < \alpha(\sigma)$, then if we set $\tau_0 =$ least $\tau \in \mathrm{Dom}(p)$ such that $\mathrm{Range}(\bar{p}) \subseteq \mathrm{Range}(p \upharpoonright \tau)$, either τ_0 is not defined or the hypothesis of the lemma holds with $\bar{p}, \bar{q}, p$ replaced by $p \upharpoonright \tau_0$, $(p - p \upharpoonright \tau_0)$, $p(\bar{d})$. In either case the arguments of the preceding two paragraphs show that $G(p(\bar{d}) \restriction (\mathrm{Dom}(p(\bar{d})) - \sigma))$ is $\mathscr{C}^{p(\bar{d}) \restriction \sigma}_{\mathrm{Dom}(p(\bar{d}))-\sigma}$-generic over $L_{\nu(\alpha)}(p)$. Thus as $\bar{q}$ is a thinning of $p(\bar{d}) - \sigma$ (via π) we need only show: If $r \in \mathscr{C}^{p(\bar{d}) \restriction \sigma}_{\mathrm{Dom}(p(\bar{d}))-\sigma} = \mathscr{C}$ and if $D \in L_{\nu(\alpha)}(p)$ is dense on $\mathscr{C}^{\bar{p}}_{Y}$, then there exists $r' \leqslant r$ in $\mathscr{C}$ such that $r' \circ \pi$ meets D.

Let $\sigma_0 =$ least σ such that $\mathrm{Range}(\bar{p}) \subseteq \mathrm{Range}(p(\bar{d}) \restriction \sigma)$. If $\sigma_0 = \sigma$, then clearly r' as above must exist, for by the density of D we can choose $\bar{r} \leqslant r \circ \pi$ to meet D and then there exists $r' \in \mathscr{C}$, $r' \circ \pi = \bar{r}$. If $\sigma_0 < \sigma$, then argue as follows: If r' does not exist, then by the $\mathscr{C}^{p(\bar{d}) \restriction \sigma_0}_{\mathrm{Dom}(p(\bar{d})) \cap [\sigma_0, \sigma)}$-genericity of $p(\bar{d}) \restriction [\sigma_0, \sigma)$ over $L_{\nu(\alpha)}(p)$, there must be $s \in G(p(\bar{d}) \restriction [\sigma_0, \sigma))$ such that $s \Vdash$ there is no $r' \leqslant r$ in $\mathscr{C}^{(p(\bar{d}) \restriction \sigma_0) \cup G}_{\mathrm{Dom}(p(\bar{d}))-\sigma}$ such that $r' \circ \pi$ meets D. We can assume that $|r| = |s|$. But by the density of D we can choose $s' \cup r' \leqslant s \cup r$ in $\mathscr{C}^{p(\bar{d}) \restriction \sigma_0}_{\mathrm{Dom}(p(\bar{d}))-\sigma_0}$ so that $s' \leqslant s$ in $\mathscr{C}^{p(\bar{d}) \restriction \sigma_0}_{\mathrm{Dom}(p(\bar{d})) \cap [\sigma_0, \sigma)}$ and $r' \circ \pi$ meets D; this contradicts the choice of s. □

Remark. The last part of the preceding proof is very similar to the argument in Lemma 1B.11(d). Of course we can now carry out that argument successfully by establishing 'enough' of Lemma 1B.12. As we remarked earlier, it is precisely this argument that necessitates our construction of the super-generic codes.

To complete our study of supergenericity we establish (g). The argument is very similar to that used in the preceding proof.

Lemma 1C.15. *Suppose $p \in \mathscr{C}^*_X$, $p \cup q \in \mathscr{C}^*_{X \cup Y}$, where $X, X \cup Y \in I_\alpha$, $|p| = |q| = |p \cup q| = \alpha \in U(\gamma)$ and if $\sigma = \min(Y)$, then $q(\sigma) = b_\nu$ for some $\nu \in T_\alpha$. Then $G(q)$ is $\mathscr{C}^p_Y$-generic over $L_\nu(p)$ (is $\mathscr{C}^p_Y$ Σ-generic over $L_\nu(p)$ if ν is p-admissible).*

Proof. By induction on α. First assume that ν is not p-admissible. Suppose $\eta \in T_\alpha \cap \nu$ is a T_α-successor and chose a $<$-successor $\tau < \eta$. By induction we know that $G((p_1 \cup q)^{\alpha(\tau)})$ is $\mathscr{C}^{p_0^{\alpha(\tau)}}_{Y_1}$-generic over $L_\tau(p_0^{\alpha(\tau)})$, where $Y_1 = \{i \in X \cup Y \mid (p \cup q)(i) \notin L_\eta\}$ and $p_0 = p \restriction (X - Y_1)$, $p_1 = p \restriction Y_1$. As η is a T_α-successor we have that τ is a $T_{\alpha(\tau)}$-successor and so $p_0^{\alpha(\tau)} \in L_\tau$. As $\pi_{\tau\eta}$ is Σ_1-elementary for such τ we get that $G(p_1 \cup q)$ is $\mathscr{C}^{p_0}_{Y_1}$-generic over $L_\eta = L_\eta(p_0)$. Thus $G(q)$ is $\mathscr{C}^p_Y$-generic over $L_\eta(p)$, by the product lemma (and induction). So we have shown that if ν is a T_α-limit, then $G(q)$ is $\mathscr{C}^p_Y$-generic over $L_\nu(p)$, as desired. If ν is a T_α-successor or T_α-minimal, use the last statement of the Genericity Lemma to inductively establish that for $\eta \in (\nu', \nu)$, $\nu' = T_\alpha$-predecessor to ν ($= \alpha$ if ν is T_α-minimal) we have: $G(q)$ is $\mathscr{C}^p_Y$-generic over $L_\eta(p)$.

For Σ-genericity note that we have $\mathrm{Range}(p) \in L_\nu$. Thus we can assume that $\nu^* = \alpha$ as otherwise Σ-genericity reduces to genericity. If ν is a $<$-limit, then we can reflect along $\{\tau \mid \tau < \nu\}$ and apply induction. Otherwise use the relation $\dashv\!\vdash$ (as in the proof of the Genericity Lemma) to see that all appropriate dense sets $\mathscr{D}(W_e)$ are met. □

The proof of Lemma 1C.15 really shows a bit more than property (h). The following will be useful in Part D.

Corollary 1C.16. *Suppose $\bar{p} \in \mathscr{C}^*_{\bar{X}}$, $\bar{p} \cup \bar{q} \in \mathscr{C}^*_{\bar{X} \cup \bar{Y}}$, $|\bar{p}| = |\bar{q}| = |\bar{p} \cup \bar{q}| = \alpha \in U(\gamma)$ and $\bar{q}(\bar{\sigma}) = b_\nu$ where $\nu \in T_\alpha$, $\bar{\sigma} = \min \bar{Y}$. Also suppose $\bar{p}$ is a thinning of p and* $\mathrm{Range}(\bar{q}) \cap \mathrm{Range}(p) = \emptyset$. *Then $G(\bar{q})$ is $\mathscr{C}^{\bar{p}}_{\bar{Y}}$-generic over $L_\nu(p)$ (is $\mathscr{C}^{\bar{p}}_{\bar{Y}}$-$\Sigma$-generic over $L_\nu(p)$ if ν is p-admissible).*

Relativization

We have defined supergeneric codes b_σ for $\sigma \in T_\alpha$, $\alpha \in U(\gamma)$. However the definition of R^s in Part D requires that we relativize the above construction so as to define codes b_s for $s \in S_\alpha$, $\alpha \in U(\gamma)$. The collection of strings S_α is defined in Part D, in analogy with our earlier definition of S^{β}_{κ}. In particular each $s \in S_\alpha$ obeys the strict definability condition: $\xi \leqslant |s| \to s \restriction \xi$ is $\Delta_1\langle L_{\mu^0_\xi}, C'_{\mu^0_\xi}\rangle$.

This last condition allows us to define a 'quasi-morass' (see [1, p. 247] as follows: $\bar{s} \leqslant s$ iff $\bar{s} \in S_{\alpha(\bar{s})}$, $s \in S_{\alpha(s)}$ and there exists $g : \mu^0_{\bar{s}} \Rightarrow \mu^0_s$ such that $g \restriction \alpha(\bar{s}) = \mathrm{id} \restriction \alpha(\bar{s})$, $g \restriction \langle L_{\mu^0_{\bar{s}}}, \bar{s} \restriction \mu^0_{\bar{s}}\rangle$ is a Q-embedding into $\langle L_{\mu^0_s}, s \restriction \mu^0_s\rangle$.

Now repeat the construction of the supergeneric codes, using quasi-morass maps rather than morass maps. Thus for $s \in S_\alpha$ one defines b_s and $\mathscr{C}^{\alpha}_{X,s}$ for $X \in I_\alpha = \{X \mid \text{for some } \bar{s} \in S_{\bar{\alpha}},\ \bar{\alpha} \in U(\gamma) \cap \alpha,\ X = X(\leqslant \bar{s}) \text{ or } X = X(< \bar{s})\}$ where $X(\leqslant \bar{s}) =$ all initial segments of $\bar{s}$ in $S_{\alpha(\bar{s})}$, $X(<\bar{s}) =$ all proper initial segments of $\bar{s}$ in $S_{\alpha(\bar{s})}$ and considers *α, s-conditions,* defined like α-conditions but with the requirement that $u \in Y_0 = \mathrm{Dom}(p_0) \to p_0(u) = b_{\pi(u)}$ where $\pi : \bar{s} < s'$ is a quasi-morass map, $\bar{s} \in S_{|Y_0|}$, $s' \subseteq s$. Then we obtain the following genericity property analogous to property (h): Suppose $\bar{p} \in \mathscr{C}^*_{\bar{X},s}$, $\bar{p} \cup \bar{q} \in \mathscr{C}^*_{\bar{X} \cup \bar{Y},s}$, $X_0(\bar{p} \cup \bar{q}) \subseteq \bar{X}$ and $|\bar{p}| = |\bar{q}| = |\bar{p} \cup \bar{q}| = \alpha \in U(\gamma)$. If $\bar{p}$ is a thinning of p, $\mathrm{Range}(\bar{q}) \cap \mathrm{Range}(p) = \emptyset$, then $G(\bar{q})$ is $\mathscr{C}^{\bar{p}}_{\bar{Y},s}$-generic over $L_{\nu(\alpha)}(p)$ (is $\mathscr{C}^{\bar{p}}_{\bar{Y},s}$-$\Sigma$-generic over $\langle L_{\nu(\alpha)}(p), s \restriction \nu(\alpha)\rangle$ if $\nu(\alpha)$ is p, s-admissible).

D. Successor cardinal coding III: The forcing R^s

In this part we define the forcing R^s and discuss its basic properties. This forcing is a type of almost disjoint forcing where supergeneric codes as constructed in Part C are used.

Fix $\beta \in \overline{\mathrm{Adm}}$ and let γ be a β-cardinal less than $\mathrm{gc}\,\beta$ (= greatest β-cardinal if exists, $= \beta$ otherwise). We use γ^+ to denote $(\gamma^+)^{L_\beta}$ and fix $s \in S^{\beta}_{\gamma^+}$. The forcing R^s is designed to code s into a subset of γ^+.

We will have use of a canonical procedure $b \mapsto S(b)$ of converting distinct subsets b_1, b_2 of γ^+ into almost disjoint subsets $S(b_1)$, $S(b_2)$ of γ^+ (we say that $c_1, c_2 \subseteq \gamma^+$ are *almost disjoint* if $c_1 \cap c_2$ is bounded in γ^+). Let $x \mapsto x^*$ be a canonical injection of $2^{<\gamma^+}$ into γ^+ (say $x \mapsto <_L\text{-rank}(x)$) and for $b \subseteq \gamma^+$, $\eta < \gamma^+$ we let $b \restriction \eta$ denote the element of 2^η defined by $(b \restriction \eta)(\xi) = 1$ iff $\xi \in b \cap \eta$. Then $S(b) = \{\langle 0, (b \restriction \eta)^*\rangle \mid \eta < \gamma^+, \eta \in b\}$. The trick of adding the clause "$\eta \in b$" will be useful in the proof of Lemma 1D.2.

Definition. A condition in R^s is a pair $(t, \bar{t})$ in $L_{\mu_s^1}$ such that:
(i) $t \in S_\gamma^\beta$, $\bar{t} \subseteq \{b_{s \upharpoonright \xi} \mid s(\xi)=1,\ \xi \in \mathcal{O}(\gamma^+)\}$.
(ii) For some $\sigma < \mu_s^0$, $\alpha(\sigma) \leq |t|$ we have that $\bar{t} \subseteq \{b_{s \upharpoonright \xi} \mid \mu_\xi \in \text{Range}(\pi_{\sigma\mu_s^0})\}$.

It is convenient to let $\bar{t}$ denote $\{\eta \mid t(\eta)=1\}$ for $t \in S_\gamma^\beta$. If $(t_1, \bar{t}_1)$ and $(t_2, \bar{t}_2)$ are conditions in R^s, we define $(t_1, \bar{t}_1) \leq (t_2, \bar{t}_2)$ if: $t_1 \supseteq t_2$, $\bar{t}_1 \supseteq \bar{t}_2$, $b \in \bar{t}_2 \to \bar{t}_1 \cap S(b) \subseteq \bar{t}_2$.

Note that the set of $(t, \bar{t}_2) \in R^s$ satisfying the following property is dense in R^s: $\bar{t}$ is of the form $\{b_{s \upharpoonright \xi} \mid s(\xi)=1,\ \xi \in \mathcal{O}(\gamma^+)$ and $\mu_\xi \in \text{range}(\pi_{\sigma\mu_s^0})\}$ for some $\sigma < \mu_s^0$, $\alpha(\sigma) \leq |t|$. The restriction on $\bar{t}$ in (ii) is necessary to show extendibility for R^s.

We will show that $p \in R^s$, $\eta < \gamma^+ \to \exists q \leq p, |q| \geq \eta$ (where if $q=(t, \bar{t})$, then $|q|=|t|$). From this it follows that if G is R^s-generic over $L_{\mu_s^1}$ then, setting $g = \bigcup\{t \mid (t, \bar{t}) \in G\}$, we have: $s(\xi)=1$ iff $\bar{g} \cap S(b_{s \upharpoonright \xi})$ is bounded in γ^+ for $\xi \in \mathcal{O}(\gamma^+)$. Moreover, g uniquely determines $G = G(g)$ by: $G = \{(t, \bar{t}) \mid t = g \upharpoonright |t|$, $\bar{g} \cap S(b) \subseteq \bar{t}$ for $b \in \bar{t}\}$. In this case we say that g is *R^s-generic* over $L_{\mu_s^1}$.

We want to establish some basic lemmas about R^s, such as extendibility. Much as in the discussion of the generic codes in Part B, the analysis of the R^s forcing requires the definition of 'localized' versions of the R^s forcing, defined at morass points below γ^+.

Note that even though we have defined $\mathcal{O}(\kappa)$, μ_ξ for $\xi \in \mathcal{O}(\kappa)$ only when $\kappa \in \beta$-Card for some $\beta \in \text{Adm}$, these definitions make sense in a much more general context. Namely for any γ let $U(\gamma)$ denote $\{\alpha \in \text{Adm} \mid L_\alpha \vDash \gamma$ is the largest cardinal$\}$ and fix $\alpha \in U(\gamma)$. First let $\mathcal{O}'(\alpha)$ consist of all $\xi \geq \alpha$ such that $L_\xi \vDash \alpha$ is a cardinal, $L_\xi \vDash \text{card}(\xi) \leq \alpha$ (we do not require that $L_\xi \vDash \alpha$ is a cardinal). Now consider the inductive definition, for $\xi \in \mathcal{O}'(\alpha)$:

$$\mu_\xi^0 = \sup\{\mu_{\xi'} \mid \xi' < \xi\} \quad (=\alpha \text{ if } \xi = \alpha),$$
$$\mu_\xi^{i+1} = \text{least p.r. closed } \mu > \mu_\xi^i \text{ s.t. } L_\mu \vDash \text{card}(\xi) = \alpha \quad \text{if such a } \mu \text{ exists},$$
$$\mu_\xi = \sup\{\mu_\xi^i \mid i < \omega\}, \quad \text{if the } \mu_\xi^i\text{'s are defined.}$$

The ordinal μ_ξ^i need not be defined for all $\xi \in \mathcal{O}'(\alpha)$ because we require that $L_{\mu_\xi^i} \vDash \alpha$ is a cardinal. Let $\mathcal{O}(\alpha) = \{\xi \in \mathcal{O}'(\alpha) \mid \mu_{\xi'}$ is defined for all $\xi' \in \mathcal{O}'(\alpha) \cap \xi\}$ $(= \{\xi \in \mathcal{O}'(\alpha) \mid \mu_\xi^0$ is defined$\})$. Clearly $\mathcal{O}(\alpha)$ has a maximum which we denote by $\xi(\alpha)$. If $\xi < \xi(\alpha)$, define $\mu_\xi^i = \mu_{\xi'}^i$, where $\xi' = \inf(\mathcal{O}(\alpha) - \xi)$.

We can now define S_α in analogy to our earlier definition of $S_\kappa^{\bar{\beta}}$. Thus let $\beta = \max(\mathcal{O}(\alpha) \cap \text{Adm})$. We have already defined S_α^β. Now S_α consists of all $s : [\alpha, |s|) \to 2$, $\alpha \leq |s| \in \mathcal{O}(\alpha)$ such that either $s \in S_\alpha^\beta$ or $\beta \leq |s|$ and:

(a) Let $X_s = \{\delta \in [\alpha, \beta) \mid s(\delta)=1\}$. Then $f(\alpha, \beta, X_s)$ is $\mathcal{P}_\alpha^\beta$-generic over L_β (if β is recursively inaccessible, then $f(\alpha, \beta, X_s)$ is $\mathcal{P}_\alpha^\beta$-Σ-generic over L_β) and $\langle L_\beta, s \upharpoonright \beta \rangle$ is inadmissible if β is a successor admissible.

(b) For all $\alpha \leq \xi \leq |s|$, $s \upharpoonright \xi$ is $\Delta_1^*(\mathcal{A}(\mu_\xi^0))$ where $\mathcal{A}(\mu_\xi^0)$, Δ_1^* are defined as in Part B.

As before we have that $s \in S_\alpha$, $\xi \in \mathcal{O}(\alpha) \to s \upharpoonright \xi \in S_\alpha$. Note that (b) need only be verified for $\xi \geq (\alpha^+)^{L_\beta}$; and for $\xi > \beta$ we have $\mathscr{A}(\mu_\xi^0) = L_{\mu_\xi^0}$. Also let $\mu_s^i = \mu_{|s|}^i$ (when defined).

Next we want to extend the definition of the forcing R^s, given above for $s \in S_{\gamma^+}^\beta$, to all $s \in S_\alpha$, $\alpha \in U(\gamma)$. In this case a condition in R^s is a pair $(t, \bar{t}) \in L_{\nu(\alpha)}$ such that:

(i) $t \in S_\gamma^\alpha$, $\bar{t} \subseteq \{b_{s \restriction \xi} \mid s(\xi) = 1\}$.

(ii) For some $\sigma < \tau \in T_\alpha$, $\alpha(\sigma) \leq |t|$ we have that $\bar{t} \subseteq \{b_{s \restriction \xi} \mid \mu_\xi \in \text{Range}(\pi_{\sigma\tau}) \cup \{\tau\}\}$.

Extension of conditions is defined as before. Note that it is possible for $\nu(\alpha)$ to equal μ_s^0, in which case we have that $\{\xi \mid b_{s \restriction \xi} \in \bar{t}\}$ is bounded in $|s|$ (if $|s|$ limit). If $\mu_s^0 < \nu(\alpha)$, then $R^s \in L_\nu$ where $\nu =$ least member of T_α greater than μ_s^0. It is easy to see that this definition generalizes the old one for $s \in S_{\gamma^+}^\beta$ in the sense that the latter is a dense subordering of the former. Also $s \subseteq t \in S_\alpha \rightarrow R^s = R^t \cap L_{\mu_s^1}$.

We now discuss extendibility for R^s.

Lemma 1D.1. *Suppose $s \in S_\alpha$, $\alpha \in U(\gamma)$ and $(t, \bar{t}) \in R^s$. Then for all $\eta < \alpha$ there exists $(t', \bar{t}') \leq (t, \bar{t})$ in R^s such that $|t'| \geq \eta$.*

This lemma is to be proved by induction on α. In order to carry out this induction we must prove a stronger statement, which we now describe.

Definition. A *labeled α, s-string* is a pair $(p, \bar{s})$ where for some $\alpha \in U(\gamma)$, $X \in I_\alpha$, $s \in S_\alpha$ we have $p \in \mathscr{C}_{X,s}$ and $\bar{s}: |\bar{s}| \rightarrow 2$, $|\bar{s}| = \sup\{|u| + 1 \mid u \in X\}$, $u \in X \rightarrow u \subseteq \bar{s}$. In this case

$$\text{Range}(p) = \{p(\bar{s} \restriction \xi) \mid \bar{s} \restriction \xi \in X\},$$
$$\text{Range}(p, \bar{s}) = \{p(\bar{s} \restriction \xi) \mid \bar{s} \restriction \xi \in X, \bar{s}(\xi) = 1\}$$

and

$$\text{Range}^*(p) = \{p(\bar{s} \restriction \xi) \restriction \eta \mid \bar{s} \restriction \xi \in X, \eta \in U(\gamma) \cap (|X|, \alpha]\},$$
$$\text{Range}^*(p, \bar{s}) = \{p(\bar{s} \restriction \xi) \restriction \eta \in \text{Range}^*(p) \mid \bar{s}(\xi) = 1\}.$$

Two labeled α, s-strings $(p_1, \bar{s}_1)$, $(p_2, \bar{s}_2)$ are *compatible* if for all $b \in \text{Range}^*(p_1) \cap \text{Range}^*(p_2)$: $b \in \text{Range}^*(p_1, \bar{s}_1)$ iff $b \in \text{Range}^*(p_2, \bar{s}_2)$.

Lemma 1D.2. *Suppose $s \in S_\alpha$, $|s| = \xi(\alpha) = \max(\mathscr{O}(\alpha))$ where $\alpha \in U(\gamma) \cap \gamma^+$. Also suppose that $t \in S_\gamma^\alpha$, F is a finite, pairwise compatible collection of labeled α, s-strings and $(p, u) \in F \rightarrow |\text{Dom}(p)| \leq |t|$. Then*:

(a) *There exists $g \subseteq R^s$, $t \subseteq g$ such that g is R^s-generic over $L_{\nu(\alpha)}$ (is R^s-Σ-generic over $\langle L_{\nu(\alpha)}, s \restriction \nu(\alpha)\rangle$ if $\nu(\alpha)$ is recursively inaccessible) and such that $\bar{g} \cap S(b) \subseteq \bar{t}$ for all $b \in \bigcup\{\text{Range}(p, u) \mid (p, u) \in F\}$. Moreover, g is $\Delta_1^*(\mathscr{A}(\alpha))$.*

(b) *For $\eta < \alpha$ there exists $t' \supseteq t$ in S^{α}_{γ} such that $|t'| \geq \eta$ and $\bar{t}' \cap S(b) \subseteq \bar{t}$ for all* $b \in \bigcup \{\text{Range}(p, u) \mid (p, u) \in F\}$.

Note. We abbreviate the final clauses of (a), (b) above by saying that "g avoids F", "t' avoids F", respectively.

Lemma 1D.2 is analogous to Lemma 1B.9 and is proved by induction on α, using an analysis much like the one used in the proof of that lemma. We need to consider labeled α, s-strings for the following reason: We are of course primarily interested in the case where F contains only a single standard α, s-string (p, u) (that is, where p is standard); this case establishes extendibility for R^s. But in the course of handling this case we are led to consider p^{η}, which is possibly nonstandard, and another (standard) labeled $\eta, \bar{s}$-string. To 'close off' this process we are forced to consider arbitrary finite collections of compatible a, s-strings.

We will also need to assume extendibility for $S_{\alpha'}$, $\alpha' \in U(\gamma) \cap \alpha$. The following will be established in Section Two:

Lemma. *Suppose $s \in S_{\beta}$, $\beta \in U(\gamma)$. Then there exists $s' \supseteq s$, $s' \in S_{\beta}$, $|s'| = \xi(\beta)$.*

Proof of Lemma 1D.2. We follow the basic outline of the proof of Lemma 1B.9. We prove (a), (b) by a simultaneous induction on α.

First we prove (b). If α is a $U(\gamma)$-limit, then we can choose $\beta \in U(\gamma) \cap \alpha$ so that $|t|$, $\eta < \beta$ and $T_{\beta} = \emptyset$. Then clearly we are done by applying induction to the forcing $R^{\emptyset}$ (viewing $\emptyset \in S_{\beta}$) and $F^{\beta} = \{(p^{\beta}, u) \mid (p, u) \in F\}$. If $\alpha = U(\gamma)$-successor to $\beta \in U(\gamma) \cup \{\gamma\}$ and $|t| < \beta$, then define $s' \in S_{\beta}$ by: $|s'| = \sup\{\xi \mid b_{\bar{s} \restriction \xi} \in \text{Range}^*(p)$ for some $\bar{s}$, some $(p, u) \in F\}$ and $s'(\xi) = 1$ iff $b_{\bar{s} \restriction \xi} \in \bigcup \{\text{Range}^*(p, u) \mid (p, u) \in F\}$ for some $\bar{s}$. It is easy to check that s' in fact belongs to S_{β}, as for some $(p, u) \in F$, $s' = \bigcup \{\pi(u \restriction \xi) \mid \xi \in \text{Dom}(u)\}$ where π is the quasi-morass map determined by $p(i) = b_{\pi(i)}$. Now by induction we can apply (a) to $R^{s'}$ and t to obtain $t' \supseteq t$, $|t'| = \beta$, t' avoids F. To arrange that t' belongs to S^{α}_{γ} we must be sure that if β is a successor admissible, then β is t'-indamissible (this is easy: let $t'(\delta) = 1$ iff $\delta = \langle 1, \delta' \rangle$ where $\delta' \in C_{\beta}$, for $\delta \in [\beta', \beta)$, $\beta = \bar{\beta}'$) and if $|s'|^- = \beta^*$ is recursively inaccessible, then $(t' \cup s') \restriction \beta^*$ is $\mathcal{P}^{\beta^*}_{\gamma} \approx \mathcal{P}^{\beta^*}_{\beta} * R^{G_{\beta}}$-$\Sigma$-generic over L_{β^*} (the above factoring, where $\mathbf{G}_{\beta}$ denotes the $\mathcal{P}^{\beta^*}_{\beta}$-generic subset of $(\beta^+)^{L_{\beta^*}}$, will follow from the definition of $\mathcal{P}$; see the Factoring Property 1A.7). Given this, we can assume that $|t| \geq \beta$, in which case (b) is trivial as we can simply define $t' \supseteq t$, $|t'| = \eta$ by setting $t'(\delta) = 0$ for $\delta \in [|t|, \eta)$. The avoidance condition is vacuous as $\bar{t}' \subseteq \bar{t}$.

To establish the Σ-genericity of $t' \cup s'$ note first that the $R^{s'}$-genericity of t' over $\langle L_{\nu(\beta)}, s' \restriction \nu(\beta) \rangle$ implies that in fact t' is $R^{s' \restriction \beta^*}$-$\Sigma$-generic over $\langle L_{\beta^*}, s' \restriction \beta^* \rangle$, thanks to the remark immediately following the definition of Σ-generic, as well as the fact that $R^{s'}$ has the $(\beta^+)^{L_{\beta^*}}$-c.c. in $L_{\beta^*}[s']$ (this is clear, using the definition of $R^{s'}$ and the fact that s' preserves β^*-cardinals). Now we are done using:

Σ-Generic Product Lemma. *Suppose $\mathcal{P} \subseteq L_\alpha$ is a $\Delta_1(L_\alpha)$-partial ordering, $\mathcal{P} \Vdash \mathbf{Q} \subseteq L_\alpha[\mathbf{G}]$ is $\Delta_1(L_\alpha[\mathbf{G}])$ and $G \subseteq \mathcal{P}$ is $\mathcal{P}$-Σ-generic over L_α, $H \subseteq \mathbf{Q}^{L_\alpha[G]}$ is $\mathbf{Q}^{L_\alpha[G]}$-Σ-generic over $L_\alpha[G]$. Also suppose that $\mathcal{P} \Vdash \mathbf{Q}$ has the Σ-c.c. in $L_\alpha[\mathbf{G}]$ (that is, $\mathcal{P} \Vdash$ any $\Sigma_1(L_\alpha[\mathbf{G}])$-predense $D \subseteq \mathbf{Q}$ can be effectively reduced to a predense set $D^* \in L_\alpha[\mathbf{G}]$, $D^* \subseteq D$) and that the $\mathcal{P}$-forcing relation is Σ_1 for ranked sentences. Then $G * H = \{(p, q) \mid p \in G, q \in H\}$ is $\mathcal{P} * \mathbf{Q}$-Σ-generic over L_α.*

Proof. Suppose $T \subseteq (\mathcal{P} * \mathbf{Q}) \times \gamma$ is $\Sigma_1(L_\alpha)$ and persistent, $\gamma < \alpha$. Consider $T_G = \{(q, \delta) \mid \text{for some } p \in G,\ ((p, q), \delta) \in T\}$. Then by the Σ-genericity of H there is $q \in H$ such that either (i) $\delta < \gamma \rightarrow (T_G)_\delta$ is dense below q, or (ii) for some $\delta < \gamma$, $r \leqslant q \rightarrow (r, \delta) \notin T_G$. As $\mathbf{Q}^{L_\alpha[G]}$ has the Σ-c.c. we can construe (i) as a $\Pi_0\Sigma_1$-sentence (bounded universal followed by Σ_1) and thus by the Σ-genericity of G, if (i) holds, then $p \Vdash$ (i) for some $p \in G$. But then $(p, q) \in \mathcal{D}(T) \cap G * H$. Similarly, (ii) is a Π_1-sentence, so if it is true, we can choose $p \in G$ so that $p \Vdash$ (ii). Then once again $(p, q) \in \mathcal{D}(T) \cap G * H$. $\square$

This completes the proof of (b).

The proof of (a) when $\alpha = U(\gamma)$-successor of $\beta \in U(\gamma) \cup \{\gamma\}$ is also trivial as by induction we can assume that $|t| \geqslant \beta$ and then let g be defined by $g(\delta) = 1$ iff $\delta = \langle 1, \delta' \rangle$ where $\delta' \in C_\alpha$, for $\delta \in [|t|, \alpha)$. The genericity property is automatic as any $\mathcal{D} \in L_\alpha$ which is predense on $R^\emptyset$ is met by $(t', \emptyset)$ for all sufficiently long $t' \in S^\alpha_\gamma$.

Now we turn to the proof of (a) when α is a $U(\gamma)$-limit. The cases are similar to those in the proof of Lemma 1B.9.

Suppose that C_α is unbounded in α and $\nu(\alpha)$ is not recursively inaccessible. Let $\alpha_0 < \alpha_1 < \cdots$ enumerate $C_\alpha \cap (|t|, \alpha)$. For each i canonically choose $f_i : \gamma_i \Rightarrow \alpha$, $\lambda(f_i) = \alpha_i$ and let $\sigma_i = \bigcup \text{Range}(f_i) < \rho(\alpha)$. Let h be the canonical Σ_1^p-Skolem function for $\mathcal{A}(\alpha)$ where $p = p(\alpha)$ and for each i let h_i be the canonical Σ_1^p-Skolem function for $\langle S_{\sigma_i}, A(\alpha) \cap \sigma_i \rangle \subseteq \mathcal{A}(\alpha)$. Let $\pi_i : T_1 \simeq h_{\sigma_i}[\omega \times \alpha_i]$, T_i transitive. Note that $(\gamma^+)^{T_i} = \alpha_i$ and that $T_i = \mathcal{A}(\alpha_i)$. We have Σ_0-embeddings $\pi_{ij} : \mathcal{A}(\alpha_i) \rightarrow \mathcal{A}(\alpha_j)$ defined by $\pi_{ij} = \pi_j^{-1} \circ \pi_i$ and these extend to $\Sigma_{n(\alpha)-1}$-embeddings $\hat{\pi}_{ij} : S_{\beta(\alpha_i)} \rightarrow S_{\beta(\alpha_j)}$. We also have $\hat{\pi}_i : S\beta(\alpha_i) \rightarrow S_{\beta(\alpha)}$. (*Note.* The S's here refer to the S-hierarchy for L.)

Define $s_i = s \circ \hat{\pi}_i$. It is easy to check that $s_i \in S_{\alpha_i}$, $|s_i| = \xi(\alpha_i)$. (This uses the definability property (b) in the definition of S_α.) Also for $i < j$, $s_i = s_j \circ \hat{\pi}_{ij}$ and the functions $\hat{\pi}_{ij} \restriction \nu(\alpha_i) = \delta_{ij}$ are morass maps from $\nu(\alpha_i)$ into some $\sigma_{ij} \in T_a$ ($\sigma_{ij} = \bigcup \text{Range}(\delta_{ij})$). Thus for each i, if $\bar{\delta}_{ij}(s_i \restriction \xi) = bs_j \restriction \delta_{ij}(\xi)$, then $(\bar{\delta}_{ij}, s_i)$ is a labeled α_j, s_j-string. We can assume that $\nu(\alpha_i)$ is not s_i-admissible for each i as $h_{\sigma_i}[\omega \times \alpha_i]$ contains a parameter witnessing that $\nu(\alpha)$ is not s-admissible for sufficiently large i.

Now we claim that for sufficiently large i, $(\bar{\delta}_{ij}, s_i)$ is compatible with (p^{α_j}, u) for all $(p, u) \in F$. Indeed, we claim that in general if (p, u), (q, v) are labeled α, s-strings, p is standard and $b_s \notin \text{Range}(p)$, then (p, u) is compatible with (q, v). This is proved by induction on α. By examining cases as in the supergeneric codes construction it is easy to produce $\beta < \alpha$ so that (p^β, u), (q^β, v) are labeled β, s_β-strings for some $s_\beta \in S_\beta$ and either $p^\beta = p^\beta \restriction X_0(p^\beta)$ is standard, or $\text{Range}^*(p^\beta) \cap \text{Range}^*(q^\beta) \subseteq \text{Range}^*(p^\beta \restriction X_0(p^\beta))$. If $b_{s_\beta} \notin \text{Range}(p^\beta)$, we are now done by induction (applied to arbitrarily large such $\beta < \alpha$). Otherwise note that we can also choose β so that $b_{s_\beta} \in \text{Range}(p^\beta) \cap \text{Range}(q^\beta) \to b_{s_\beta} \in \text{Range}(p^\beta, u)$ iff $b_{s_\beta} \in \text{Range}(q^\beta, v)$. This proves our claim. Finally to establish the compatibility of $(\bar{\delta}_{ij}, s_i)$, (p^{α_j}, u) for sufficiently large i, note that the above shows that $(\bar{\delta}_i, s_i)$, (p, u) are compatible; by applying $\hat{\pi}_j^{-1}$ we get that $(\bar{\delta}_{ij}, s_i)$ and (p^{α_j}, u) are compatible (for i and j sufficiently large).

Now define $t_1 \geq t_2 \geq \cdots$ inductively below t as follows. For $i \geq 0$, $t'_{i+1} =$ least $t' \leq t_i$ which is $R^{s_{i+1}}$-generic over $L_{\nu(\alpha_{i+1})}$ such that t' is $\Delta_1^*(\mathscr{A}(\alpha_{i+1}))$ and $(p, u) \in F \cup \{(\bar{\delta}_{i,i+1}, s_i)\}$, $b \in \text{Range}(p, u) \to \bar{t}' \cap S(b) \subseteq \bar{t}_i$. This is possible by the induction hypothesis. Let $t_{i+1} = t'_{i+1} \cup s_{i+1}$. For limit λ let $t_\lambda = \bigcup \{t_i \mid i < \lambda\} \cup s_\lambda$. Note that t_λ avoids F as $b \in \text{Range}(p) \to \bar{b} \subseteq \hat{U}(\gamma)$ when $p \in \mathscr{C}_X^\alpha$ (see Fact 7 from Part C).

We must check that for limit λ, $t' = \bigcup \{t_i \mid i < \lambda\}$ is R^{s_λ}-generic. Then t_i is a member of $S_\gamma^{\alpha_{i+1}}$ for all i and the above induction defines the desired R^s-generic t_{λ_0}, $\lambda_0 = \text{ordertype}(C_\alpha)$. So suppose $\mathscr{D} \in L_{\nu(\alpha_\lambda)}$ is predense on R^{s_λ}. Then $\mathscr{D} \in \text{Range}(\hat{\pi}_{i\lambda} \restriction L_{\nu(\alpha_i)})$ for some $i < \lambda$ and so by induction $(u_i, \bar{t}_i)$ meets $\hat{\pi}_{i\lambda}^{-1}(\mathscr{D})$ for some $u_i \subseteq t'_i$, $\bar{t}_i \subseteq \{b_{s_i \restriction \xi} \mid s_i(\xi) = 1\}$, $\bar{t}_i \in L_{\nu(\alpha_i)}$. But by definition of t_{i+1}, $t_{i+2}, \ldots$ we have that $(t_j, \hat{\pi}_{ij}(\bar{t}_i)) \leq (u_i, \hat{\pi}_{ij}(\bar{t}_i))$ for $i \leq j < \lambda$ and the latter condition meets $\hat{\pi}_{j\lambda}^{-1}(\mathscr{D})$. (This is where we use the fact that t'_{j+1} 'avoids' $(\bar{\delta}_{j,j+1}, s_j)$.) So $(t_\lambda, \hat{\pi}_{i\lambda}(\bar{t}_i)) \leq (u_i, \hat{\pi}_{i\lambda}(\bar{t}_i))$ and $G(t'_\lambda)$ meets $\mathscr{D}$.

Next we consider the situation where C_α is bounded in α, $\nu(\alpha)$ not recursively inaccessible. First suppose that $\rho(\alpha) > \alpha$ is the limit of p.r. closed ordinals. As C_α is bounded in α we can take a canonical $\Delta_1^*(\mathscr{A}(\alpha))$ ω-sequence $\rho_0 < \rho_1 < \cdots$ cofinal in $\rho(\alpha)$ and let $H_i = \Sigma_1^{p(\alpha)}$-Skolem Hull of $\gamma \cup \{A(\alpha) \cap \rho_i, s \restriction \{\xi \mid \mu_\xi < \rho_i\}\}$ inside $L_{\rho_{i+1}}$, $\alpha_i = H_i \cap \alpha$ for each i. (We are assuming that the ρ_i's are p.r. closed, $p(\alpha) \in L_{\rho_0}$ and $A(\alpha) \cap \rho_i$, $s \restriction \{\xi \mid \mu_\xi < \rho_i\} \in L_{\rho_{i+1}}$ for each i.) Then the transitive collapse of H_i is $L_{\beta(\alpha_i)}$ and $n(\alpha_i) = 1$ for all i. Also $\bigcup_i H_i = L_{\rho(\alpha)}$ and $\bigcup_i \alpha_i = \alpha$. Let $\pi_i : L_{\beta(\alpha_i)} \cong H_i$, $\pi_i(\alpha_i) = \alpha$ and set $\pi_{ij} = \pi_j^{-1} \circ \pi_i$ for $i < j$. Then $\delta_{ij} = \pi_{ij} \restriction \nu(\alpha_i)$ $(= \pi_{ij} \restriction \beta(\alpha_i))$ is a morass map from $\nu(\alpha_i)$ into some $\sigma_{ij} \in T_{\alpha_j}$. Define $s_i = s \circ \pi_i \restriction \{\xi \mid \mu_\xi < \rho_i\}$ and $s'_i \supseteq s_i$, $|s'_i| = \xi(\alpha_i)$. Then for each i, j, $(\bar{\delta}_{ij}, s_i)$ is a labeled α_j-string, where $\bar{\delta}_{ij}(s_i \restriction \xi) = bs_j \restriction \delta_{ij}(\xi)$.

Now define $t_1 \geq t_2 \geq \cdots$ inductively below t as follows. For $i \geq 0$ let $t'_{i+1} =$ least $t' \leq t_i$ which is $R^{s'_{i+1}}$-generic over $L_{\nu(\alpha_{i+1})}$ such that t' is $\Delta_1^*(\mathscr{A}(\alpha_{i+1}))$ and $\bar{t}' - \bar{t}_i$ 'avoids' $F \cup \{(\bar{\delta}_{i,i+1}, s_i)\}$. Then $t_{i+1} = t'_{i+1} \cup s'_{i+1}$ and $t'_\omega = \bigcup \{t_i \mid i < \omega\}$. We need to check that t'_ω is R^s-generic over $L_{\nu(\alpha)}$. This is clear for if $\mathscr{D} \in L_{\nu(\alpha)}$ is predense on R^s, then $\mathscr{D} \in \text{Range}(\pi_i \restriction L_{\nu(\alpha_i)})$ for some i (as $\beta(\alpha_i)$ is p.r. closed) and so

$(u_i, \bar{t}_i) \in R^{s_i}$ meets $\pi_i^{-1}(\mathscr{D})$ for some $(u_i, \bar{t}_i)$ so that $(u_i, \pi_i(\bar{t}_i))$ is extended by $(t_j, \pi_i(\bar{t}_i))$ for all $j > i$. So $(u_i, \pi_i(\bar{t}_i)) \in G(t'_\omega)$ and t'_ω is generic.

Now consider the possibility that $\rho(\alpha) = \alpha$. If $\beta(\alpha) = \alpha$, then R^s is just S_γ^α and we can easily build the desired R^s-generic t' by choosing a $\Pi_1(L_\alpha)$-sequence of admissible ordinals $\langle \alpha_i \mid i < \omega \rangle$ cofinal in α and picking $t_1 \geqslant t_2 \geqslant \cdots$ below t, $|t_i| \geqslant \alpha_i$, $t' = \bigcup \{t_i \mid i < \omega\}$. Note that this covers the case: $n(\alpha) = 1$.

Next if $n(\alpha) > 1$, $\beta(\alpha) > \alpha$ we do a construction similar to that for the case $\rho(\alpha) > \alpha$ a limit of p.r. closed ordinals, but working with $\mathscr{A}'(\alpha)$ instead of $\mathscr{A}(\alpha)$, where $\mathscr{A}'(\alpha) = \langle S_{\rho'(\alpha)}, A'(\alpha) \rangle$ is defined by $\rho'(\alpha) = \rho_j^{\beta(\alpha)}$, $A'(\alpha) = A_j^{\beta(\alpha)}$ with j so that $\rho_{j+1}^{\beta(\alpha)} = \rho(\alpha) = \alpha$. Let $p'(\alpha) =$ least $p \in S_{\rho'(\alpha)}$ such that $\mathscr{A}'(\alpha)$ is the Σ_1-hull of $\alpha \cup \{p\}$ and choose a $\Delta_1^*(\mathscr{A}(\alpha))$-sequence $\langle \alpha_i \mid i < \omega \rangle$ cofinal in α so that $\alpha_i = H_i \cap \alpha$ for each i, where $H_i = \Sigma_1$-Skolem hull of $\alpha'_i \cup \{p'(\alpha)\}$ inside $\mathscr{A}'(\alpha)$ for some $\alpha'_i < \alpha_i$. We again have $\pi_i : T_i \cong H_i$, T_i transitive and $T_i = \mathscr{A}(\alpha_i)$ as π_i is a Σ_1-embedding into $\mathscr{A}'(\alpha) = \mathscr{A}(\nu(\alpha))$ and $\alpha'_i < \alpha_i$. We define $s_i = s \circ \pi_i$ and $\delta_{ij} = \pi_{ij} \restriction \nu(\alpha_i)$ and $\bar{\delta}_{ij}(s_i \restriction \xi) = bs_j \restriction \delta_{ij}(\xi)$. Then define $t_1 \geqslant t_2 \geqslant \cdots$ below t by choosing t'_{i+1} extending t_i to be $R^{s_{i+1}}$-generic over $L_{\nu(\alpha_{i+1})}$, $\Delta_1^*(\mathscr{A}(\alpha_{i+1}))$ and so that $\bar{t}'_{i+1} - \bar{t}_i$ avoids $F \cup \{((\bar{\delta}_{i,i+1}), s_i)\}$; $t_{i+1} = t'_{i+1} \cup s_{i+1}$. Set $t'_\omega = \bigcup \{t_i \mid i < \omega\}$. $G(t'_\omega)$ is R^s-generic as if $\mathscr{D} \in L_{\nu(\alpha)}$ is predense on R^s, then $\mathscr{D} \in \text{Range}(\pi_i \restriction L_{\nu(\alpha_i)})$ for some i (as π_i is a Σ_1-elementary function into $L_{\rho'(\alpha)}$ and $\rho'(\alpha) \geqslant \nu(\alpha)$) and so $(u_i, \bar{t}_i) \in R^{s_i}$ meets $\pi_i^{-1}(\mathscr{D})$ for some (u_i, t_i) so that $(u_i, \pi_i(\bar{t}_i))$ is extended by $(t_j, \pi_i(\bar{t}_i))$ for all $j > i$. So $(u_i, \pi_i(\bar{t}_i)) \in G(t'_\omega)$ and we are done.

We are left with the case: $\rho(\alpha) > \alpha$ is not the limit of p.r. closed ordinals (and the cases where $\nu(\alpha)$ is recursively inaccessible). The argument for the case ($\rho(\alpha) > \alpha$ is the limit of p.r. closed ordinals) actually succeeds whenever $\beta(\alpha) > \nu(\alpha)$: The main point was to get a given $\mathscr{D} \in L_{\nu(\alpha)}$ into $\text{Range}(\pi_i \restriction L_{\nu(\alpha_i)})$ for some i so that induction could be applied; but if $\beta(\alpha) > \nu(\alpha)$, then we can arrange this as $\nu(\alpha) \in \text{Range}(\pi_i)$ for sufficiently large i. (Also, if $\beta(\alpha) = \rho(\alpha)$ is not a limit of limit ordinals, then one must use a $\Sigma_i^{p(\alpha)}$-Skolem hull in $S_{\beta(\alpha)-\omega}$ in defining H_i: otherwise choosing the ρ_i's to be limit ordinals and taking Σ_1-hulls will suffice.) We are therefore left with the case: $\nu(\alpha) = \beta(\alpha)$ is a T_α-successor.

This is the first of the 'active' cases. First assume that $F = \emptyset$ and $\nu = (T_\alpha$-predecessor of $\nu(\alpha))$ is not a T_α-limit (ν could be α). We begin with the following.

Claim. *R^s is $(<\alpha)$-distributive in $L_{\nu(\alpha)}$.*

Proof of Claim. Suppose $\langle \mathscr{D}_i \mid i < \gamma \rangle \in L_{\nu(\alpha)}$ are predense on R^s. Note that $\{b_{s \restriction \xi} \mid s(\xi) = 1\} \subseteq L_\nu$ (indeed $s \in L_\nu$) as each μ_ξ is a T_α-limit. Choose $\nu_0 \geqslant \nu$ so that $\langle \mathscr{D}_i \mid i < \gamma \rangle \in L_{\nu_0}$ and set $\nu_i = \nu_0 + \alpha \cdot i$ for $i \leqslant \gamma$. Now inductively define $H_0 = \Sigma_1$-Skolem hull of $\gamma \cup \{p(\beta), s, \langle \mathscr{D}_i \mid i < \gamma \rangle\}$ inside L_{ν_0}, $\alpha_0 = H_0 \cap \alpha$; $H_{i+1} = \Sigma_1$-Skolem hull of $H_i \cup \{\alpha_j \mid j \leqslant i\}$ inside $L_{\nu_{i+1}}$, $\alpha_{i+1} = H_{i+1} \cap \alpha$; $H_\lambda = \bigcup \{H_i \mid i < \lambda\}$, $\alpha_\lambda = H_\lambda \cap \alpha$ for limit $\lambda \leqslant \gamma$. Also let $\pi_i : T_i \cong H_i$, T_i transitive and $s_i = s \circ \pi_i \in L_{\nu(\alpha_i)}$.

Now given $(t_0, \bar{t}_0) \in R^s$ inductively define $(t_1, \bar{t}_1) \geqslant (t_2, \bar{t}_2) \geqslant \cdots$ as follows. For all $i \geqslant 0$, $(t_{i+1}, \bar{t}_{i+1})$ is the least $(t, \bar{t}) \leqslant (t_i, \bar{t}_i)$ in R^s so that $|t| \geqslant \alpha_i$, $\bar{t} \supseteq \{b_{s \restriction \xi} \mid s_i(\pi_i^{-1}(\xi)) = 1\}$, $(t, \bar{t})$ meets $\mathscr{D}_i$. Then $(t, \bar{t})$ exists since $s \restriction \text{Range}(\pi_i) \in L_{\nu(\alpha)}$. For limit λ define $(t'_\lambda, \bar{t}_\lambda) = (\bigcup \{t_i \mid i < \lambda\}, \bigcup \{\bar{t}_i \mid i < \lambda\})$ and $t_\lambda = t'_\lambda \cup s_\lambda$. We claim that $(t_\lambda, \bar{t}_\lambda)$ is a condition for limit $\lambda \leqslant \gamma$. The thing to check is that t'_λ is R^{s_λ}-generic over $L_{\nu(\alpha_\lambda)}$. But this follows as before, using the fact that $\bar{t}_{j+1} - \bar{t}_j$ avoids $\{b_{s_\lambda \restriction \xi} \mid s_\lambda(\xi) = 1,\ \xi \in \text{Range}(\pi_{i\lambda})\}$ at all stages $j \geqslant i$. □ (Claim).

Given the Claim we can finish the (special) case at hand. Let $\beta_0 < \beta_1 < \cdots$ be a final segment of $C'_{\nu(\alpha)}$ with $p(\alpha)$, $(t, \bar{t}) \in L_{\beta_0}$ and $C'_\alpha = \{\alpha_0 < \alpha_1 < \cdots\}$. Now define $H'_0 = \Sigma_1$-Skolem hull of $\gamma \cup \{p(\alpha)\}$ in L_{β_0}, $H'_{i+1} = \Sigma_1$-Skolem hull of $\gamma \cup \{\alpha'_i, \alpha_i\}$ inside $L_{\beta_{i+1}}$, $\alpha'_i = H'_i \cap \alpha$ for all $i < \omega$. Next define $(t_1, \bar{t}_1) \geqslant (t_2, \bar{t}_2) \geqslant \cdots$ below $(t, \bar{t})$ in R^s so that $(t_{i+1}, \bar{t}_{i+1})$ meets all predense $\mathscr{D}$ on R^s, $\mathscr{D} \in H'_i$. This is possible by the Claim. Then $t'_\omega = \bigcup \{t_i \mid i < \omega\}$ is R^s-generic over $L_{\nu(\alpha)}$.

Now to extend the above argument to the general case it will suffice to show the following.

Sublemma 1D.3. *Given $(t, \emptyset)$ in R^s, $\mathscr{D} \in L_{\nu(\alpha)}$ predense on R^s and F a finite pairwise compatible collection of labeled α, s-strings, $(p, u) \in F \rightarrow |\text{Dom}(p)| \leqslant |t|$ there exists $(t', \bar{t}') \leqslant (t, \emptyset)$ in $\mathscr{D}^*$ such that $\bar{t}' - \bar{t}$ avoids F (i.e. $(p, u) \in F$, $b \in \text{Range}\ (p, u) \rightarrow \bar{t}' \cap S(b) \subseteq \bar{t}$).*

Proof of Sublemma. Here is the second key use of the genericity property of the supergeneric codes (the other was Lemma 1B.12). We can assume that $\nu(\alpha)$ is not a T_α-limit and not a $<$-limit as otherwise let $\nu \leqslant \nu(\alpha)$ be T_α-least so that $\mathscr{D}$, $(t, \bar{t}) \in L_\nu$ and choose $\bar{\nu} < \nu$, $\bar{\nu}$ a $<$-successor, so that $\{b_{s \restriction \xi} \mid s(\xi) = 1,\ \mu_\xi < \nu\}$, $\mathscr{D}$, $(t, \bar{t}) \in \text{Range}\ \pi_{\bar{\nu}\nu}$. Then by applying the Sublemma to $R^{\bar{s}}(\bar{s} = s \circ \pi_{\bar{\nu}\nu})$, $\bar{\mathscr{D}} = \pi_{\bar{\nu}\nu}^{-1}(\mathscr{D})$ and $(t, \pi_{\bar{\nu}\nu}^{-1}(\bar{t}))$, we obtain it for R^s, $\mathscr{D}$, $(t, \bar{t})$.

If α is not a $U(\gamma)$-limit, say $\beta = U(\gamma)$-predecessor of α ($= \gamma$ if $\alpha = \min(U(\gamma))$), then the Sublemma follows by the induction hypothesis of the lemma: We need only extend $(t, \bar{t}) = (t, \emptyset)$ to $(t', \emptyset)$ avoiding F of arbitrarily large length $|t'| < \alpha$. But, if we can achieve $|t'| \geqslant \beta$, then we are done for then we can let $t'(\eta) = 0$ for all $\eta \in [\beta, |t'|)$ and the avoidance condition is trivial. To extend t to t' of length β we need only avoid the finite collection of compatible β-strings $\{(p^\beta, u) \mid (p, u) \in F\}$ which is possible by the induction hypothesis of the lemma.

Now as we are assuming that $\nu(\alpha)$ is not a T_α-limit, not a $<$-limit we can form $F_0 \in \bar{\mathscr{D}}_0(\alpha, s)$ by replacing each $(p, u) \in F$ by some α, s-name $\bar{c}_p$ such that p is a thinning of $p(\bar{c}_p)$ and $b_s \notin \text{Range}(\text{first component of } \bar{c}_p)$. In fact we assume that $p = p(\bar{c}_p)$ and that $i < \text{length}(\bar{c}_p) \rightarrow \bar{c}_p(\leqslant i) \in F_0$. (If $p \neq p(\bar{c}_p)$ for some p, then a small modification of the argument below will suffice.) Assume that the Sublemma fails. Then by the $\mathscr{P}(\alpha, F_0)$-genericity of the assignment $G: \bar{c} \mapsto (g_1, \ldots, g_n)$, $\bar{c} \in F_0$ (see Lemma 1C.13) there must be a condition $r: \bar{c} \mapsto (q_1, \ldots, q_n) \in \mathscr{C}^*(\bar{c})$, $\bar{c} \in F_0$ such that $r \Vdash_{\mathscr{P}(\alpha, F_0)}$ "No $(t', \bar{t}') \leqslant (t, \emptyset)$ meeting $\mathscr{D}$

avoids $\bigcup\{\text{Range}(\mathbf{G}(\bar{c}_p), u) \mid (p, u) \in F\}$". (Here we are using $\text{Range}(\mathbf{G}(\bar{c}_p), u)$ to denote $\{g'_n(u \upharpoonright \xi) \mid u(\xi) = 1$ and $u \upharpoonright \xi \in \text{Dom}(g'_n)\}$ where $g'_{i+1} = \bar{g}_{i+1} \cup g_{i+1}$, $\bar{g}_{i+1}$ thins g'_i as $\bar{p}_{i+1}$ thins $p(\bar{c}_p(\leq i))$, $\bar{c}_p = (p_0, (\bar{p}_1, p_1), \ldots, (\bar{p}_n, p_n))$.) We can assume that $|q_i| < \eta$ for all $(q_1, \ldots, q_n) \in \text{Range}(r)$ where η is some element of $U(\gamma)$ greater than $|t|$ such that $\bar{c} \in F_0 \to \bar{c}^\eta$ is an η-name. To each $\bar{c} \in F_0$, $\bar{c} = (p_0, (\bar{p}_1, p_1), \ldots, (\bar{p}_n, p_n))$, associate the η-name $\bar{d} = (p_0^\eta, (\bar{p}_1^\eta, q_1), \ldots, (\bar{p}_n^\eta, q_n))$ where $r(\bar{c}) = (q_1, \ldots, q_n)$. Then by induction we can extend $(t, \emptyset)$ to $(t', \emptyset)$ avoiding $\bigcup\{\text{Range}(p(\bar{d}), u)) \mid (p, u) \in F$, $\bar{d}$ associated to $\bar{c}_p$ as above$\}$ and so that $|t'| \geq \eta$. By the predensity of $\mathcal{D}$ choose $(t^*, \bar{t}^*) \leq (t', \emptyset)$ below some element of $\mathcal{D}$; let $\eta^* = \text{length}(t^*)$. Finally consider the condition $r^* \in \mathcal{P}(\alpha, F_0)$ defined as follows: Let f have domain $[\eta, \eta^*)$ and be defined by $f(i) = 0$ for all $i \in [\eta, \eta^*)$. Now if $\bar{c} = (p_0, (\bar{p}_1, p_1), \ldots, (\bar{p}_n, p_n)) \in F_0$ define $r^*(\bar{c}) = (p_1^*, \ldots, p_n^*)$ where $p_i^*(w) = p(\bar{d}(\leq i))(w) * f$, where $\bar{d}$ is the η-name associated to $\bar{c}$. Then $r^* \leq r$. But the definition of $S(b)$ implies that $r \Vdash (t^*, \bar{t}^*) \leq (t, \emptyset)$ meets $\mathcal{D}$ and avoids $\bigcup\{\text{Range}(\mathbf{G}(\bar{c}_p), u) \mid (p, u) \in F\}$. This is a contradiction. □ (Sublemma)

The Sublemma allows us to complete the case: $\nu(\alpha) = \beta(\alpha)$ is a T_α-successor. Indeed the $(<\alpha)$-distributivity of R^s in $L_{\nu(\alpha)}$ can now be shown by repeating the proof in the special case except now we use the Sublemma to choose t_{i+1} avoiding $\{b_{s \upharpoonright \xi} \mid s_i(\pi_i^{-1}(\xi)) = 1\}$ and avoiding F. (b_{s_i} is also avoided if $b_{s \upharpoonright \xi} \in \bar{t}_i$, $\nu(\alpha) = T_\alpha$-successor of μ_ξ.) One should note that the definability property (b) from the definition of S_α follows for t_λ as the definition of $p(\bar{c})$ for proper α-names $\bar{c}$ is $\Delta_1^*(\mathcal{A}(\alpha))$. The last part of the argument is the same, using the $(<\alpha)$-distributivity (with 'F-avoidance') to build the desired R^s-generic.

Now suppose that $\nu(\alpha) = \alpha$ is recursively inaccessible. We can assume that α is Σ_1-projectible as otherwise Σ-genericity for $R^\emptyset \subseteq L_\alpha$ reduces to ordinary genericity: If $W \subseteq R^\emptyset \times \gamma$ is $\Sigma_1(L_\alpha)$, then α not Σ_1-projectible$\to$for some $\beta < \alpha$, any condition $t \in R^\emptyset$, $|t| \geq \beta$ meets $\mathcal{D}(W^*) \cap L_\beta =$ "$\mathcal{D}(W^* \cap L_\beta)$ in the sense of L_β". (Recall that $W^* = \{(q, \delta) \mid q \leq p$ for some $(p, \delta) \in W\}$.)

Thus C_α is unbounded in α and $\beta \in C_\alpha \to \beta$ is inadmissible. First suppose that $F = \emptyset$. The following is analogous to our earlier distributivity claim for R^s.

Claim. *Suppose $\langle W_i \mid i < \gamma' \rangle$ is a uniformly $\Sigma_1(L_\alpha)$-sequence of persistent subsets of $R^\emptyset \times \gamma$ (i.e., $(p, \delta) \in W_i$ and $p' \leq p \to (p', \delta) \in W_i$) and $\gamma' < \gamma$. Then for any $t \in R^\emptyset$ there exists $t' \leq t$ such that $t' \in \mathcal{D}(W_i)$ for all $i < \gamma'$.*

Proof of Claim. First assume that γ is α-regular and suppose that the Claim fails. Now define $t = t_0 \geq t_1 \geq \cdots$ inductively by: $t_{i+1} =$ least $t' \leq t_i$ in $R^\emptyset$ so that for some $j < \gamma'$, $\delta < \gamma$ we have $\{t'\} \times \delta \subseteq W_j^{|t'|}$ but $\{t_i\} \times \delta \nsubseteq W_j^{|t_i|}$; $t_\lambda = \bigcup\{t_i \mid i < \lambda\}$ for limit λ. Then $|t_\lambda|$ is inadmissible for limit λ and so t_λ is a member of S_γ^α for limit λ as genericity for t_λ follows automatically. The desired $t' \leq t$ is t_{i_0} where i_0 is least so that t_{i_0+1} is not defined. Such an i_0 exists as otherwise γ is the α-finite union of γ'-many sets of α-cardinality $< \gamma$.

If γ is singular in L_α, then notice that we can assume that $W_i \subseteq R^\emptyset \times \gamma_0$ for all i, where $\gamma_0 < \gamma$. Then proceed as above. □ (Claim)

Remark. It follows from the Claim that $R^\emptyset$ is Σ-distributive over L_α.

Given the Claim and $t \in S^\alpha_\gamma$ we build a $R^\emptyset$-Σ-generic $g \supseteq t$ as follows. Write $C_\alpha = \{\alpha_i \mid i < \bar{\gamma}\}$ and canonically choose $f_i : \gamma_i \Rightarrow \alpha$ so that $\lambda(f_i) = \alpha_i$. Let $t_0 = t$ and $t_{i+1} =$ least $t' \leqslant t_i$ in $R^\emptyset$ so that $|t'| \in C_\alpha$ and $t' \in \mathcal{D}(W^{\alpha^*}_e)$ for all $e \in \text{Range}(f_i)$ (where $\langle W^\alpha_e \mid e < \alpha \rangle$ is a canonical enumeration of the sets $W \subseteq L_\alpha$ which are $\Sigma_1(L_\alpha)$); $t_\lambda = \bigcup \{t_i \mid i < \lambda\}$. Then $|t_\lambda|$ is inadmissible for limit λ and so t_λ is a condition in $R^\emptyset$ as before. Finally $g = \bigcup \{t_i \mid i < \bar{\gamma}\}$ is the desired Σ-generic.

Now to carry out this argument successfully when $F \neq \emptyset$ we only need a strengthened form of the Claim where we require that t' avoids F. To prove the stronger Claim, form $F_0 \in \bar{\mathcal{D}}_0(\alpha)$ as before by replacing each $(p, u) \in F$ by an α-name $\bar{c}_p$ such that p is a thinning of $p(\bar{c}_p)$; we assume that $p = p(\bar{c}_p)$ and $i < \text{length}(\bar{c}_p) \to \bar{c}_p(\leqslant i) \in F_0$. Let $G \subseteq \mathcal{P}(\alpha, F_0)$ be the $\mathcal{P}(\alpha, F_0)$-Σ-generic defined by Lemma 1C.13, $G : \bar{c} \mapsto (g_1, \ldots, g_n)$. Modify the definition of t_{i+1} when γ is α-regular as follows: $t_{i+1} =$ least $t' \leqslant t_i$ in $R^\emptyset$ such that t' avoids F and either (a) for some $i < \gamma'$, $\delta < \gamma$ we have $\{t\} \times \delta \subseteq W_i^{|t'|}$, $\{t_i\} \times \delta \nsubseteq W_i^{|t_i|}$, or (b) for some $p \leqslant G^{|t_i|}$ (= that condition in $\mathcal{P}(\alpha, F_0)$ defined by $G^{|t_i|}(\bar{c}) = (g_1 \restriction |t_i|, \ldots, g_n \restriction |t_i|)$) we have $|p| \leqslant |t'|$ and $p \Vdash t_i$ can be extended to t'' obeying (a). Thus the idea is to keep extending t_i if either (a) occurs or can be forced to occur by some condition extending $G^{|t_i|}$ of length $> |t_i|$.

Now note that α is admissible relative to G (by the Σ-genericity of G) so there is a stage i_0 so that t_{i_0} has no extension t' avoiding F which obeys (a). So again by Σ-genericity, there is $\eta < \alpha$ so that this fact is forced by G^η. It follows that if $t' \leqslant t_{i_0}$ is defined to avoid F and have length η, then t' is as desired. The case where γ is α-singular can be similarly handled.

Finally we consider the case: $\nu(\alpha) > \alpha$ is recursively inaccessible. Then $\langle L_{\nu(\alpha)}, s \restriction \nu(\alpha) \rangle$ is admissible as either $|s|^- = \nu(\alpha)$ (and so $s \restriction \nu(\alpha)$ is $\mathcal{P}^{\nu(\alpha)}_\alpha$-$\Sigma$-generic over $L_{\nu(\alpha)}$, hence admissibility follows by the Σ-distributivity of $\mathcal{P}^{\nu(\alpha)}_\alpha$) or $|s|^- > \nu(\alpha)$ (in which case the result follows from the fact that $s \restriction |s|^-$ preserves cardinals).

Claim. *Suppose $\langle W_i \mid i < \alpha' \rangle$ is a uniformly $\Sigma_1\langle L_{\nu(\alpha)}, s \restriction \nu(\alpha) \rangle$-sequence of persistent subsets of $R^s \times \alpha$ and $\alpha' < \alpha$. Then for any $(t, \bar{t}) \in R^s$ there exists $(t', \bar{t}') \leqslant (t, \bar{t})$ such that $(t', \bar{t}') \in \mathcal{D}(W_i)$ for all $i < \alpha'$.*

Proof of Claim. We define $(t, \bar{t}) = (t_0, \bar{t}_0) \geqslant (t_1, \bar{t}_1) \geqslant \cdots$ and $\nu_0 < \nu_1 < \cdots < \nu(\alpha)$ inductively as follows: Suppose $(t_i, \bar{t}_i)$ and ν_i are defined and $\alpha_i = |t_i|$. Let $H_i \doteq \Sigma_1$-Skolem hull of $\alpha_i \cup \{p\}$ inside $\langle L_{\nu(\alpha)}, s \restriction \nu(\alpha) \rangle$ where p is a parameter both for defining $\langle W_i \mid i < \alpha' \rangle$ and for defining a $\Sigma_1\langle L_{\nu(\alpha)}, s \restriction \nu(\alpha) \rangle$-injection of $\nu(\alpha)$ into α. (Note that we can assume that $\nu(\alpha)$ is Σ_1-projectible relative to $s \restriction \nu(\alpha)$ as otherwise Σ-genericity for R^s reduces to ordinary genericity.) We assume that $H_i \cap \alpha = \alpha_i$ and $\nu_i = \bigcup (H_i \cap \mathrm{ORD})$, for $i > 0$. Choose $\bar{t}_i' = \bar{t}_i \cup \{b_{s \restriction \xi} \mid s(\xi) = 1,\ \xi \in H_i\}$ and if i is even, let $(t_{i+1}', \bar{t}_{i+1}) \leqslant (t_i, \bar{t}_i')$ be least so that either for some $j < \alpha'$, $\delta < \alpha$ we have that $(t', \bar{t}') \leqslant (t_{i+1}', \bar{t}_{i+1}) \to (t', \bar{t}') \notin (W_j)_\delta$, or for some $j < \alpha'$, $W \subseteq W_j$, $W \in L_{\nu(\alpha)}$ we have that $(W)_\delta$ is predense below $(t_{i+1}', \bar{t}_{i+1})$ for all $\delta < \alpha$ (and this is not true if $(t_{i+1}', \bar{t}_{i+1}) = (t_i, \bar{t}_i')$). Now let $H_{i+1} = \Sigma_1$-Skolem hull of $\alpha_i \cup \{p,\ (t_{i+1}', \bar{t}_{i+1})\}$ inside $\langle L_{\nu(\alpha)},\ s \restriction \nu(\alpha)\rangle$, $\alpha_{i+1} = H_{i+1} \cap \alpha$ and obtain $(t_{i+1}, \bar{t}_{i+1})$ from $(t_{i+1}', \bar{t}_{i+1})$ by extending so that $|t_{i+1}| = \alpha_{i+1}$. Also $\nu_{i+1} = \bigcup (H_{i+1} \cap \mathrm{ORD})$. If i is odd, then let $(t_{i+1}, \bar{t}_{i+1}) \leqslant (t_i, \bar{t}_i')$ be the least so that $\bar{t}_{i+1} = \bar{t}_i'$ and if $\alpha_{i+1} = |t_{i+1}|$, then $\alpha_{i+1} = H_{i+1} \cap \alpha$ where $H_{i+1} = \Sigma_1$-Skolem hull of $\alpha_{i+1} \cup \{p,\ (t_i, \bar{t}_i')\}$ inside $\langle L_{\nu(\alpha)},\ s \restriction \nu(\alpha)\rangle$ and such that if $\nu_{i+1} = \bigcup (H_{i+1} \cap \mathrm{ORD})$, then $s \restriction \nu_{i+1}$ meets all $\mathscr{D}(T^*) \subseteq \mathscr{P}_\alpha^{\nu(\alpha)}$ where $T \subseteq \mathscr{P}_\alpha^{\nu(\alpha)} \times \alpha$ is $\Sigma_1(L_{\nu(\alpha)})$ with parameter $x \in H_{i+1}$.

For limit λ take unions. Note that as Π_1-cof$\langle L_{\nu(\alpha)},\ s \restriction \nu(\alpha)\rangle = \alpha$ it must be that α_i is undefined for some $i < \alpha$. In fact the same argument shows that for limit λ if Π_1-cof$\langle L_{\nu(\alpha_\lambda)},\ s \circ \pi_\lambda\rangle = \alpha_\lambda$ (where $\pi_\lambda : L_{\nu(\alpha_\lambda)} \simeq H_\lambda$), then α_i is undefined for some $i < \lambda$. We conclude that $\langle L_{\nu(\alpha_\lambda)},\ s \circ \pi_\lambda\rangle$ is inadmissible and hence $\nu(\alpha_\lambda)$ is not recursively inaccessible whenever λ is a limit ordinal so that α_λ is defined, as otherwise by construction $s \circ \pi_\lambda$ is $\mathscr{P}_{\alpha_\lambda}^{\nu(\alpha_\lambda)}$-$\Sigma$-generic over $L_{\nu(\alpha_\lambda)}$ and hence $\langle L_{\nu(\alpha_\lambda)},\ s \circ \pi_\lambda\rangle$ would be admissible. It is now easy to verify that α_i defined$\to$ $(t_i, \bar{t}_i)$ defined and that if $(t', \bar{t}') = \bigcup \{(t_i, \bar{t}_i) \mid \alpha_i \text{ is defined}\}$, then $(t', \bar{t}') \in \mathscr{D}(W_i)$ for all $i < \alpha'$. (We are using the fact that R^s has the Σ-c.c. in $\langle L_{\nu(\alpha)},\ s \restriction \nu(\alpha)\rangle$.) This proves the Claim. □

Given the Claim, we can dispose of the case at hand, assuming $F = \emptyset$. Indeed, let $\bar{C}_\alpha =$ all $\beta \in C_\alpha$ such that (Σ_1-Skolem hull of $\beta \cup \{p\}$ in $\langle L_{\nu(\alpha)},\ s \restriction \nu(\alpha)\rangle) \cap \alpha = \beta$, where $p =$ least p such that there is a $\Sigma_1\langle L_{\nu(\alpha)},\ s \restriction \nu(\alpha)\rangle$-injection of $\nu(\alpha)$ into α with parameter p. If $\bar{C}_\alpha$ is unbounded in α, then define $(t_0, \bar{t}_0) = (t, \emptyset)$, $(t_{i+1}, \bar{t}_{i+1}) \leqslant (t_i, \bar{t}_i)$ so that t_{i+1} avoids $\{b_{s \restriction \xi} \mid s(\xi) = 1,\ \xi \in H_i\}$ and $(t_{i+1}, \bar{t}_{i+1})$ meets all $\mathscr{D}(W_e)$ for $e < \alpha_i$ (where $\bar{C}_\alpha = \langle \alpha_i \mid i < \gamma_0\rangle$, $H_i = \Sigma_1$-Skolem hull of $\alpha_i \cup \{p\}$ in $\langle L_{\nu(\alpha)},\ s \restriction \nu(\alpha)\rangle$ and $\langle W_e \mid e < \alpha\rangle$ is a canonical Σ_1-listing of the $\Sigma_1\langle L_{\nu(\alpha)},\ s \restriction \nu(\alpha)\rangle$-persistent subsets of $R^s \times \alpha$). It is easy to verify that $g = \bigcup \{t_i \mid i < \gamma_0\}$ is as desired. If $\bar{C}_\alpha$ is bounded, then use a $\Delta_1^*(\mathscr{A}(\alpha))$ ω-sequence $\alpha_0 < \alpha_1 < \cdots$ to guide the $|t_i|$'s. (This is possible as $s \restriction \nu(\alpha)$ is $\Delta_1\langle L_{\nu(\alpha)},\ C_{\nu(\alpha)}\rangle$.)

To carry out this argument successfully when $F \neq \emptyset$ we only need a strengthened form of the Claim where we require that t' avoids F. To prove the stronger Claim, we proceed in a manner similar to the way we handled the case $\nu(\alpha) = \alpha$: form F_0 and $G \subseteq \mathscr{P}(\alpha, F_0)$ as in that case. Assuming for the moment that Σ_1-projectum$(\nu(\alpha)) = \alpha$, we can then assume that $F_0 \in \bar{\mathscr{D}}_0(\alpha, s)$ as $\sigma < \nu(\alpha) \to$ $\pi_{\sigma\nu(\alpha)}$ is bounded in $\nu(\alpha)$. Now proceed as in the proof of the preceding claim with the following modification: choose $(t_{i+1}', \bar{t}_{i+1})$ so that t_{i+1}' avoids F and the condition of the earlier proof holds or is forced to hold by some $p \leqslant G^{|t_i|}$, $|p| \leqslant |t_{i+1}'|$. As $\nu(\alpha)$ is admissible relative to G (see the version of property (h) stated at the end of Part C), there must be $i<\alpha$ so that the condition of the earlier proof does not hold; by the Σ-genericity of G this fact is forced by some $G^{\alpha_{i_0}}$. Then if $(t', \bar{t}') \leqslant (t_{i_0}, \bar{t}_{i_0})$ is defined to avoid F and have length α_{i_0} we see that $(t', \bar{t}') \in \mathscr{D}(W_i)$ for all $i<\alpha'$.

It now remains only to show that Σ_1-projectum $\langle L_{\nu(\alpha)}, s \restriction \nu(\alpha)\rangle = \alpha$ implies Σ-projectum$(L_{\nu(\alpha)}) = \alpha$. To prove this we use the fact that $\mathscr{P}^{\nu(\alpha)}_\alpha$ is equivalent to $S^{\nu(\alpha)}_\alpha$ (in fact the latter is a dense open subset of the former) and for $s_1, s_2 \in S^{\nu(\alpha)}_\alpha$, s_1 is stronger than s_2 in $\mathscr{P}^{\nu(\alpha)}_\alpha$ exactly if s_1 extends s_2. Now given this, we can show: If Σ_1-projectum$(L_{\nu(\alpha)}) = \nu(\alpha) > \alpha$, $s_0 \in \mathscr{P}^{\nu(\alpha)}_\alpha$ and $\langle W_i \mid i<\alpha\rangle$ is a uniformly Σ_1-sequence of subsets of $\mathscr{P}^{\nu(\alpha)}_\alpha$, then there exists $s^* \leqslant s_0$ in $\mathscr{P}^{\nu(\alpha)}_\alpha$ so that $s^* \in \mathscr{D}^*(W_i) = \{p \mid p \leqslant \text{some } q \in W_i \text{ or } q \leqslant p \to q \notin W_i\}$ for all $i<\alpha$. Indeed, define $s_0 \geqslant s_1 \geqslant s_2 \cdots$ effectively by letting $s_{j+1} =$ least $s' \leqslant s_j$ such that for some $i<\alpha$, $L_{|s'|} \models (s'' \in W_i$ for some $s'' \geqslant s'$, $s_j \notin W_i)$. It is easy to check that $s_\lambda = \bigcup_{j<\lambda} s_j$ is a member of $S^{\nu(\alpha)}_\alpha$ for limit λ. Now this induction must terminate in some $s^* = \bigcup\{s_i \mid \text{all } i\}$, else there is a cofinal Σ_1-partial function from α into $\nu(\alpha)$ in contradiction to the hypothesis that $\nu(\alpha)$ is Σ_1-nonprojectible.

Having shown the preceding, we can now use the Σ-genericity of s to argue that $\langle L_{\nu(\alpha)}, s\rangle$ is not Σ_1-projectible. For, if f^s is (a definition for) a partial $\Sigma_1\langle L_{\nu(\alpha)}, s\rangle$-function from α into $\nu(\alpha)$ we can let $W_i = \{s' \mid L_{|s'|} \models f^{s'}(i) = \beta$, some $\beta\}$ and the preceding implies that $\mathscr{D} = \{s' \mid \text{for some } \beta, \text{Range}(f^{s''}) \subseteq \beta \text{ for all } s'' \leqslant s'\}$ is dense open and π_1 over $L_{\nu(\alpha)}$. But the Σ-genericity of s implies that s' meets $\mathscr{D}$ for some $s' \geqslant s$ and we have shown that Range(f^s) is bounded in $\nu(\alpha)$. This completes the proof of our assertion and of Lemma 1D.2. □

The following was demonstrated in the course of proving Lemma 1D.2.

Lemma 1D.4. *R^s is $(<\alpha)$-distributive in $L_{\nu(\alpha)}$. If $\alpha = \nu(\alpha)$ is admissible then $R^\emptyset$ is Σ-distributive over $L_{\nu(\alpha)}$.*

We now discuss the antichain property for R^s. A similar property will also be demonstrated for the limit coding, but via a more difficult argument. In both cases the key fact to establish is a form of the following genericity property.

Lemma 1D.5 (Genericity Property for R^s). *Suppose $s \subseteq t$ belong to S_α, $\alpha \in U(\gamma)$ and $\mathscr{D} \in L_{\mu_s}$ is predense on R^s. Then $\mathscr{D}$ is predense on R^t.*

Proof. Suppose not. Let $(u, \bar{u}_0 \cup \bar{u}_1) \in R^t$ be incompatible with each element of $\mathscr{D}$, where $\bar{u}_0 \subseteq \{b_{s\restriction\xi} \mid \xi < |s|\}$ and $\bar{u}_1 \subseteq \{b_{t\restriction\xi} \mid \xi \geqslant |s|\}$. Then $(u, \bar{u}_0) \in R^s$ and we can assume that Range$(\bar{u}_0) \subseteq$ Range(p), Range$(\bar{u}_1) \subseteq$ Range$(q) \subseteq \{b_{t\restriction\xi} \mid \xi \geqslant |s|\}$ where $p \cup q \in \mathscr{C}_{X,t}$ is standard, $X \in I_\alpha$, $|p \cup q| = \alpha$. By Lemma 1C.15, $G(q)$ is $\mathscr{C}^{\mathscr{P}}_{\text{Dom}(q),t}$-generic over L_{μ_s}. But then as in the proof of Sublemma 1D.3, the genericity of $G(q)$ implies that $(i, \bar{u}_0 \cup \bar{u}_1)$ must be compatible with some element of $\mathscr{D}$: Let $\bar{s} = \bigcup X$ and $\bar{q} \in G(q)$ force that "$(u, \bar{u}_0 \cup \{\mathbf{G}(\bar{s} \restriction \xi) \mid \bar{s}(\xi) = 1\})$ is incompatible with each element of $\mathcal{D}$". But as in Sublemma 1D.3 there exists $\bar{q}' \leq \bar{q}$ which forces the negation of the preceding statement. This contradiction establishes the Lemma. □

Lemma 1D.6 (Chain Condition for R^s). *Suppose $\alpha \in U(\gamma)$, $s \in S_\alpha$ and $\beta \geq \mu_s^0$ where β is admissible, $L_\beta \models \mu_s^0$ is a cardinal. Also suppose that β is admissible relative to $g = s \restriction \mu_s^0$ and $(\alpha^+)^{L_\beta} = (\alpha^+)^{L_\beta[g]}$. Then R^g has the Σ-α^+-c.c. in $L_\beta[g]$; i.e., if $\mathcal{D} \subseteq R^g = \bigcup \{R^s \mid s \in S_\alpha,\ s \subseteq g\}$ is Σ_1 over $L_\beta[g]$ and predense on R^g, then $\mathcal{D} \cap L_\xi$ is predense on R^g for some $\xi < (\alpha^+)^{L_\beta}$.*

Proof. First note that if $(u_1, \bar{u}_1)$, $(u_2, \bar{u}_2) \in R^g$ are incompatible, then $u_1 \neq u_2$ (and so R^g has the $(\alpha^+)^{L_\beta}$-c.c. in $L_\beta[g]$). For, on page 61 we showed that any two standard labeled α, s-strings (p_1, u_1), (p_2, u_2) are compatible, provided $b_s \notin \text{Range}(p_1) \cap \text{Range}(p_2)$. Thus, if $u_1 = u_2$, then by Lemma 1D.2 we can extend $(u_1, \bar{u}_1)$ to $(u_1', \bar{u})$ avoiding $\bar{u}_2$, where $\bar{u} \supseteq \bar{u}_1 \cup \bar{u}_2$. So $(u_1', \bar{u})$ is a common extension of $(u_1, \bar{u}_1)$, $(u_2, \bar{u}_2)$.

If $\beta > (\alpha^+)^{L_\beta}$ and $\mathcal{D} \subseteq R^g$ is $\Sigma_1(L_\beta[g])$ and predense it must be that $\mathcal{D}$ contains a predense $\mathcal{D}^* \in L_\beta[g]$, as $R^g \in L_\beta[g]$. Now pick $\xi < (\alpha^+)^{L_\beta}$ so that L_ξ contains a maximal antichain $M \subseteq \{p \mid p \leq \text{some element of } \mathcal{D}^*\}$. It follows that $\mathcal{D} \cap L_\xi$ is predense on R^g.

The interesting case is where $\beta = (\alpha^+)^{L_\beta}$. Then we can pick $\xi < \beta$ so that $(\mathcal{D})^{L_\xi[g \restriction \xi]}$ is predense on $R^{g \restriction \xi}$ (where $(\mathcal{D})^{L_\xi[g \restriction \xi]}$ is obtained by relativizing the $\Sigma_1(L_\beta[g])$]definition of $\mathcal{D}$ to $L_\xi[g \restriction \xi]$). But $(\mathcal{D})^{L_\xi[g \restriction \xi]}$ is an element of L_{μ_ξ} so by Lemma 1D.5, $\mathcal{D} \cap L_\xi$ is predense on R^g. □

The preceding Lemma is what one needs on order to show that admissibility is preserved by the successor cardinal coding. An important step in that argument is the following.

Lemma 1D.7 (Δ_1-Definability of Forcing for R^s). *Suppose that α, β, g are as in the preceding lemma. Then the relation $p \Vdash \phi$ ($p \in R^g$, ϕ a ranked sentence from $L_\beta[g]$) is Δ_1 over $L_\beta[g]$.*

Proof. The case $\beta > (\alpha^+)^{L_\beta}$ is clear as then R^g is a set forcing over $L_\beta[g]$. Otherwise we show the following by a Σ_1-induction on ϕ: Given $p \in R^g$ we can effectively extend p to $q \leq p$ to decide ϕ (i.e., $q \Vdash \phi$ or $q \Vdash \sim\phi$, and we know which one). If ϕ has no (ranked) quantifiers, then the result is easy. If ϕ is a negation, then the result follows trivially by induction. The interesting case is where ϕ is of the form $\forall x_\gamma \psi(x_\gamma)$ where $\gamma < \beta$ and x_γ ranges over sets of rank less than γ. By induction we can effectively build dense $\Sigma_1(L_\beta[g])$-sets $\mathcal{D}_\tau$ for τ a term of rank $< \gamma$ so that $q \in \mathcal{D}_\tau \to q$ decides $\psi(\tau)$. By the proof of Lemma 1D.6 we can effectively produce $\xi < \beta$ so that $\mathcal{D}_\tau \cap L_\xi$ is predense for each τ. Thus either $p \Vdash \phi$ or we can effectively find $q \leq p$, τ so that $q \in \mathcal{D}_\tau \cap L_\xi$, $q \Vdash \sim\psi(\tau)$. We have completed the induction. Finally note that we can define $p \Vdash \phi$ as follows: First effectively build a dense $\Sigma_1(L_\beta[g])$-set $\mathscr{D} \subseteq R^g$ so that $q \in \mathscr{D} \to q$ decides ϕ and then effectively find $\xi < \beta$ so that $\mathscr{D} \cap L_\xi$ is predense. Then $p \Vdash \phi$ iff $q \leq p$, $q \in \mathscr{D} \cap L_\xi \to q \Vdash \phi$. $\square$

E. Limit cardinal coding

Suppose $\beta \in \overline{\mathrm{Adm}}$, κ is a limit β-cardinal and $s \in S^\beta_\kappa$. We now consider how to code s into a subset of κ. This coding is similar to the one used in Beller–Jensen–Welch [1]; however, due to the requirement that our forcing preserve recursively inaccessibles we must introduce some further restraint.

We begin by reviewing the basic coding strategy from [1]. The following follows from results proved there.

Lemma (Jensen) ($V = L$). *One can associate to each singular β-cardinal κ a continuous cofinal increasing sequence of β-cardinals $\gamma^\kappa_0 < \gamma^\kappa_1 < \cdots$ of length $\lambda_\kappa < \kappa$ such that:*

(a) *$\langle \gamma^\kappa_i \mid i < \lambda_\kappa \rangle$ is definable as an element of L_μ whenever $\mu > \kappa$ is p.r. closed, $L_\mu \models \kappa$ singular. This definition is uniform in κ.*

(b) *If κ' is a limit point of $\langle \gamma^\kappa_i \mid i < \lambda_\kappa \rangle$, then $\gamma^{\kappa'}_i = \gamma^\kappa_i$ for $i < \lambda_{\kappa'}$.*

(c) *If κ' is a successor point of $\langle \gamma^\kappa_i \mid i < \lambda_\kappa \rangle$, then κ' is a successor β-cardinal.*

We shall use the singular sequences provided by the Lemma in defining the coding at singular β-cardinals.

Now suppose that κ is a singular β-cardinal and $s \in S^\beta_\kappa$. The forcing $\mathscr{P}^s \subseteq L_{\mu_s}$ is designed to code s into a subset of $\bigcup\{((\gamma^\kappa_i)^+, (\gamma^\kappa_i)^{++}) \mid i < \lambda_\kappa\}$. A condition $p \in \mathscr{P}^s$ will be defined to be a special type of function from β-Card $\cap\, \kappa$ into L_κ which associates a pair $p(\gamma) = (p_\gamma, \bar{p}_\gamma)$ to each $\gamma \in \mathrm{Dom}(p)$. In addition we will have $p(\gamma) \in R^{p_{\gamma^+}}$, $p \restriction \gamma \in \mathscr{P}^{p_\gamma}$ for all γ as well as some special requirements at limit β-cardinals $\gamma \in \mathrm{Dom}(p)$. Extension is defined by: $p \leq q$ iff $p(\gamma) \leq q(\gamma)$ in $R^{p_{\gamma^+}}$ for all γ. Our coding strategy at singular cardinals is to define almost disjoint codes $\langle b_\xi \mid \xi < |s| \rangle$ so that $b_\xi \subseteq \bigcup\{((\gamma^\kappa_i)^+, (\gamma^\kappa_i)^{++}) \mid i < \lambda_\kappa\}$ and to arrange that $s(\xi) = 0$ iff b_ξ is almost disjoint from $\bar{p} = \bigcup\{\bar{p}_\gamma \mid \gamma \in \mathrm{Dom}(p)\}$. (In this context we say that $x, y \subseteq \kappa$ are *almost disjoint* if $x \cap y$ is bounded in κ.)

Now we must define the b_ξ's. Suppose that κ is a limit β-cardinal, $\gamma \in \beta$-Card $\cap\, \kappa$ and $\xi \in \mathscr{O}(\kappa) \cap \beta$. For any $n \leq \omega$ let $M^n_{\xi\gamma}$ = transitive collapse of the Σ_1-Skolem hull of $\gamma \cup \{\kappa\}$ inside $L_{\mu^n_\xi}$ (for $n = \omega$, $\mu^n_\xi = \mu^\omega_\xi = \mu_\xi = \sup\{\mu^n_\xi \mid n < \omega\}$). Then $\rho^n_{\xi\gamma}$ denotes the canonical $<_L$-code for $M^n_{\xi\gamma^+}$. For L_{μ_ξ}-singular κ we let $b_\xi = \{\langle 1, \rho_{\xi\gamma_i\kappa} \rangle \mid i < \lambda_\kappa\}$ where $\rho_{\xi\gamma} = \rho^\omega_{\xi\gamma}$. Thus $b_\xi \subseteq \bigcup\{((\gamma^\kappa_i)^+, (\gamma^\kappa_i)^{++}) \mid i < \lambda_\kappa\}$. For L_{μ_ξ}-inaccessible κ it is convenient to define b^n_ξ to be the function on $\kappa \cap \beta$-Card defined by $b^n_\xi(\gamma) = \langle 1, \rho^n_{\xi\gamma} \rangle$. Our strategy for coding at L_{μ_ξ}-

inaccessibles is to arrange that $s(\xi)=0$ iff for sufficiently large $n<\omega$, $\{\gamma \mid b^n_\xi(\gamma) \in \bar{p}\}$ is nonstationary in L_{μ_ξ}.

Having described the coding strategy at limit cardinals, we now turn to the major restraint that we wish to impose on our conditions. Suppose κ is a limit β-cardinal, $s \in S^\beta_\kappa$. For $|s|>\kappa$ let $s^*=\{\mu_{s\restriction\xi} \mid \xi \in \mathcal{O}(\kappa) \cap |s|\}$ and $\mathcal{P}^{<s}=\bigcup\{\mathcal{P}^{s\restriction\xi} \mid \xi \in \mathcal{O}(\kappa)\cap|s|\}$. Also define $\nu_s=\max(\mu_s^-, \mu_s^0)$. We require that if $p \in \mathcal{P}^s - \mathcal{P}^{<s}$, then $\{q \in P^{<s} \mid p \leq q\}$ is $\mathcal{P}^{<s}$-generic over $L_{\nu_s}[s^*]$, in a weak sense.

Definition. For $p \in \mathcal{P}^s$ and $\gamma<\kappa$ let $(p)_\gamma = p \restriction (\mathrm{Dom}(p)-\gamma)$ and $(p)^\gamma = p \restriction (\mathrm{Dom}(p)\cap\gamma)$. Suppose $\mathcal{D} \subseteq \mathcal{P}^{<s}$ is predense on $\mathcal{P}^{<s}$. Then $p \in \mathcal{P}^s$ *reduces* $\mathcal{D}$ if for some $\gamma<\kappa$, $\{q \in \mathcal{P}^{p_\gamma} \mid q \leq (p)^\gamma \text{ and } q \cup (p)_\gamma \leq \text{some element of } \mathcal{D}\}$ is dense below $(p)^\gamma$ in $\mathcal{P}^{p_\gamma}$.

Now it is certainly too much to require that $p \in \mathcal{P}^s - \mathcal{P}^{<s}$ be $\mathcal{P}^{<s}$-generic over $L_{\nu_s}[s^*]$ as this implies for example that for all $\eta<\gamma^+$, $|p_\gamma| \geq \eta$ which is ridiculous. However, it is reasonable to require that p at least reduces each predense $\mathcal{D} \subseteq \mathcal{P}^{<s}$, $\mathcal{D} \in L_{\nu_s}[s^*]$ and this is precisely what we do. This will enable us to establish properties for the limit codings $\mathcal{P}^{<s}$ which are analogous to those for the forcings R^s.

In order to deal with the special case $\nu_s=\mu_s^0$ is recursively inaccessible we must build a bit more 'predensity reduction' into the definition of our conditions. This further requirement is based on an effective version of $\Diamond$.

Lemma 1E.1. *Suppose α is admissible. There is a $\Delta_1(L_\alpha)$-sequence $\langle D_\beta \mid \beta<\alpha\rangle$ such that:*

(a) *$D_\beta \subseteq \beta$ for all β.*

(b) *If $D \subseteq \alpha$ is $\Delta_1(L_\alpha)$ and $C \subseteq \alpha$ is $\Sigma_1(L_\alpha)$ and closed unbounded in α, then there exists $\beta \in C$ such that $D \cap \beta = D_\beta$ (i.e., $\{\beta \mid D_\beta = D\cap\beta\}$ is Σ_1-stationary over L_α).*

Proof. It suffices to consider $\Delta_1(L_\alpha)$ closed unbounded C in (b) as any $\Sigma_1(L_\alpha)$CUB C contains a $\Delta_1(L_\alpha)$CUB set. Now let $\phi_0, \phi_1, \ldots$ be an effective listing of the $\Sigma_1(L_\alpha)$-formulas of one free variable and let $f:\alpha \to \alpha^*$ be a $\Sigma_1(L_\alpha)$-injection where $\alpha^*=\Sigma_1$-projectum of α. Define $\langle D_\beta \mid \beta<\alpha\rangle$ as follows. At stage β, pick the least j so that $f^{-1}(j)=i$ is defined by stage β, $\beta \in \mathrm{Lim}(\phi_{i_1})^{L_\beta}$ and for no $\beta' \in \mathrm{Lim}(\phi_{i_1})^{L_\beta} \cap \beta$ do we have $(\phi_{i_0})^{L_\beta} \cap \beta' = D_{\beta'}$. (We use $(\phi_i)^{L_\beta}$ to denote $\{\gamma<\beta \mid L_\beta \models \phi_i(\gamma)\}$ and $i=\langle i_0, i_1\rangle$.) If there is no such j let $D_\beta=\emptyset$ and otherwise let $D_\beta=(\phi_{i_0})^{L_\beta}$. This completes the construction.

Now say that $j<\alpha^*$ is *active* at stage β if action is taken as above at stage β to define $D_\beta=(\phi_{i_0})^{L_\beta}$ where $f^{-1}(j)=\langle i_0, i_1\rangle$. Now as the function $j \mapsto$(stage at which j is active) is a partial $\Sigma_1(L_\alpha)$-function it follows that for any $j<\alpha^*$ there is a stage β_j after which no $j'<j$ is active.

Suppose that $D \subseteq \alpha$ is $\Delta_1(L_\alpha)$ and $C \subseteq \alpha$ is both $\Delta_1(L_\alpha)$ and CUB. To verify

(b) we can assume that we have Σ_1-formulas ϕ_{i_0}, ϕ_{i_1} which define D, C, respectively and that $\beta \in C \rightarrow (\phi_{i_1})^{L_\beta} = C \cap \beta$, $(\phi_{i_0})^{L_\beta} = D \cap \beta$ (this may require thinning C). Let $j = f(\langle i_0, i_1 \rangle)$ and pick a stage $\beta \in \text{Lim}(C)$ greater than β_j. Then either $D \cap \beta' = D_{\beta'}$ for some $\beta' < \beta$ or $D_\beta = (\phi_{i_0})^{L_\beta} = D \cap \beta$. □

We next improve the result of Lemma 1E.1 in two ways. The first improvement is to require that the sequence of D_β's 'live on' some Δ_1-stationary set $E \subseteq \alpha$. ($E \subseteq \alpha$ is *Δ_1-stationary* if $E \cap C \neq \phi$ whenever $C \subseteq \alpha$ is both CUB and $\Delta_1(L_\alpha)$.) This is not difficult to arrange. The more serious improvement is to arrange that $\langle D_\beta \mid \beta < \alpha \rangle$ be independent of α, in the sense that $\alpha_1 < \alpha_2 \rightarrow$ the D_β-sequence for α_1 is an initial segment of that for α_2. Notice that the proof of Lemma 1E.1 made use of a $\Sigma_1(L_\alpha)$-injection $\alpha \rightarrow \alpha^*$ which requires a parameter $p(\alpha)$. The key idea for obtaining the desired independence of α is to 'guess' at this parameter.

Lemma 1E.2. *Suppose $E \cap \alpha$ is Δ_1-stationary and uniformly Δ_1 over L_α for all admissible α. Then there exists a sequence $\langle D_\beta \mid \beta \in E \rangle$ such that*

(a) *$D_\beta \subseteq \beta$ for all $\beta \in E$.*

(b) *$\{\beta \in E \mid D \cap \beta = D_\beta\}$ is Δ_1-stationary over L_α whenever $D \subseteq \alpha$ is $\Delta_1(L_\alpha)$, for all admissible α.*

(c) *$\langle D_\beta \mid \beta \in E \cap \alpha \rangle$ is uniformly $\Delta_1(L_\alpha)$ for admissible α.*

Proof. Repeat the proof of Lemma 1E.1 except only consider β's which belong to E and at stage β replace f by f^β defined as follows: Let $h : \omega \times L \rightarrow L$ be the canonical Σ_1-Skolem function for L and define $f^\beta(i) =$ least $\langle n, j \rangle < \beta^*$ so that $h(n, \langle j, p(\beta) \rangle) = i$, where 'least' is in the sense of the canonical enumeration of Graph(h). If $\beta^* = \beta$ we can assume that $f =$ identity. In addition, when defining D_β one should only consider $\beta' < \beta$ in E such that $p(\beta') = p(\beta)$, $(\beta')^* = \beta^*$.

Now the assertion of the second paragraph should be weakened to say that for any $j < \alpha^*$ there is a stage $\beta_j < \alpha$ such that no $j' < j$ is active at a stage $\beta \in (\beta_j, \alpha)$ such that $\beta^* = \alpha^*$, $p(\beta) = p(\alpha)$. This has content only if $\alpha^* < \alpha$, but note that it suffices to establish (b) for this case, as the general case can be reduced to it using reflection by considering successor α-stables (which are Σ_1-projectible).

Finally note that if $\alpha^* < \alpha$, then $\{\beta < \alpha \mid \beta^* = \alpha^*, p(\beta) = p(\alpha)\}$ is $\Delta_1(L_\alpha)$ and CUB. Thus in the final argument we can choose $\beta \in \text{Lim}(C) \cap E$ to belong to this set and thereby complete the proof. □

We have a particular Δ_1-stationary set E in mind and this is described in the next lemma.

Lemma 1E.3. *Suppose α is admissible. Then $E = \{\beta < \alpha \mid C_\beta$ is bounded in $\beta\}$ is Δ_1-stationary over L_α.*

Proof. We can assume that α is Σ_1-projectible ($\alpha^* < \alpha$) as otherwise the lemma can be reduced to this case by considering successor α-stables. Thus $\alpha^* = \kappa =$ largest α-cardinal.

Now given a $\Delta_1(L_\alpha)$ CUB set C pick a parameter $\gamma < \kappa$ so that C has a $\Delta_1(L_\alpha)$-definition with parameter $\langle p(\alpha), \gamma \rangle$. (This is possible as Σ_1-Skolem hull $(\kappa \cup \{p(\alpha)\})$ in $L_\alpha = L_\alpha$.) We can pick $\beta < \alpha$ so that $p(\beta) = p(\alpha)$, L_α-cofinality (β) is greater than γ and $C \cap \beta$ is CUB in β, $\Delta_1(L_\beta)$ with parameter $\langle p(\alpha), \gamma \rangle$. Let $\beta' = (\gamma+1)$st element of C_β, when C_β is enumerated in increasing order.

We claim that $\beta' \in C$. Indeed, it follows from Lemma 6.29(b) of Beller–Jensen–Welch [1] that $\beta' = \sup \Sigma_1$-Skolem hull$(\{\beta'', p(\beta)\})$ in L_β, where $\beta'' = \gamma$th element of C_β. But clearly $\gamma \in \Sigma_1$-Skolem hull$(\{\beta'', p(\beta)\})$ in L_β as $\gamma = \mathrm{ot}(C_{\beta''})$ so $C \cap \beta$ is $\Delta_1(L_\beta)$ with parameter from Σ_1-Skolem hull$(\{\beta'', p(\beta)\})$ in L_β. It follows that β' is the limit of elements of C and hence belongs to C. $\square$

Our last improvement of the preceding version of effective $\diamondsuit(E)$ is to adapt it to amenable structures $\langle L_\beta, s \rangle$ where β is p.r. closed and $s : \beta \to 2$. Thus let $\mathbf{a}, \mathbf{b}, \ldots$ range over structures of this form and set $\mathbf{E} = \{\mathbf{a} \mid C_\mathbf{a}$ is bounded in $\mathrm{ORD}(\mathbf{a})\}$ where $C_\mathbf{a}$ is defined as in [1, p. 210]. We wish to define $\langle D_\mathbf{b} \mid \mathbf{b} \in \mathbf{E} \rangle$ with properties analogous to those of Lemma 1E.2. This 'tree version' of $\diamondsuit$ is summarized in the next lemma.

Lemma 1E.4. (1) *If* $\mathbf{a}$ *is admissible, then for any* CUB $C \subseteq \mathrm{ORD}(\mathbf{a})$, C $\Delta_1(\mathbf{a}) \to \mathbf{a} \restriction \beta \in \mathbf{E}$ *for some* $\beta \in C$. ("$\mathbf{E} \cap \mathbf{a}$ *is* Δ_1*-stationary over* $\mathbf{a}$".)

(2) *There exists a sequence* $\langle D_\mathbf{b} \mid \mathbf{b} \in \mathbf{E} \rangle$ *such that*:

(a) $D_\mathbf{b} \subseteq \mathrm{ORD}(\mathbf{b})$ *for all* $\mathbf{b} \in \mathbf{E}$.

(b) *If* $\mathbf{a}$ *is admissible,* $D \subseteq \mathrm{ORD}(\mathbf{a})$, D $\Delta_1(\mathbf{a})$ *then* $\{\mathbf{b} \in E \mid \mathbf{b} \subseteq \mathbf{a},\ D \cap \mathrm{ORD}(\mathbf{b}) = D_\mathbf{b}\}$ *is* Δ_1*-stationary over* $\mathbf{a}$.

(c) $\langle D_\mathbf{b} \mid \mathbf{b} \subseteq \mathbf{a}, \mathbf{b} \in \mathbf{E} \rangle$ *is uniformly* $\Delta_1(\mathbf{a})$ *for admissible* $\mathbf{a}$.

Proof. Just like Lemmas 1E.2, 1E.3. $\square$

Remark. We are primarily interested of course in structures $\mathbf{b} = \langle L_\beta, s \rangle$ as above where $\beta \in \mathcal{O}(\kappa)$, $s \in S_\kappa$, $|s| = \beta$. Actually this 'tree version' could be discarded for the linear one if we only had a method for guaranteeing that $\alpha \in \mathcal{O}(\kappa)$, $s \in S_\kappa$, $|s| = \alpha$ and α recursively inaccessible $\to s$ is $\Pi\Sigma$-generic over L_α. This would enable us to arrange that any CUB $C \subseteq \alpha$ which is $\Delta_1 \langle L_\alpha, s \rangle$ in fact contains a CUB $C' \subseteq \alpha$ which is $\Delta_1(L_\alpha)$; this in turn would enable us to do all of our guessing with a fixed, uniformly $\Delta_1(L_\alpha)$ $\diamondsuit$-sequence.

We can now describe the added 'predensity reduction' requirement that we need. Fix κ a limit β-cardinal. Also fix the $\diamondsuit(\mathbf{E})$-sequence $\langle D_\mathbf{b} \mid \mathbf{b} \in \mathbf{E} \rangle$ of Lemma 1E.4 where $\mathbf{E} = \{\mathbf{b} \mid C_\mathbf{b}$ is bounded in $\mathrm{ORD}(\mathbf{b})\}$. For any $p \in \mathcal{P}^s$ where $s \in S_\kappa$, $\mu^0_\beta = \beta$, $\langle L_\beta, s \restriction \beta \rangle = \mathbf{b} \in \mathbf{E}$ we require the following: Suppose $D_\mathbf{b} =$ (the collection of ordinal codes for) a predense subset of $\mathscr{P}^{<s \upharpoonright \beta}$. Then we require that p reduces $D_{\mathbf{b}}$.

The purpose of the above restriction is to enable us to establish properties for the $\mathscr{P}^{<s}$ forcings analogous to those of Part D for R^s, especially the fact that the forcing relation for ranked sentences is Δ_1 for $\mathscr{P}^{<s}$ when μ_s^0 is recursively inaccessible. We now begin a discussion of these properties.

Lemma 1E.5 (Distributivity for $\mathscr{P}_\gamma^{<s}$). *Suppose* $\gamma \in \beta$-Card $\cap\, \kappa$ *and* $\mathscr{P}_\gamma^{<s} = \{(p)_\gamma \mid p \in \mathscr{P}^{<s}\}$. *Then* $\mathscr{P}_\gamma^{<s}$ *is* $(<\gamma^+)$*-distributive in* $L_{\nu_s}[s^*]$, *whenever* ν_s *is recursively inaccessible.*

Proof. Deferred.

Lemma 1E.6 (Genericity Property for $\mathscr{P}^{<s}$). *Suppose* $s \subseteq t$ *belong to* S_κ *and* $\mathscr{D} \in L_{\nu_s}[s^*]$ *is predense on* $\mathscr{P}^{<s}$. *Then* $\mathscr{D}$ *is predense on* $\mathscr{P}^t$.

Proof. It suffices to show that if $p \in \mathscr{P}^s - \mathscr{P}^{<s}$, then p is compatible with some element of $\mathscr{D}$. By definition, p reduces $\mathscr{D}$ so there is $\gamma \in \beta$-Card $\cap\, \kappa$ so that $\{q \in \mathscr{P}^{p_\gamma} \mid q \leq (p)^\gamma$ and $q \cup (p)_\gamma \leq$ some element of $\mathscr{D}\}$ is dense below $(p)^\gamma$ in $\mathscr{P}^{p_\gamma}$. Thus in fact there exists $p' \leq p$ so that $(p')_\gamma = (p)_\gamma$ and $p' \leq$ some element of $\mathscr{D}$. So p is compatible with some element of $\mathscr{D}$ (in a strong sense). □

Lemma 1E.7 (Chain Condition for $\mathscr{P}^{<s}$). *Suppose* $\mu_s^0 \leq \beta'$ *where* β' *is recursively inaccessible,* $\beta' = \mu_s^0$ *or* $L_{\beta'} \models \mu_s^0$ *is a cardinal. Also suppose that* β' *is admissible relative to* $s \upharpoonright \mu_s^0$ *and* $\beta' = \mu_s^0$ *or* $L_{\beta'}[s \upharpoonright \mu_s^0] \models \mu_s^0$ *is a cardinal. Then* $\mathscr{P}^{<s}$ *has the* Σ-κ^+-c.c. *in* $L_{\beta'}[s \upharpoonright \mu_s^0]$.

Remark. We will eventually be able to conclude that the second hypothesis is redundant, for recursively inaccessible β'

Proof. As in the proof of Lemma 1D.6 in the case $\beta' > \mu_s^0$ we have that $\mathscr{P}^{<s}$ is an element of $L_{\beta'}[s \upharpoonright \mu_s^0]$ and thus any $\Sigma_1(L_{\beta'}[s \upharpoonright \mu_s^0])$ predense $\mathscr{D}$ contains a predense $\mathscr{D}^* \in L_{\beta'}[s \upharpoonright \mu_s^0]$. But then for some $\xi = \mu_\xi^0 < \mu_s^0$ we must have that $\mathscr{D}^* \cap \mathscr{P}^{<s \upharpoonright \xi}$ belongs to $L_{\nu_{s \upharpoonright \xi}}[(s \upharpoonright \xi)^*]$ and is predense on $\mathscr{P}^{<s \upharpoonright \xi}$. Now it follows from predensity reduction that $\mathscr{D}^* \cap \mathscr{P}^{<s \upharpoonright \xi} \subseteq \mathscr{D} \cap L_\xi$ is predense on $\mathscr{P}^{<s}$.

So suppose that $\beta' = \mu_s^0$ and $\mathscr{D} \subseteq \mathscr{P}^{<s}$ is $\Delta_1(L\mu_s^0[s \upharpoonright \mu_s^0])$ and predense on $\mathscr{P}^{<s}$. Then $C = \{\xi < \mu_s^0 \mid \xi = \mu_\xi^0$ and $\mathscr{D} \cap L_\xi = \mathscr{D}^{L_\xi}$ is predense on $\mathscr{P}^{<s \upharpoonright \xi}\}$ is $\Delta_1(L_{\mu_s^0}[s \upharpoonright \mu_s^0])$ and CUB (in μ_s^0). By the $\diamondsuit(\mathbf{E})$-property of $\langle D_{\mathbf{b}} \mid \mathbf{b} \in \mathbf{E}\rangle$ we can find $\beta \in C$ so that $\langle L_\beta, s \upharpoonright \beta\rangle = \mathbf{b} \in \mathbf{E}$ and $D_{\mathbf{b}} =$ (the collection of ordinal codes for) $\mathscr{D} \cap L_\beta$. Then $\mathscr{D} \cap L_\beta$ is predense on $\mathscr{P}^{<s \upharpoonright \beta}$ so by predensity reduction $\mathscr{D} \cap L_\beta$ is predense on $\mathscr{P}^{<s}$. As any $\Sigma_1(L\mu_s^0[s \upharpoonright \mu_s^0])$ predense $\mathscr{D} \subseteq \mathscr{P}^{<s}$ contains a $\Delta_1(L\mu_s^0[s \upharpoonright \mu_s^0])$ predense set, we are done. □

Lemma 1E.8 (Δ_1-Definability of Forcing for $\mathcal{P}^{<s}$). *Suppose that* s, β' *are as in the preceding lemma. Then the relation* $p \Vdash \phi$ ($p \in \mathcal{P}^{<s}$, ϕ *a ranked sentence of* $L_{\beta'}[s \restriction \mu_s^0]$) *is* Δ_1 *over* $L_{\beta'}[s \restriction \mu_s^0]$.

Proof. Just as in Lemma 1D.7. □

F. The definition of $\mathcal{P}_\gamma^s$

We can now define the desired forcing. We first define $\mathcal{P}_\gamma^s$ for $s \in S_\kappa^\beta$ where $\gamma, \kappa \in \beta$-Card, $\gamma < \kappa$ by induction on β, $\underline{\kappa, |s|}$. In addition the definition of $\mathcal{P}_\gamma^s$ will depend on that of $\mathcal{P}_{\gamma'}^{\beta'}$ for $\beta' \in \overline{\text{Adm}} \cap \beta$, $\gamma' \in \beta'$-Card and the latter definition will be given afterward. (This is actually a double induction.) It is worthwhile to first isolate a few properties of the forcing.

(a) A condition $p \in \mathcal{P}_\gamma^s$ will be a function on β-Card $\cap [\gamma, \kappa)$ which assigns to each δ a pair $(p_\delta, \bar{p}_\delta) \in R^{p_{\delta+}}$.

(b) Each $p \in \mathcal{P}_\gamma^\beta$ will be of the form $p' \cup \{(\kappa', (s', \emptyset))\}$ for some $p' \in \mathcal{P}_\gamma^{s'}$, $s \in S_{\kappa'}^{\beta'}$, $\beta' \leq \beta$. (We also allow p' to be empty, $\kappa' = \gamma$.)

(c) If $p, q \in \mathcal{P}_\gamma^s$, then $p \leq q$ iff $p(\delta) \leq q(\delta)$ in $R^{p_{\delta+}}$ for each $\delta < \kappa$.

(d) If $p, q \in \mathcal{P}_\gamma^\beta$, then we define $p \leq q$ as follows: Let $\delta = \max(\text{Dom}(p) \cap \text{Dom}(q))$. If $\delta = \max(\text{Dom}(q))$, then we say $p \leq q$ iff $(p)^\delta \leq (q)^\delta$ in $\mathcal{P}_\gamma^{p_\delta}$, $q_\delta \subseteq p_\delta$. (Recall that $(r)^\delta = r \restriction [\gamma, \delta)$ and $(r)_\delta = r - (r)^\delta$.) Otherwise we must have $q \in \mathcal{P}_\gamma^{\beta'}$ with $\beta' \leq |p_\delta|$ and we insist that both $(p)^\delta \leq (q)^\delta$ in $\mathcal{P}_\gamma^{p_\delta}$ and that $(q)_\delta \in \mathcal{P}_\delta^{\beta'}$ is a condition in the $\mathcal{P}_\delta^{\beta'}$-generic determined by $p_\delta \restriction \beta'$. (Note that as $p_\delta \in S_\delta^\beta$ it follows that $p_\delta \restriction \beta'$ is $\mathcal{P}_\delta^{\beta'}$-generic over $L_{\beta'}$.)

We now define $\mathcal{P}_\gamma^s$ when $s \in S_\kappa^\beta$, $\kappa \in \beta$-Card, $\gamma \in \beta$-Card $\cap \kappa$. A condition $p \in \mathcal{P}_\gamma^s$ is a function in L_{μ_s} that assigns to each $\delta \in \beta$-Card $\cap [\gamma, \kappa)$ a pair $(p_\delta, \bar{p}_\delta)$ such that either $p \in \mathcal{P}_\gamma^{<s} = \bigcup \{\mathcal{P}_\gamma^{s \restriction \xi} \mid \xi \in \mathcal{O}(\kappa) \cap |s|\} \cup \mathcal{P}_\gamma^\kappa$ or:

(1) (Smoothness) $(p)^\delta \in \mathcal{P}_\gamma^{p_\delta}$ for all $\delta \in \text{Dom}(p)$, $\delta > \gamma$. If γ is a limit β-cardinal in Dom(p), $\delta > \gamma$, then $(p)^\delta \in \mathcal{P}_\gamma^{p_\delta} - \mathcal{P}_\gamma^{<p_\delta}$.

(2) If $\kappa = \delta^+$ is a successor β-cardinal, then $p(\delta) \in R^s$.

(3) (Predensity Reduction) If κ is a limit β-cardinal, then:

(a) p reduces all predense $\mathcal{D} \subseteq \mathcal{P}^{<s}$ which belong to $L_{\nu_s}[s^*]$. If ν_s is recursively inaccessible, $T \subseteq \mathcal{P}^{<s} \times \gamma'$, $\gamma' < \nu_s$ is persistent and $\Sigma_1 \langle L_{\nu_s}[s^*], s^* \rangle$, then p reduces $\mathcal{D}(T)$.

(b) If $\mathbf{b} = \langle L_{\mu_s^0}, s \restriction \mu_s^0 \rangle \in \mathbf{E}$ and $D_\mathbf{b}$ is predense on $\mathcal{P}^{<s}$, then p reduces $D_\mathbf{b}$.

(4) (Coding) (i) If $L_{\mu_\xi} \models \kappa$ is singular, then $s(\xi) = 0$ iff $b_\xi \cap \bigcup \{\bar{p}_\delta \mid \delta \in \text{Dom}(p)\}$ is bounded in κ, for all $\xi \in \mathcal{O}(\kappa) \cap |s|$.

(ii) If $L_{\mu_\xi} \models \kappa$ is inaccessible, then $s(\xi) = 0$ iff for sufficiently large $i < \omega$, $\{\delta \in \text{Dom}(p) \mid b_\xi^i(\delta) \in \bar{p}_{\delta+}\}$ is non-stationary in L_{μ_ξ}, for all $\xi \in \mathcal{O}(\kappa) \cap |s|$.

(5) (Code Thinning) If κ is a limit β-cardinal and p is $\Sigma_n(L_n)$, κ $\Sigma_n(L_\eta)$-regular, then there is a CUB $C \subseteq \kappa$, C $\Sigma_n(L_\eta)$ so that $\delta \in C \to \bar{p}_\delta = \emptyset$.

(6) (Growth Condition) If κ is a limit β-cardinal, $\kappa\, \Sigma_n(L_\eta)$-singular, p is

$\Sigma_n(L_\eta) - \Sigma_{n-1}(L_\eta)$, then there is a CUB $C \subseteq \kappa \cap \beta$-Card, C $\Sigma_n(L_\eta)$ so that $\delta \in C \to C \cap \delta \in \Sigma_{n(\delta)}(L_{\eta(\delta)})$ (where $p \restriction \delta \in \Sigma_{n(\delta)}(L_{\eta(\delta)}) - \Sigma_{n(\delta)-1}(L_{\eta(\delta)})$) and such that given any pair $(\bar\eta, \bar n) < (\eta, n)$ (lexicographically) and parameter $x \in L_{\bar\eta}$, $|p_\delta| \geqslant H_\delta \cap \delta^+$ for sufficiently large $\delta \in C$, where $H_\delta = \Sigma_{\bar n}$-Skolem hull of $\delta \cup \{\kappa, x\}$ in $L_{\bar\eta}$.

(7) (Restriction) If κ is a limit β-cardinal, then $\xi \in \mathcal{O}(\kappa) \cap |s| \to p \leqslant q$ for some $q \in \mathcal{P}^{s \restriction \xi}_\gamma - \mathcal{P}^{<s \restriction \xi}_\gamma$.

This completes the definition of $\mathcal{P}^s_\gamma$. Now we define $\mathcal{P}^\beta_\gamma$ ($\beta \in \overline{\mathrm{Adm}}$, $\gamma \in \beta$-Card) to consist of all conditions of the form $p' \cup \{(\kappa', (s', \emptyset))\}$ where for some $\beta' \leqslant \beta$, $\kappa' \in \beta'$-Card, $\kappa' \geqslant \gamma$ and $s' \in S^{\beta'}_{\kappa'}$ we have $p' \in \mathcal{P}^{s'}_\gamma$ and κ' a limit β'-cardinal$\to$ $p' \notin \mathcal{P}^{<s'}_\gamma$. We allow the possibility $p' = \emptyset$, $\kappa' = \gamma$. Thus if $\gamma = \mathrm{gc}\,\beta$, then $\mathcal{P}^\beta_\gamma$ is essentially the same as S^β_γ. Extension of conditions is defined as in (c), (d) above.

We also introduce the following notation. If $p \in \mathcal{P}^s_\gamma$, then $|p|$ denotes the least ξ such that $p \in \mathcal{P}^{s \restriction \xi}_\gamma$. Note that we require in (1) of the definition of $\mathcal{P}^s_\gamma$ that for limit β-cardinals $\delta \in \mathrm{Dom}(p)$, $\delta > \gamma$: $|(p)^\delta| = |p_\delta|$.

Finally set $\mathcal{P}^s$, $\mathcal{P}^\beta$ equal to $\mathcal{P}^s_0$, $\mathcal{P}^\beta_0$, respectively, and $\mathcal{P}_\gamma = \bigcup \{\mathcal{P}^\beta_\gamma \mid \beta \in \overline{\mathrm{Adm}}\}$, $\mathcal{P} = \mathcal{P}_0$. Our aim is to show that $\mathcal{P} \Vdash \mathrm{ZFC}$ and $\exists R \subseteq \omega$ (R-admissibles = Recursively Inaccessibles).

Remarks. (i) The Restriction Property (7) implies that Predensity Reduction holds in the stronger form: If $\xi = \mathcal{O}(\kappa)$, $\xi \leqslant |s|$, then p reduces all predense $\mathcal{D} \subseteq \mathcal{P}^{<s \restriction \xi}$ which belong to $L_{\nu_\xi}[(s \restriction \xi)^*]$. And, if $\mathbf{b} = \langle L_{\mu^0_\xi}, s \restriction \mu^0_\xi \rangle \in \mathbf{E}$, $\xi \leqslant |s|$, then $D_\mathbf{b}$ predense on $\mathcal{P}^{<s \restriction \xi} \to p$ reduces $D_\mathbf{b}$.

(ii) Both the Growth Condition and Code Thinning are useful in the proof of the Extension Lemma in the next section.

(iii) As in [1, p. 45] if $s \in S^\beta_\kappa$ where κ is a limit β-cardinal and $p \in \mathcal{P}^s_\gamma$, $|p| = \xi$, then for sufficiently large $n < \omega$ there exists $\gamma_0 < \kappa$ such that $\delta \in \mathrm{Dom}(p) - \gamma_0 \to$ $\rho^n_{\xi\delta} > |p_{\delta+}|$. Thus p has not yet coded $s(\xi)$.

The rest of this part is devoted to establishing earlier lemmas whose proofs have been deferred, with the assumption of distributivity and extendibility. Both distributivity and the following form of extendibility will be established in Section Two.

Lemma 1F.1 (Extension Lemma). *Suppose $p \in \mathcal{P}^\beta_\kappa$ and $f \in L_\beta$, $\mathrm{Dom}(f) \subseteq \beta$-Card is thin in L_β ($L_\beta \vDash \mathrm{Dom}(f) \cap \kappa'$ is nonstationary for regular κ'), $f(\gamma) < (\gamma^+)^{L_\beta}$ for each $\gamma \in \mathrm{Dom}(f)$. Then there exists $q \leqslant p$ in $\mathcal{P}^\beta_\kappa$ such that $\gamma \in \mathrm{Dom}(f)$, $\gamma \geqslant \kappa \to$ $\gamma \in \mathrm{Dom}(q)$, $|q| \geqslant f(\gamma)$. The same is true for $\mathcal{P}^s_\kappa$, with L_β replaced by L_{μ_s}.*

Thus we wish to reduce the proof of our theorem to that of Lemmas 1A.6 and 1F.1. We must establish Lemmas 1A.1, 1A.2, 1A.3, 1A.5, 1A.7, 1A.8, 1B.5, 1B.6, the lemma immediately preceding the proof of Lemma 1D.2 and Lemma 1E.5.

Proof of Lemma 1A.1. (a) and the first statement of (b) are clear from the definition of $\mathcal{P}_\kappa^\beta$. (Note that in fact there is a uniformly $\Delta_1(L_\beta)$-definition of $\mathcal{P}_\kappa^\beta$ for β's which are limits of admissibles: $\mathcal{P}_\kappa^\beta = \bigcup\{\mathcal{P}_\kappa^{\beta'} \mid \beta' \in \beta \cap \mathrm{Adm}\}$.) To prove the second statement of (b), suppose that $p, q \in \mathcal{P}_\kappa^{\beta'}$ are compatible in $\mathcal{P}_\kappa^\beta$; let $r \leqslant p, q$ belong to $\mathcal{P}_\kappa^\beta$. If $\mathrm{Dom}(p) = \mathrm{Dom}(q)$, then in fact p, q have a lower bound r_0 defined by $r_0(\delta) = (r_{0\delta}, \bar{p}_\delta \cup \bar{q}_\delta)$ where $r_{0\delta} \supseteq p_\delta \cup q_\delta$ and $r_0 \in \mathcal{P}_\kappa^{\beta'}$. Otherwise assume without loss of generality that $\delta = \max(\mathrm{Dom}(p) \cap \mathrm{Dom}(q)) < \max(\mathrm{Dom}(p))$. If $\delta < \max(\mathrm{Dom}(q))$, then assume without loss of generality that $\delta' =$ (least $\delta' \in \mathrm{Dom}(q)$ greater than δ) is greater than $\max(\mathrm{Dom}(p))$. (This follows if $\delta' >$ least $\gamma \in \mathrm{Dom}(p)$ greater than δ.) But then we can assume that $(r)_{\delta'} = (q)_{\delta'}$ and hence $r \in \mathcal{P}_\kappa^{\beta'}$. Finally suppose that $\delta = \max(\mathrm{Dom}(q))$. If $|q_\delta| < \gamma =$ least $\gamma \in \mathrm{Dom}(p)$ greater than δ, then p, q have the lower bound r_0 as defined earlier. If $|q_\gamma| \geqslant \gamma$, then in fact $|q_\gamma| \geqslant \max(\mathrm{Dom}(p))$ and p, q have the lower bound r_1 defined by $\mathrm{Dom}(r_1) = \mathrm{Dom}(q)$, $r_1(\eta) = (r_{1\eta}, \bar{p}_\eta \cup \bar{q}_\eta)$ where $r_{1\eta} \supseteq p_\eta \cup q_\eta$ for $\eta < \delta$, $r_1(\delta) = q(\delta)$. The proof of the last statement in (b) is similar. □

Proof of Lemma 1A.2. We must describe the 'decoding process'. Suppose $X_0 \subseteq [\kappa, (\kappa^+)^{L_\beta})$. We first describe the decoding function $f'(\kappa, \beta, X_0)$ which produces $X \subseteq [\kappa, \beta)$. By induction on $\xi \in [\kappa, \beta)$ we define $c(\xi)$, where $c =$ characteristic function of X. If $\xi < (\kappa^+)^{L_\beta}$, then $c(\xi) = 1$ iff $\xi \in X_0$. Otherwise let $\mu \geqslant \xi$ be least so that either $\mu \in \mathrm{Adm}$ or $L_\mu \vDash \xi \in \mathcal{O}(\gamma)$, where $\gamma \in \beta$-Card. Such a μ exists as if the former case fails, then $\beta = \tilde{\beta}'$ where $\beta' \leqslant \xi < \beta$ and $\xi \in \mathcal{O}(\mathrm{gc}\,\beta)$. In the former case let $\gamma = \mu\text{-card}(\xi) = \mathrm{gc}\,\mu$. Then $\xi \in \mathcal{O}(\gamma)$ and if γ is a successor μ-cardinal, then $c(\xi) = 1$ iff b_ξ is almost disjoint from $X \cap \gamma$; if γ is a limit μ-cardinal, then we decode $c(\xi)$ inside $L_{\mu_{\xi+1}}$ as in (4) of the definition of $\mathcal{P}_\kappa^s$. In the latter case the decoding is the same, using the latter definition of the β-cardinal γ.

It is easy to see that $X \subseteq [\kappa, \beta)$ so defined is $\Delta_1\langle L_\beta[X_0], \beta\text{-Card}\rangle$, uniformly in κ, β. Notice that it is important when decoding $c(\xi)$ to isolate an admissible μ and μ-cardinal γ so that $\xi \in \mathcal{O}(\gamma)$.

Now to define $f(\kappa, \beta, X)$. If $X \neq f'(\kappa, \beta, X \cap (\kappa^+)^{L_\beta})$, then $f(\kappa, \beta, X) = \emptyset$. Otherwise $p \in f(\kappa, \beta, X)$ iff for all $\delta \in \mathrm{Dom}(p)$, $p_\delta \subseteq c =$ characteristic function of X and $b \in \bar{p}_\delta \to S(b) \cap X \subseteq \bar{p}_\delta$. Thus $f(\kappa, \beta, X)$ is uniformly Δ_1 over $\langle L_\beta[X \cap (\kappa^+)^{L_\beta}], \beta\text{-Card}\rangle$. If G is $\mathcal{P}_\kappa^\beta$-generic over L_β, then let $X_G = \bigcup\{\bar{p}_\delta \mid p \in G, \delta \in \mathrm{Dom}(p)\}$. Clearly $p \in G \to p \in f(\kappa, \beta, X_G)$ and X_G is determined by the condition $f(\kappa, \beta, X_G) = G$. Also $c(\eta) = 1$, $\eta < \beta' \in \mathrm{Adm} \to \exists p \in G \cap L_{\beta'}$, $\bar{p}(\eta) =$ 1: For, if $p_\delta(\eta) = 1$, $p \in G$, then either $(p)^\delta \cup \{(\delta, (p_\delta, \emptyset))\} \in G \cap L_{\beta'}$ or $p_\delta \restriction \beta'$ is $\mathcal{P}_\delta^{\beta'}$-generic over $L_{\beta'}$, in which case $(p)^\delta \cup q \in G \cap L_{\beta'}$ for some $q \in \mathcal{P}_\delta^{\beta'}$, $\eta \in \bar{q}_\delta$. The same is true with 1 replaced by 0. Thus, we can conclude that $X_G \cap \beta'$ is uniformly Δ_1 over $L_{\beta'}[G]$ for admissible $\beta' \leqslant \beta$.

To show that $p \in f(\kappa, \beta, X_G) \to p \in G$ we need only show that $p, q \in f(\kappa, \beta, X_G) \to p, q$ are compatible (see condition (ii) in the definition of $\mathcal{P}$-

generic). We establish this by induction on β. If $\mathrm{Dom}(p)=\mathrm{Dom}(q)$, then it is easy to check that there is a condition r defined by $r(\delta)=(r_{1\delta}, \bar{p}_\delta \cup \bar{q}_\delta)$ where $r_{1\delta} \supseteq p_\delta \cup q_{\delta'}$. Otherwise suppose without loss of generality that $\delta = \max(\mathrm{Dom}(p) \cap \mathrm{Dom}(q)) < \max(\mathrm{Dom}(p))$ and let $\gamma = \text{least } \gamma \in \mathrm{Dom}(p)$ greater than δ. We can assume without loss of generality that either $\delta = \max(\mathrm{Dom}(q))$ or $\gamma' = (\text{least } \gamma' \in \mathrm{Dom}(q) \text{ greater than } \delta)$ is greater than $\max(\mathrm{Dom}(p))$. If $|q_\delta| \geqslant \gamma$, then for some $\beta' \leqslant \beta$, $p \in \mathcal{P}_\kappa^{\beta'}$ and $q_\delta \restriction \beta'$ is $\mathcal{P}_\delta^{\beta'}$-generic over $L_{\beta'}$, hence it follows by induction that $(p)_\delta \in$ generic determined by $q_\delta \restriction \beta'$. In this case $r \leqslant p, q$ where $r(\eta)=(r_\eta, \bar{p}_\eta \cup \bar{q}_\eta)$ and $r_\eta \supseteq p_\eta \cup q_\eta$ for $\eta < \delta$, $(r)_\delta = (q)_\delta$. If $|q_\delta| < \gamma$ and $\delta = \max(\mathrm{Dom}(q))$, then $r \leqslant p, q$ where r is defined as in the case $\mathrm{Dom}(p) = \mathrm{Dom}(q)$.

There remains the case where $\delta < \max(\mathrm{Dom}(q))$ (and so $\gamma' > \max(\mathrm{Dom}(p))$ is defined) yet $|q_\delta| < \gamma$. By the Extension Lemma 1F.1 choose $r^* \in G$ so that for some $\delta' \leqslant \delta$, $|r^*_{\delta'}| \geqslant \max(\mathrm{Dom}(p))$. Then $r^*_{\delta'} \restriction \beta'$ is $\mathcal{P}_{\delta'}^{\beta'}$-generic over $L_{\beta'}$ where $\beta' = \sup(|r^*_{\delta'}| \cap \mathrm{Adm})$. It follows that $r^*_{\delta'} \restriction [\delta, \beta')$ is $\mathcal{P}_\delta^{\beta'}$-generic over $L_{\beta'}$ and hence by induction $(p)_\delta$, $(q)_\delta \restriction \beta'$-Card belong to the generic determined by $r^*_{\delta'} \restriction [\delta, \beta')$. Finally let r be defined by $r(\eta)=(r_\eta, \bar{p}_\eta \cup \bar{q}_\eta)$ and $r_\eta \supseteq p_\eta \cup q_\eta$ for $\eta < \delta'$; $r(\delta') = r^*(\delta')$ if $\gamma' < \beta'$, $r(\delta') = q(\delta')$ if $\gamma' \geqslant \beta'$ (and hence $\delta' = \delta$); $(r)_{\gamma'} = (q)_{\gamma'}$ if $\gamma' \geqslant \beta'$. Then $r \leqslant p, q$. □

Before turning to Lemma 1A.3, we first establish the Factoring property.

Proof of Lemma 1A.7. We let $\mathbf{G}_\kappa$ denote a name for $\bigcup\{p_\kappa \mid p \in G\}$ (where $\mathbf{G}$ names the generic object for $\mathcal{P}_\kappa^\beta$) and $\mathcal{P}_\gamma^{\mathbf{G}_\kappa}$ denotes $\bigcup\{\mathcal{P}_\gamma^s \mid s \in S_\kappa^\beta,\ s \subseteq \mathbf{G}_\kappa\}$. The fact that $\mathcal{P}_\gamma^\beta$ is equivalent to $\mathcal{P}_\kappa^\beta * \mathcal{P}_\gamma^{\mathbf{G}_\kappa}$ is clear. The fact that $\mathcal{P}_\kappa^\beta \Vdash \mathcal{P}_\gamma^{\mathbf{G}_\kappa}$ has the Σ-κ^+-c.c. if β is recursively inaccessible follows from Lemmas 1D.6, 1E.7: Use 1E.7 if κ is a limit β-cardinal and 1D.6 together with a factoring $\mathcal{P}_\kappa^\beta * R^{\mathbf{G}_\kappa} * \mathcal{P}_\gamma^{\mathbf{G}_\delta}$ if $\kappa = \delta^+$. In both cases we must know that if $s : [\kappa, \beta) \to 2$ is $\mathcal{P}_\kappa^\beta$-generic over L_β, then s preserves $(\kappa^+)^{L_\beta}$ and if in addition s is $\mathcal{P}_\kappa^\beta$-Σ-generic over L_β, then s preserves admissibility. The former assertion follows from distributivity. To prove the latter assertion first note that by Σ-distributivity, if $\kappa' = \mathrm{gc}\,\beta$, then $s \restriction [\kappa', \beta)$ preserves admissibility as it is Σ-generic and the forcing relation for $\mathcal{P}_{\kappa'}^\beta$ is clearly Δ_1 when restricted to ranked sentences. And $\mathcal{P}_\kappa^{G_{\kappa'}}$ has the $(\kappa')^+$-Σ-c.c. in $L_\beta[G_{\kappa'}]$ where $G_{\kappa'} =$ generic corresponding to $s \restriction [\kappa', \beta)$. By Lemmas 1D.7, 1E.8 the forcing relation for $\mathcal{P}_\kappa^{G_{\kappa'}}$ is Δ_1 when restricted to ranked sentences. Now we can show admissibility-preservation by the usual antichain argument: Suppose $\mathbf{f}$ is a name for a $\Sigma_1(L_\beta[G_{\kappa'}])$-function from κ' into β and $p \in \mathcal{P}_\kappa^{G_{\kappa'}}$, $p \Vdash \mathbf{f}$ is total. By the $(\kappa')^+$-Σ-c.c. of $\mathcal{P}_\kappa^{G_{\kappa'}}$, for each $\gamma < \kappa'$ there is a predense below p set $D_\gamma \in L_\beta[G_{\kappa'}]$ such that $q \in D_\gamma \to$ for some $\delta < \beta$, $q \Vdash \mathbf{f}(\gamma) = \delta$. By the admissibility of β, $p \Vdash \mathrm{Range}(\mathbf{f}) \subseteq \delta_0$, for some $\delta_0 < \beta$. We have shown that $\mathcal{P}_{\kappa'}^\beta \Vdash KP$ and $\mathcal{P}_{\kappa'}^\beta \Vdash (\mathcal{P}_\kappa^{G_{\kappa'}} \Vdash KP)$. So $\mathcal{P}_\kappa^\beta \Vdash KP$ and the Σ-genericity of s implies that s preserves admissibility. □

We return now to the Genericity Lemma.

Proof of Lemma 1A.3. It suffices to show that for $\beta \in \overline{\mathrm{Adm}}$, $\kappa \in \tilde{\beta}$-Card $\kappa < \beta$ we have G $\mathcal{P}_\kappa^{\tilde{\beta}}$-generic over $L_{\tilde{\beta}} \rightarrow G \cap \mathcal{P}_\kappa^\beta$ is $\mathcal{P}_\kappa^\beta$-generic over L_β, and if in addition β is recursively inaccessible, then G $\mathcal{P}_\kappa^{\tilde{\beta}}$-generic over $L_{\tilde{\beta}} \rightarrow G \cap \mathcal{P}_\kappa^\beta$ is $\mathcal{P}_\kappa^\beta$-Σ-generic over L_β. First suppose that $\kappa' = \mathrm{gc}\,\tilde{\beta} < \beta$. Then $\mathcal{P}_\kappa^{\tilde{\beta}}$ is equivalent to $\mathcal{P}_{\kappa'}^{\tilde{\beta}} * \mathcal{P}_\kappa^{\mathbf{G}_{\kappa'}}$ (the latter is empty if $\kappa = \kappa'$) and $\mathcal{P}_\kappa^\beta$ is equivalent to $\mathcal{P}_{\kappa'}^\beta * \mathcal{P}_\kappa^{\mathbf{G}_{\kappa'}^\beta}$. If $s : [\kappa', \tilde{\beta}) \rightarrow 2$ is $\mathcal{P}_{\kappa'}^{\tilde{\beta}}$-generic over $L_{\tilde{\beta}}$, then by definition of $S_{\kappa'}^{\tilde{\beta}}$, $s \upharpoonright [\kappa', \beta)$ is $\mathcal{P}_{\kappa'}^\beta$-generic over L_β (is $\mathcal{P}_{\kappa'}^\beta$-$\Sigma$-generic over L_β if β is recursively inaccessible) and by Lemmas 1D.5, 1E.6 any $\mathcal{P}_\kappa^{<s}$-generic over $L_{\tilde{\beta}}[s]$ is $\mathcal{P}_\kappa^{<s \upharpoonright [\kappa', \beta)}$-generic over $L_\beta[s]$. So we conclude that any $\mathcal{P}_\kappa^{\tilde{\beta}}$-generic over $L_{\tilde{\beta}}$ is $\mathcal{P}_\kappa^\beta$-generic over L_β (when intersected with $\mathcal{P}_\kappa^\beta$). Also if β is recursively inaccessible, then we get $\mathcal{P}_\kappa^\beta$-Σ-genericity over L_β, using the second statement of part (a) of Predensity Reduction (when κ' is a limit $\tilde{\beta}$-cardinal). If $\mathrm{gc}\,\tilde{\beta} = \beta$, then we can again use the second statement of part (a) of Predensity Reduction to obtain the desired result (when β is a limit $\tilde{\beta}$-cardinal. Note that in this case $\nu_{s \upharpoonright [\kappa', \beta)} = \nu_\emptyset = \beta$, β is recursively inaccessible and $\mathcal{P}_\kappa^{<s \upharpoonright [\kappa', \beta)} = \mathcal{P}_\kappa^\beta$.) If β is a successor $\tilde{\beta}$-cardinal, then use Factoring and Lemma 1D.5 when $s \in S_\alpha$, $|s| = \alpha$ and thus $R^s = \mathcal{P}_\gamma^\alpha$. □

Proof of Lemma 1A.5. This follows immediately from the extension Lemma. □

Proof of Lemma 1A.8. By induction on β. If $L_\beta \models$ (there is a largest cardinal), then we are done by Lemmas 1D.7, 1E.8 and Factoring. (If $\mathrm{gc}\,\beta = \kappa'$ is a successor β-cardinal, then factor $\mathcal{P}^\beta$ as $\mathcal{P}_{\kappa'}^\beta * \mathcal{P}_\delta^{\mathbf{G}_{\kappa'}} * \mathcal{P}^{\mathbf{G}_\delta}$, $\kappa' = \delta^+$. The Δ_1-definability of ranked forcing is clear for $\mathcal{P}_{\kappa'}^\beta$, follows from Lemma 1D.7 for $\mathcal{P}_\delta^{\mathbf{G}_{\kappa'}} = R^{\mathbf{G}_{\kappa'}}$ and is clear for the set forcing $\mathcal{P}^{\mathbf{G}_\delta}$). If there is no largest β-cardinal, then use Lemma 1A.3 and induction to write $p \Vdash \phi \leftrightarrow (p \Vdash \phi$ in $\mathcal{P}_\kappa^{\beta'})$ for an admissible $\beta' < \beta$, p, $\phi \in L_{\beta'}$. □

Proof of Lemma 1B.5. This is clear from the definition of $\mathcal{P}_\gamma^\beta$. □

Proof of Lemma 1B.6. This is clear from the Extension Lemma. □

Proof of Lemma immediately preceding Proof of Lemma 1D.2. This is clea from the Extension Lemma. □

Proof of Lemma 1E.5. This follows from Distributivity (Lemma 1A.6), Lemm 1E.6 and Factoring. □

SECTION TWO: EXTENDIBILITY

A. Introduction

Our proof of Extendibility comes in two parts. In the first part, Extendibility I, we establish Lemma 1E.5 and the last statement of Lemma 1F.1 by a simultaneous induction on $|s|$, assuming Extendibility for all $\beta' < \beta$. Then we establish Extendibility II, which asserts extendibility for S^β_κ for all $\kappa \in \beta$-Card: $s \in S^\beta_\kappa$, $\eta < (\kappa^+)^{L_\beta} \rightarrow \exists t \in S^\beta_\kappa$, $s \subseteq t$, $|t| \geq \eta$. From this full Extendibility will be derived.

Suppose κ is a limit β-cardinal, $s \in S^\beta_\kappa$, $p \in \mathcal{P}^{<s}_\gamma$, $\gamma \in \beta\text{-Card} \cap \kappa$ and we wish to extend p to $q \in \mathcal{P}^s_\gamma$ such that $|q| = |s|$. (If κ is a successor β-cardinal, then the desired results follow easily from Extendibility for R^s, distributivity for R^s and induction.) Our treatment of Predensity Reduction is closely modelled on the proof of Lemma 1B.9 where if μ^0_s belongs to $\overline{\text{Adm}}$, then μ^0_s, ν_s play the role of α, $\nu(\alpha)^-$. Our treatment of Coding when $|s| = \xi + 1$ requires that we first establish the last statement of Lemma 1F.1, by a subinduction on the level of L at which f is defined. These arguments split into two cases, depending on whether or not $L_{\mu^0_s} = \mathcal{A}^0_s \models \kappa$ is singular.

As in Jensen [7] the key technique for building transfinite sequences of conditions is to meet certain auxiliary dense sets Σ^p_g. In fact we shall prove extendibility for $\mathcal{P}^s_\gamma$, distributivity for $\mathcal{P}^{<s}_\gamma$ and density for Σ^p_g, $g \in \mathcal{A}_s = L_{\mu_s}$ by a simultaneous induction.

We now define the sets Σ^p_g. Say that $X \subseteq \kappa \cap \beta$-Card is *thin in* L_{μ_s} if $L_{\mu_s} \models X \cap \delta$ is nonstationary in δ for all L_{μ_s}-regular $\delta \leq \kappa$. Then $g \in \mathbb{F}(s)$ if $g \in L_{\mu_s}$, $\text{Dom}(g)$ is thin in L_{μ_s} and for all $\delta \in \text{Dom}(g)$, $L_\kappa \models g(\delta)$ has cardinality $\leq \delta$. And $q \in \Sigma^p_g$ if q is incompatible with p or ($q \leq p$ and $\delta \in \text{Dom}(g)$, D predense on $R^{p_{\delta^+}}$, $D \in g(\delta) \cap L_{\mu^1_{p_{\delta^+}}} \rightarrow q(\delta)$ meets D).

Our definition of Σ^p_g is somewhat simpler than Jensen's due to the fact that we have built Predensity Reduction into our definition of $\mathcal{P}^s_\gamma$ at limit cardinals. However, unlike [7] we cannot separate the Distributivity and Extendibility arguments; in fact both depend on the density of the Σ^p_g's.

First we establish the density of the Σ^p_g's, assuming extendibility. The following lemma will be needed in our inductive proof of Extendibility I in Parts B, C.

Lemma 2A.1. *Suppose* $\eta < \mu_s$, $p \in \mathcal{P}^s \cap L_\eta$, $g \in \mathbb{F}(s) \cap L_\eta$ *and if* κ *is* L_η-*inaccessible,* $L_\eta \models \text{Dom}(g)$ *is nonstationary in* κ. *If* κ *is singular in* L_η, *then also suppose that for all* $f \in L_\eta$, $f(\delta) < \delta^+$ *for all* $\delta \in \kappa \cap \beta$-Card *there exists* $q \leq p$,

$q \in \mathcal{P}^s \cap L_\eta$ and CUB $C \subseteq \kappa$ so that $\delta \in C \to |q_\delta| \geq f(\delta)$, $\delta \in C \cup \{\kappa\} \to C \cap \delta$ is $\Sigma_{n(\delta)}(L_{\eta(\delta)})$ (where $q \restriction \delta \in \Sigma_{n(\delta)}(L_{\eta(\delta)}) - \Sigma_{n(\delta)-1}(L_{\eta(\delta)})$). Then there exists $q \leq p$, $q \in \mathcal{P}^s \cap L_\eta$, $q \in \Sigma_g^p$.

Proof. Without loss of generality assume that $|p| = |s|$. We first suppose that κ is the limit of limit β-cardinals. Then choose $\eta' < \eta$ and $k \in \omega$ so that p, g are $\Delta_k(L_{\eta'})$ with parameter $x \in L_{\eta'}$. We claim that we can choose $q' \leq p$ so that $q' \in \mathcal{P}^s \cap L_\eta$ and a CUB $C \subseteq \kappa$ so that $\delta \in C \cup \{\kappa\} \to C \cap \delta$ is $\Sigma_{n(\delta)}(L_{\eta(\delta)})$ (where $q' \restriction \delta \in \Sigma_{n(\delta)}(L_{\eta(\delta)}) - \Sigma_{n(\delta)-1}(L_{\eta(\delta)})$) and $\delta \in C \to |q'_\delta| \geq h(\delta) =$ Transitive Collapse (Σ_k-Skolem hull of $\delta \cup \{\kappa, x\}$ in $L_{\eta'}$) $\cap$ ORD. This is clear if κ is L_η-singular by hypothesis. Otherwise let $D = \{\delta \mid \delta = \kappa \cap [\Sigma_k\text{-Skolem hull of } \delta \cup \{\kappa, x\} \text{ in } L_{\eta'}]\}$ and $C =$ limits of D. Then $C \in L_\eta$ is CUB and note that $\delta \in C \to \pi_\delta(s) = p_\delta$ (where π_δ is the transitive collapse map for Σ_k-Skolem hull of $\delta \cup \{\kappa, x\}$ in $L_{\eta'}$) since $p \restriction \delta$ codes p_δ just as p codes s. Now by induction successively extend $p \restriction \delta$ for $\delta \in C$ to $q' \restriction \delta \in \mathcal{P}^{p_\delta}$ so that $q' \restriction \delta \in \Sigma_{k+1}(L_{h(\delta)}) - \Sigma_k(L_{h(\delta)})$. Doing this in the L-least way guarantees that this last property also obtains at limit points of C. Thus we have constructed the desired q', C.

Now define $C = \{\kappa_0 < \kappa_1 < \cdots\}$ and q_i by induction on i. First we set $q_0 = p$. If q_i has been defined, then obtain q_{i+1} from q_i by extending $q_i \restriction [\kappa_i, \kappa_{i+1})$ so that $q_{i+1} \restriction \kappa_{i+1}^+ \in \Sigma_{g \restriction \kappa_{i+1}}^{q' \restriction \kappa_{i+1}^+}$. This is possible by induction. For limit λ let $q_\lambda(\delta) = \bigcup \{q_i(\delta) \mid i < \lambda\}$. We must verify that q_λ is a condition for limit λ. But this is clear as $\langle q_i \mid i < \lambda \rangle$ is $\Sigma_{n(\kappa_\lambda)}(L_{\eta(\kappa_\lambda)})$ where $q' \restriction \kappa_\lambda \in \Sigma_{n(\kappa_\lambda)}(L_{\eta(\kappa_\lambda)})$. Finally let $q = \bigcup \{q_i \mid i < \text{ordertype}(C)\}$.

If κ is not the limit of limit β-cardinals, then the result is easy, using induction and distributivity for the $R^{p_{\delta^+}}$-forcings. □

We next describe how the Σ_g^p's are used by establishing a version of the key lemma for constructing transfinite sequences of conditions.

Lemma 2A.2. *Suppose $\mu_s^0 < \eta < \mu_s$ and $L_\eta \vDash \text{card}(\mu_s^0) = \kappa$. Let $\langle g_i \mid i < \lambda \rangle$, $\langle p_i \mid i < \lambda \rangle$ be sequences of elements of $\mathbb{F}(s) \cap L_\eta$, $\mathcal{P}^s \cap L_\eta$ respectively so that $|p_0| = |s|$, $i < j \to p_j \leq p_i$, $p_{i+1} \in \Sigma_{g_i}^{p_i}$, $p_{i+1}(\delta) = p_i(\delta)$ if $g_i(\delta) \subseteq \bigcup \{g_j(\delta) \mid j < i\}$ for each i, $\eta' < \eta$, $k \in \omega \to p_i \notin \Sigma_k(L_{\eta'})$ for some i and for each $\delta \in \beta\text{-Card} \cap \kappa$, $\bigcup \{g_i(\delta) \mid i < \lambda\} = H_\delta = \Sigma_1$-Skolem hull of $\delta \cup \{q, s^*\}$ in L_η whenever $\delta \in H_\delta$ (where $q \in L_\eta$ is a fixed parameter). If $\langle p_i \mid i < \lambda \rangle$ is $\Sigma_1(L_\eta)$ with parameter q, then there exists $p \in \mathcal{P}^s$, $p \leq p_i$ for each i.*

Proof. (Notation: $i(n) = i_n$, $\delta(n) = \delta_n$ for $n = 0, 1, \ldots$.) Let $\pi_\delta : L_{\eta_\delta} \cong H_\delta$. We claim that if $\delta \in H_\delta$, then $p'_\delta = \bigcup \{p_{i_\delta} \mid \underline{i < \lambda}\}$ is $\mathcal{P}_\delta^{\beta_\delta}$-generic over L_{β_δ}, where $\pi_\delta(\beta_\delta) = \eta^-$. (Recall that $\eta^- = \sup(\eta \cap \overline{\text{Adm}})$.) Note that $L_{\eta^-} \vDash \mu_s^0 = \kappa^+$ (or $\eta^- \leq \mu_s^0$).

Suppose that $\delta \in H_\delta$. If $D_\delta \in L_{\beta_\delta}$ is predense on $\mathcal{P}_\delta^{\beta_\delta}$, then $D = \pi_\delta(D_\delta)$ is predense on $\mathcal{P}_\delta^{\eta^-}$ and for some $i(0) < \lambda$, $D \in g_{i(0)}(\delta)$. Now by Predensity

Reduction for p_0 and the $\mathcal{P}_\kappa^{\eta^-}$-genericity of $s \upharpoonright \eta^-$ there must exist $\delta(0) \in \beta$-Card $\cap (\delta, \kappa)$ so that $D((p_0)_{\delta(0)}) = \{q \in \mathcal{P}_\delta^{p_0{}_{\delta(0)}} \mid q \cup (p_0)_{\delta(0)} \leq$ some element of $D\}$ is dense below $(p_0)^{\delta(0)}$ on $\mathcal{P}_\delta^{p_0{}_{\delta(0)}}$. Note that we can choose $\delta(0) \in H_{\delta(0)}$ and hence $D((p_0)_{\delta(0)})$ belongs to $g_{i(1)}(\delta(0))$ for some $i(1) < \lambda$. We can also assume that $D_{\delta(0)}((p_0)_{\delta(0)})$ belongs to $\mathcal{A}^1_{p_{i(1)_{\delta(0)}}}$ where $D_{\delta(0)}((p_0)_{\delta(0)}) = \pi^{-1}_{\delta(0)}(D((p_0)_{\delta(0)}))$, as $\bigcup \{|p_{i_{\delta(0)}}| \mid i < \lambda\} = H_{\delta(0)} \cap \delta(0)^+$. Moreover by the Genericity Lemma, $D_{\delta(0)}((p_0)_{\delta(0)})$ is predense below $(p_0)^{\delta(0)}$ (and hence below $(p_{i(1)})^{\delta(0)}$) on $\mathcal{P}_\delta^{p_{i(1)_{\delta(0)}}}$. If $\delta(0)$ is a limit β-cardinal we can apply the same reasoning to $D_{\delta(0)}((p_0)_{\delta(0)})$, $p_{i(1)}$ as above to obtain $\delta(1) \in \beta$-Card $\cap (\delta, \delta(0))$ so that $D_{\delta(0)}((p_0)_{\delta(0)})((p_{i(1)})_{\delta(1)})$ is dense below $(p_{i(1)})^{\delta(1)}$ on $\mathcal{P}_\delta^{p_{i(1)_{\delta(1)}}}$. If $\delta(0)$ is a successor β-cardinal, then the distributivity of $R^{p_{i(1)_{\delta(0)}}}$ and the chain condition for $\mathcal{P}_\delta^G$ (G $R^{p_{i(1)_{\delta(0)}}}$-generic over $\mathcal{A}^1_{p_{i(1)_{\delta(0)}}}$) imply that if $\delta < \delta(1) \in \beta$-Card, $\delta(1)^+ = \delta(0)$, then $\{q \in R^{p_{i(1)_{\delta(0)}}} \mid D_{\delta(0)}((p_0)_{\delta(0)})(q)$ is predense below $(p_{i(1)})^{\delta(1)}$ on $\mathcal{P}_\delta^{q_{\delta(1)}}\}$ is dense on $R^{p_{i(1)_{\delta(0)}}}$ below $p_{i(1)}(\delta(1))$. So by definition of $\Sigma^{p_{i(1)}}_{g_{i(1)}}$ we have that $p_{i(1)+1}(\delta(1))$ meets the latter dense set and hence $D_{\delta(0)}((p_0)_{\delta(0)})((p_{i(1)+1})_{\delta(1)})$ is dense below $(p_{i(1)+1})^{\delta(1)}$ on $\mathcal{P}_\delta^{p_{i(1)+1_{\delta(1)}}}$. If we continue in this way we either produce an infinite descending sequence $\delta(0) > \delta(1) > \cdots$ or we obtain $\delta(n) = \delta^+$ for some n. But then $p_{i(n)+1}$ meets D. We have shown that $G_\delta = \{p \in \mathcal{P}_\delta^{\beta_\delta} \mid \pi_\delta(p) \leq p_i \cup q$ for some $i < \lambda$, $q \in G(s \upharpoonright \eta^-)\}$ is $\mathcal{P}_\delta^{\beta_\delta}$-generic over L_{β_δ} (where $G(s \upharpoonright \eta^-)$ denotes the $\mathcal{P}_\kappa^{\eta^-}$-generic associated with $s \upharpoonright \eta^-$). But clearly $p'_\delta = \bigcup \{p_\delta \mid p \in G_\delta\}$ so we are done. The same argument demonstrates the $\mathcal{P}_\delta^{\beta_\delta}$-$\Sigma$-genericity of p'_δ over L_{β_δ} if β_δ is recursively inaccessible using the Σ-genericity of $s \upharpoonright \eta^-$ and Predensity Reduction.

Now suppose that $\delta \notin H_\delta$. Then $p'_\delta = \pi^{-1}_{\delta\gamma}(p'_\gamma)$ where $\gamma \in \beta$-Card, $\gamma \leq \kappa$ is least so that $\delta < \gamma \in H_\delta \cap \mathrm{ORD}$ and $\pi_{\delta\gamma} = \pi_\gamma^{-1} \circ \pi_\delta$. For, p'_δ is coded by $\bigcup \{\bar{p}_{i_{\bar{\delta}}} \mid i < \lambda,$ $\bar{\delta} < \delta\}$ just as p'_γ is coded by $\bigcup \{\bar{p}_{i_{\bar{\gamma}}} \mid i < \lambda,\ \bar{\gamma} < \gamma\}$ and $\pi_{\delta\gamma}(p_i \upharpoonright \delta) = p_i \upharpoonright \gamma$.

Now we can define the desired condition p. Write $G_\delta = f(\delta, \beta_\delta, X_\delta)$ for $\delta \in H_\delta$ and let p''_δ be the characteristic function of X_δ, restricted to $[\delta, \beta_\delta)$. Then $p_\delta = p''_\delta \cup \pi^{-1}_\delta(s)$ for $\delta \in H_\delta$ and $p_\delta = \pi^{-1}_{\delta\gamma}(p_\gamma)$ for $\delta \notin H_\delta$, $\gamma = \min[(H_\delta - \delta) \cap \mathrm{ORD}]$. And $\bar{p}_\delta = \bigcup \{\bar{p}_{i_\delta} \mid i < \lambda\}$, $p(\delta) = (p_\delta, \bar{p}_\delta)$. We show that $p \upharpoonright \delta \in \mathcal{P}^{p_\delta}$ by induction on $\delta \in \beta$-Card, $\delta \leq \kappa$.

First note that $p_\delta \in S_\delta$. Indeed by the above we need only verify that p_δ is $\Delta_1^*(\mathcal{A}(\mu^0_{p_\delta}))$ when $\delta \in H_\delta$. But $\mathcal{A}(\mu^0_{p_\delta}) = L_{\eta_\delta}$ and p'_δ is $\Delta_1(L_{\eta_\delta})$. By induction p''_δ can be decoded from p'_δ (over L_{β_δ}) and thus p_δ is $\Delta_1(L_{\eta_\delta})$. As in general any $\Delta_1(\mathcal{A}(\nu))$-subset of ν is $\Delta_1^*(\mathcal{A}(\nu))$ it follows that p_δ is $\Delta_1^*(\mathcal{A}(\mu^0_{p_\delta}))$.

We now verify properties (1)–(7) in the definition of $\mathcal{P}^s_\gamma$ for $p \upharpoonright \delta$. If δ' is a limit β-cardinal less than δ, then smoothness for $p \upharpoonright \delta'$ is clear: $p \upharpoonright \delta' \in \mathcal{P}^{p_{\delta'}}$ as $p \upharpoonright \delta' \in L_{\eta_{\delta'}+1}$ and $p \upharpoonright \delta' \notin \mathcal{P}^{<p_{\delta'}}$ as the function $\delta'' \mapsto |p_{\delta''}|$ eventually dominates all functions on $\delta' \cap \beta$-Card in L_{η_δ}. To see that $p(\delta') \in R^{p_\delta}$ if $\delta = (\delta')^+$ just notice that $\bar{p}_{\delta'}$ is included in $\{b_{p_\delta \upharpoonright \xi} \mid \xi \in \mathrm{Range}\, \pi_\delta^{-1} \circ \pi_{\delta'}\}$ and $\pi_\delta^{-1} \circ \pi_{\delta'} \upharpoonright \{p'_{\delta'} \upharpoonright \xi \mid \xi < |p'_{\delta'}|\}$ is a quasi-morass map.

Predensity Reduction (a) follows as $\nu_{p_\delta} = \pi_\delta^{-1}(\nu_s)$ and so a predense $\mathcal{D} \subseteq \mathcal{P}^{<p_\delta}$, $\mathcal{D} \in L_{\nu_{p_\delta}}[p^*_\delta]$ is reduced since for some $\xi < \mu^0_{p_\delta}$ we have that $\mathcal{D} \cap L_\xi = \mathcal{D} \cap L_{\mu^0_\xi}$ is predense on $\mathcal{P}^{<p_\delta \restriction \mu_{\bar{\xi}}}$, $\mathcal{D} \cap L_{\mu^0_\xi} \in L_{\nu_{p_\delta} \restriction \mu_{\bar{\xi}}}[(p_\delta \restriction \mu_{\bar{\xi}})^*]$ and hence $\mathcal{D}$ is reduced by virtue of the fact that $|p_i \restriction \delta| \geq \xi$ for some $i < \lambda$ (if $\delta \notin H_\delta$, use the fact that $p_i \restriction \gamma$ is a condition in $\mathcal{P}^{p_\gamma}$). Also note that any $\Sigma_1\langle L_{\nu_{p_\delta}}[p^*_\delta], p^*_\delta\rangle$-subset of $\mu^0_{p_\delta}$ is actually an element of $L_{\nu_{p_\delta}}[p^*_\delta]$ for $\delta \in H_\delta$. Both of these facts use distributivity for $\mathcal{P}^{\nu_{p_\delta}}_\delta$ to argue that $\mu^0_{p_\delta}$ is a cardinal in $L_{\nu_{p_\delta}}[p_\delta]$. Predensity Reduction (b) is vacuous in this case as the definition of $D_{\mathbf{b}}$ in Part E of Section 1 implies that $D_{\mathbf{b}}$ is $\Delta_1(\mathbf{b})$.

Coding for $p \restriction \delta$ follows from the fact that each p_i is a condition and $p \restriction \delta \in L_{\mu_{p_\delta}}$. For Code Thinning just notice that if δ is a limit β-cardinal, δ $\Sigma_2(L_{\eta_\delta})$-regular, then we can choose for each $i < \lambda$ a CUB $C_i \subseteq \beta$-Card $\cap\, \delta$ so that $\bar{p}_{i_\gamma} = \emptyset$ for $\gamma \in C_i$ and $\langle C_i \mid i < \lambda\rangle$ is $\Sigma_1(L_{\eta_\delta})$; then let $C = \bigcap\{C_i \mid i < \lambda\} \in \Sigma_2(L_{\eta_\delta})$. The growth condition is clear for we can use the CUB set $C_\delta = \beta$-Card $\cap\, \delta$. Restriction is clear as the p_i's are conditions. □

When used in conjunction, Lemmas 2A.1 and 2A.2 supply the needed method for the inductive extension of conditions, in the successor case (Part B). The limit case (Part C) will make use of an altered version of Lemma 2A.2 whose proof is virtually the same.

B. Extendibility I: The successor case

We assume in this part that $|s|$ is a successor ordinal $\xi + 1$ and if $t = s \restriction \xi \in S_\kappa$ we show that each $p \in \mathcal{P}^t$, $|p| = \xi$ has an extension q in $\mathcal{P}^s$, $|q| = \xi + 1$. There are two subcases, defined according to whether or not $\mathcal{A}_t = L_{\mu_t} \models \kappa$ is regular.

Subcase 1: *$\mathcal{A}_t \models \kappa$ regular (hence $\mathcal{A}_t \models \kappa$ inaccessible).*

In this subcase we closely follow the argument of Jensen [7].

By definition of $\mathcal{P}^t$ we can pick a least CUB $C \in L_{\mu_\xi}$ so that $\delta \in C \rightarrow \bar{p}_\delta = \emptyset$. Also choose n_0 so that $p, C \in L_{\mu^{n_0}_\xi}$. For each $n \geq n_0$ set $C_n = \{\delta < \kappa \mid \delta = \kappa \cap H^n_{\xi\delta}$ where $H^n_{\xi\delta} = (\Sigma_\omega$-) Skolem hull of $\delta \cup \{p, C\}$ inside $L_{\mu^n_\xi}\}$. Then C_n is CUB as $L^{n+1}_{\mu_\xi} \models \kappa$ regular and $n_0 \leq n < m \rightarrow C_m \subseteq$ limit points (C_n), $C_n \subseteq C$.

Recall that our goal is to define $q \leq p$ so that $t(\xi) = 0$ iff for sufficiently large n, $\{\delta \mid b^n_\xi(\delta) \in \tilde{q}\}$ is nonstationary in L_{μ_ξ}, where $b^n_\xi(\delta) = \langle 1, \rho^n_{\xi\delta}\rangle$ and $\rho^n_{\xi\delta}$ is the ordinal code for $M^n_{\xi\delta^+} =$ transitive collapse (Skolem hull of $\delta^+ \cup \{\kappa\}$ in $L_{\mu^n_\xi}$). Notice that for large enough $\delta \in C_{n_0}$, $|p_{\delta^+}| < b^{n_0}_\xi(\delta)$ since $p \in$ Skolem hull of $\delta^+ \cup \{\kappa\}$ in $L_{\mu^{n_0}_\xi}$ for large enough $\delta \in C_{n_0}$.

Now define $q'_n \in \mathcal{P}^t$ by induction on $n \geq n_0$: If $\gamma \in \beta$-Card $\cap\, \kappa$ is not of the form δ^+, $\delta \in C_{n_0}$, then set $q'_{n_0}(\gamma) = p(\gamma)$. Do the same if $\gamma = \delta^+$, $\delta \in C_{n_0}$ but $|p_{\delta^+}| \geq b^{n_0}_\xi(\delta)$. Now if $\gamma = \delta^+$, $\delta \in C_{n_0}$, $|p_{\delta^+}| < b^{n_0}_\xi(\delta)$, then let $\bar{q}'_{n_{0_\gamma}} = \bar{p}_\gamma$ and choose $q'_{n_{0_\gamma}} \in S_\gamma$ to be least so that $(q'_{n_{0_\gamma}}, \bar{p}_\gamma) \leq (p_\gamma, \bar{p}_\gamma)$ in $R^{p_{\gamma^+}}$ and $q'_{n_{0_\gamma}}(b^{n_0}_\xi(\delta)) = s(\xi)$. This is possible since $\bar{p}_\gamma$ restricts ordinals of the form $\langle 0, \eta\rangle$ from entering $\tilde{q}'_{n_{0_\gamma}}$ but not of the form $\langle 1, \eta\rangle$ such as $b^{n_0}_\xi(\delta)$. More generally define

$q'_{n+1} \leqslant q'_n (n \geqslant n_0)$ in precisely the same manner as above but with p replaced by q'_n, q'_{n_0} by q'_{n+1} and C_{n_0} by C_{n+1}.

We claim that $q'_n \in \mathscr{P}^t$ for each $n \geqslant n_0$. But note that q'_{n-1} belongs to the Skolem hull of $\delta \cup \{p, C\}$ inside $L_{\mu^n_\xi}$ for $\delta \in C_n$ and hence $q'_{n-1} \restriction \delta$, $C_n \cap \delta$ belongs to $L_{\bar{\mu}^n_\xi+1}$ where $L_{\bar{\mu}^n_\xi}$ = transitive collapse of this Skolem hull. But then $q'_n \restriction \delta \in L_{\bar{\mu}^n_\xi+1} \subseteq L_{\mu_{p_\delta}}$ as t collapses to p_δ (see the argument in Lemma 2A.2). The remaining properties of a condition are now easy to verify for q'_n as if $\delta \notin C_n$ then q'_n, q'_{n-1} differ only on a bounded part of δ.

Now note that the hypothesis of Lemma 2A.1 holds in this context, with s replaced by t. Thus we know that for any $q \in \mathscr{P}^t$, $g \in \mathbb{F}(t)$ there exists $q' \leqslant q$ in $\mathscr{P}^t$ so that $q' \in \Sigma^q_g$, by Lemma 2A.1. Now modify the construction of $q'_{n_0}, q'_{n_0+1}, \ldots$ to $q_{n_0}, q_{n_0+1}, \ldots$ so as to require in addition that $q_{n+1} \in \Sigma^{q_n}_{g_n}$ where $g_n(\delta^+)$ = Skolem hull of $\delta^+ \cup \{p, C\}$ in $L_{\mu^n_\xi}$ for all $\delta \in C_n$; $g_n(\gamma)$ undefined otherwise. As before we can verify that q_n is a condition for each n.

Now we argue as in the proof of Lemma 2A.2 to show that there is a condition $q \leqslant q_i$ for each i, $q \in \mathscr{P}^s$, $|q| = \xi + 1 = |s|$. Indeed if $\delta \notin C_\omega = \bigcap_n C_n$, then we set $q'(\delta) = \bigcup_n q_n(\delta)$ and then we can argue as in 2A.2 that $\bigcup_n q_{n_\delta} = q'_\delta$ is $\mathscr{P}^{\beta_\delta}_\delta$-generic over L_{β_δ}. Thus we set $q(\delta) = (q_\delta, \bar{q}'_\delta)$ where $q_\delta = q''_\delta \cup \pi_\delta^{-1}(t)$, as in Lemma 2A.2. If $\delta \in C_\omega$, then $q_n(\delta) = p(\delta)$ for all δ (as $q_{n_\delta} = \pi^{-1}(t)$ for all n) and so we set $q(\delta) = (p_\delta \cup \{(\bar{\xi}, s(\xi))\}, \bar{p}_\delta)$ where $\pi_\delta(\bar{\xi}) = \xi$. Note that $\bar{p}_\delta = \emptyset$ for $\delta \in C_\omega$ so this is in fact an element of $R^{p_{\delta^+}}$. It is clear that $q \restriction \delta \in \Sigma_2(L_{\mu_\xi}) - \Sigma_1(L_{\mu_\xi})$ as $q \restriction \delta$ can be defined relative to $L_{\mu_{p_\delta}}$ just as q was defined relative to $L_{\mu_\xi} = L_{\mu_t}$. Predensity Reduction for $q \restriction \delta$ is trivial as $\mu^0_{\xi+1} = \mu_\xi$ is not a fixed point of $\eta \mapsto \mu^0_\eta$ (so $\langle L_{\mu^0_{\xi+1}}, q_\delta \rangle \notin \mathbf{E}$ and $\nu_{q_\delta} = \mu^0_{q_\delta}$ is not recursively inaccessible). Clearly $q \restriction \delta$ codes q_δ properly as $q_\delta(\bar{\xi}) = 0$ iff $s(\xi) = 0$ iff $q_{\bar{\delta}^+}(b^n_\xi(\bar{\delta})) = 0$ for $\bar{\delta} \in C_n \cap \delta$ and each $C_n \cap \delta$ belongs to L_{μ_ξ}. Lastly, if $L_{\mu_\xi} \models \delta$ is Σ_2-regular, then $C_\omega \cap \delta$ is CUB, belongs to $\Sigma_2(L_{\mu_\xi})$ and $\bar{\delta} \in C_\omega \cap \delta \rightarrow \bar{q}_{\bar{\delta}} = \bar{p}_{\bar{\delta}} = \emptyset$. The Growth Condition and Restriction are easily verified. This completes the proof that $q \restriction \delta \in \mathscr{P}^{q_\delta}$ for $\delta \in C_\omega \cup \{\kappa\}$ and thus $q \in \mathscr{P}^s$.

Subcase 2: *$\mathscr{A}_t \models \kappa$ is singular.*

The main thing that we must demonstrate in this subcase is that $p \in \mathscr{A}_t$, $f \in \mathscr{A}_t$, $f(\delta) < \delta^+$ for all $\delta \in \beta$-Card $\cap\, \kappa \rightarrow$ there exists $q \leqslant p$ in $\mathscr{P}^t$ and a CUB $C \subseteq \kappa$ so that $\delta \in C \rightarrow |q_\delta| \geqslant f(\delta)$, $\delta \in C \cup \{\kappa\} \rightarrow C \cap \delta$ is $\Sigma_{k(\delta)}(L_{\eta(\delta)})$ (where $q \restriction \delta \in \Sigma_{k(\delta)}(L_{\eta(\delta)}) - \Sigma_{k(\delta)-1}(L_{\eta(\delta)})$). (This is as in Lemma 2A.1.) To do this we first extend p to $q \in \mathscr{P}^t$ so that $q \in \Sigma_k(L_\eta) - \Sigma_{k-1}(L_\eta)$, κ $\Sigma_k(L_\eta)$-singular and then proceed by induction on the least pair (η, k) so that $f \in \Sigma_k(L_\eta) - \Sigma_{k-1}(L_\eta)$.

So suppose that κ is $\Sigma_{\bar{k}}(L_{\bar{\eta}})$-regular where $(\bar{\eta}, \bar{k})$ is least so that p is $\Sigma_{\bar{k}}(L_{\bar{\eta}})$ and let $(\eta, k) > (\bar{\eta}, \bar{k})$ be least so that κ is $\Sigma_k(L_\eta)$-singular. First suppose that $k > 1$. Now for each $\delta \in \kappa \cap \beta$-Card let $H_\delta = \Sigma_{k-1}$-Skolem hull of $\delta \cup \{\kappa, x\}$ in L_η where x is a parameter for defining p as a $\Sigma_{k-1}(L_\eta)$-function. Then $C = \{\delta \mid \delta = \kappa \cap H_\delta\}$ is closed; if it is unbounded, then as in the proof of Lemma 2A.1 construct $q \leqslant p$ so that $\delta \in H_\delta \rightarrow |q_\delta| \geqslant H_\delta \cap \delta^+$: List $C = \{\kappa_0, \kappa_1, \ldots\}$ and inductively extend $p \restriction (\kappa_i, \kappa_{i+1})$ so that $\delta \in \beta\text{-Card} \cap (\kappa_i, \kappa_{i+1}) \to |q_\delta| \geq H_\delta \cap \delta^+$. This is possible by induction, using the fact that $\pi_\delta^{-1}(s) = p_\delta$ for $\delta \in C$ and the Σ_k-definability of $C \cap \delta$ over transitive collapse (H_δ) for δ a limit point of C. Then clearly $q \in \Sigma_k(L_\eta) - \Sigma_{k-1}(L_\eta)$ and it can be verified as in the proof of Lemma 2A.1 that $q \in \mathcal{P}^t$. If C is bounded in κ, then as κ is $\Sigma_{k-1}(L_\eta)$-regular it must be that $\Sigma_k(L_\eta)$-cofinality of κ equals ω. Then choose $\kappa_0 < \kappa_1 < \cdots$ to be a cofinal sequence of successor β-cardinals below κ and extend $p(\kappa_i)$ successively so that $|q_{\kappa_i}| \geq H_{\kappa_i} \cap \kappa_i^+$. Once again the resulting q belongs to $(\Sigma_k(L_\eta) - \Sigma_{k-1}(L_\eta)) \cap \mathcal{P}^t$.

If $k = 1$, η limit, then choose a continuous $\Sigma_1(L_\eta)$-cofinal sequence $\eta_0 < \eta_1 < \cdots$ below η of length $\Sigma_1\text{-cof}(\eta) = \kappa_0 < \kappa$. (This is possible as we know that $\eta > \beta(\mu_t^0)$ and hence $\Sigma_1\text{-cof}(\eta) \leq \kappa$; as κ is $\Sigma_1(L_\eta)$-singular we get $\Sigma_1\text{-cof}(\eta) < \kappa$.) Also note that by the leastness of $(\eta, k) = (\eta, 1)$ it must be that $\kappa_0 = \Sigma_1(L_\eta)$-cofinality of κ. If $\kappa_0 = \omega$, then the desired result is easily obtained by extending $p(\kappa_i)$ successively so that $|q_{\kappa_i}| \geq H^i_{\kappa_i} \cap \kappa_i^+$, where $\kappa_0 < \kappa_1 < \cdots$ is a $\Sigma_1(L_\eta)$-cofinal sequence of successor β-cardinals below κ, $H^i_{\kappa_i} = \Sigma_1$-Skolem hull of $\kappa_i \cup \{\kappa, p\}$ in L_{η_i}. Similarly, if $\kappa_0 > \omega$, let $C = \{\kappa_0 < \kappa_1 < \cdots\}$ be a closed $\Sigma_1(L_\eta)$-cofinal sequence of β-cardinals below κ; we also assume that $\langle \kappa_i \mid i < \lambda \rangle$ is uniformly $\Sigma_1(L_{\eta_\lambda})$ and that $\kappa_\lambda \notin H^\lambda_{\kappa_\lambda} = \Sigma_1$-Skolem hull of $\kappa_\lambda \cup \{\kappa, p\}$ in L_{η_λ}. This is possible as we can choose CUB sets $C_i \in L_{\eta_{i+1}}$ so that $\delta \in C_i \to \delta \notin H^i_\delta$ and $C_\lambda = \bigcap_i C_i$ for limit λ; then let $\kappa_i = \min(C_i)$. Now extend $p \restriction [\kappa_i, \kappa_{i+1})$ successively so that $\delta \in \beta\text{-Card} \cap [\kappa_i, \kappa_{i+1}) \to |q_\delta| \geq H^i_\delta \cap \delta^+$. Note that $p(\kappa_\lambda)$ need not be extended for limit $\lambda < \kappa_0$. The resulting q obeys the growth condition at κ_λ, as witnessed by $\langle \kappa_i \mid i < \lambda \rangle$, so $q \in \mathcal{P}^t$, $q \in \Sigma_1(L_\eta) - L_\eta$ as desired. Finally the case η a successor, $k = 1$ can be treated just like the case $\kappa_0 = \omega$ above, using $H^i_{\kappa_i} = \Sigma_i$-Skolem hull of $\kappa_i \cup \{\kappa, p\}$ in $L_{\eta-1}$.

Thus we can assume now that we are given $p \in \mathcal{A}_t$, $p \in \Sigma_{\bar{k}}(L_{\bar{\eta}}) - \Sigma_{\bar{k}-1}(L_{\bar{\eta}})$, κ $\Sigma_{\bar{k}}(L_{\bar{\eta}})$-singular and our goal now is to show that for any $f \in \mathcal{A}_t$, $f(\delta) < \delta^+$ for all $\delta \in \beta\text{-Card} \cap \kappa$, there exists $q \leq p$ in $\mathcal{P}^t$ and a CUB $C \subseteq \kappa$ so that $\delta \in C \to |q_\delta| \geq f(\delta)$, $\delta \in C \cup \{\kappa\} \to C \cap \delta$ is $\Sigma_{k(\delta)}(L_{\eta(\delta)})$ (where $q \restriction \delta \in \Sigma_{k(\delta)}(L_{\eta(\delta)}) - \Sigma_{k(\delta)-1}(L_{\eta(\delta)})$). Now in fact it suffices to show the following: Given p as above and $(\eta, k) \in \mu_t^0 \times (\omega - \{0\})$ there exists $q \leq p$ in $\mathcal{P}^t \cap [\Sigma_{k+1}(L_\eta) \cup \Sigma_{\bar{k}}(L_{\bar{\eta}})]$ so that for some CUB $C \subseteq \beta\text{-Card} \cap \kappa$ we have that:

(a) $\delta \in C \cup \{\kappa\} \to C \cap \delta$ is $\Sigma_{k(\delta)}(L_{\eta(\delta)})$, where $q \restriction \delta \in \Sigma_{k(\delta)}(L_{\eta(\delta)}) - \Sigma_{k(\delta)-1}(L_{\eta(\delta)})$.

(b) For any $x \in L_\eta$, $|q_\delta| \geq H^x_\delta \cap \delta^+$ for sufficiently large $\delta \in C$, where $H^x_\delta = \Sigma_k$-Skolem hull of $\delta \cup \{\kappa, x\}$ in L_η.

We prove the above assertion by induction on (η, k). Note that the Growth Condition on p implies the desired conclusion when (η, k) is less than $(\bar{\eta}, \bar{k})$ (letting $q = p$). So assume that $(\bar{\eta}, \bar{k}) \leq (\eta, k)$ and fix a continuous $\Sigma_{\bar{k}}(L_{\bar{\eta}})$-cofinal κ_0-sequence $\kappa_0 < \kappa_1 < \cdots$ of β-cardinals below κ, $\kappa_0 = \Sigma_{\bar{k}}(L_{\bar{\eta}})\text{-cof}(\kappa)$.

First suppose that $k = 1$, η limit. Choose a continuous $\Sigma_1(L_\eta)$-sequence $\eta_0 < \eta_1 < \cdots$ cofinal in η of ordertype $\Sigma_1\text{-cof}(\eta) = \gamma_0$. Let $x \in L_\eta$ be a parameter so that $\langle\kappa_i \mid i<\kappa_0\rangle$, $\langle\eta_i \mid i<\gamma_0\rangle$, p are $\Sigma_1(L_\eta)$ with parameter x. If $(\bar\eta, \bar k) = (\eta, k)$, we then can conclude that $\gamma_0 = \kappa_0$ and in this case we assume that the η_i's are chosen so that $\langle\kappa_j \mid j<i\rangle$, $p \restriction \kappa_i$ are $\Sigma_1(L_{\eta_i})$ with parameter x, uniformly for $i<\gamma_0$. Now define $H^i_\delta = \Sigma_1$-Skolem hull of $\delta \cup \{\kappa, x\}$ in L_{η_i} for each $\delta \in \beta$-Card $\cap\, \kappa$ and let $g_i \in \mathbb{F}(t) \cap L_{\eta_{i+1}}$ be defined by $g_i(\delta) = H^i_\delta$ for $\delta \in H^i_\delta$, $\delta \geq \kappa_0$. By Lemma 2A.1 and induction, $\Sigma^q_{g_i}$ is dense in $\mathcal{P}^t \cap L_{\eta_{i+1}}$ for each $q \in \mathcal{P}^t \cap L_{\eta_{i+1}}$ and each $i<\gamma_0$.

If $(\bar\eta, \bar k) < (\eta, k)$, then we can assume that $\langle\kappa_j \mid j<\kappa_0\rangle$, $p \in H^0_0$ and we choose $p = q_0 \geq q_1 \geq \cdots$ successively so that $q_{i+1} \in \mathcal{P}^t \cap L_{\eta_{i+1}}$, $q_{i+1} \in \Sigma^{q_i}_{g_i}$. By Lemma 2A.2, $q_\lambda \in \mathcal{P}^t$ for limit $\lambda \leq \gamma_0$. Let $q = q_{\gamma_0} \in \Sigma_2(L_\eta) \cap \mathcal{P}^t$. Then $\kappa_j \in H_{\kappa_j} = \bigcup_i H^i_{\kappa_j}$ for each $j<\kappa_0$ and so q is as desired, letting $C = \{\kappa_i \mid i<\kappa_0\}$. If $(\bar\eta, \bar k) = (\eta, k)$, then we choose $p = q_0 \geq q_1 \geq \cdots$ successively so that $(q_{i+1})_{\kappa_i^+} = (p)_{\kappa_i^+}$ and $q_{i+1} \restriction \kappa_i \in \Sigma^{q_i \restriction \kappa_i}_{g_i \restriction \kappa_i}$ and so that uniformly for limit $\lambda \leq \gamma_0$, $\langle q_i \restriction \kappa_i \mid i<\lambda\rangle$ is $\Sigma_1(L_{\eta_\lambda})$ in parameter x. This is possible by induction and by the fact that $\langle\kappa_i \mid i<\lambda\rangle$, $p \restriction \kappa_\lambda$ are uniformly $\Sigma_1(L_{\eta_\lambda})$ with parameter x. It is then easy to verify as in Lemma 2A.2 that q_λ is a condition in $\mathcal{P}^t$ for limit λ. (Note however that we do not have $q_\lambda \in L_\eta$ as $p \notin L_\eta$.) Then $q = q_{\gamma_0}$ is the desired condition.

A similar argument can be used when $k = 1$, η successor using Σ_i-Skolem hulls in $L_{\eta-1}$ instead of Σ_1-Skolem hulls in L_{η_i}.

Now we consider the case $k>1$. By induction (if $(\bar\eta, \bar k) < (\eta, k)$) or by the Growth Condition for p (if $(\bar\eta, \bar k) = (\eta, k)$) choose $q' \leq p$ in $\Sigma_k(L_\eta)$ and a CUB $C \subseteq \beta$-Card $\cap\, \kappa$, $C \in \Sigma_k(L_\eta)$ so that $\delta \in C \cup \{\kappa\} \to C \cap \delta$ is $\Sigma_{k(\delta)}(L_{\eta(\delta)})$ (where $q' \restriction \delta \in \Sigma_{k(\delta)}(L_{\eta(\delta)}) - \Sigma_{k(\delta)-1}(L_{\eta(\delta)})$) and for any $x \in L_\eta$, $|q'_\delta| \geq H^x_\delta \cap \delta^+$ for sufficiently large $\delta \in C$, where $H^x_\delta = \Sigma_{k-1}$-Skolem hull of $\delta \cup \{\kappa, x\}$ in L_η. Now let $x \in L_\eta$ be a parameter so that C, q' are $\Sigma_k(L_\eta)$ with parameter x and for each $\delta \in C$, $\delta' \in \beta$-Card $\cap\, \delta$ let $H^\delta_{\delta'} = \Sigma_k$-Skolem hull of $\delta' \cup \{\kappa, x\}$ in H^x_δ. Then $\bigcup\{H^\delta_{\delta'} \mid \delta \in C\} = \Sigma_k$-Skolem hull of $\delta' \cup \{\kappa, x\}$ in L_η, for each $\delta' \in \beta$-Card $\cap\, \kappa$. By induction we can successively extend $q' \restriction \delta$ for sufficiently large $\delta \in C$ so that $q \restriction \delta \in \Sigma_k(H^x_\delta)$ and $|q_{\delta'}| \geq H^\delta_{\delta'} \cap (\delta')^+$ for $\delta' \in \beta$-Card $\cap\, \delta$, for $\delta \in C$. (We have assumed that $C \cap \delta$, $q' \restriction \delta$ are $\Sigma_k(H^x_\delta)$ for $\delta \in$ Limits Points(C), a property easily arranged by thinning C.) We must of course require that if $\kappa_0 < \kappa_1 < \cdots$ are the elements of C, then q' is extended to $q_0 \geq q_1 \geq \cdots$ where $q_{i+1} \restriction \kappa_i \in \Sigma^{q_i \restriction \kappa_i}_{g_i}$ and $g_i(\delta') = H^{\kappa_i}_{\delta'}$ for $\delta' \in \beta$-Card $\cap\, [\kappa_0, \kappa_i)$, $\delta' \in H^{\kappa_i}_{\delta'}$. This guarantees that as in Lemma 2A.2, q_λ is a condition for limit $\lambda \leq \gamma_0 =$ ordertype(C). Finally let $q = q_{\gamma_0}$, the desired extension.

Having now established the property stated at the beginning of this subcase we can now easily prove extendibility. Indeed the above argument shows the following: say that $q' \in L_{\mu^0_s}$ belongs to $\hat{\mathcal{P}}^s$ if q' obeys all the conditions for belonging to $\mathcal{P}^s$ with the possible exception of the Coding property (clause (4) in the definiton of $\mathcal{P}^s$). Then given $p \in \mathcal{P}^t$ the above shows that there exists $q' \leq p$, $q' \in \hat{\mathcal{P}}^s$ so that $q' \in \Sigma_2(L_{\mu_t+1})$ and for some $x \in L_{\mu_t+1}$, $|q'_\delta| =$ Transitive Collapse$(H^x_\delta) \cap$ ORD for all sufficiently large $\delta \in \beta$-Card $\cap\, \kappa$, $\delta \in H^x_\delta$ (where $H^x_\delta = \Sigma_1$-Skolem hull of $\delta \cup \{\kappa, x\}$ in L_{μ_t+1}).

Thus q' is our desired extension of p except for the fact that we may not have $s(\xi)=0$ iff $\bar{q}'=\bigcup\{\bar{q}'_\delta \mid \delta\in\beta\text{-Card}\cap\kappa\}$ is almost disjoint from $b_\xi=\{\langle 1,\rho_{\xi\gamma}\rangle \mid \gamma=\gamma_i^\kappa$ for some $i<\lambda_\kappa\}$. But note that $\langle 1,\rho_{\xi\gamma}\rangle\notin\text{Dom}(q'_{\gamma^+})$ for sufficiently large $\gamma\in\beta\text{-Card}\cap\kappa$ so extend q' to q by extending each q'_{γ^+} to q_{γ^+} for $\gamma=\gamma_i^\kappa$ so that $s(\xi)=0$ iff $q_{\gamma^+}(\langle 1,\rho_{\xi\gamma}\rangle)=0$. The only thing to check is that for $\delta=\gamma_\lambda^\kappa$, λ limit we have that $q\restriction\delta\in\mathcal{P}^{q_\delta'}=\mathcal{P}^{q_\delta}$. But this is clear as $\langle\rho_{\xi\gamma}\mid\gamma<\delta\rangle$ is Σ_2-definable over Transitive Collapse(H_δ^x) (and thus so is $q\restriction\delta$) and we can assume that $\lambda<\delta$ (and so $\delta\in H_\delta^x$).

C. Extendibility I: The limit case

The proof here is similar in outline to that of Lemma 1D.2 (whose proof in turn is based on that of Lemma 1B.9). As in that proof we divide into two subcases.

Subcase 1: $\nu_s=\max(\mu_s^0,\mu_s^-)$ *is not recursively inaccessible.*

We are given $p\in\mathcal{P}^{<s}$ and we wish to construct $q\leqslant p$, $q\in\mathcal{P}^s-\mathcal{P}^{<s}$. First we make some observations concerning the simplest possibility: $\nu_s=\mu_s^0$. If $\mathbf{b}=\langle L_{\mu_s^0}, s\restriction\mu_s^0\rangle\in\mathbf{E}$ and $D_\mathbf{b}$ is predense on $\mathcal{P}^{<s}$ we can extend p to $q'\in\mathcal{P}^{<s}$ so that q' meets $D_\mathbf{b}$. Otherwise note that Predensity Reduction is trivial for $q\in\mathcal{P}^s-\mathcal{P}^{<s}$. Thus if $\nu_s=\mu_s^0$ we need only extend p to $q\in L_{\mu_s}$ so as to meet the other conditions for belonging to $\mathcal{P}^s$.

Now we begin with the case: $C_{\mu_s^0}$ is unbounded in μ_s^0, ν_s a limit of admissibles. Let $\mu_0<\mu_1<\cdots$ enumerate a final segment of $C_{\mu_s^0}$ so that $\nu_{s\restriction\beta(\mu_i)^-}$ is inadmissible. There are canonical $\Sigma_{n(\mu_s^0)-1}$ elementary embeddings $\pi_{ij}:S_{\beta(\mu_i)}\to S_{\beta(\mu_j)}$ for $i\leqslant j\leqslant\lambda_0=\text{ordertype}(C_{\mu_s^0})$. Now for each i define $g_i\in\mathbb{F}(s\restriction\beta(\mu_i)^-)$ by: $g_i(\delta)=H_\delta^i=\Sigma_{n(\mu_s^0)}$-Skolem hull of $\delta\cup\{\kappa,p(\mu_i)\}$ inside $\mathcal{A}(\mu_i)$, if $\delta\in H_\delta^i$. Then by Lemma 2A.1, $\Sigma_{g_i}^q$ is dense on $\mathcal{P}^{<s}$ for $q\in\mathcal{P}^{<s}$. Now define $p_0\geqslant p_1\geqslant\cdots$ inductively by: $p_0=p$, $p_{i+1}=$ least $q\leqslant p_i$ such that $q\in\Sigma_{g_i}^{p_i}(q(\delta)=p_i(\delta)$ if $g_i(\delta)=\bigcup\{g_j(\delta)\mid j<i\})$, $p_\lambda=\bigcup\{p_i\mid i<\lambda\}$ (as in Lemma 2A.2) for limit $\lambda\leqslant\lambda_0$. We must verify that this is a valid induction at limit stages λ. Note that p_λ is $\Sigma_{n(\mu_\lambda)+1}(S_{\beta(\mu_\lambda)})$; using this the Growth Condition is easily verified. The main thing to check is Predensity Reduction (the other properties follow as in the proof of Lemma 2A.2). So suppose $D\in L_{\nu_{s_\lambda}}[s_\lambda^*]$ is predense on $\mathcal{P}^{<s_\lambda}$ where $s_i=s\restriction\beta(\mu_i)^-$. We must show that p_λ reduces D. This is clear if $\nu_{s_\lambda}=\mu_{s_\lambda}^0$, for then $D\in L_{\nu_{s_i}}[s_i^*]$ for some $i<\lambda$ and hence p_{i+1} reduces D. Otherwise let $\kappa'=(\mu_{s_\lambda}^0)^+$ in the sense of $L_{\nu_{s_\lambda}}$ ($\kappa'=\nu_{s_\lambda}$ if $L_{\nu_{s_\lambda}}\models\mu_{s_\lambda}^0$ is the largest cardinal) and choose a limit ordinal $\xi\in[\mu_{s_\lambda}^0,\kappa')$ so that $D\in L_{\mu_\xi}[s_\lambda^*]$ (viewing $\xi\in\mathcal{O}(\mu_{s_\lambda}^0)$ in the definition of μ_ξ). Now $s_\lambda\restriction[\kappa,\mu_{s_\lambda}^0)$ is $R^{s_\lambda\restriction[\mu_{s_\lambda}^0,\xi)}$-generic over L_{μ_ξ} as it is R^G-generic over $L_{\nu_{s_\lambda}}[G]$, $G=\mathcal{P}_{\mu_{s_\lambda}^0}^{\nu_{s_\lambda}}$-generic corresponding to $s_\lambda\restriction[\mu_{s_\lambda}^0,\nu_{s_\lambda})$. So we can pick $q\in H=R^{G\restriction\xi}$-generic corresponding to $s_\lambda\restriction[\kappa,\mu_{s_\lambda}^0]$ so that $q\Vdash\mathbf{D}$ is predense on $\mathcal{P}^{<H}=\bigcup\{\mathcal{P}^{<s}\mid(s,\bar{s})\in H$ for some $\bar{s}\}$, where $\mathbf{D}$ is a name for D.

Now as ν_{s_λ} (and hence κ') is a limit of admissibles we can choose $i<\lambda$ so that

$\xi, q, \mathbf{D} \in \text{Range}(\pi_{i\lambda})$ and in addition there is an admissible $\alpha \in (\xi, \nu_{s_\lambda}) \cap \text{Range}(\pi_{i\lambda})$. Then $q \Vdash \pi_{i\lambda}^{-1}(\mathbf{D})$ is predense on $\mathcal{P}^{<\bar{H}}$ where $\bar{H} = R^{\bar{G} \restriction \bar{\xi}}$-generic corresponding to $s_i \restriction [\kappa, \mu_i)$ and $\bar{G} = \mathcal{P}_{\mu_i}^{\bar{\alpha}}$-generic corresponding to $s_i \restriction [\mu_i, \bar{\alpha})$, $(\bar{\alpha}, \bar{\xi}) = (\pi_{i\lambda}^{-1}(\alpha), \pi_{i\lambda}^{-1}(\xi))$.

But then $q \Vdash \mathbf{D} \cap L_{\mu_i}$ is predense on $\mathcal{P}^{<H}$ by Predensity Reduction for elements of $\mathcal{P}^{<H}$ and the elementariness of $\pi_{i+1,\lambda}$. So we have shown that any predense $D \subseteq \mathcal{P}^{<s_\lambda}$, $D \in L_{\nu_{s_\lambda}}[s_\lambda^*]$ contains a predense $D' \in L_{\mu_{s_\lambda}^0}$. Thus as ν_{s_λ} is inadmissible, Predensity Reduction follows from the simple case $\mu_{s_\lambda}^0 = \nu_{s_\lambda}$. (Also note that $\langle L_{\mu_{s_\lambda}^0}, s_\lambda \restriction \mu_{s_\lambda}^0 \rangle \notin \mathbf{E}$ for limit λ.) The above argument also works in the case: $C_{\mu_s^0}$ unbounded in μ_s^0, $\nu_s' = ((\mu_s^0)^+$ in the sense of $L_{\nu_s})$ is a limit of admissible ordinals.

Suppose next that $C_{\mu_s^0}$ is bounded in μ_s^0 and $\rho(\mu_s^0) > \mu_s^0$ is the limit of admissible ordinals. As $C_{\mu_s^0}$ is bounded in μ_s^0 we can choose a $\Pi_1(\mathcal{A}(\mu_s^0))$-cofinal ω-sequence $\rho_0 < \rho_1 < \cdots$ of admissible ordinals below $\rho(\mu_s^0)$; also arrange that $A(\mu_s^0) \cap \rho_i \in L_{\rho_{i+1}}$, where $A(\mu_s^0) = \Sigma_{n(\mu_s^0)-1}$-Master Code for $\beta(\mu_s^0)$. Let $H_\delta^i = \Sigma_1$-Skolem hull of $\delta \cup \{\kappa, p(\mu_s^0), A(\mu_s^0) \cap \rho_{i-1}\}$ inside L_{ρ_i} and $\mu_i = H_\kappa^i \cap \mu_s^0$ for all $i \in \omega$, $\delta \in \beta\text{-Card} \cap \mu_s^0$. Define $g_i(\delta) = H_\delta^i$ for $\delta \in H_\delta^i$ and construct $p = p_0 \geqslant p_1 \geqslant \cdots$ in $\mathcal{P}^{<s}$ by requiring $p_{i+1} =$ least $q \leqslant p_i$ in $\Sigma_{g_i}^{p_i}$. Then $q = \bigcup \{p_i \mid i \in \omega\}$ (as in Lemma 2A.2) is a member of $\mathcal{P}^s$: We need only check that each predense $D \in L_{\nu_s}[s^*]$ is reduced by p. But as in the preceding case, we can use the genericity of $s \restriction [\kappa, \mu_s^0)$ to show that $D \in L_{\nu_s}[s^*]$, D predense on $\mathcal{P}^{<s} \to D \supseteq$ a predense $D' \in L_{\mu_s^0}$ and thus p reduces D.

The above arguments also work if $\rho(\mu_s^0)$ is not the limit of admissible ordinals but $\rho(\mu_s^0) > \nu_s' = (\mu_s^0)^+$ in the sense of L_{ν_s}. For if $C_{\mu_s^0}$ is bounded in μ_s^0, then we can choose $\rho_0 < \rho_1 < \cdots$ cofinally in $\rho(\mu_s^0)$ (if $\rho(\mu_s^0)$ is a limit of limit ordinals) so that $\langle \rho_i \mid i \in \omega \rangle$ is $\Sigma_1(S_{\rho(\mu_s^0)})$, $\nu_s' < \rho_0$ and let $H_\delta^i = \Sigma_1$-Skolem hull of $\delta \cup \{\kappa, p(\mu_s^0), \nu_s'\}$ in L_{ρ_i}. Then we can still argue that $D \in L_{\nu_s}[s^*]$, D predense$\to D \supseteq$ a predense D', $D' \in L_{\mu_s^0}$ since $\nu_s' \in \text{Range}(\pi_i)$ for all i. (If $\rho(\mu_s^0)$ is not the limit of limit ordinals, then write $\rho(\mu_s^0) = \rho' + \omega$ and let $H_\delta^i = \Sigma_i$-Skolem hull of $\delta \cup \{\kappa, x\}$ in $L_{\rho'}$ where $p(\mu_s^0)$, ν_s' can be assumed to be $L_{\rho'}$-definable with parameter x.) If $C_{\mu_s^0}$ is unbounded in μ_s^0, then proceed as in the case where ν_s is a limit of admissible ordinals except note that we can choose $\mu_0 < \mu_1 < \cdots$ to be a final segment of $C_{\mu_s^0}$ so that $\nu_s' \in \text{Range}(\pi_{i_{\lambda_0}})$ for all i.

So we are left with the cases: $\rho(\mu_s^0) = \mu_s^0$, $\rho(\mu_s^0) = \nu_s'$ is not the limit of admissibles. The latter case is contained in the former one as it implies that $\rho(\mu_s^0) = \beta(\mu_s^0) = \nu_s'$ is a successor admissible and hence either $\beta(\mu_s^0) = \mu_s^0$ or Σ_1-projectum$(\beta(\mu_s^0)) = \mu_s^0$ (contradicting the definition of $\rho(\mu_s^0)$). The former case is easily handled if $n(\mu_s^0) = 1$ for then $\nu_s = \beta(\mu_s^0) = \mu_s^0$ and Predensity Reduction is trivial for $q \in \mathcal{P}^s - \mathcal{P}^{<s}$; thus if $C_{\mu_s^0}$ is bounded in μ_s^0, we can simply choose a $\Pi_1(L_{\mu_s^0})$ ω-sequence $\mu_0 < \mu_1 < \cdots$ cofinal in μ_s^0, let $H_\delta^i = \Sigma_1$-Skolem hull of $\delta \cup \{\kappa, p(\mu_s^0)\}$ in L_{μ_i} and proceed as before. Then $D \in L_{\nu_s}[s^*] = L_{\nu_s} = L_{\mu_s^0} \to D \in L_{\mu_i} = L_{\nu(\mu_i)}$ for some i so D is reduced by p_{i+1}. (We assume that $\mu_i = \mu_{t_i}^0$ for some $t_i \in S_\kappa$, $t_i \subseteq s$.) A similar argument suffices if $C_{\mu_s^0}$ is unbounded in μ_s^0. So assume that $n(\mu_s^0) > 1$ and consider $\mathcal{A}'(\mu_s^0) = \langle L_{\rho'(\mu_s^0)}, A'(\mu_s^0) \rangle$ where $\rho'(\mu_s^0) =$

$\rho_j^{\beta(\mu_s^0)} > \mu_s^0$, $\rho_{j+1}^{\beta(\mu_s^0)} = \mu_s^0$, $A'(\mu_s^0) = \Sigma_j$-Master Code for $\beta(\mu_s^0)$. Let $p'(\mu_s^0) =$ least p so that $\mathscr{A}'(\mu_s^0) = \Sigma_1$-Skolem hull of $\mu_s^0 \cup \{p\}$ in $\mathscr{A}'(\mu_s^0)$. Now if $C_{\mu_s^0}$ is bounded in μ_s^0 we can choose a $\Pi_1(\mathscr{A}(\mu_s^0))$ ω-sequence $\mu_0 < \mu_1 < \cdots$ cofinal in μ_s^0 so that $\mu_i = \mu_s^0 \cap (\Sigma_1$-Skolem hull of $\mu_i' \cup \{\kappa, p'(\mu_s^0)\}$ in $\mathscr{A}'(\mu_s^0))$ for some $\mu_i' < \mu_i$ and define $H_\delta^i = \Sigma_1$-Skolem hull of $\delta \cup \{\kappa, p(\mu_i)\}$ inside $\mathscr{A}(\mu_i)$ for $\delta \in \beta$-Card $\cap\, \mu_s^0$, $i \in \omega$. Note that $\mathscr{A}(\mu_i) =$ Transitive Collapse of Σ_1-Skolem hull of $\mu_i \cup \{\kappa, p'(\mu_s^0)\}$ in $\mathscr{A}'(\mu_s^0)$; let $\pi_i : \mathscr{A}(\mu_i) \to \mathscr{A}'(\mu_s^0)$ be the collapsing map. Then if we define $p_0 \geqslant p_1 \geqslant \cdots$ by $p_0 = p$, $p_{i+1} =$ least $q \leqslant p_i$, $q \in \Sigma_{g_i}^{p_i}(g_i(\delta) = H_\delta^i$ for $\delta \in H_\delta^i)$ we see that $q = \bigcup\{p_i \mid i \in \omega\}$ (as in Lemma 2A.2) is a condition, provided $\rho'(\mu_s^0)$ is either greater than ν_s' or the limit of admissibles; for in that case we can show that any predense $D \in L_{\nu_s}[s^*]$ contains a predense $D' \in L_{\nu_s^0}$ and hence is reduced by q, using the fact that $\nu_s' \leqslant \rho'(\mu_s^0)$, $L_{\rho'(\mu_s^0)} \subseteq \bigcup\{H_\kappa^i \mid i \in \omega\}$.

If $C_{\mu_s^0}$ is unbounded in μ_s^0 but $\rho(\mu_s^0) = \mu_s^0$, $n(\mu_s^0) > 1$ and $\rho'(\mu_s^0)$ is either greater than ν_s' or the limit of admissibles, then a similar argument suffices. (Indeed, if $\rho'(\mu_s^0)$ is the limit of admissibles, then so is ν_s' and the argument has already been provided.)

Finally consider the case: $\nu_s' = \rho'(\mu_s^0)$ is a successor admissible. It follows that $\nu_s = \nu_s'$ and Σ_1-projectum$(\nu_s) = \mu_s^0 < \nu_s$. (We may have $n(\mu_s^0) > 2$.) This case will be easy to handle after we establish the following.

Claim. *Suppose $\mathscr{S} \in L_{\nu_s}[s^*]$ is a collection of predense sets on $\mathscr{P}^{<s}$, $L_{\nu_s}[s^*] \vDash$ card$(\mathscr{S}) = \kappa$. Then for all $p \in \mathscr{P}^{<s}$ there exists $q \leqslant p$ in $\mathscr{P}^{<s}$ so that q reduces D for all $D \in \mathscr{S}$.*

Proof. By the $R^{s \upharpoonright [\mu_s^0, \nu_s)}$-genericity of $s \upharpoonright [\kappa, \mu_s^0)$ it suffices to show that if $r = (r_\kappa, \bar{r}_\kappa) \in R^{s \upharpoonright [\mu_s^0, \nu_s)} = R$ and $p \in \mathscr{P}^{r_\kappa}$, then there exists $r' \leqslant r$ and $q \leqslant p$ so that $r' \Vdash q$ meets D for all $D \in \mathscr{S}$, where $\mathscr{S}$ is a name for $\mathscr{S}$. Choose a limit ordinal $\beta < \nu_s$ so that rank$(\mathscr{S}) < \beta$, L_{ν_s}-cof$(\beta) = \kappa$. Now define κ-sequences $p_0 \geqslant p_1 \geqslant \cdots$, $r_0 \geqslant r_1 \geqslant \cdots$ and $\beta_0 < \beta_1 < \cdots$ as follows: $p_0 = p$, $r_0 = r$, $\beta_0 =$ least ordinal $< \beta$ so that rank$(\mathscr{S}) < \beta_0$. If p_1, r_i, β_i have been defined, then first define (r_{i+1}, p_{i+1}') to be least so that $p_{i+1}' \in \mathscr{P}^{r_{i+1\kappa}}$, $p_{i+1}' \leqslant p_i$, $r_{i+1}' \leqslant r_i$ in $R^{s \upharpoonright [\mu_s^0, \beta)}$ and $r_{i+1}' \Vdash p_{i+1}'$ reduces the ith element of $\mathscr{S}$ (in the least well-ordering of $\mathscr{S}$ of length κ). Also require that $(p_{i+1}')^{i^+} = (p_i)^{i^+}$. Then choose β_{i+1} to be the least limit ordinal greater than β_i, $r_{i+1}' \in R^{s \upharpoonright [\mu_s^0, \beta_{i+1})}$ and set $H_\delta^i = \Sigma_1$-Skolem hull of $\delta \cup \{\kappa, p_{i+1}', r_{i+1}', \mathscr{S}, s \upharpoonright [\mu_s^0, \beta_{i+1})\}$ in $L_{\mu_{\beta_{i+1}}}$, $g_i(\sigma) = H_\delta^i$ if $\delta \in H_\delta^i$. Then choose $p_{i+1} \leqslant p_{i+1}'$ to be least so that $p_{i+1} \in \Sigma_{g_i}^{p_{i+1}'}$, $(p_{i+1})^{i^+} = (p_i)^{i^+}$. Also define $r_{i+1} \leqslant r_{i+1}'$ so that r_{i+1} meets all predense $d \in H_\kappa^i$. For limit λ let $r_\lambda = \bigcup\{r_i \mid i < \lambda\}$ (as in Lemma 1B.9), $p_\lambda = \bigcup\{p_i \mid i < \lambda\}$ (as in Lemma 2A.2) and $\beta_\lambda = \bigcup\{\beta_i \mid i < \lambda\}$. Then $q = p_\kappa$, $r' = r_\kappa = \bigcup\{r_i \mid i < \kappa\}$ are as desired. This proves the Claim.

Now to build the desired $q \leqslant p$, $q \in \mathscr{P}^s - \mathscr{P}^{<s}$ when $C_{\mu_s^0}$ is bounded in μ_s^0, choose a $\Pi_1(\mathscr{A}(\mu_s^0))$ ω-sequence $\mu_0 < \mu_1 < \cdots$ cofinal in μ_s^0, let $\beta_0 < \beta_1 < \cdots$ be defined by $\beta_i = \bigcup[(\Sigma_1$-Skolem hull of $\mu_i \cup \{p(\nu_s)\}$ in $L_{\nu_s}) \cap \text{ORD}]$ and $\mathscr{S}_i =$ all predense $D \subseteq \mathcal{P}^{<s}$, $D \in \Sigma_1$-Skolem hull of $\mu_i \cup \{s^*, p(\nu_s)\}$ in $L_{\beta_i}[s^*]$. Define $p_0 \geqslant p_1 \geqslant \cdots$ as follows: $p_0 = p$; if $g_i(\delta) = H^i_\delta = \Sigma_1$-Skolem hull of $\delta \cup \{\kappa, p_i, \mu_i, \mathcal{S}_i, s^*\}$ in $L_{\beta_{i+1}}[s^*]$ for $\delta \in H^i_\delta$, then $p_{i+1} \leqslant p_i$ is least so that $p_{i+1} \in \Sigma^{p_i}_{g_i}$, p_{i+1} reduces all predense $D \in \mathcal{S}_i$. Then $q = \bigcup \{p_i \mid i \in \omega\}$ (as in Lemma 2A.2) is a condition in $\mathcal{P}^s - \mathcal{P}^{<s}$.

If $C_{\mu^0_s}$ is unbounded in μ^0_s, then let $\mu_0 < \mu_1 < \cdots$ list $C_{\mu^0_s}$ and let $\beta_0 < \beta_1 < \cdots$ be defined by $\beta_i = \bigcup [(\Sigma_1$-Skolem hull of $\mu_i \cup \{p(\nu_s)\}$ in $L_{\nu_s}) \cap \mathrm{ORD}]$; set $\mathcal{S}_i =$ all predense $D \subseteq \mathcal{P}^{<s}$, $D \in \Sigma_1$-Skolem hull of $\mu_i \cup \{s^*, p(\nu_s)\}$ in $L_{\beta_i}[s^*]$ and define $p_0 \geqslant p_1 \geqslant \cdots$ as follows: $p_0 = p$; if $g_i(\delta) = H^i_\delta = \Sigma_1$-Skolem hull of $\delta \cup \{\kappa, p_i, \mu_i, \mathcal{S}_i, s^*\}$ in $L_{\beta_{i+1}}[s^*]$ for $\delta \in H^i_\delta$, then $p_{i+1} \leqslant p_i$ is least so that $p_{i+1} \in \Sigma^{p_i}_{g_i}$, p_{i+1} reduces all $D \in \mathcal{S}_i$, $p_{i+1}(\delta) = p_i(\delta)$ whenever $\delta < i$ and $g_i(\delta) \subseteq \bigcup \{g_j(\delta) \mid j < i\}$; for limit $\lambda \leqslant \lambda_0 =$ order type$(C_{\mu^0_s})$ set $p_\lambda = \bigcup \{p_i \mid i < \lambda\}$ (as in Lemma 2A.2). Then $q = p_{\lambda_0}$ is a condition in $\mathcal{P}^s - \mathcal{P}^{<s}$. This completes Subcase 1.

Subcase 2: ν_s *is recursively inaccessible.*

We begin with the following observation: We can assume that $\rho = \Sigma_1$-projectum $L_{\nu_s}[s^*] \leqslant \mu^0_s$. For, if $\gamma' < \nu_s$, $T \subseteq \mathcal{P}^{<s} \times \gamma'$, T $\Sigma_1(L_{\nu_s}[s^*])$, then the κ^+-c.c. of $\mathcal{P}^{<s}$ in $L_{\nu_s}[s^*]$ implies the Σ-κ^+-c.c. of $\mathcal{P}^{<s}$ in $L_{\nu_s}[s^*]$ if $\rho > \mu^0_s$ and hence $\mathcal{D}(T) = \{p \in \mathcal{P}^{<s} \mid (\forall \eta < \gamma'\ T_\eta$ is dense below $p)$ or $(\exists \eta < \gamma' \forall q \leqslant p$ $(q, \eta) \notin T)\}$ is $(\Sigma_1 \vee \Pi_1)$-definable over $L_{\nu_s}[s^*]$. But again, if $\rho > \mu^0_s$, then $\mathcal{D}(T)$ is an element of $L_{\nu_s}[s^*]$ and thus Σ-genericity follows from Subcase 1. From the proof of Lemma 1D.2 we can actually conclude that $\nu^*_s = \Sigma_1$-projectum$(L_{\nu_s}) \leqslant \mu^0_s$. So $\nu_s = \mu^0_s$ or $L_{\nu_s} \vDash \mu^0_s$ is the largest cardinal.

First suppose the former and that κ is regular in L_{ν_s}. We use the following.

Claim. *Suppose $\langle W_i \mid i < \delta_0 \rangle$ is a uniformly $\Sigma_1 \langle L_{\nu_s}, s^* \rangle$-sequence of persistent subsets of $\mathcal{P}^{<s} \times \kappa$ and $\delta_0 < \kappa$. Then for any $p \in \mathcal{P}^{<s}$ there exists $q \leqslant p$, q reduces $\mathcal{D}(W_i)$ for each $i < \delta_0$.*

Proof of Claim. First fix $i_0 < \delta_0$ and we describe a certain procedure for extending p to q so that $q \in \mathcal{D}(W_{i_0})$. Define $p_0 = p$ and if p_i has been defined, construct p_{i+1} as follows: First locate the least pair $(\eta, r) \in L_\delta$, $\delta = \beta$-card(i) so that $r \leqslant (p_i)^\delta$, $r \in \mathcal{P}^{p_i}{}_\delta$ and for no $r' \leqslant r$ do we have $r' \cup (p_i)_\delta \in (W_{i_0})^{|p_i|}_\eta =$ amount of $(W_{i_0})_\eta$ enumerated over $\langle L_{\nu_s}, s^* \rangle$ by stage $|p_i|$. If (η, r) does exist when p_i is replaced by p' for all $p' \leqslant p_i$, then p_{i+1} is undefined. Otherwise let $p'_{i+1} =$ least extension of p_i for which this is not true and such that $(p'_{i+1})^\delta = (p_i)^\delta$; then obtain p_{i+1} from p'_{i+1} by defining $g_i(\gamma) = H^i_\gamma = \Sigma_1$-Skolem hull of $\gamma \cup \{\kappa, p'_{i+1}, x\}$ in $L_{\mu_{|p'_{i+1}|}}$ for $\gamma \in H^i_\gamma$ (where x is a parameter for defining $\langle W_i \mid i < \delta_0 \rangle$) and letting p_{i+1} be the least extension of p'_{i+1} in $\Sigma^{p'_{i+1}}_{g_i}$, $p_{i+1}(\gamma) = p'_{i+1}(\gamma)$ if $g_i(\gamma) = \bigcup \{g_j(\gamma) \mid j < i\}$; for limit λ let $p_\lambda = \bigcup \{p_i \mid i < \lambda\}$ (as in Lemma 2A.2). If i is least so that p_{i+1} is not defined, then it must be that $p' \leqslant p_i$ as above does not exist.

(Note that $|p_\lambda|$ is inadmissible relative to s^* for limit λ, so Σ-genericity need not be shown.)

Now there are two possibilities. If $i=\kappa$ then $p_\kappa \in \mathscr{D}(W_{i_0})$ because for each $\eta<\kappa$, $(W_{i_0})_\eta^{p_{\kappa\eta^+}} = \{r \in \mathscr{P}^{p_{\kappa\eta^+}} \mid r \cup (p_\kappa)_{\eta^+} \in (W_{i_0})_\eta\}$ is dense by construction below $(p_\kappa)^{\eta^+}$ and hence $(W_{i_0})_\eta$ is dense below p_κ. If $i<\kappa$, then for some $\delta \in \beta\text{-Card} \cap \kappa$ and $\eta<\delta$ we can extend $(p_\kappa)^\delta$ to $r \in \mathscr{P}^{p_{\kappa_\delta}}$ so that no extension of $r \cup (p_\kappa)_\delta$ belongs to $(W_{i_0})_\eta$. Thus $r \cup (p_\kappa)_\delta \in \mathscr{D}(W_{i_0})$.

We need a refinement of the above procedure. Namely fix $\delta \in \beta\text{-Card} \cap \kappa$ and $p_0 \in \mathscr{P}^{<s}$. Then the above procedure provides a method of extending p_0 to q s.t. $(q)^\delta=(p_0)^\delta$ and such that for some extension $r \leqslant (p_0)^\delta$ in $\mathscr{P}^{q_\delta}$ we have $r \cup (q)_\delta \in \mathscr{D}(W_{i_0})$. Now we iterate this procedure for all $p_0^* \leqslant p$, $(p_0^*)_\delta=(p)_\delta$. (Thus we obtain $q \leqslant p$, $(q)^\delta=(p)^\delta$ so that $\mathscr{D}(W_{i_0})^{(q)_\delta} = \{r \in \mathscr{P}^{q_\delta} \mid r \cup (q)_\delta \in \mathscr{D}(W_{i_0})\}$ is dense below $(p)^\delta$ in $\mathscr{P}^{q_\delta}$.) The difficulty will be that the procedure $p_0 \mapsto r$, q is $\Pi_1\langle L_{\nu_s}, s^*\rangle$ and thus we have definability problems at limit stages. So as in Subcase 1 we resort to the genericity properties of s: it is enough to show that for $t \in S_\kappa$, $p \in \mathscr{P}^t$ there exists $t' \leqslant t$, $q \leqslant p$ such that $q \in \mathscr{P}^{t'}$ and $(q)^\delta=(p)^\delta$, $\mathscr{D}(W_{i_0})^{(q)_\delta}$ is dense below $(p)^\delta$ in $\mathscr{P}^{q_\delta}$, as s is Σ-generic. But this is not difficult as we can define $p_0 \geqslant p_1 \geqslant \cdots$ and $t_0 \geqslant t_1 \geqslant \cdots$ as follows: $p_0=p$, $t_0=t$; if p_i, t_i have been defined then first pick $p'_{i+1} \leqslant p_i$, $t'_{i+1} \leqslant t_i$ to be least so that $p'_{i+1} \in \mathscr{P}^{t'_{i+1}}$ and $(p'_{i+1})^\delta=(p_i)^\delta$, $r \cup (p'_{i+1})_\delta \in \mathscr{D}(W_{i_0})$ where $r \leqslant (p_0^*)^\delta$ and $p_0^* \leqslant p$, $(p_0^*)_\delta=(p)_\delta$ is least so that this fails with p'_{i+1} replaced by p_i. Then pick $t_{i+1} \leqslant t'_{i+1}$, $p_{i+1} \in \mathscr{P}^{t_{i+1}}$, $p_{i+1} \leqslant p'_{i+1}$ so that $p_{i+1} \in \Sigma_{g_i}^{p'_{i+1}}$, $(p_{i+1})^\delta=(p)^\delta$ where $g_i(\gamma)=H^i_\gamma=\Sigma_1$-Skolem hull of $\gamma \cup \{\kappa, t'_{i+1}, x\}$ in L_{ν_s}, for $\gamma \in H^i_\gamma$. Then p_{i+1} is undefined for some $i<\delta^{++}$ as the function $i \mapsto p_i$ is $\Pi_1(L_{\nu_s})$ and hence bounded on $\delta^{++} > \beta\text{-Card}\{p_0^* \leqslant p \mid (p_0^*)_\delta=(p)_\delta\}$. If $q=p_i$ where i is least so that p_{i+1} is undefined, then q is as desired.

It is now easy to prove the Claim. In fact we can apply the above argument so as to show that for any $\delta \in \beta\text{-Card} \cap \kappa$, $\delta>\delta_0$ and any $p \in \mathscr{P}^{<s}$ there exists $q \leqslant p$ such that $(q)^\delta=(p)^\delta$ and $i<\delta_0 \to \mathscr{D}(W_i)^{(q)_\delta}$ is dense below $(p)^\delta$. The construction is identical to the previous one except we consider all the $\mathscr{D}(W_i)$, $i<\delta_0$. This is possible as $\delta_0<\delta$. This completes the proof of the Claim.

We can now construct the desired $q \leqslant p$, $q \in \mathscr{P}^s - \mathscr{P}^{<s}$ using the Claim. Note that we can assume that Σ_1-projectum $\langle L_{\nu_s}, s^*\rangle=\kappa$ and hence (as in the proof of Lemma 1D.2) Σ_1-projectum $(L_{\nu_s})=\kappa$. As ν_s is admissible, we have that C_{ν_s} is unbounded in ν_s and $\nu \in C_{\nu_s} \to \nu$ is inadmissible. Now construct $p_0 \geqslant p_1 \geqslant \cdots$ as follows: $p_0=p$; if p_i is defined, then p'_{i+1} is the least extension of p_i such that $|p'_{i+1}| \in C_{\nu_s}$, $(p'_{i+1})^{i^{++}}=(p_i)^{i^{++}}$, $s \restriction |p'_{i+1}| \Vdash p'_{i+1}$ reduces $\mathscr{D}(W_j)$ for all $j \in \Sigma_1$-Skolem hull of $i^+ \cup \{\kappa, p_i, p(\nu_s)\}$ in $L_{\nu_s}[\mathbf{s}]$ (where $\mathbf{s}$ is a name for the $\mathscr{P}^{\nu_s}$-generic s, $W_j \subseteq \mathscr{P}^{<\mathbf{s}} \times \kappa$ is the $\Sigma_1(L_{\nu_s}[\mathbf{s}])$-set with index j) and $p_{i+1} \leqslant p'_{i+1}$ is defined to be least so that $|p_{i+1}| \in C_{\nu_s}$, $p_{i+1} \in \Sigma_{g_i}^{p'_{i+1}}$ where $g_i(\delta)=H^i_\delta=\Sigma_1$-Skolem hull of $\delta \cup \{\kappa, p(\nu_s), p'_{i+1}\}$ in $L_{\mu_{|p'_{i+1}|}}$ for $\delta \in H^i_\delta$, $p_{i+1}(\delta)=p_i(\delta)$ if $\delta<i^+$, $g_i(\delta)=\bigcup\{g_j(\delta) \mid j<i\}$. Then $p_\lambda=\bigcup\{p_i \mid i<\lambda\}$ is a condition for limit $\lambda \leqslant \kappa$ and we set $q=p_\kappa$. This completes the case: $\mu_s^0=\nu_s$, κ regular in L_{ν_s}.

If $\mu_s^0 = \nu_s$ and κ is singular in L_{ν_s}, then it suffices to reduce sets of the form $\mathscr{D}(W)$ where $W \subseteq \mathscr{P}^{<s} \times \gamma$, $\gamma < \kappa$ is $\Sigma_1\langle L_{\nu_s}, s^*\rangle$, for if $W \subseteq \mathscr{P}^{<s} \times \kappa$, then consider $W' \subseteq \mathscr{P}^{<s} \times \gamma_0$, $\gamma_0 = L_{\nu_s}$-cof(κ) defined by $(p, \eta) \in W'$ iff $\forall i < f(\eta)\ (W)_i$ is dense below p and then $q \in \mathscr{D}(W') \to (W)_i$ is dense below q for all $i < \kappa$ or $\forall q' \leq q \sim \forall i < f(\eta)\ (W)_i$ is dense below q', for some η. Thus if q reduces all $\mathscr{D}(W')$, $W' \subseteq \mathscr{P}^{<s} \times \gamma$, $\gamma < \kappa$, then q also reduces all $\mathscr{D}(W)$, $W \subseteq \mathscr{P}^{<s} \times \kappa$ $(W, W'\ \Sigma_1\langle L_{\nu_s}, s^*\rangle)$. Now using our previous construction it is easy to establish the following version of the previous Claim: If $\langle W_i \mid i < \delta_0\rangle$ is a uniformly $\Sigma_1\langle L_{\nu_s}, s^*\rangle$-sequence of persistent subsets of $\mathscr{P}^{<s} \times \delta_0$, $\delta_0 < \kappa$, then for all $p \in \mathscr{P}^{<s}$ there exists $q \leq p$, q reduces $\mathscr{D}(W_i)$ for each $i < \delta_0$. Indeed we can insist that $(q)^{\delta_0} = (p)^{\delta_0}$. The new point here is that we cannot use the fact that Π_1-cof$(L_{\nu_s}) = \kappa$ to argue that the sequence $\langle p_i \mid i < \delta_0^{++}\rangle$ is an element of L_{ν_s}; instead just note that the collection of stages at which p_{i+1} is defined is the range of a partial $\Sigma_1(L_{\nu_s})$-function on a subset of $L_{\delta_0^+}$.

Given this new version of the Claim we can proceed as before to define $q \leq p$, except now the construction takes L_{ν_s}-cof$(\kappa) = \gamma_0$ steps instead of κ many. (Note that Π_1-cof$\langle L_{\nu_s}, s^*\rangle = \gamma_0$.)

We now turn to the case $\nu_s > \mu_s^0$. We wish to establish the following Claim analogous to the earlier one in the case $\nu_s = \mu_s^0$. Let μ denote μ_s^0.

Claim. *Suppose $\langle W_i \mid i < \kappa\rangle$ is a uniformly $\Sigma_1(L_{\nu_s}[s^*])$-sequence of persistent subsets of $\mathscr{P}^{<s} \times \mu$. Then for any $p \in \mathscr{P}^{<s}$ there exists $q \leq p$, q reduces $\mathscr{D}(W_i)$ for each $i < \kappa$.*

To establish this Claim we must first show the following: Given $W \subseteq \mathscr{P}^{<s} \times \mu$, $W\ \Sigma_1(L_{\nu_s}[s^*])$ and $p \in \mathscr{P}^{<s}$, $\delta \in \beta$-Card $\cap\ \kappa$, there exists $q \leq p$, $(q)^{\delta^+} = (p)^{\delta^+}$ and q reduces $\mathscr{D}(W)$. As in Lemma 1D.2 we use the fact that Π_1-cof$(L_{\nu_s}[s^*]) = \mu$ to do this. (We also use the Σ-genericity of $s \restriction \mu$ to deal with definability at limit stages.) Define $p_0 \geq p_1 \geq \cdots$ as follows: $p_0 = p$; if p_i is defined, then $p'_{i+1} \leq p_i$ is least so that $(p'_{i+1})^{\delta^+} = (p)^{\delta^+}$ and for some $r \leq (p)^{\delta^+}$ in $\mathscr{P}^{p_{i\delta^+}}$ we have $r'' \leq r$ so that either $q' \leq r'' \cup (p'_{i+1})_{\delta^+} \to q' \notin (W)_\eta$ for some $\eta < \mu$ or for some $w \subseteq W$ such that $w \in L_{\nu_s}[s^*]$, $(w)_\eta$ is predense below $r'' \cup (p'_{i+1})_{\delta^+}$ for all $\eta < \mu$ (and this is not true if $p'_{i+1} = p_i$). Then let $p_{i+1} \leq p'_{i+1}$ be least so that $|p_{i+1}| = \mu \cap (\Sigma_1$-Skolem hull of $|p_{i+1}| \cup \{p'_{i+1}, \mu, x\}$ in $\langle L_{\nu_s}, s \restriction [\mu, \nu_s)\rangle)$, $s \restriction |p_{i+1}| \Vdash p'_{i+1}$ has the above properties (where x is a parameter for a $\Sigma_1\langle L_{\nu_s}, s \restriction [\mu, \nu_s)\rangle$-injection of L_{ν_s} into μ and $\Vdash$ refers to $\mathscr{P}_\kappa^{s \restriction [\mu, \nu_s]} = R^{s \restriction [\mu, \nu_s)}$) and if $g_i(\gamma) = H^i_\gamma = (\Sigma_1$-Skolem hull of $\gamma \cup \{\kappa, p'_{i+1}\}$ in $L_{\mu_{|p'_{i+1}|}})$ for $\gamma \in H^i_\gamma$, $\gamma > \delta$, then $p_{i+1} \in \Sigma_{g_i}^{p'_{i+1}}$. Set $p_\lambda = \bigcup\{p_i \mid i < \lambda\}$ (as in Lemma 2A.2) at limit stages λ. Then $q = p_i$ where i is least so that p_{i+1} is not defined satisfies that $\mathscr{D}(W)^{(q)_{\delta^+}} = \{r \in \mathscr{P}^{q_{\delta^+}} \mid r \cup (q)_{\delta^+} \in \mathscr{D}(W)\}$ is predense on $\mathscr{P}^{q_{\delta^+}}$ and $(q)^{\delta^+} = (p)^{\delta^+}$. Note that we are using Π_1-cof$(L_{\nu_s}[s^*]) > \delta^+$ to argue that $q_\lambda \in L_\mu$ for limit λ and the μ-Σ-c.c. of $\mathscr{P}^{<s}$ over $L_{\nu_s}[s^*]$ to argue that in fact $\mathscr{D}(W)$ is reduced by q.

Now we can establish the Claim. Indeed define $p_0 \geq p_1 \geq \cdots$ by setting $p_0 = p$; if p_i has been defined choose $p'_{i+1} \leq p_i$ to be least so that $(p'_{i+1})^{i^+} = (p_i)^{i^+}$ and

$\mathscr{D}(W_i)^{(p'_{i+1})_{i^+}}$ is predense below $(p_i)^{i^+}$ on $\mathscr{P}^{p'_{i+1}i^+}$. Then let $p_{i+1} \leqslant p'_{i+1}$ be least so that $|p_{i+1}| = \mu \cap (\Sigma_1\text{-Skolem hull of } |p_{i+1}| \cup \{p'_{i+1}, \mu, x\} \text{ in } \langle L_{\nu_s}, s \upharpoonright [\mu, \nu_s)\rangle)$, $s \upharpoonright |p_{i+1}| \Vdash p'_{i+1}$ is as above and if $g_i(\gamma) = H^i_\gamma = \Sigma_1$-Skolem hull of $\gamma \cup \{\kappa, p'_{i+1}\}$ in $L_{\mu_{|p'_{i+1}|}}$) for $\gamma \in H^i_\gamma$, then $p_{i+1} \in \Sigma^{p'_{i+1}}_{g_i}$, $p_{i+1}(\gamma) = p'_{i+1}(\gamma)$ if $g_i(\gamma) = \bigcup \{g_j(\gamma) \mid j < i\}$. For limit λ set $p_\lambda = \bigcup \{p_i \mid i < \lambda\}$ (as in Lemma 2A.2). Then $p_\kappa \in \mathscr{P}^{<s}$ is a condition since $\Pi_1\text{-cof}(L_{\nu(\alpha)}[s^*]) = \mu > \kappa$ and $p_\kappa = q$ satisfies the Claim.

We now can complete the case at hand. First suppose that $\bar{C}_\mu = \{\bar{\mu} \in C_\mu \mid \bar{\mu} = \mu \cap \Sigma_1\text{-Skolem hull of } \bar{\mu} \cup \{\mu, x\} \text{ in } \langle L_{\nu_s}, s \upharpoonright [\mu, \nu_s)\rangle)\}$ is unbounded in μ. Note that the proof of the Claim in fact shows that for any $\gamma \in \beta\text{-Card} \cap \kappa$ we can in fact require in the conclusion of the Claim that $(p)^{\gamma^+} = (q)^{\gamma^+}$. Now inductively define $p_0 \geqslant p_1 \geqslant \cdots$ as follows: $p_0 = p$; if p_i has been defined then $p'_{i+1} \leqslant p_i$ is least so that $(p'_{i+1})^{i^+} = (p_i)^{i^+}$, p'_{i+1} reduces all $\mathscr{D}(W_j)$ for $j \in \Sigma_1$-Skolem hull of $\kappa \cup \{p_i, x\}$ in $L_{\nu_s}[s^*]$ (where $\langle W_j \mid j \in L_{\nu_s}[s^*]\rangle$ is a $\Sigma_1(L_{\nu_s}[s^*])$-listing of the $\Sigma_1(L_{\nu_s}[s^*])$-subsets of $\mathscr{P}^{<s} \times \mu$). Then $p_{i+1} \leqslant p'_{i+1}$ is least so that $s \upharpoonright |p_{i+1}| \Vdash p'_{i+1}$ is as above; $|p_{i+1}| \in \bar{C}_\mu$ and if $g_i(\gamma) = H^i_\gamma = \Sigma_1$-Skolem hull of $\gamma \cup \{\kappa, p'_{i+1}\}$ in $L_{\mu_{|p'_{i+1}|}}$ for $\gamma \in H^i_\gamma$, then $p_{i+1} \in \Sigma^{p'_{i+1}}_{g_i}$, $p_{i+1}(\gamma) = p'_{i+1}(\gamma)$ if $g_i(\gamma) = \bigcup \{g_j(\gamma) \mid j < i\}$. For limit λ let $p_\lambda = \bigcup \{p_i \mid i < \lambda\}$ (as in Lemma 2A.2) and then $q = p_\lambda$ where λ is least so that $|p_\lambda| = |s|$ is the desired extension. If $\bar{C}_\mu$ is bounded in μ, then use a $\Delta^*_1(\mathscr{A}(\mu))$-cofinal ω-sequence $\mu_0 < \mu_1 < \cdots$ below μ in place of $\bar{C}_\mu$. This completes the case $\mu^0_s < \nu_s$.

D. Extendibility II

In this part we use the ideas of Extendibility I to show the following: Suppose $\beta \in \overline{\text{Adm}}$, $\kappa \in \beta\text{-Card}$, $\eta < (\kappa^+)^{L_\beta}$ and $s \in S^\beta_\kappa$. Then there exists $t \supseteq s$, $t \in S^\beta_\kappa$, $|t| \geqslant \eta$. We show this by induction on β. Clearly it suffices to treat the case $\beta = \tilde{\alpha}$, $\alpha \in \overline{\text{Adm}}$, $\kappa = \text{gc}\,\beta$. We can also assume that $\kappa < \alpha$ as otherwise the result is easily established (let $t = s \cup s'$ where $\text{Dom}(s') = [|s|, \eta)$ and $s'(\eta') = 0$ for all $\eta' \in \text{Dom}(s')$). Thus in fact our real goal is to obtain a $\mathscr{P}^\alpha_\kappa$-generic G ($\mathscr{P}^\alpha_\kappa$-$\Sigma$-generic G if α is recursively inaccessible) so that G is 'sufficiently' definable. (More precisely if $\gamma = (\kappa^+)^{L_\alpha}$, then $G = f(\kappa, \alpha, X)$ where $X \cap \gamma$ is $\Delta^*_1(\mathscr{A}(\gamma))$.)

The key technical tool for doing this is the method of 'critical projecta' from Friedman [5]. We begin by reviewing those aspects of that method which are relevant here.

If $\alpha \leqslant \beta'$, $n \in \omega - \{0\}$, then the *(n, β')-projectum* of α is the least γ such that there is a $\Sigma_n(S_{\beta'})$-injection of α into γ. The pair (n, β') is *α-critical* if (n, β')-projectum of α is less than α and $(n', \beta'') < (n, \beta') \rightarrow (n', \beta'')$-projectum of $\alpha > (n, \beta')$-projectum of α (where $(n', \beta'') < (n, \beta')$ means that $\beta'' < \beta'$ or $\beta'' = \beta'$ and $n' < n$). There are only finitely many α-critical pairs $(n_1, \beta_1) < (n_2, \beta_2) < \cdots < (n_k, \beta_k)$ and we have that (n_k, β_k)-projectum of α equals

$\kappa = \mathrm{gc}\,\beta$. It is easily proved (see Lemma 9 of [5]) that $\rho_{n_i}^{\beta_i} = (n_i, \beta_i)$-projectum of α, $\kappa \leqslant \rho_{n_i}^{\beta_i} < \alpha$.

It is convenient to define $\rho_0^{\beta'} = \beta'$ for all β'. Then we set $\rho_i = (n_i, \beta_i)$-projectum of α, $\rho_i' = \rho_{n_i-1}^{\beta_i}$, $\mathscr{A}_i = \langle S_{\rho_i'}, A_i \rangle$ where A_i is a Σ_{n_i-1}-master code for β_i ($A_i = \emptyset$ if $n_i = 1$). Thus Σ_1-projectum$(\mathscr{A}_i) = \rho_1^{\mathscr{A}_i} = \rho_i$. The following lemma is very useful.

Lemma 2D.1. (a) *The Π_1-cofinality of $\mathscr{A}_i$ is at most $\rho_i = \rho_1^{\mathscr{A}_i}$.*

(b) *If γ is a regular α-cardinal, $\rho_i < \gamma < \rho_{i-1}$, then $\Pi_1(\mathscr{A}_i)$-cof$(\gamma) = \Pi_1$-cof$(\mathscr{A}_i)$.*

Proof. (a) Let $f : \rho_i' \to \rho_i$ be 1–1 and $\Sigma_1(\mathscr{A})_i$ with parameter x. We are certainly done if Σ_1-cof$(\mathscr{A}_i) < \rho_i$. Otherwise consider $g(\gamma) = \sup(H_\gamma \cap \mathrm{ORD})$ where $H_\gamma = \Sigma_1$-Skolem hull of $\gamma \cup \{x\}$ in $\mathscr{A}_i$. Then g is $\Pi_1(\mathscr{A}_i)$ and cofinal.

(b) Choose a cofinal $\Pi_1(\mathscr{A}_i)$ $g : \gamma_i \to \rho_i'$ where $\gamma_i = \Pi_1$-cof$(\mathscr{A}_i)$. As in (a), choose $f : \rho_i' \to \rho_i$ to be 1–1 and $\Sigma_1(\mathscr{A}_i)$ with parameter x. Given γ as in the hypothesis consider $h : \gamma_i \to \gamma$ as defined by $h(j) = \sup(\gamma \cap H_j)$ where $H_j = \Sigma_1$-Skolem hull of $\rho_i \cup \{x\}$ in $\langle L_{g(j)}, A_i \cap L_{g(j)} \rangle$ (if the latter structure is amenable; otherwise let $H_\gamma = \Sigma_1$-Skolem hull of $\rho_i \cup \{x, A_i \cap L_{g(j)}\}$ in $L_{g(j+1)}$ and assume that $A_i \cap L_{g(j)} \in L_{g(j+1)}$ for all j). Then $h(j) < \gamma$ for all j as otherwise γ is not $\mathscr{A}_i$-regular, hence not L_α-regular since (n_i, β_i) is α-critical and $\gamma < \rho_{i-1}$. But $\bigcup \mathrm{Range}(h) = \gamma$ as $\bigcup \{H_j \mid h < \gamma_i\} = \mathscr{A}_i$ and $\gamma \leqslant \rho_i'$ (in fact $\gamma < \rho_i'$). Finally h is $\Pi_1(\mathscr{A}_i)$ so we are done. □

The above lemma helps to explain our strategy for building the desired $\mathscr{P}_\kappa^\alpha$-generic G. First we build a $\mathscr{P}_{\rho_1}^\alpha$-generic G_1. (The construction will be $\Delta_1^*(\mathscr{A}_1)$ and hence G_1 will obey the desired definability condition.) This is done much as in Extendibility I but where instead of building a condition $q = \bigcup \{p_i \mid i < \lambda\}$ we actually build an entire generic set G_1. This is reasonable because $\Pi_1(\mathscr{A}_1)$-cof$(\gamma^+) = \Pi_1$-cof$(\mathscr{A}_1)$ for all $\gamma \in \alpha$-Card, $\gamma \geqslant \rho_1$ and hence we can arrange $\bigcup \{|p_{i\gamma}| \mid i < \lambda\} = \lambda^+$ for all such γ, $\lambda = \pi_1$-cof$(\mathscr{A}_1)$. Next let $f(\rho_1, \alpha, X_1) = G_1$, $s_1 : [\rho_1, \rho_1^+) \to 2$ be defined by $s_1(\eta) = 1$ iff $\eta \in X_1$ and build a $\mathscr{P}_{\rho_2}^{<s_1}$-generic G_2 in π_1-cof$(\mathscr{A}_2)$ steps, where $\mathscr{P}_{\rho_2}^{<s_1} = \bigcup \{\mathscr{P}_{\rho_2}^t \mid t \subseteq s_1,\ t \in S_{\rho_1}^\alpha\}$. Then we have a $\mathscr{P}_{\rho_2}^\alpha = \mathscr{P}_{\rho_1}^\alpha * \mathscr{P}_{\rho_2}^{G_1}$-generic set $G_1 * G_2$. Continuing in this way we finally obtain the desired $G = G_1 * G_2 * \cdots * G_k$.

Step 1. We build G_1. Let $\lambda_1 = \Pi_1$-cof$(\mathscr{A}_1) \leqslant \rho_1$. We obtain G_1 as the 'union' (as in Lemma 2A.2) of a sequence $\langle p_i \mid i < \lambda_1 \rangle$ where the p_i's are not necessarily conditions in $\mathscr{P}_{\rho_1}^\alpha$ but instead 'quasiconditions'.

Definition. $p \in S_{\rho_1'}$ is a *$\mathscr{P}_{\rho_1}^\alpha$-quasicondition* if Dom$(p)$ is an initial segment of α-Card $\cap [\rho_1, \alpha)$ and p obeys properties (1)–(7) in the definition of $\mathscr{P}_{\rho_1}^s$ at all α-cardinals $\delta \in (\rho_1, \alpha)$. If p, q are $\mathscr{P}_{\rho_1}^\alpha$-quasiconditions, then $p \leqslant q$ iff $p \restriction \delta \leqslant q \restriction \delta$ for all $\delta < \alpha$.

We also define $\mathbb{F}_1$ to consist of all $g \in S_{\rho_1'}$ so that $\mathrm{Dom}(g) \subseteq \alpha\text{-Card} \cap [\rho_1, \alpha)$ is thin in $S_{\rho_1'}$, $\mathrm{Range}(g) \subseteq L_\alpha$ and for all $\delta \in \mathrm{Dom}(g)$, $L_\alpha \vDash \mathrm{card}(g(\delta)) \leq \delta$. If $p \in S_{\rho_1'}$ is a $\mathscr{P}^\alpha_{\rho_1}$-quasicondition and $g \in \mathbb{F}_1$, then Σ^p_g is defined as before: $q \in \Sigma^p_g$ iff q is incompatible with p or ($q \leq p$ and $\delta \in \mathrm{Dom}(g)$, D predense on $R^{p_{\delta^+}}$, $D \in g(\delta) \cap L_{\mu^1_{p_{\delta^+}}} \rightarrow q(\delta)$ meets D).

First assume that α is not recursively inaccessible. Choose a $\Delta^*_1(\mathscr{A}_1)$-cofinal $g_1 : \gamma_1 \rightarrow \rho_1'$, $\gamma_1 = \Pi_1\text{-cof}(\mathscr{A}_1)$. Define a γ_1-sequence $\langle p_i \mid i < \gamma_1 \rangle$ of $\mathscr{P}^\alpha_{\rho_1}$-quasiconditions as follows: Let x be a parameter so that (Σ_1-Skolem hull of $\rho_1 \cup \{x\}$ in $\mathscr{A}_1$) equals $\mathscr{A}_1$. Then $p_0 = \emptyset$; if p_i has been defined, then let $g_i(\delta) = \Sigma_1$-Skolem hull of $\delta \cup \{x\}$ in $\langle S_{g_1(i)}, A_1 \cap S_{g_1(i)} \rangle = H^i_\delta$ for $\delta \in H^i_\delta$ and let $p_{i+1} \leq p_i$ be least so that $p_{i+1} \in \Sigma^{p_i}_{g_i}$, $p_{i+1}(\delta) = p_i(\delta)$ if $H^i_\delta = \bigcup \{H^j_\delta \mid j < i\}$ (if g_1 can be defined so that the structures $\langle S_{g_1(i)}, A_1 \cap S_{g_1(i)} \rangle$ are amenable; otherwise let $H^i_\delta = \Sigma_1$-Skolem hull of $\delta \cup \{x, A_1 \cap S_{g_1(i)}\}$ in $S_{g_1(i+1)}$ and assume that $A_1 \cap S_{g_1(i)} \in S_{g_1(i+1)}$ for all i). For limit $\lambda < \gamma_1$ let $p_\lambda = \bigcup \{p_i \mid i < \lambda\}$ (as in Lemma 2A.2).

Now exactly as in Extendibility I it can be shown that if p is a $\mathscr{P}^\alpha_{\rho_1}$-quasicondition, then for each $i < \gamma_1$ there exists $q \leq p$ such that $|q_\delta| \geq H^i_\delta \cap \delta^+$ for all $\delta \in \alpha\text{-Card} \cap [\rho_1, \alpha)$. Thus exactly as in Lemma 2A.1 the $\Sigma^{p_i}_{g_i}$'s are dense in the partial order of $\mathscr{P}^\alpha_{\rho_1}$-quasiconditions. And as in Lemma 2A.2 the above induction is well-defined at limit stages (using Lemma 2D.1(b) for successor α-cardinals γ).

If $\lambda = \gamma_1$, define $p_\lambda = G_1 = \{r \in \mathscr{P}^\alpha_{\rho_1} \mid p_i \leq r$ for some $i < \lambda\}$. The proof of Lemma 2A.2 shows that $s = \bigcup \{r_{\rho_1} \mid r \in G_1\}$ is $\mathscr{P}^{\bar\alpha}_{\rho_1}$-generic over $L_{\bar\alpha}$ where $\pi : \langle L_{\bar\rho}, \bar A \rangle \simeq \bigcup \{H^i_{\rho_1} \mid i < \gamma_1\}$ is the transitive collapse and $\pi(\bar\alpha) = \alpha$. But π is the identity since $\bigcup \{H^i_{\rho_1} \mid i < \gamma_1\} = \mathscr{A}_1$ and thus s (equivalently: G_1) is $\mathscr{P}^\alpha_{\rho_1}$-generic over L_α.

If α is a successor admissible, we must also guarantee that G_1 kills the admissibility of α. Note that in this case $\rho_1 = \mathrm{gc}\,\alpha$, $(n_1, \beta_1) = (1, \alpha)$ and then the construction of G_1 is particularly simple: we are just choosing successively longer elements $s_0 \subseteq s_1 \subseteq \cdots$ of $S^\alpha_{\rho_1}$ using a $\Pi_1(L_\alpha)$-cofinal $g_1 : \gamma_1 = \Pi_1\text{-cof}(\alpha) \rightarrow \alpha$. To guarantee that the admissibility of α is killed, if necessary replace $s = \bigcup \{s_i \mid i < \gamma_1\}$ by $s \vee t$ where $t : [\rho_1, \alpha) \rightarrow 2$ is the characteristic function of C_α and $s \vee t(2\eta) = s(\eta)$, $s \vee t(2\eta + 1) = t(\eta)$.

Finally suppose that α is recursively inaccessible. If $\alpha^* = \alpha$ (α is nonprojectible), then Σ-genericity for $\mathscr{P}^\alpha_{\rho_1}$ reduces to regular genericity as in that case α is the limit of α-stables. If $\alpha^* < \alpha$, then $\rho_1 = \mathrm{gc}\,\alpha$, $(\eta_1, \beta_1) = (1, \alpha)$ and as above we are just choosing successively longer elements $s_0 \subseteq s_1 \subseteq \cdots$ of $S^\alpha_{\rho_1}$ using a $\Pi_1(L_\alpha)$-cofinal $g_1 : \gamma_1 = \Pi_1\text{-cof}(L_\alpha) \rightarrow \alpha$. We must modify this construction as follows: First we Claim that if ρ_1 is L_α-regular, $\langle W_i \mid i < \delta \rangle$ is a uniformly $\Sigma_1(L_\alpha)$-sequence of persistent subsets of $\mathscr{P}^\alpha_{\rho_1} \times \rho_1$ and $\delta < \rho_1$, then for any $p \in \mathscr{P}^\alpha_{\rho_1}$ there exists $q \leq p$, $q \in \mathscr{D}(W_i)$ for all $i < \delta$. To prove this define $p_0 \geq p_1 \geq \cdots$ by setting $p_0 = p$ and, if p_i has been defined, let $p_{i+1} \leq p_i$ be least so that for some $j < \delta$, $\eta \leq \rho_1$, $\{p_{i+1}\} \times \eta \subseteq W_j^{|p_{i+1}|}$ but $\{p_i\} \times \eta \nsubseteq W_j^{|p_i|}$, where

W_j^σ = amount of W_j enumerated by stage $\sigma < \alpha$; for limit λ let $p_\lambda = \bigcup\{p_i \mid i < \lambda\}$. Then p_{i+1} is undefined for some $i < \alpha$ as α is admissible $\to$ the function ($j \mapsto$ last stage at which W_j is considered) is totally defined and Π_1-cof$(L_\alpha) = \rho_1 > \delta$. Also note that λ limit $\to |p_\lambda|$ is inadmissible as otherwise by the same argument, p_{i+1} would be undefined for some $i < \lambda$. Thus $q = \bigcup\{p_i \mid p_i$ is defined$\}$ is as desired. Given the Claim we see how to modify our earlier construction of the sequence $s_0 \subseteq s_1 \subseteq \cdots$ when ρ_1 is L_α-regular. Namely, choose s_{i+1} so that $s_{i+1} \in \mathcal{D}(W_j)$ for all $j \in \Sigma_1$-Skolem hull of $\bar{g}_1(i) \cup \{x\}$ in L_α, where $\bar{g}_1 : \gamma_1 \to \rho_1$ is $\Delta_1^*(L_\alpha)$-cofinal and $\langle W_j \mid j \in L_\alpha\rangle$ is a uniformly $\Sigma_1(L_\alpha)$-listing of the $\Sigma_1(L_\alpha)$ persistent subsets of $\mathcal{P}_{\rho_1}^\alpha \times \rho_1$. Then $\bigcup\{s_i \mid i < \gamma_1\}$ is as desired, using the fact that Π_1-cof$(L_{|s_\lambda|}) < \rho_1$ at limit stages λ. If ρ_1 is L_α-singular, then in the Claim replace $\mathcal{P}_{\rho_1}^\alpha \times \rho_1$ by $\mathcal{P}_{\rho_1}^\alpha \times \gamma$, $\gamma < \rho_1$, and in the construction of the s_i's replace $s_{i+1} \in \mathcal{D}(W_j)$ by $s_{i+1} \in \mathcal{D}(W_j^i)$ where $\langle W_j^i \mid j \in L_\alpha\rangle$ is a uniformly $\Sigma_1(L_\alpha)$-listing of the $\Sigma_1(L_\alpha)$ persistent subsets of $\mathcal{P}_{\rho_1}^\alpha \times g_1(i)$. This completes Step 1.

Step $j+1$. This step is handled in two parts. First we build a '$\mathcal{P}_{\rho_{j+1}}^{s_j}$-quasicondition' p where $s_j = \bigcup\{r_{\rho_j} \mid r \in G_j\}$. This p will code all of s_j and reduce all predense $D \in L_{\nu_{s_j}}[s_j^*]$. Then much as in Step 1 we build a Π_1-cof$(\mathcal{A}_{j+1}) = \gamma_{j+1}$-sequence $p = p_0 \geqslant p_1 \geqslant \cdots$ of '$\mathcal{P}_{\rho_{j+1}}^{s_j}$-quasiconditions' so that $G_{j+1} = \{r \in \mathcal{P}_{\rho_{j+1}}^{<s_j} \mid p_i \leqslant r$ for some $i < \gamma_{j+1}\}$ is the desired $\mathcal{P}_{\rho_{j+1}}^{<s_j}$-generic.

Definition. $\mathcal{P}_{\rho_{j+1}}^{<s_j} = \bigcup\{\mathcal{P}_{\rho_{j+1}}^t \mid t \subseteq s_j,\ t \in \mathcal{P}_{\rho_j}^\alpha\}$. A $\mathcal{P}_{\rho_{j+1}}^{s_j}$-*quasicondition* is a function $p : \alpha\text{-Card} \cap [\rho_{j+1}, \rho_j) \to L_\alpha$ such that p is $\Sigma_{n_{j+1}}(S_{\beta_{j+1}})$ and p obeys conditions (1)–(7) in the definition of $\mathcal{P}_{\rho_{j+1}}^{s_j}$ at all α-cardinals $\delta \in (\rho_{j+1}, \rho_j]$.

Now exactly as in Extendibility I we can construct a $\mathcal{P}_{\rho_{j+1}}^{s_j}$-quasicondition $p \notin \mathcal{P}_{\rho_{j+1}}^{<s_j}$. Indeed, were $s_j \in S_{\rho_j}^{\alpha'}$ for some $\alpha' \in \overline{\text{Adm}}$, then the construction of such a p is precisely the construction of $p \in \mathcal{P}_{\rho_{j+1}}^{s_j}$, $|p| = |s_j|$. But that construction was $\Delta_1^*(\mathcal{A}(\rho_j^+))$ and hence only required that s_j was $\mathcal{P}_{\rho_j}^\alpha$-generic and the fact that ρ_j is a cardinal in the sense of $\mathcal{A}(\rho_j^+) = \mathcal{A}_j$. Note that p is $\Sigma_{n_{j+1}}(S_{\beta_j})$ and hence $\Sigma_{n_{j+1}}(S_{\beta_{j+1}})$. Also note that we obtain $|p_\delta| < \delta^+$ for $\delta \in \alpha\text{-Card} \cap [\rho_{j+1}, \rho_j)$ (as required by the definition of quasicondition) as if $H_\delta = \Sigma_1$-Skolem hull of $\delta \cup \{x\}$ in $\mathcal{A}_j$, $x \in S_{\rho_j'}$, then $H_\delta \cap \delta^+$ is an ordinal less than δ^+.

Finally to build the desired G_{j+1}, a $\mathcal{P}_{\rho_{j+1}}^{<s_j}$-generic over $L_\alpha[s_j^*]$ (a $\mathcal{P}_{\rho_{j+1}}^{<s_j}$-$\Sigma$-generic over $L_\alpha[s_j^*]$ if α is recursively inaccessible) it suffices to define a $\Delta_1^*(\mathcal{A}_{j+1})$ γ_{j+1}-sequence $p = p_0 \geqslant p_1 \geqslant \cdots$ of $\mathcal{P}_{\rho_{j+1}}^{s_j}$-quasiconditions so that for $\delta \in \alpha\text{-Card} \cap [\rho_{j+1}, \rho_j)$, $\bigcup\{|p_{i_\delta}| \mid \{|p_{i_\delta}| \mid i < \gamma_{j+1}\} = \delta^+$ (where $\gamma_{j+1} = \Pi_1$-cof$(\mathcal{A}_{j+1})$). For, as p is a $\mathcal{P}_{\rho_{j+1}}^{s_j}$-quasicondition, $p \notin \mathcal{P}_{\rho_{j+1}}^{<s_j}$, it must be that all predense $D \in L_{\nu_{s_j}}[s_j^*] = L_\alpha[s_j^*]$ (all predense $\mathcal{D}(W)$, W a $\Sigma_1\langle L_\alpha[s_j^*], s_j^*\rangle$ persistent subset of $\mathcal{P}_{\rho_{j+1}}^{<s_j} \times \gamma$, $\gamma < \alpha$) are reduced by p and hence met by some p_i.

To build the sequence $p = p_0 \geqslant p_1 \geqslant \cdots$ proceed as follows: Define $\mathbb{F}_{j+1}$ to consist of all $g \in S_{\rho_{j+1}'}$ so that $\text{Dom}(g) \subseteq \alpha\text{-Card} \cap [\rho_{j+1}, \rho_j)$ is thin in $S_{\rho_{j+1}'}$, $\text{Range}(g) \subseteq L_\alpha$ and for all $\delta \in \text{Dom}(g)$, $L_\alpha \models \text{card}(g(\delta)) \leqslant \delta$. If q is a $\mathcal{P}_{\rho_{j+1}}^{s_j}$-

quasicondition, $q \in S_{\rho'_{j+1}}$ and $g \in \mathbb{F}_{j+1}$ then Σ^q_g is defined as before: $r \in \Sigma^q_g$ iff r is incompatible with q or $r \leqslant q$ and $\delta \in \mathrm{Dom}(g)$, D predense on $R^{q_{\delta^+}}$, $D \in g(\delta) \cap L_{\mu^1_{q_{\delta^+}}} \rightarrow r(\delta)$ meets D. If $\rho'_{j+1} > \rho_j$, then as in Extendibility I we can show that the Σ^q_g's are dense in the partial order of all $\mathcal{P}^{s_j}_{\rho'_{j+1}}$-quasiconditions belonging to $S_{\rho'_{j+1}}$. If $\rho'_{j+1} = \rho_j$, then note that $g \in L_{\rho_j}$ and so Σ^q_g is dense by induction.

Now choose a $\Delta^*_1(\mathcal{A}_{j+1})$-cofinal function $g: \gamma_{j+1} \rightarrow \rho'_{j+1}$, $\gamma_{j+1} = \Pi_1\text{-cof}(\mathcal{A}_{j+1})$ and if $\rho'_{j+1} = \rho_j$ we assume that $p \restriction g(\lambda)$ is $\Sigma\langle L_{g(\lambda)}, \mathcal{A}_{j+1} \cap L_{g(\lambda)}\rangle$ for limit $\lambda \leqslant \gamma_{j+1}$. (This is possible as p is $\Sigma_{n_{j+1}}(S_{\beta_{j+1}})$ and hence $\Sigma_1(\mathcal{A}_{j+1})$.) Set $p_0 = p$; if p_i has been defined, then let $p_{i+1} \leqslant p_i$ be least so that $p_{i+1} \in \Sigma^{p_i}_{g_i}$, $p_{i+1}(\delta) = p_i(\delta)$ if $g_i(\delta) = \bigcup\{g_j(\delta) \mid j < i\}$, where $g_i(\delta) = \Sigma_1$-Skolem hull of $\delta \cup \{x\}$ in $\langle S_{g(i)}, A_{j+1} \cap S_{g(i)}\rangle = H^i_\delta$ for $\delta \in H^i_\delta$, $\delta \geqslant \rho_{j+1}$ (where $x \in S_{\rho'_{j+1}}$ is such that Σ_1-Skolem hull of $\rho_{j+1} \cup \{x\}$ in $\mathcal{A}_{j+1} = \mathcal{A}_{j+1}$. Also, if these structures are not amenable, then use the Σ_1-Skolem hull of $\delta \cup \{x, \mathcal{A}_{j+1} \cap S_{g(i)}\}$ in $S_{g(i+1)}$, having arranged that $\mathcal{A}_{j+1} \cap S_{g(i)} \in S_{g(i+1)}$.) Then $p_\lambda = \bigcup\{p_i \mid i < \lambda\}$ (as in Lemma 2A.2) is a $\mathcal{P}^{s_j}_{\rho'_{j+1}}$-quasicondition for limit $\lambda < \gamma_{j+1}$. If $\lambda = \gamma_{j+1}$, then $p_\lambda = G_{j+1} = \{r \in \mathcal{P}^{<s_j}_{\rho_{j+1}} \mid p_i \leqslant r$ for some $i < \lambda\}$ is the desired $\mathcal{P}^{<s_j}_{\rho_{j+1}}$-generic (the desired Σ-generic if α is recursively inaccessible).

By the Factoring Lemma 1A.7 we have that $G_1 * G_2 * \cdots * G_k = G$ is $\mathcal{P}^\alpha_\kappa$-generic over L_α (is $\mathcal{P}^\alpha_\kappa$-$\Sigma$-generic over L_α if α is recursively inaccessible). As G is $\Delta^*_1(\mathcal{A}_k) = \Delta^*_1(\mathcal{A}(\mu^0_\gamma))$, $\gamma = (\kappa^+)^{L_\alpha}$, G also meets the required definability condition. This completes the proof of Extendibility II.

E. Conclusion

We now establish Lemmas 1A.6 (Distributivity) and 1F.1 (Extendibility).

Proof of Lemma 1A.6. If β-Card $\cap [\kappa, \beta)$ is finite, then the result follows from the Σ-distributivity of the R^s-forcings and Factoring. We can also assume that β is Σ_1-projectible as otherwise the result follows by induction using a sufficiently large β-stable ordinal. So let $\gamma =$ largest limit β-cardinal; by the above remarks we need only show the Σ-distributivity of $\mathcal{P}^{<s}_\kappa$ where $s: [\gamma, \gamma^+) \rightarrow 2$ is $\mathcal{P}^\beta_\gamma$-Σ-generic over L_β, γ^+ denoting $(\gamma^+)^{L_\beta} \leqslant \beta$. An inspection of the Claims in Subcase 2 of Part C reveals that given $p \in \mathcal{P}^{<s}_\kappa$ and a $\Sigma_1\langle L_\beta[s^*], s^*\rangle$-sequence $\langle D_i \mid i < \kappa\rangle$ of predense sets there exists $q \leqslant p$ so that q reduces each D_i and in fact for some fixed $\delta \in (\kappa, \gamma)$, $D_i^{(q)_\delta}$ is predense on $\mathcal{P}^{q_\delta}_\kappa$ for each i. We also know by admissibility of $\langle L_\beta[s^*], s^*\rangle$ that the D_i's can be thinned to $\bar{D}_i$'s so that $\langle \bar{D}_i \mid i < \kappa\rangle \in L_\beta[s^*]$ and each $\bar{D}_i$ is predense. So we can in fact assume that $D_i = \bar{D}_i$ and thus $D_i^{(q)_\delta} \in L_{\nu_{q_\delta}}[q^*_\delta]$ for each i. Then by induction choose $r \leqslant q$, $r \restriction \delta$ meets $D_i^{(q)_\delta}$ for each i. So $r \leqslant p$, r meets each D_i. □

Proof of Lemma 1F.1. We can assume that Dom(f) has a maximum element δ. By Extendibility II we can extend p so that $|p_\delta| \geqslant f(\delta)$, Dom($f$) is thin in $L_{\mu_{p_\delta}}$.

Then we are reduced to the last statement of the lemma. But that statement is precisely Extendibility I. □

Note that the proof of Lemma 1A.6 actually shows that if $\beta \leqslant \mathrm{ORD}$ is Σ_n-admissible $\kappa \in \beta$-Card, then $\mathcal{P}^\beta_\kappa$ is Σ_n-distributive over L_β. Thus by Factoring, if $\langle D_i \mid i<\kappa\rangle$ is a $\Sigma_n(L)$-sequence of predense subclasses of $\mathcal{P} = \mathcal{P}_0^{\mathrm{ORD}} = \bigcup \{\mathcal{P}^\beta_0 \mid \beta \in \overline{\mathrm{Adm}}\}$ and $p \in \mathcal{P}$, then there exists $q \leqslant p$ and $\langle d_i \mid i<\kappa\rangle \in L$ such that $d_i \subseteq D_i$, d_i is predense below q for each i. This is enough to prove that $\mathcal{P} \Vdash \mathrm{ZFC}$. The proof of our Theorem is now complete.

Before listing some open questions we describe how a technique from Jensen [7] can be used in the present context to obtain a $\mathcal{P}$ 'pseudo-generic' below $0^\#$.

Theorem. *There is a real $R \in L[0^\#]$ such that $\Lambda(R) =$ Recursively Inaccessibles.*

Proof. We construct $s:[\omega, \infty) \to 2$ definably over $L[0^\#]$ so that $s \restriction [\omega, \kappa)$ is $\mathcal{P}_\omega^{s\restriction[\kappa,\kappa^+)}$-generic over L_{κ^+} for each $\kappa \in L\text{-Card} \cap (\omega, \infty)$. This is accomplished in ω steps; at stage i we define $\langle s^{i\nu} \mid \nu \in I\rangle$, $\langle p^{i\nu} \mid \nu \in I\rangle$ where $I =$ canonical Silver indiscernibles, $s^{i\nu} \in S_{\nu^+}$ $(= S_{\nu^+}^{\nu^{++}})$ and $p^{i\nu} \in \mathcal{P}_\omega^{s^{i\nu}}$. Moreover we have the following coherence properties:

(a) $\nu < \tau$ in $I \to s^{i\nu} = p^{i\tau}_{\nu^+}$, $\bar{p}^{i\tau}_{\nu^+} = \emptyset$, $p^{i\tau} \restriction \nu = p^{i\nu} \restriction \nu$.

(b) $j \leqslant i \to s^{j\nu} \subseteq s^{i\nu}$ and $p^{i\nu} \leqslant p^{j\nu}$ in $\mathcal{P}_\omega^{s^{i\nu}}$.

(c) $p^{i\nu}$ is uniformly $\Sigma_1(L)$ with parameter $\langle \nu, \tau_1, \ldots, \tau_i\rangle$ for any $\nu < \tau_1 < \cdots < \tau_i$ in I.

To begin let $s^{0\nu} = \emptyset$, $p^{0\nu}(\delta) = (\emptyset, \emptyset)$ for all $\nu \in I$, $\delta \in L$-Card. Now suppose that $s^{i\nu}, p^{i\nu}$ are defined for $\nu \in I$. Pick $\nu \in I$ and we shall define $s^{(i+1)\nu}$, $p^{(i+1)\nu}$. Choose any $\tau_1 < \cdots < \tau_i$ from I so that $\nu < \tau_1$ and let $g_{i\nu}(\delta) = \Sigma_1$-Skolem hull of $\delta \cup \{\nu, \tau_1, \ldots, \tau_i\}$ in L for $\delta \in L\text{-Card} \cap \nu$, $\delta \in q_{i\nu}(\delta)$. Then $p^{(i+1)\nu}$ is least so that $p^{(i+1)\nu} \leqslant p^{i\nu}$ in $\mathcal{P}_\omega^{s^{(i+1)\nu}}$ and $p^{(i+1)\nu} \in \Sigma^{p^{i\nu}}_{g_i}$. Define $s^{(i+1)\nu} = p^{(i+1)\tau}_{\nu^+}$ for $\tau \in I$, $\tau > \nu$.

Now by indiscernibility, $\nu < \tau$ in $I \to p^{i\tau} \restriction \nu = p^{i\nu} \restriction \nu$. Also $p^{i\tau}_\nu = p^{i\nu}_\nu$ and $s^{i\nu} = p^{i\tau}_{\nu^+}$ (the latter by definition, the former by coding). Moreover, $\bar{p}^{i\tau}_\nu = \emptyset$ since if C_ν, C_τ are the least CUB's contained in $L\text{-Card} \cap \nu$, $L\text{-Card} \cap \tau$, respectively so that $\gamma \in C_\nu \to \bar{p}^{i\nu}_\gamma = \emptyset$, $\gamma \in C_\tau \to \bar{p}^{i\tau}_\gamma = \emptyset$, then we get that $C_\nu = C_\tau \cap \nu$ and hence $\nu \in C_\tau$.

Let $p^i = \bigcup \{p^{i\nu} \mid \nu \in I\}$. Note that for $\delta \in L$-Card we have $\bigcup \{|p^i_\delta| \mid i \in \omega\} = \delta^+$ since $\bigcup \{g_{i\nu}(\delta) \mid i \in \omega\} \supseteq \delta^+$ for any $\nu \in I$, $\nu > \delta$. Now consider $G =$ all $q \in \mathcal{P}_\omega$ such that for some i, $\hat{p}^i \leqslant q$ where $\hat{p}^i$ differs from p^i only on a finite $u \subseteq I$, where $\nu \in u \to \hat{p}^i_\nu = p^{i\nu}_\nu$. (Thus the only differences between $\hat{p}^i$ and p^i is that $\bar{\hat{p}}^i_\nu$ may be nonempty for $\nu \in u$.) We claim that G is P_ω-generic over L. Indeed suppose $D \in L$ is the least predense set not met by G. Pick the least L-cardinal δ so that $D \in L_{\delta^+}$. Choose $q \in G$ so that $D \in L_{\nu_{q_\delta}}$. To get a contradiction we need only show that $G \cap P_\omega^{<q_\delta}$ is $P_\omega^{<q_\delta}$-generic over $L_{\nu_{q_\delta}}$. If δ is a limit L-cardinal, then $q \restriction \delta$ reduces D and so by the minimality of δ and the definition of G, G meets D. If $\delta = \gamma^+$ is a successor L-cardinal, then we can find $q' \leqslant q$ in G so that $q'(\gamma)$

reduces D (this is where the definition of G comes in): Namely, choose q' so that $q'(\gamma) = p^{i\nu}(\gamma)$ where $\nu \geq \gamma$ is the min of $I - \gamma$ and $D \in g_{i\nu}(\gamma)$. Again we have contradicted the minimality of δ.

If $D \subseteq [\omega, (\omega_1)^L)$ is defined as $\bigcup \{\bar{p}^i_\omega \mid i \in \omega\}$, then choose $R \subseteq \omega$ to be $\mathcal{P}^D$-generic over $L[D]$ where $\mathcal{P}^D = \bigcup \{\mathcal{P}^s \mid s \subseteq p^i_\omega$ for some $i\}$. This can be done in $L[0^\#]$ as $\omega_1^{L[D]} = \omega_1^L$ is countable in $L[0^\#]$. Then R is P-generic over L and hence $\Lambda(R) =$ Recursively Inaccessibles. □

We conclude with some open questions.

(1) Which L-definable classes can occur as $\Lambda(R)$ for some real R?

(2) Which L-definable classes can be Δ_1-definable over $L[R]$ for some real R, $0^\# \notin L[R]$?

(3) Is there a model of ZFC + Post's Problem is false for HC, obtainable as a generic extension of L?

The above questions appear to require further improvements on the strong coding method.

References

[1] A. Beller, R. Jensen and P. Welch, Coding the Universe (Cambridge University Press, Cambridge, 1982).
[2] R. David, A functorial Π^1_2 singleton, Advances in Math. (1984).
[3] K. J. Devlin, Aspects of Constructibility (Springer, Berlin, 1973).
[4] H.-D. Donder, Another look at Gap-1 Morasses (1984).
[5] S. D. Friedman, Uncountable admissibles I: Forcing, Trans. AMS, (1982).
[6] R. Jensen, The fine structure of the constructible hierarchy, Annals Math. Logic 4 (1972) 229–308.
[7] R. Jensen, Coding the Universe, handwritten, (1975).
[8] L. J. Stanley, A short course on gap-1 morasses with a review of the fine structure of L, Proc. of the 1978 Cambridge Summer School in Set Theory (1982).
[9] D. J. Velleman, Simplified morasses with linear limits (1984).
[10] G. E. Sacks and S. G. Simpson, The α-finite injury method, Annals Math. Logic 4 (1972) 343–368.