Universally Baire sets and definable well-orderings of the reals*†

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June 21, 1999

Abstract

Let n be a positive integer. We show that it is consistent (relative to the consistency of finitely many strong cardinals) that every $\mathbf{\Sigma}^1_n$-set of reals is universally Baire yet there is a (lightface) projective well-ordering of the reals. The proof uses "David's trick" in the presence of inner models with strong cardinals.

1 Introduction.

Let $\Gamma \subset \Gamma' \subset \mathcal{P}(\mathbb{R})$ be pointclasses, where Γ' is not too far away from Γ. There is a conflict between every set in Γ being "nice" (being Lebesgue measurable, having the property of Baire, being Ramsey, each of which contradicts certain doses of choice) and Γ' providing choice-like principles for Γ (every non-empty set in Γ contains a Γ'-singleton or there is a well-ordering of $\mathbb{R}$ in Γ'). For example, Woodin has shown that if every projective set of

*Math. Subj. Class. 03E35, 03E45, and 03E55.

†Keywords and phrases: Descriptive set theory, large cardinals, inner models.

‡The first author was supported by NSF Grant Number 9625997-DMS

§The second author gratefully acknowledges a fellowship from the Deutsche Forschungsgemeinschaft (DFG). The results in this paper were obtained while he was visiting the UC Berkeley. He would like to thank Greg Hjorth, Vladimir Kanovei, John Steel, Philip Welch, and Hugh Woodin for providing illuminating information about the questions addressed in this paper.

reals is Lebesgue measurable and has the property of Baire and every projective relation on $\mathbb{R}^2$ can be uniformized by a function with a projective graph then $\boldsymbol{\Pi}^1_1$-determinacy holds (c.f. [14]).

The present paper also deals with this conflict at the projective level. Let $n \geq 2$, $\Gamma = \boldsymbol{\Sigma}^1_n$ and $\Gamma' = \boldsymbol{\Delta}^1_{n+1}$. Of course, if every Γ-set of reals is Lebesgue measurable then there cannot be a well-ordering of $\mathbb{R}$ in Γ. But we may ask whether nevertheless there can be a projective well-ordering of the reals, or one in Γ' for that matter.

An answer to this question can be found in the literature. Moschovakis (cf. [9]) showed that if Projective Determinacy holds then there is an inner model M^n with a Σ^1_{n+1}-well-ordering of $\mathbb{R}$ and in which $\boldsymbol{\Delta}^1_{n-1}$-determinacy holds (hence if n is odd then in M^n every set in Γ is Lebesgue measurable and has the property of Baire). Moreover, if M_{n-1} denotes the minimal sufficiently iterable inner model with $n-1$ Woodin cardinals then in M_{n-1} there is a Δ^1_{n+1}-well-ordering of $\mathbb{R}$ and $\boldsymbol{\Pi}^1_{n-1}$-determinacy holds (hence in M_{n-1} every set in Γ is Lebesgue measurable and has the property of Baire; cf. [12]).

Let us consider the following question.

Question. Let $n \geq 3$. Suppose that every $\boldsymbol{\Sigma}^1_n$-set of reals is Lebesgue measurable and has the property of Baire, and that there is a lightface projective well-ordering of the reals. Does $\boldsymbol{\Delta}^1_{n-1}$-determinacy hold?

For the case $n = 3$ or 4 this is refuted by a couple of theorems due to the first author of the present paper. He showed (cf. [5]): starting from a Mahlo cardinal in L (or, alternatively, from an inaccessible cardinal plus $\sharp$'s), one can construct a forcing extension with a Δ^1_4-well-ordering of $\mathbb{R}$ in which all $\boldsymbol{\Sigma}^1_3$-sets of reals are Lebesgue measurable and have the property of Baire; and starting from a Mahlo cardinal plus $\sharp$'s, one can construct a forcing extension with a Δ^1_5-well-ordering of $\mathbb{R}$ in which all $\boldsymbol{\Sigma}^1_4$-sets of reals are Lebesgue measurable and have the property of Baire. (David had earlier shown that if L has an inaccessible then there is a forcing extension with a Δ^1_3-well-ordering of $\mathbb{R}$ in which all $\boldsymbol{\Sigma}^1_2$-sets of reals are Lebesgue measurable and have the property of Baire; cf. [2].

We here answer the above question negatively for all $n < \omega$, as follows.

Theorem 1.1 *Let $n \geq 3$. It is consistent, relative to the existence of $n-2$ strong cardinals, that every $\boldsymbol{\Sigma}^1_n$-set of reals is Lebesgue measurable and has* *the property of Baire, and yet there is a lightface projective well-ordering of the reals.*

Recall that by a theorem of Woodin Δ^1_2-determinacy implies the existence of an inner model with a Woodin cardinal, and hence the existence of transitive models with infinitely many strong cardinals, so that Gödel's theorem shows that 1.1 provides a negative answer to the above question, granting the consistency of strong cardinals.

We shall obtain 1.1 as an immediate corollary to the following two theorems.

Theorem 1.2 *Let $n \in \omega$. Let $L[E^n]$ denote the minimal inner model closed under the $\sharp$-operation if $n = 0$, viz. the minimal fully iterable inner model with n strong cardinals if $n > 0$.*

Then there is a real a ($a = 0$ if $n = 0$), set-generic over $L[E^n]$, such that in $L[E^n][a]$ every $\mathbf{\Sigma}^1_{n+2}$-set of reals is universally Baire, there is a $\Delta^1_{n+3}(a)$-well-ordering of the reals, and a is a Π^1_{n+4}-singleton (and hence there is a Δ^1_{n+5}-well-ordering of $\mathbb{R}$).

We shall in fact see that a may be chosen in such a way that every Σ^1_{n+3}-set of reals is Lebesgue measurable and has the property of Baire. Refining this observation we can also show:

Theorem 1.3 *Let $n > 0$, and let $L[E^n]$ be the minimal fully iterable inner model with n strong cardinals. Suppose that in $L[E^n]$ there is an inaccessible cardinal above the strong cardinals.*

Then there is a set-generic extension of $L[E^n]$ in which every $\mathbf{\Sigma}^1_{n+2}$-set of reals is universally Baire, every $\mathbf{\Sigma}^1_{n+3}$-set of reals is Lebesgue measurable and has the property of Baire, and there is a Δ^1_{n+5}-well-ordering of $\mathbb{R}$.

Recall that a set $A \subset \mathbb{R}$ is called universally Baire if for every topological space $\mathcal{X}$ and every continuous $f : \mathcal{X} \to \mathbb{R}$ it is the case that $f^{-1}{}''A$ has the property of Baire (in $\mathcal{X}$). If $A \subset \mathbb{R}$ is universally Baire then A is Lebesgue measurable, is Ramsey, and has the Bernstein property (and, trivially, has the property of Baire, cf. [3] Theorems 2.2 and 2.3). In the following, as in the statements of 1.2 and 1.3, we shall always suppose that $L[E^n]$ as well as enough generics exist.

We don't know whether the models of 1.2 and 1.3 have a Δ^1_{n+4}-well-ordering of their reals. We hence have to leave unanswered the strengthening of the above question in which "projective" is replaced by Δ^1_{n+1} (for $n \geq 5$).

We also don't know whether the large cardinals used for constructing the models in 1.2 and 1.3 are actually necessary, although the 3^{rd} section will provide partial evidence in favor of this. It is open as how to get more than an inaccessible cardinal in L from the assumption of the above question.

The paper is organized as follows. Section 2 provides the necessary inner model theory, and states a crucial technical lemma due to Woodin. Sections 3 and 4 contain proofs of 1.2 and 1.3, respectively, using heavily ideas of R. David (cf. [1], [2], and also [4]). We shall in fact only prove 1.2 and 1.3 for the case $n > 0$, as the case $n = 0$ is easily seen to be given by [2] (or may be derived by simplifying the arguments to follow). The short Section 5 lists three key open problems.

We want to emphasize that this paper contains yet another example in which inner model theory (in fact, core model theory proper) is used for obtaining *upper* bounds for the consistency strength of set theoretical hypotheses. Traditionally, core model theory is designed for getting lower bounds.

2 Preliminaries.

Woodin has seen how strong cardinals may be used to propagate universal Baireness in certain generic extensions. More precisely, he has shown the following theorem which will become crucial for the construction of our models.

Theorem 2.1 (**Woodin**) *Let $0 < n < \omega$, and let $\kappa_1 < ... < \kappa_n$ be strong cardinals. Let G be P-generic over V for some $P \in V$, and suppose that $(2^{2^{\kappa_n}})^V$ becomes countable in $V[G]$. Then in every set generic extension of $V[G]$, every $\mathbf{\Sigma}^1_{n+2}$-set of reals is universally Baire.*

In fact, in $V[G]$ there is a definable sequence $(T_m, S_m \colon 2 \leq m \leq n+2)$ of proper class sized trees on $\omega \times OR$ such that:

(a) S_2 is the Shoenfield tree for a universal Σ^1_2-set of reals in every (set-generic) extension,

(b) for $3 \leq m \leq n+2$, S_m is T_{m-1} reorganized so as to have $p[S_m] \approx \exists^{\mathbb{R}}\ p[T_{m-1}]$, and

(c) for $2 \leq m \leq n+2$*, for all p.o.'s* $Q \in V[G]$*,*

$$Q \;\Vdash\; p[\hat{T}_m \restriction \alpha] = \mathbb{R} \setminus p[\hat{S}_m \restriction \alpha]$$

for sufficiently large α*.*

Notice that the existence of the sequence $(T_m, S_m \colon 2 \leq m \leq n+2)$ implies that every $\mathbf{\Sigma}^1_{n+2}$-set of reals is universally Baire (in every set-generic extension) by the main characterization of universal Baireness from [3]. For a proof of 2.1 see [15] (cf. also [3] p.240).

We now turn to the inner model theory. We shall presuppose that the reader is familiar to a certain extent with [13]. In order to compute the complexity of the canonical well-ordering of the reals in the models we are about to construct, we shall also have to use some of the machinery of [8].

In the sections to follow we shall make heavy use of the fact that the ground model we are starting with will be the core model of all of its set-generic extensions. This is true if the ground model is a minimal fully iterable inner model for a given large cardinal assuption (roughly) below one Woodin cardinal. In particular, it will be true if the ground model happens to be $L[E^n]$, some $n < \omega$, the minimal fully iterable inner model with n strong cardinals.

Suppose that V is closed under $\sharp$'s (i.e., for any *set* $x \subset OR$, $x^\sharp$ exists), but there is no inner model with a Woodin cardinal. For any set $x \subset OR$, let $K^{x^\sharp}$ (of height the least x-indiscernible) denote the core model built inside $x^\sharp$. By a "local definability" property, it is true that for all $\alpha < \infty$, $\mathcal{J}_\alpha^{K^{x^\sharp}}$ is stable on a cone of x's, so that we may define a global core model K by letting

$$\mathcal{J}_\alpha^K = \mathcal{J}_\alpha^{K^{x^\sharp}},$$

for x having a large enough constructibility degree. (This "trick" is well-known among inner model theorists.) Throughout this section, the letter K will be reserved for denoting the model constructed in this fashion.

Now let $n > 0$, and suppose that $V = L[E^n]$. In particular, then, V is closed under $\sharp$'s. Moreover, it is not too hard to check that $K = V$ in this case. (This quickly reduces to some absoluteness of iterability fact.) Also, $K^{V[G]} = K$ for any G being set-generic over V. (Cf. [13].)

Let $n < \omega$. For our purposes, a premouse $\mathcal{M}$ is called n-full iff there is a universal weasel $W \rhd \mathcal{M}$ having the definability property (see [13] 4.4) at all $\kappa \in \mathcal{M}$ such that $\mathcal{J}_\kappa^{\mathcal{M}} \models$ "there are $< n$ many strong cardinals." It is straightforward to verify that if $W \triangleright \mathcal{M}$ witnesses that $\mathcal{M}$ is n-full then W has the hull-property (see [13] 4.2) at all $\kappa \in \mathcal{M}$ such that $\mathcal{J}_\kappa^{\mathcal{M}} \models$ "there are $\leq n$ many strong cardinals" (cf. [8] 1.3). One of the main results of [8], namely Corollary 2.18 (a) which was shown by the second author of the present paper, is that the set of all reals coding n-full premice is Π^1_{n+3}. (The informed reader will notice that the concept of "n-fullness" of [8] is just a bit stronger than the one defined above.)

Lemma 2.2 *Suppose that V is closed under $\sharp$'s. Let $1 \leq n < \omega$, and suppose that there is no inner model with n strong cardinals and a measurable above them all. Let α be an infinite cardinal of K, and let $\mathcal{M} \trianglerighteq \mathcal{J}_\alpha^K$ be a premouse with $\mathcal{M} \models$ "α is the largest cardinal."*

Then $\mathcal{M} \trianglelefteq \mathcal{J}_{\alpha^{+K}}^K$ iff $\mathcal{M}$ is $(n-1)$-full. Moreover, the set of reals coding $\mathcal{J}_{\alpha^{+K}}^K$ is Π^1_{n+3} in any code for $\mathcal{J}_\alpha^K$.

PROOF. As to the first part, "$\Rightarrow$" is trivial, so let us show "$\Leftarrow$. Let $W \triangleright \mathcal{M}$ witness that $\mathcal{M}$ is $(n-1)$-full, and let K' be a very soundness witness for $\mathcal{J}_{\alpha^{+K}}^K$. Let Q denote the common coiterate of W, K'.

Claim. The iteration is above α along the main branch on the W-side.

PROOF. Suppose not. Let π_{WQ} and $\pi_{K'Q}$ be the respective maps obtained from the main branches on the W- and K'-side. Set $\kappa = c.p.(\pi_{WQ})$, so that $\kappa < \alpha$ by assumption. Let Γ be a class of fixed points under both π_{WQ} and $\pi_{K'Q}$ which is thick in W, K', and Q (see [13] 3.8 through 3.11).

Of course, $\mathcal{J}_\kappa^W$ has $< n$ many strong cardinals, because otherwise we would end up with an inner model with n strong cardinals and a measurable above. By the above remarks, W hence has the hull- and definability property at all $\bar\kappa < \kappa$ which are strong in $\mathcal{J}_\kappa^W$, and W has the hull property at κ. Moreover, K' has the hull- and definability property at all $\gamma < \alpha^{+K}$.

Suppose $\bar\kappa = c.p.(\pi_{K'Q}) < \kappa$, so that $\bar\kappa$ is easily seen to be strong in $\mathcal{J}_\kappa^{K'} = \mathcal{J}_\kappa^W$. Then $\bar\kappa = \tau^W[a,b]$ where τ is a term, $a \in [\bar\kappa]^{<\omega}$, and $b \in [\Gamma]^{<\omega}$. Hence $\pi_{K'Q}(\bar\kappa) = \tau^Q[a,b] \in ran(\pi_{K'Q})$. Contradiction!

Repeating the same argument with $\kappa = \tau^{K'}[a,b]$ for some term τ, $a \in [\kappa]^{<\omega}$, and $b \in [\Gamma]^{<\omega}$ shows that actually we must have $\kappa = c.p.(\pi_{K'Q})$. This readily implies that W, Q, and K' all have the same $\mathcal{P}(\kappa)$, just written $\mathcal{P}(\kappa)$ in what follows.

Let $X \in \mathcal{P}(\kappa)$. As W has the hull-property at κ we have $X = \tau^W[a,b] \cap \kappa$ where τ is a term, $a \in [\kappa]^{<\omega}$, and $b \in [\Gamma]^{<\omega}$. For $\xi < \kappa$ we have that $\xi \in \tau^W[a,b]$ iff $\xi \in \tau^Q[a,b]$ iff $\xi \in \tau^{K'}[a,b]$, so that also $X = \tau^{K'}[a,b] \cap \kappa$. But then for $\beta < min\{\pi_{WQ}(\kappa), \pi_{K'Q}(\kappa)\}$ we get that $\beta \in \pi_{WQ}(X)$ iff $\beta \in \tau^Q[a,b]$ iff $\beta \in \pi_{K'Q}(X)$. This finally would mean that the two first extenders used for getting π_{WQ} and $\pi_{K'Q}$ are compatible. Contradiction!

$\square$ (Claim)

In particular, $\mathcal{M} \trianglelefteq \mathcal{J}^W_{\alpha^{+W}} = \mathcal{J}^Q_{\alpha^{+Q}}$, as α is the largest cardinal of $\mathcal{M}$. Hence we are done in the case that $Q = K'$.

Otherwise let ν be the index of the first extender used along the main branch on the K'-side. Of course, $\nu > \alpha$, and because ν will be a cardinal in Q we have that $\nu \geq \alpha^{+Q}$, and thus $\mathcal{M} \trianglelefteq \mathcal{J}^Q_{\alpha^{+Q}} \trianglelefteq \mathcal{J}^Q_\nu = \mathcal{J}^{K'}_\nu$. But $\alpha^{+K} = \alpha^{+K'} \geq \alpha^{+Q}$, so that in fact $\mathcal{M} \triangleleft K$, as desired.

This proves the first part. But now we have that $\mathcal{M} = \mathcal{J}^K_{\alpha^{+K}}$ iff $\mathcal{M} \triangleright \mathcal{J}^K_\alpha$, $\mathcal{M} \models$ "α is the largest cardinal," and $\mathcal{M}$ is $(n-1)$-full, and for all $\mathcal{N}$ such that $\mathcal{N} \triangleright \mathcal{J}^K_\alpha$, $\mathcal{N} \models$ "α is the largest cardinal," and $\mathcal{N}$ is $(n-1)$-full we have that $\mathcal{N} \trianglelefteq \mathcal{M}$. As $(n-1)$-fullness is Π^1_{n+2} in the codes, this proves the second part.

$\square$ (2.2)

As a simple corollary to 2.2 we get that under the hypotheses of 2.2 $K \cap HC$ is Σ^1_{n+4}. (This generalizes a result of Jensen and Mitchell, cf. [13] p. 85f.) Despite of [7] 3.6 (proving a weaker statement), both 2.2 and this corollary seem to be new. Let us state the latter in the following way.

Corollary 2.3 *Suppose that V is closed under $\sharp$'s. Let $1 \leq n < \omega$, and suppose that there is no inner model with n strong cardinals and a measurable above them all.*

Then $\mathcal{M} \triangleleft \mathcal{J}^K_{\omega_1^V}$ and $\mathcal{M} \cap OR$ is a cardinal in K iff [$\mathcal{M}$ is $(n-1)$-full, and IF $\alpha < \mathcal{M} \cap OR$ is a cardinal of $\mathcal{M}$ and $\mathcal{N} \triangleright \mathcal{J}^\mathcal{M}_\alpha$ is $(n-1)$-full with largest cardinal α THEN $\mathcal{N} \trianglelefteq \mathcal{M}$].

In particular, the set of all reals coding some $\mathcal{M} \triangleleft \mathcal{J}^K_{\omega_1^V}$ with $\mathcal{M} \cap OR$ being a K-cardinal is Π^1_{n+3}. Moreover, $K \cap HC$ is Σ^1_{n+4} in the codes.

PROOF. Straightforward, using 2.2 and the fact that $(n-1)$-fullness is Π^1_{n+2} in the codes.

$\square$ (2.3)

We shall need later:

Corollary 2.4 *Suppose that V is closed under $\sharp$'s. Let $n < \omega$, and suppose that there is no inner model with n strong cardinals and a measurable above them all. Assume that K has n strong cardinals $\kappa_1 < \ldots < \kappa_n$ such that $\lambda = \kappa_n^{++K} < \omega_1^V$.*

Then the set of reals coding $\mathcal{J}^K_\lambda$ is Π^1_{n+4}.

PROOF. It is clear that that $\mathcal{J}^K_\lambda$ is the longest initial segment of K with height a K-cardinal and satisfying "there are n strong cardinals, the largest of which is the second largest cardinal." But then 2.3 easily gives the result.

$\square$ (2.4)

We shall be able to arrange later that under certain circumstances there is a Π^1_{n+4}-singleton coding $\mathcal{J}^K_\lambda$.

We now have to turn towards condensation. In general, the condensation properties provable for K are much weaker than the ones provable for L. However, in the very special case that $K = L[E^n]$ for some $n < \omega$ we get that K satisfies an "L-like" condensation lemma. We state it in the form in which we shall need it.

Lemma 2.5 *Let $0 < n < \omega$, and set $E = E^n$. Let $\kappa_1 < \ldots < \kappa_n$ denote the strong cardinals of $L[E]$. Let $\alpha > \kappa_n^{+L[E]}$ be s.t. $J_\alpha[E]$ is cardinal-correct in $L[E]$, i.e., all cardinals $< \alpha$ in $J_\alpha[E]$ are also cardinals in $L[E]$. Let $\sigma\colon \mathcal{M} \to_{\Sigma_1} J_\alpha[E]$ where $\mathcal{M}$ is transitive and $\sigma \upharpoonright \kappa_n^{+L[E]} + 1 = id$.*

Then $\mathcal{M} = J_{\bar\alpha}[E]$ for some $\bar\alpha \leq \alpha$.

PROOF. In fact 2.5 is a consequence of the argument for Lemma 6.1 of [11]. We may of course assume w.l.o.g. that $\sigma \neq id$, and let δ denote the critical point of σ. Using σ, any iteration of the phalanx $\mathcal{P} = ((J_\alpha[E], \mathcal{M}), \delta)$ can be copied onto $J_\alpha[E]$ to give an iteration of $J_\alpha[E]$, so that in particular $\mathcal{P}$ is iterable.

We may hence coiterate $\mathcal{P}$ with $J_\alpha[E]$, getting iteration trees $\bar{\mathcal{T}}$ on $\mathcal{P}$ and $\mathcal{U}$ on $J_\alpha[E]$. By copying $\bar{\mathcal{T}}$ onto $J_\alpha[E]$ we get $\mathcal{T}$ on $J_\alpha[E]$ together with an embedding $\tilde{\sigma}\colon \mathcal{M}^{\bar{\mathcal{T}}}_\infty \to \mathcal{M}^{\mathcal{T}}_\infty$. In the case that $\pi^{\bar{\mathcal{T}}}_{0\infty}$ exists and $\mathcal{M}^{\bar{\mathcal{T}}}_\infty$ is above $J_\alpha[E]$ we also have that $\pi^{\mathcal{T}}_{0\infty} = \tilde{\sigma} \circ \pi^{\bar{\mathcal{T}}}_{0\infty}$.

Claim 1. $\mathcal{M}^{\bar{\mathcal{T}}}_\infty$ is above $\mathcal{M}$.

Proof. Suppose not, so $\mathcal{M}^{\bar{\mathcal{T}}}_\infty$ is above $J_\alpha[E]$. If $\mathcal{M}^{\bar{\mathcal{T}}}_\infty \rhd \mathcal{M}^{\mathcal{U}}_\infty$ or there were a drop on the $\mathcal{P}$-side then there wouldn't be a drop on the $J_\alpha[E]$-side and the map $\tilde{\sigma} \circ \pi^{\mathcal{U}}_{0\infty}$ would give a contradiction with the Dodd-Jensen Lemma. Similarily, if $\mathcal{M}^{\bar{\mathcal{T}}}_\infty \lhd \mathcal{M}^{\mathcal{U}}_\infty$ or there were a drop on the $J_\alpha[E]$-side then there wouldn't be a drop on the $\mathcal{P}$-side and the map $\pi^{\bar{\mathcal{T}}}_{0\infty}$ would give a contradiction with Dodd-Jensen.

Hence $\mathcal{M}^{\bar{\mathcal{T}}}_\infty = \mathcal{M}^{\mathcal{U}}_\infty$ and there's no drop on either side. Let ξ be an ordinal. We now have $\pi^{\mathcal{U}}_{0\infty}(\xi) \leq \pi^{\bar{\mathcal{T}}}_{0\infty}$ by Dodd-Jensen. Similarily, we have $\pi^{\mathcal{T}}_{0\infty}(\xi) \leq \tilde{\sigma} \circ \pi^{\mathcal{U}}_{0\infty}(\xi)$ by Dodd-Jensen; but $\pi^{\mathcal{T}}_{0\infty} = \tilde{\sigma} \circ \pi^{\bar{\mathcal{T}}}_{0\infty}$, so $\tilde{\sigma} \circ \pi^{\bar{\mathcal{T}}}_{0\infty}(\xi) \leq \tilde{\sigma} \circ \pi^{\mathcal{U}}_{0\infty}(\xi)$, and hence $\pi^{\bar{\mathcal{T}}}_{0\infty}(\xi) \leq \pi^{\mathcal{U}}_{0\infty}(\xi)$. We have shown that $\pi^{\bar{\mathcal{T}}}_{0\infty} = \pi^{\mathcal{U}}_{0\infty}$, giving the usual contradiction.

□ (Claim 1)

Claim 2. $\pi^{\bar{\mathcal{T}}}_{0\infty}$ exists, and in fact $\mathcal{M}^{\bar{\mathcal{T}}}_\infty = \mathcal{M}$.

Proof. If there were a drop on the $\mathcal{P}$-side of the comparison then $\pi^{\mathcal{U}}_{0\infty}$ would exist and the map $\tilde{\sigma} \circ \pi^{\mathcal{U}}_{0\infty}$ would contradict the Dodd-Jensen Lemma. Hence $\pi^{\bar{\mathcal{T}}}_{0\infty}$ exists.

Now suppose that $\mathcal{M}^{\bar{\mathcal{T}}}_\infty \neq \mathcal{M}$, let F be the first extender used along $[0,\infty]_{\bar{\mathcal{T}}}$, and let μ be its critical point. By Claim 1 and what has been shown so far we have that $\mu \geq \delta$ and μ is a cardinal in $\mathcal{M}$. Then $\sigma(\mu)$ is a cardinal in $J_\alpha[E]$, hence in $L[E]$ by cardinal-correctness, which implies that every κ_i, $0 < i \leq n$, is strong in $J_{\sigma(\mu)}[E]$. So using σ every κ_i, $0 < i \leq n$, is strong in $\mathcal{J}^{\mathcal{M}}_\mu$.

But this is now easily seen to imply that the model where F is taken from provides a sharp for an inner model with n strong cardinals. This contradicts the choice of $L[E]$ as the minimal (fully iterable) inner model with n strong cardinals.

□ (Claim 2)

Notice that the second part of Claim 2 immediately gives that $\mathcal{M}^{\mathcal{U}}_\infty \trianglerighteq \mathcal{M}$: this is clear if there is a drop on the main branch of $\mathcal{U}$; but if not we have $\mathcal{M}^{\mathcal{U}}_\infty \cap OR \geq \alpha \geq \mathcal{M} \cap OR$.

Claim 3. $\mathcal{M}^{\mathcal{U}}_\infty = J_\alpha[E]$.

Proof. Suppose not, and let F be the first extender used along $[0,\infty]_{\mathcal{U}}$, and let μ be its critical point.

Let us first assume that $\mu < \delta$. Using a "minimality of $L[E]$" argument as above it is then straightforward to check that $\mu = \kappa_i$ for some $0 < i \leq n$. Let ν be the index of F. Then ν is a cardinal in $\mathcal{M}^{\mathcal{U}}_\infty$, and hence in $\mathcal{M}$ by $\mathcal{M}^{\mathcal{U}}_\infty \trianglerighteq \mathcal{M}$. Moreover, $\mathcal{J}^{\mathcal{M}}_\nu$ does not satisfy that κ_i is strong (minimality again!), so that κ_i is not strong in $J_{\sigma(\nu)}[E]$.

But using σ we have that $\sigma(\nu)$ is a cardinal in $J_\alpha[E]$, and hence in $L[E]$ by cardinal-correctness. So after all κ_i must be strong in $J_{\sigma(\nu)}[E]$. Contradiction!

So $\mu > \delta$. But we can now again just vary the "minimality of $L[E]$" argument from above. We shall have that μ is a cardinal in $\mathcal{M}^{\mathcal{U}}_\infty$, and hence of $\mathcal{M}$ by $\mathcal{M}^{\mathcal{U}}_\infty \trianglerighteq \mathcal{M}$. Thus $\sigma(\mu)$ is a cardinal in $J_\alpha[E]$, and hence of $L[E]$ as well. But then every κ_i, $0 < i \leq n$, is strong in $J_{\sigma(\mu)}[E]$, and hence is strong in $\mathcal{J}^{\mathcal{M}}_\mu$ as well. But then the model where F is taken from provides a sharp for an inner model with n strong cardinals. Contradiction!

$\Box$ (Claim 3)

$\Box$ (2.5)

The reader will have noticed that by the above proof the hypothesis of 2.5 can be further weakened.

3 Proof of 1.2.

Throughout this section we fix some $n < \omega$, $n > 0$, and we assume $L[E^n]$, the minimal fully iterable inner model with n strong cardinals, exists. We shall write $L[E] = L[E^n]$. Let $\kappa_1 < ... < \kappa_n$ be the strong cardinals of $L[E]$, and set $\lambda = \kappa_n^{++L[E]}$. As explained above, $L[E]$ is the core model of all set-generic extensions of $L[E]$.

To a certain extent, the construction to be described closely follows [1]. However, there are some complications here, as we force over $L[E]$ rather than L.

PROOF of 1.2. To begin with, we define a sequence $(T_k: k < \omega)$ of λ^+-Suslin trees inside $L[E]$. Given a tree T and an ordinal α we write T^α for the α^{th} level of T. We define the T_k's by simultaneously constructing all T_k^α's by induction on $\alpha < \lambda^{+L[E]}$. We shall have that $T_k^\alpha \subset {}^\alpha 2$, and $T_k = \bigcup_{\alpha<\lambda^+} T_k^\alpha$ (where $\lambda^+ = \lambda^{+L[E]}$), ordered by $\subset$.

Work inside $L[E]$ until further notice. We set $x \in T_k^0$ iff $x = \emptyset$, and $x \in T_k^{\alpha+1}$ iff $x = y^\frown 0$ or $= y^\frown 1$ for some $y \in T_k^\alpha$. If α is a limit ordinal of cofinality $< \lambda$ then we let $x \in T_k^\alpha$ iff $x \restriction \beta \in T_k^\beta$ for all $\beta < \alpha$ (noticing that we only get $\leq \lambda^{<\lambda} = \lambda$ many branches).

Now suppose that α is a limit ordinal of cofinality λ. Let $\eta = \eta_\alpha$ be least such that $(T_k^\beta: k < \omega, \beta < \alpha) \in J_\eta[E]$, every set has cardinality $\leq \lambda$ in $J_\eta[E]$, $cf(\eta) = \lambda$, and $J_\alpha[E] \models ZF^-$. Inside $J_\eta[E]$, let us consider the forcing

$$P_\alpha = \{p \in {}^{\omega\times\lambda}V: p(k,\xi) \in \bigcup_{\beta<\alpha} T_k^\beta \text{ for all } (k,\xi) \in dom(p)\},$$

ordered by $p' \leq_{P_\alpha} p$ iff $p'(k,\xi) \supset p(k,\xi)$ for all $(k,\xi) \in dom(p)$. In V (which is $L[E]$ for the moment), we may pick some P_α-generic over $J_\eta[E]$ (notice ${}^{<\lambda}J_\eta[E] \subset J_\eta[E]$, and P_α is $< \lambda$-closed). Any such generic gives λ many branches for each $\bigcup_{\beta<\alpha} T_k^\beta$. We let $(T_k^\alpha: k < \omega)$ be the result of adding these branches at level α, for the $<_{L[E]}$-least P_α-generic over $J_\eta[E]$.

This defines $(T_k: k < \omega)$. For $X \subset \omega$ we write P^X for $\prod_{k\in X} T_k$, and we write $P = P^\omega$. So forcing with P adds cofinal branches thru the T_k's.

Claim 1. Let $X \subset \omega$. Then P^X is $< \lambda$-closed and has the λ^+-c.c. In particular, T_l is a Suslin tree in $L[E]^{P^{\omega\setminus\{l\}}}$ for any $l < \omega$.

PROOF. $< \lambda$-closedness is trivial. Let $A \subset P^X$ be a maximal antichain in P^X. Let $\sigma: J_\tau[E] \to J_{\lambda^{++}}[E]$ be elementary such that $\sigma \restriction \lambda = id$, $\tau < \lambda^+$, and $A \in ran(\sigma)$. (Such a map exists by 2.5.) Let $\alpha = c.p.(\sigma)$. We may assume that $cf(\alpha) = \lambda$.

It is easy to see that $(T_k^\beta: k < \omega, \beta < \alpha) \in J_\tau[E]$. But also $\eta = \eta_\alpha > \tau$, because $\alpha = \lambda^+$ in $J_\tau[E]$, whereas every set has size $\leq \lambda$ in $J_\eta[E]$. In particular, $P_\alpha \in J_\eta[E]$, and using the elementarity of σ we get that every $f \in \prod_{k\in X} \bigcup_{\beta<\alpha} T_k^\beta$ is compatible with some element of $\sigma^{-1}(A)$.

So if $p \in P_\alpha$, we can easily find a $q \leq_{P_\alpha} p$ with the same domain as p such that for all $(k,\xi) \in dom(p)$ with $k \in X$, $q(k,\xi)$ extends some element of $\sigma^{-1}(A)$. Thus by a straightforward density argument, every $x \in T_k^\alpha$, for $k \in X$, extends some element of $\sigma^{-1}(A)$.

Thus $\sigma^{-1}(A)$ is maximal, $A = \sigma^{-1}(A)$, and A has size $\leq \lambda$.

$\square$ (Claim 1)

Stepping out of $L[E]$, we now force with (P, Q), where $Q = Col(\omega, \lambda)$. Fix a P-generic over $L[E]$, and let $B = (B_k : k < \omega)$ be the sequence of cofinal branches obtained from the generic (essentially, B *is* the generic). Pick G being Q-generic over $L[E][B]$. Then $\lambda^{+L[E]} = \lambda^{+L[E][B]} = \omega_1^{L[E][B][G]}$, which we shall from now on denote by ω_1.

From Claim 1 we easily get:

Claim 2. Let $X \subset \omega$, $X \in L[E]$. Then P^X has the c.c.c. in $L[E][G]$. In particular, forcing with $(Q, P^{\omega \setminus \{l\}})$ over $L[E]$ does not destroy Suslinness of T_l, for any $l < \omega$.

We may fix some recursive bijection $e: \omega \to {}^{<\omega}2$. We have $({}^{<\omega}2, \subset) \in J_{\omega+\omega}$ is a tree, any two cofinal branches of which give a pair of almost disjoint (a.d.) subsets of ω via e. Let us fix $(a_k : k < \omega) \in L$, obtained from the first (in $<_L$) ω many branches in L thru $({}^{<\omega}2, \subset)$. Then $(a_k : k < \omega)$ is definable (without parameters) inside any transitive structure $\mathcal{S} \supset J_{\omega+\omega}$.

Let $x \subset \omega$ be any real. We then let

$$x^{dec} = \{k < \omega : x \cap a_k \text{ is finite }\}.$$

For $\mathcal{S}$ as above and $x \in \mathcal{S}$ we have that $(x^{dec})^{\mathcal{S}} = x^{dec}$. We also want to have a notation at hand for a second decoding device. Given $x \subset \omega$, we define $E \subset \omega \times \omega$ by $(k, l) \in E$ iff $\Gamma(k, l) \in x$ (Γ being Gödel's pairing function), and we let

$$\mathcal{M}_x = \text{ the transitive collpse of } (\omega, E),$$

provided that E is well-founded and extensional (if not, we let $\mathcal{M}_x$ be undefined). Hence if $\mathcal{S}$ is admissible and $x \in \mathcal{S}$ then $(\mathcal{M}_x)^{\mathcal{S}} = \mathcal{M}_x$ (if it exists). We shall also have to deal with the function sending x to $\mathcal{M}_{x^{dec}}$. Let us write $\mathcal{M}_x^\star$ for $\mathcal{M}_{x^{dec}}$.

Now pick a real $g \subset \omega$ (inside $L[E][B][G]$) such that $\mathcal{M}_g = J_\lambda[E]$. We may and shall assume that $L[E][B][G] = L[E][B][g]$. We want to force over $L[E][B][g]$ to obtain a real a such that $g = a^{dec}$ (hence $\mathcal{M}_a^\star = J_\lambda[E]$), and a

is a Π^1_{n+4}-singleton inside $L[E][a]$. It will then be easy to see that $L[E][a]$ is as desired.

Let us fix $(a_i : i < \omega_1) \in L[E][g]$, obtained from the first (in $<_{L[E][g]}$) ω_1 many branches in $L[E][g]$ thru $({}^{<\omega}2, \subset)$. Notice that for $k < \omega$, a_k has now been defined twice, but the point is that both definitions yield the same object. In particular, the a_i's form a family of a.d. subsets of ω. The forcing R (for adding a) consists of conditions $p = (l(p), r(p))$ where $l(p) : k \to 2$ for some $k < \omega$ and $r(p)$ is a finite subset of ω_1. We set $q = (l(q), r(q)) \leq_R p = (l(p), r(p))$ iff $l(q) \supset l(p)$, $r(q) \supset r(p)$, and the following holds true:

$$\forall k \; [k < dom(l(p)) \wedge k \in g \Rightarrow$$

$$\{m \in dom(l(q)) \setminus dom(l(p)) : l(q)(m) = 1\} \cap a_k = \emptyset],$$

$$\forall k \; \forall \alpha \; [k < dom(l(p)) \wedge l(p)(k) = 1 \wedge \alpha \in r(p) \cap B_{2k} \Rightarrow$$

$$\{m \in dom(l(q)) \setminus dom(l(p)) : l(q)(m) = 1\} \cap a_{\alpha+\omega+2k} = \emptyset], \text{ and}$$

$$\forall k \; \forall \alpha \; [n < dom(l(p)) \wedge l(p)(k) = 0 \wedge \alpha \in r(p) \cap B_{2k+1} \Rightarrow$$

$$\{m \in dom(l(q)) \setminus dom(l(p)) : l(q)(m) = 1\} \cap a_{\alpha+\omega+2k+1} = \emptyset].$$

Let H be R-generic over $L[E][g]$, and let $a \subset \omega$ be such that $\bigcup_{p \in H} \; l(p)$ is its characteristic function. Clearly:

Claim 3. $\forall k \; (k \in g \Leftrightarrow a \cap a_k$ is finite $)$, i.e., $g = a^{dec}$, and $J_\lambda[E] = \mathcal{M}^\star_a$.

Setting $D_k = \{\alpha : a \cap a_{\alpha+\omega+k}$ is finite $\}$, we also easily get

Claim 4. $\forall k \; (k \in a \Rightarrow D_{2k} = B_{2k} \wedge D_{2k+1} = \emptyset)$,

and

Claim 5. $\forall k \; (k \notin a \Rightarrow D_{2k} = \emptyset \wedge D_{2k+1} = B_{2k+1})$.

As in [1], the following two claims are crucial.

Claim 6. $\forall k \; (k \in a \Rightarrow T_{2k+1}$ is Suslin in $L[E][a]$ $)$.

Claim 7. $\forall k \; (k \notin a \Rightarrow T_{2k}$ is Suslin in $L[E][a]$ $)$.

Proof. We give the proof of Claim 6, that of Claim 7 being identical modulo notational changes. Suppose that $l \in a$, but T_{2l+1} is no longer a Suslin tree in $L[E][a]$. Set $T = T_{2l+1}$. Essentially, $(G, (B_k\colon k \neq 2l+1))$ is $(Q, P^{\omega \setminus \{2l+1\}})$-generic over $L[E]$. Moreover, there is a canonical forcing $R' \in L[E][G][(B_k\colon k \neq 2l+1)]$ such that

$$(P, Q) \star \dot{R} = [(Q, P^{\omega \setminus \{2l+1\}}) \star \dot{R}'] \star \dot{T}.$$

(R' is defined exactly as R except that we require that $l(p)(l) = 1$ and rewrite the definition of $\leq_R$ so as not to mention B_{2k+1}.) It now suffices to show that T is still Suslin in $L[E][(G, (B_k\colon k \neq 2l+1), a)]$.

By Claim 2, T is still Suslin in $L[E][(G, (B_k\colon k \neq 2l{+}1))]$. It hence remains to show that forcing with R' over this model does not add an antichain $A \subset T$ of size ω_1.

So let $\dot{A}$ be a name for a maximal antichain A in T, and let $p \in R'$ be such that

$$p \Vdash \dot{A} \text{ is a maximal antichain in } \hat{T}.$$

Let $A' = \{x \in T\colon q \Vdash \hat{x} \in \dot{A},\ \text{some } q \leq_{R'} p\}$. As $A' \supset A$, it suffices to show that A' is countable in $L[E][G][(B_k\colon k \neq 2l+1)]$.

Let us work in $L[E][G][(B_k\colon k \neq 2l+1)]$, and suppose that A' is uncountable. For any $x \in A'$ we may pick $q_x \leq_{R'} p$ with $q_x \Vdash x \in \dot{A}$. Of course, $Q = \{q_x\colon x \in A'\}$ cannot be countable, as otherwise there were an uncountable $A'' \subset A'$ such that $q_x = q_{x'}$ for all $x, x' \in A''$. But such A'' would also be an antichain in T.

So Q is uncountable. But then there is an uncountable $A^\star \subset A'$ such that $l(q_x) = l(q_{x'})$ for all $x, x' \in A^\star$. In particular, any two conditions q_x, $q_{x'}$ in $A^\star$ are compatible, which implies that x, x' itself are incompatible. But now we get that $\{x \in T\colon q_x \in A^\star\}$ is an uncountable antichain. Contradiction!

We have thus shown that A' and hence A must be countable, so that T is still Suslin in $L[E][(G, (B_k\colon k \neq 2l+1), a)]$.

□ (Claims 6, 7)

We are now going to write down a formula showing that a is a Π^1_{n+4}-singleton inside $L[E][a]$. In order to do this we have to relativize the construction of $(T_k\colon k < \omega)$, our sequence of Suslin trees in $L[E]$, as well as $(a_i\colon i < \omega_1)$, our sequence of pairwise a.d. subsets of ω.

Let $\mathcal{N}$ be a premouse with a largest cardinal η which actually happens to be a double successor cardinal in $\mathcal{N}$. We may then, working inside $\mathcal{N}$, construct a sequence $(T_k^{\mathcal{N}}: k < \omega)$ of trees of height η by using a word for word repetition of how $(T_k: k < \omega)$ was constructed in $L[E]$, but with every occurence of "$L[E]$" replaced by "$\mathcal{N}$," and with "λ" replaced by "the predecessor of η in $\mathcal{N}$." Further, if x is any real with $\mathcal{N}[x]$ admissible such that $\omega_1^{\mathcal{N}[x]}$ exists then we shall write $(a_i^{\mathcal{N},x}: i < \omega_1^{\mathcal{N}[x]})$ for that sequence of pairwise a.d. subsets of ω obtained from the first (along $<_{\mathcal{N}[x]}$) $\omega_1^{\mathcal{N}[x]}$ many branches in $\mathcal{N}[x^{dec}]$ thru $({}^{<\omega}2, \subset)$.

We now consider the following formula, abbreviated $\Phi(x)$:

"$\mathcal{M}_x^\star = J_\lambda[E]$, and

IF (a) $\mathcal{N}$ is $(n-1)$-full, $\mathcal{N} \rhd \mathcal{M}_x^\star$,
(b) $\mathcal{M}_x^\star \cap OR$ is the second largest cardinal of $\mathcal{N}$,
(c) $\mathcal{N}[x] \models ZF^-$,
(d) $(T_n^{\mathcal{N}}: n < \omega)$ and $(a_i^{\mathcal{N},x}: i < \omega_1^{\mathcal{N}[x]})$ are as described above, and
(e) $(B_n^{\mathcal{N},x}: n < \omega)$ is such that $B_k^{\mathcal{N},x} = \{\alpha: x \cap a_{\alpha+k}^{\mathcal{N},x} \text{ is finite }\}$,
THEN we have that:
(a)' if $k \in x$ then $B_{2k}^{\mathcal{N},x}$ is a cofinal branch thru $T_{2k}^{\mathcal{N}}$, and
(b)' if $k \notin x$ then $B_{2k+1}^{\mathcal{N},x}$ is a cofinal branch thru $T_{2k+1}^{\mathcal{N}}$."

Claim 8. $\{x: \Phi(x)\}$ is a Π^1_{n+4}-set of reals.

PROOF. This readily follows from 2.4.

□ (Claim 8)

Claim 9. In $L[E][a]$, for all $x \in \mathbb{R}$ we have that $\Phi(x)$ iff $x = a$.

PROOF. We work inside $L[E][a]$. First let $x \in \mathbb{R}$ be given such that $\Phi(x)$ holds. Suppose that $x \neq a$, and suppose w.l.o.g. that there is $l < \omega$ such that $l \in x$, yet $l \notin a$, so that in particular T_{2l} is a Suslin tree by Claim 5. (Otherwise we can pick $l \in a \setminus x$ and consider T_{2l+1}, being Suslin by Claim 4.)

We may now pick $\sigma: \mathcal{N}[x] \to J_{\omega_2}[E][x]$ with $\mathcal{N}$ being countable and $c.p.(\sigma) > \lambda$. By 2.5 we have that $\mathcal{N} = J_\tau[E]$ where $\lambda < \tau < \omega_1$. Notice that $\lambda^{+\mathcal{N}} = c.p.(\sigma)$, which is sent to ω_1 by σ.

We have that $\mathcal{M}_x^\star = J_\lambda[E]$ by the first part of $\Phi(x)$, so that $\mathcal{N}$ and x certainly satisfy the IF part of $\Phi(x)$. We have that $\sigma(T_{2n}^{J_\tau[E]}) = T_{2n}$.

From (a') we now get that $B_{2l}^{\mathcal{N},x} \in \mathcal{N}[x]$ is a cofinal branch thru $\mathcal{T}_{2l}^{\mathcal{N}}$, so that by the elementarity of σ there is a cofinal branch (in $J_{\omega_2}[E][x]$) thru T_{2l}. Contradiction!

Conversely, we want to show that $\Phi(a)$ holds. Well, the first part of $\Phi(a)$ is fulfilled by Claim 3. Moreover, for any $\mathcal{N}$ as in (a) through (c) of the IF part of $\Phi(a)$ we have by 2.2 that $\mathcal{N} = J_\tau[E]$ where $\lambda < \tau < \omega_1$.

But then $(a_i^{\mathcal{N},a}\colon i < \omega_1^{\mathcal{N}[a]}) = (a_i\colon i < \omega_1^{\mathcal{N}[a]})$ and $B_k^{\mathcal{N},a} = B_k \cap \mathcal{N}$ are claer. Moreover, we claim that $T_{2n}^{\mathcal{N}} = T_{2n} \cap \mathcal{N}$, in fact that $(T_k^{\mathcal{N}}\colon k < \omega) = (T_k \cap \mathcal{N}\colon k < \omega)$.

To verify this, one has to show $(T_k^\alpha)^{\mathcal{N}} = T_k^\alpha$ for all $k < \omega$ and all $\alpha < \omega_1^{\mathcal{N}[a]} = \lambda^{+\mathcal{N}}$ by induction on α. Notice that ${}^\lambda\alpha \cap L[E] \subset \mathcal{N}$, so that the only non-trivial case is when α has cofinality λ (both in $\mathcal{N}$ and in $L[E]$). But then $\eta_\alpha < \lambda^{+\mathcal{N}}$ is easily seen, so that $(T_k^\alpha)^{\mathcal{N}} = T_k^\alpha$ follows from the choice of T_k^α.

But now (a') and (b') are clear.

□ (Claim 9)

Now by virtue of 2.1 and Claims 8 and 9, in order to finish the proof of 1.2 it suffices to show:

Claim 10. In $L[E][a]$, there is a $\Delta^1_{n+3}(a)$-well-ordering of $\mathbb{R}$.

Proof. Using the fact that $(P, Q) * \dot{R}$ has the λ^+-c.c., it is easily seen that $\mathbb{R} \subset J_{\omega_1}[E][a]$. Setting $\mathcal{P} = J_{\omega_1}[E][a]$, the reals of $L[E][a]$ may hence be well-ordered by $<_{\mathcal{P}}$, the order of constructibility of $\mathcal{P}$.

As $J_\lambda[E] = \mathcal{M}_a^\star$, 2.2 gives that for any $x, y \in \mathbb{R} \cap L[E][a]$, $x <_{\mathcal{P}} y$ iff

$$\exists \mathcal{N}\ (\mathcal{N} \text{ is } (n-1)\text{–full}, \ \rho_\omega(\mathcal{N}) = OR \cap \mathcal{M}_a^\star$$

$$\mathcal{N} \trianglerighteq \mathcal{M}_a, \text{ and } x <_{\mathcal{N}} y).$$

This is a $\Sigma^1_{n+3}(a)$-relation. Hence $<_{\mathcal{P}}$ is a $\Delta^1_{n+3}(a)$-well-ordering of $\mathbb{R} \cap L[E][a]$.

□ (Claim 10)
□ (1.2)

4 Proof of 1.3.

As in the last section we fix $n < \omega$, $n > 0$, and we assume $L[E^n]$, the minimal fully iterable inner model with n strong cardinals, to exist. However, we shall now assume that $L[E^n]$ has an inaccessible cardinal above its strong cardinals. (This is for example the case if in V there is an inaccessible cardinal above the strong cardinals of $L[E^n]$.) Again, we shall write $L[E] = L[E^n]$, we let $\kappa_1 < ... < \kappa_n$ be the strong cardinals of $L[E]$, and we let $\eta > \kappa_n$ be the least inaccessible in $L[E]$ above κ_n.

The construction to follow will absorb the construction of the previous section, and it will heavily use the key idea of [2] (for a general formulation of David's trick, cf. [4]). We shall make use of the following little lemma (which is well-known).

Lemma 4.1 *Let $A \subset \mathbb{R}$, and suppose that there is an inner model W with countably many reals and a tree (on $\omega \times \kappa$ say, for some ordinal κ) $T \in W$ such that $A = p[T]$ (in V). Then A is Lebesgue measurable and has the property of Baire.*

PROOF. For a real x we have that $x \in A$ iff $x \in p[T]$ iff $W[x] \models x \in p[T]$, so A is Solovay over W. But the set of all reals not being random over W is null, and the set of all reals not being Cohen over W is meager (by $Card(\mathbb{R} \cap W) = \aleph_0$), and hence A is Lebesgue measurable and has the property of Baire.

$\square$ (4.1)

It is a simple observation that 4.1 can be used to get a real a, set-generic over $L[E]$ such that in $L[E][a]$ every (lightface) Σ^1_{n+3}-set of reals is universally Baire whereas there is a $\Delta^1_{n+3}(a)$-well-ordering of the reals. (For example, just let a be a code for some $Col(\omega, \kappa_n^{+++L[E]})$-generic over $L[E]$.) Being familiar with the methods of the preceding section one may then find some such a being a Π^1_{n+4}-singleton.

This idea can be exploited a bit further to give a

PROOF of 1.3. This time, we have to start from a sequence $(T^i_k \colon i < \eta \wedge k < \omega)$ of η^+-Suslin trees inside $L[E]$. In fact, we construct this sequence in exactly the same way as we had constructed $(T_k \colon k < \omega)$ in the proof of 1.2, except that λ is replaced by η, and we want to obtain η many trees instead of just ω many. We shall not repeat the details of the construction here.

For $X \subset \eta \times \omega$ we write P^X for $\prod_{(i,k)\in X} T^i_k$, and we write $P = P^{\eta\times\omega}$. We shall leave it to the reader to formulate and verify analogues to Claims 1 and 2 in the previous section. They play the same role here as they played there.

Forcing with $P^{\{(i,k)\}}$ over $L[E]$ gives a generic B^i_k, a cofinal branch thru the tree T^i_k. We want to code B^i_k "nicely" by A^i_k, a certain bounded subset of η. Before actually doing this we want to illustrate the method by describing a simplified version of the forcing which is to come.

Fix $(a_i : i < \eta^+) \in L[E]$, a canonical sequence of pairwise a.d. subsets of η, obtained in a fashion as in the previous section. Let $\bar{Q}^i_k$ be the standard a.d. forcing for coding B^i_k by a subset of η, using $(a_i : i < \eta^+)$. Forcing with $\bar{Q}^i_k$ over $L[E][B^i_k]$ adds $\bar{A}^i_k \subset \eta$ coding B^i_k, and $\bar{Q}^i_k$ is $< \eta$-closed and has the η^+-c.c.

We now let Θ denote the theory $ZF^- +$ "there is exactly one inaccessible cardinal, also being the second largest cardinal." If $\mathcal{N} \models \Theta$ we denote by $\eta^{\mathcal{N}}$ its inaccessible cardinal. We may also denote by $(a^{\mathcal{N}}_i : i < (\eta^{\mathcal{N}})^{+\mathcal{N}}) \in \mathcal{N}$ a canonical sequence of pairwise a.d. subsets of $\eta^{\mathcal{N}}$. Moreover, as in the previous section, we may let $((T^i_k)^{\mathcal{N}} : i < \eta^{\mathcal{N}}, k < \omega)$ denote the sequence of $(\eta^{\mathcal{N}})^{+\mathcal{N}}$-Suslin trees being defined in $\mathcal{N}$ in exactly the same way as $(T^i_k : i < \eta, k < \omega)$ is defined in $L[E]$.

We now consider a forcing Q^i_k for adding $\hat{A}^i_k$, defined as follows. We let conditions be functions $p : \delta \to 2$ for some $\delta < \eta$ and such that the following holds true:

$$\forall \mathcal{N} \triangleleft L[E] \ (\ \mathcal{N} \models \Theta \wedge \mathcal{N}[\bar{A}^i_k \cap \eta^{\mathcal{N}}, p \restriction \eta^{\mathcal{N}}] \models \Theta \wedge i < \eta^{\mathcal{N}} \ \Rightarrow$$
$$\{\xi \in (T^i_k)^{\mathcal{N}} : \ a^{\mathcal{N}}_i \cap \bar{A}^i_k \text{ is bounded in } \eta^{\mathcal{N}}\}$$
$$\text{is a cofinal branch thru } (T^i_k)^{\mathcal{N}} \ ).$$

Claim 1. For all $p \in Q^i_k$ and all $\delta < \eta$ there is some $q \leq_{Q^i_k} p$ with $dom(q) \geq \delta$.

PROOF. Easy. Just pick q such that $dom(q) \geq max\{\delta, dom(p) + \omega\}$ and such that $\{n < \omega : q(dom(p) + n) = 1\}$ codes a well-ordering of length δ.

□ (Claim 1)

Claim 2. Q^i_k is $< \eta$-distributive (in $L[E][\bar{A}^i_k]$).

PROOF. Let $(D_\alpha : \alpha < \bar{\eta} < \eta) \in L[E][\bar{A}^i_k]$ be a sequence of open dense subsets of Q^i_k, and let $p \in Q^i_k$. Notice that $\{p, (D_\alpha : \alpha < \bar{\eta})\} \subset J_{\eta^+}[E][\bar{A}^i_k]$.

We define $(X_\alpha : \alpha \leq \bar{\eta})$ by the following recursion: X_0 = the smallest $X \prec J_{\eta^{++}}[E][\bar{A}^i_k]$ with $\{\kappa^+_n, i, p, (D_\alpha : \alpha < \eta)\} \subset X$ and $\eta \cap X$ being transitive, $X_{\alpha+1}$ = the smallest $X \prec J_{\eta^+}[E][\bar{A}^i_k]$ with $X_\alpha \cup \{X_\alpha\} \subset X$ and $\eta \cap X$ being transitive, and $X_\lambda = \bigcup_{\alpha<\lambda} X_\alpha$ for a limit ordinal $\lambda \leq \bar{\eta}$. By 2.5, all X_α's condense to models of the form $J_\gamma[E][\bar{A}^i_k \cap \beta]$. I.e., we get

$$\sigma_\alpha : \mathcal{N}_\alpha = J_{\gamma_\alpha}[E][\bar{A}^i_k \cap \beta_\alpha] \simeq X_\alpha \prec J_{\eta^{++}}[E][\bar{A}^i_k],$$

where β_α is the critical point of σ_α, and $\sigma_\alpha(\beta_\alpha) = \eta$. Notice $\beta_\alpha = \eta^{\mathcal{N}_\alpha}$.

Next, we aim to define a sequence $(p_\alpha : \alpha \leq \bar{\eta})$ of conditions such that $p_0 = p$, $p_{\alpha+1}$ = the least $q \leq_{Q^i_k} p_\alpha$ with $q \in X_{\alpha+1}$, $dom(q) \geq \beta_\alpha$, and $q \in D_\alpha$, and for limit ordinals $\lambda \leq \bar{\eta}$, $p_\lambda = \bigcup_{\alpha<\lambda} p_\alpha$.

It remains to show that this latter recursion does not break down, i.e., that $p_{\bar{\eta}} \in Q^i_k$ is defined. Well, the successor step does not cause any problems due to Claim 1 above. So let $\lambda \leq \bar{\eta}$ be a limit ordinal. Notice that $\beta_\alpha \leq dom(p_{\alpha+1}) < \beta_{\alpha+1}$ for $\alpha < \lambda$, so that $dom(p_\lambda) = \beta_\lambda$.

Let $\mathcal{N} \triangleleft L[E]$ be as in the definition of the conditions. Then $\mathcal{N} \cap OR \leq \gamma_\lambda$, because $(\beta_\alpha : \alpha < \lambda)$ is definable over $\mathcal{N}_\lambda$ and hence if $\mathcal{N} \cap OR > \gamma_\lambda$ then $(\beta_\alpha : \alpha < \lambda) \in \mathcal{N}$ would witness that β_λ is singular in $\mathcal{N}$, contradicting $\mathcal{N}[A^i_k \cap \beta_\lambda, p_\lambda] \models \Theta$.

But then $\eta^{\mathcal{N}} = \eta^{\mathcal{N}_\lambda} = \beta_\lambda$, and $(T^i_k)^{\mathcal{N}} = (T^i_k)^{\mathcal{N}_\lambda} \cap \mathcal{N}$ by a reasoning as in the proof of Claim 9 of the previous section. Moreover, we clearly also have $a^{\mathcal{N}}_i = a^{\mathcal{N}_\lambda}_i$ for $i < \beta^{+\mathcal{N}}_\lambda$.

By elementarity, $\{\xi \in (T^i_k)^{\mathcal{N}_\lambda} : a^{\mathcal{N}_\lambda}_i \cap \bar{A}^i_k$ is bounded in $\beta_\lambda\}$ is a cofinal branch thru $(T^i_k)^{\mathcal{N}_\lambda}$, from which we may conclude by the previous paragraph that $\{\xi \in (T^i_k)^{\mathcal{N}} : a^{\mathcal{N}}_i \cap \bar{A}^i_k$ is bounded in $\eta^{\mathcal{N}}\}$ is a cofinal branch thru $(T^i_k)^{\mathcal{N}}$, as desired.

□ (Claim 2)

We now have to turn towards the forcing which we shall actually use for constructing our model. Because we have to eventually code B^i_k "down to a real" without destroying the inaccessibility of η (to be able to apply 4.1), we have to incorporate more advanced Jensen-like coding techniques, due to the first author (cf. [6]), to vary the above forcing construction. However, whereas Jensen coding itself achieves a "coding into L," we have to code into K instead - otherwise we would end up with a $\mathbf{\Delta}^1_2$-well-ordering of the reals!

Let $(\kappa^i : i < \eta)$ enumerate the cardinals of $L[E]$ in the half-open interval $[\kappa_n^{++}, \eta)$. By combining the above approach with [6], inside $L[E][B^i_k]$ there is a forcing S^i_k adding a subset A^i_k of $[\kappa^{i+1}, \kappa^{i+2})$ such that the following holds true:

Claim 3. For all $\mathcal{N}$ with $J_{\kappa^{i+1}}[E] \lhd \mathcal{N} \lhd L[E]$ and $\mathcal{N} \models \Theta$ as well as $\mathcal{N}[A^i_k \cap (\kappa^{i+1})^{+\mathcal{N}}] \models \Theta$ we have that if $A^i_k \cap (\kappa^{i+1})^{+\mathcal{N}}$ is decoded inside $\mathcal{N}[A^i_k \cap (\kappa^{i+1})^{+\mathcal{N}}]$ - using a coding device as in [6] - then a cofinal branch thru $(T^i_k)^{\mathcal{N}}$ is obtained.

Proof Sketch. Code relative to $L[E]$ as one codes relative to L, using the "almost disjoint codes" provided by the natural wellordering of $L[E]$. Our "coding structures" will be initial segments of $L[E]$. Simultaneously with the coding we will be forcing the branches through the trees T^i_k. We require that our coding structure at an ordinal $\alpha < \eta^+$ be tall enough to contruct the restrictions of our branches to α, relative to E. These coding structures are cardinal-correct in $L[E]$.

We are not trying here to preserve large cardinals properties, but only to verify distributivity for the forcing. As a result, our only concern is that we have enough condensation to do so. However the only condensations that take place are within our coding structures, which are cardinal-correct initial segments of $L[E]$, using hulls which contain $\kappa_n^{+L[E]}+1$. Condensation of this form follows from 2.5.

$\square$ (Claim 3)

We shall also need that that S^i_k preserves cofinalities, and that in fact more is true. Let

$$S = \prod_{i<\eta, k<\omega} P^i_k \star \dot{S}^i_k,$$

so that S adds $(B^i_k : i < \eta, k < \omega)$, a sequence of branches thru the T^i_k's, together with codes $A^i_k \subset [\kappa^{i+1}, \kappa^{i+2})$ for $i < \eta$ and $k < \omega$. We shall need that S preserves cofinalities, and that analogues to Claims 1 and 2 in the previous section are still valid.

Next we want to add reals r^i by forcings R^i in such a way that r^i collapses κ^{i+1} to ω and such that r^i "codes" $(A^i_k: k < \omega)$ in much the same way as we had that a "codes" $(A_k: k < \omega)$ in the previous section. We let R^i be $Col(\omega, J_{\kappa^{i+1}}[E])\star$ the forcing R from the previous section, but with ω_1 replaced by κ^{i+2} and with g being canonically obtained from the the $Col(\omega, J_{\kappa^{i+1}}[E])$-generic (and naturally called g^i now). We shall denote

$$R = \prod_{i<\eta} R^i.$$

We denote by $(r^i: i < \eta)$ the sequence of reals obtained by forcing with R over $L[E][(A^i_k: i < \eta, k < \omega)]$. Our model witnessing 1.3 shall be $L[E][(r^i: i < \eta)]$.

Claim 4. In $L[E][(A^i_k: i < \eta, k < \omega)][(r^i: i < \eta)]$, for any $i < \eta$ there are $g^i \subset \omega$ and $(D^i_k: k < \omega)$ with:

(a) $(r^i)^{dec} = g^i$, and $\mathcal{M}^\star_{r^i} = J_{\kappa^{i+1}}[E]$, in fact g^i is $Col(\omega, J_{\kappa^{i+1}}[E])$-generic over $L[E][(A^i_k: i < \eta, k < \omega)]$, and

(b) if $(a_l: l < \kappa^{i+2})$ is the "least" sequence of pairwise a.d. subsets of ω in $L[E][g^i]$ then, setting $D_l = \{\alpha: r^i \cap a_{\alpha+\omega+l}$ is finite $\}$, we have that

$$\forall l(l \in r^i \Rightarrow D_{2l} = A^i_{2l} \wedge D_{2l+1} = \emptyset) \text{ and}$$

$$\forall l(l \notin r^i \Rightarrow D_{2l} = \emptyset \wedge D_{2l+1} = A^i_{2l+1}.$$

We now consider the model $L[E][\vec{r}]$, where we write $\vec{r} = (r^i: i < \eta)$. We again have the following:

Claim 5. $\forall i \ \forall k \ (k \in r^i \Rightarrow T^i_{2k+1}$ is Suslin in $L[E][\vec{r}])$.

Claim 6. $\forall i \ \forall k \ (k \notin r^i \Rightarrow T^i_{2k}$ is Suslin in $L[E][\vec{r}])$.

These two claims are verified in the same fashion as were Claims 6 and 7 of the previous section. In fact, the proof also shows that $\eta = \omega_1^{L[E][\vec{r}]}$, which we shall denote by ω_1 from now on.

Claim 7. For any $x \in \mathbb{R} \cap L[E][\vec{r}]$ there is some $\alpha < \omega_1$ with $x \in J_\alpha[E][\vec{r} \upharpoonright \alpha]$, and η is inaccessible in $L[E][x]$.

Proof. Let $x \in J_\rho[E][\vec{r}]$, where $\rho > \omega_1$ is a cardinal of $L[E]$. By 2.5 we may pick some

$$\pi \colon J_\alpha[E][\vec{r} \restriction \tau] \to J_\rho[E][\vec{r}]$$

with $\tau < \alpha < \omega_1$, $c.p.(\pi) = \tau$, and $\pi(\tau) = \omega_1$. But then $x \in J_\alpha[E][\vec{r} \restriction \alpha]$.

That η is still inaccessible in $J_\alpha[E][\vec{r} \restriction \alpha]$ (and hence in $L[E][x]$) follows from the fact that $\vec{r} \restriction \alpha$ is obtained by forcing with $\prod_{i<\alpha} R^i$ over $L[E][(B^i_k \colon i < \eta, k < \omega)]$.

□ (Claim 7)

Now Claim 7 together with 4.1 immediately buys us that in $L[E][\vec{r}]$, every $\mathbf{\Sigma}^1_{n+3}$-set of reals is Lebesgue measurable and has the property of Baire. Moreover, 2.1 tells us that in $L[E][\vec{r}]$, every $\mathbf{\Sigma}^1_{n+2}$-set of reals is universally Baire.

We are hence left with having to verify that $L[E][\vec{r}]$ has a Δ^1_{n+5}-well-ordering of its reals. The key for being able to do this is the following:

Claim 8. For any $i < \omega_1$, r^i is uniformly Π^1_{n+4} in any code for $J_{\kappa^i}[E]$.

Proof. We consider the following formula, abbreviated $\Phi(x, J_{\kappa^i}[E])$:

"$\mathcal{M}^\star_x = J_{\kappa^{i+1}}[E]$, and

IF $\mathcal{N}$ is such that (a) $\mathcal{M}^\star_x \lhd \mathcal{N} \lhd L[E]$, and
(b) $\mathcal{N} \models \Theta$, and $\mathcal{N}[x] \models \Theta$
THEN we have that
(a)' if $k \in x$ then there is a cofinal branch in $\mathcal{N}[x]$ thru $(T^i_{2k})^{\mathcal{N}}$, and
(b)' if $k \notin x$ then there is a cofinal branch in $\mathcal{N}[x]$ thru $(T^i_{2k+1})^{\mathcal{N}}$."

By 2.2, "$\mathcal{M}^\star_x = J_{\kappa^{i+1}}[E]$" can be written uniformly as a Π^1_{n+3}-formula in any code for $J_{\kappa^i}[E]$, and by 2.3, the second conjunct is certainly uniformly Π^1_{n+4} in any code for $J_{\kappa^i}[E]$.

Using Claim 3 above we can then verify that $\Phi(x, J_{\kappa^i}[E])$ holds iff $x = r^i$ in much the same way as we had verified Claim 9 in the last section, but this time by using Claim 3 above.

□ (Claim 8)

We finally obtain the following:

Claim 9 In $L[E][\vec{r}]$, there is a Δ^1_{n+5}-well-ordering of $\mathbb{R}$.

PROOF. Set $\mathcal{P} = J_{\omega_1}[E][\vec{r}]$. By Claim 7, $\mathbb{R} \cap L[E][\vec{r}] \subset \mathcal{P}$, so that we may well-order the reals by $<_\mathcal{P}$, the order of constructibility of $\mathcal{P}$.

Well, we now clearly have that for any $x, y \in \mathbb{R} \cap L[E][\vec{r}]$, $x <_\mathcal{P} y$ iff

$$\exists \mathcal{N}_0 \ \exists \mathcal{N} \ \exists (s^i : i < \mathcal{N} \cap OR) \ [\ \mathcal{N}_0 = J_{\kappa^0}[E] \ \wedge \mathcal{N}_0 \triangleleft \mathcal{N} \triangleleft J_{\omega_1}[E] \wedge$$

$$\Phi(s^0, \mathcal{N}_0) \wedge \forall i < \mathcal{N} \cap OR \ \Phi(s^{i+1}, \mathcal{M}^\star_{s^i}) \ ].$$

Here, $\Phi(-,-)$ is the formula from the proof of Claim 8.

An inspection shows that, using 2.4 and 2.3 together with Claim 8, the displayed formula can be rewritten in a Σ^1_{n+5}-way. Hence $<_\mathcal{P}$ is a Δ^1_{n+5}-well-ordering of $\mathbb{R} \cap L[E][\vec{r}]$.

$\square$ (Claim 9)

This finishes the proof of 1.3.

$\square$ (1.3)

5 Open problems.

We want to finish this paper by stating three key open problems.

(1) Let $n < \omega$. Starting only from an inaccessible, can you construct a model in which every $\mathbf{\Sigma}^1_{n+3}$-set of reals is Lebesgue measurable and has the property of Baire, yet there is a (lightface) projective (ideally, Δ^1_{n+4}) well-ordering of the reals?

(2) Do the conclusions of 1.2 and 1.3 imply the consistency of strong cardinals? (Cf. [3].)

(3) Is there a Δ^1_{n+4}-well-ordering of $\mathbb{R}$ in the model of 1.3 or a variant thereof?

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