Set Forcing

Suppose M is a countable transitive model of ZFC
Then M has many set-generic extensions (uncountably many)

Turn this around:
M is a *set-generic restriction* of V iff
V is a set-generic extension of M

Questions:
1. How many set-generic restrictions does a countable V have?
2. Can we "characterise" the set-generic restrictions of V?

Laver's Theorem

In fact, a countable V has only countably many set-generic restrictions:

Theorem

(Laver) Suppose that V is a set-generic extension of M. Then M is a definable inner model of V (with parameters).

Proof. Choose a V-regular κ so that P belongs to $H(\kappa)^M$, where V is P-generic over M. We need three facts:

1. M κ-covers V: Any subset X of M in V of size $< \kappa$ in V is a subset of such a set in M.

This is because if f maps some ordinal $\alpha < \kappa$ onto X then for each $i < \alpha$ there are $< \kappa$ possibilities for $f(i)$, given by the $< \kappa$ different forcing conditions.

Laver's Theorem

2. M κ-approximates V: If X is a subset of M in V all of whose size $< \kappa$ M-approximations (i.e., intersections with size $< \kappa$ elements of M) belong to M, then X also belongs to M.

This is because if $\dot{X}$ is forced not to be in M then we can choose for each condition a set in M whose membership in $\dot{X}$ is not decided by that condition; no condition can force the intersection of $\dot{X}$ with the resulting size $< \kappa$ set of elements of M to be in M.

Laver's Theorem

3. If N is an inner model which κ-covers and κ-approximates V such that M, N have the same $H(\kappa^+)$ then $M = N$.

By κ-approximation it's enough to show that any set X of ordinals of size $< \kappa$ in M also belongs to N (and vice-versa). Build a κ-chain $X = X_0 \subseteq X_1 \subseteq \cdots$ of sets of size $< \kappa$ such that $X_{2\alpha+1}$ belongs to M and $X_{2\alpha+2}$ belongs to N. If Y is the union of the X_α's then by κ-approximation, Y belongs to $M \cap N$. But as M, N have the same $H(\kappa^+)$ they also have the same subsets of the ordertype of Y and therefore the same subsets of Y. It follows that X belongs to N.

Finally: All of this holds with M, V replaced by $H(\lambda)^M, H(\lambda)$ for V-regular cardinals $\lambda > \kappa^+$. So $H(\lambda)^M$ is definable in V from λ, $H(\kappa^+)^M$ uniformly in λ, so M is V-definable.

Global Covering

Another easy consequence of set-genericity is the following.

Proposition

Suppose that V is a set-generic extension of M. Then M globally covers V: For some V-regular κ, if $f : \alpha \to M$ belongs to V then there is $g : \alpha \to M$ in M such that $f(i) \in g(i)$ and $g(i)$ has V-cardinality $< \kappa$ for all $i < \alpha$.

To see this define $g(i)$ to be the set of possible values of $f(i)$ given by the different forcing conditions. We can choose any κ so that the forcing is κ-cc.

Suprisingly, we now know enough to characterise set-generic restrictions.

Bukovsky's Theorem

Theorem

(Bukovsky) Suppose that M is a definable inner model which globally covers V. Then V is a set-generic extension of M.

I'll give a proof of this and discuss some refinements and open questions.

First suppose that $V = M[A]$ for some set of ordinals A; we'll get rid of this extra hypothesis later.

Fix a V-regular κ such that A is a subset of κ and M globally κ-covers V, i.e., if $f : \alpha \to M$ in V then there is $g : \alpha \to M$ in M so that $f(i) \in g(i)$ and $g(i)$ has V-cardinality $< \kappa$ for each $i < \alpha$.

Bukovsky's Theorem

The language $\mathcal{L}_\kappa^{QF}(M)$

The formulas of $\mathcal{L}_\kappa^{QF}(M)$ are defined inductively by:

1. *Basic formulas* $\alpha \in \dot{A}$, $\alpha \notin \dot{A}$ for $\alpha < \kappa$.
2. If $\Phi \in M$ is a size $< \kappa$ set of formulas then so are $\bigvee \Phi$ and $\bigwedge \Phi$.

Each formula can be regarded as an element of $H(\kappa)^M$. The set of formulas forms a κ-complete Boolean algebra in M, denoted by $\mathcal{B}_\kappa^M$.

$A \subseteq \kappa$ *satisfies* φ iff φ is true when $\dot{A}$ is replaced by A.

$T \vDash \varphi$ iff for all $A \subseteq \kappa$ (in a set-generic extension of M), if A satisfies all formulas in T then A also satisfies φ.

The above is expressible in M for T, φ in M and by Lévy absoluteness, $T \vDash \varphi$ in M iff $T \vDash \varphi$ in V.

Bukovsky's Theorem

Quotients of $\mathcal{B}^M_\kappa$: Suppose that T is a set of formulas in $\mathcal{B}^M_{(2^{<\kappa})^+}$. Then $\mathcal{I}_T$ is the ideal of formulas in $\mathcal{B}^M_\kappa$ which are inconsistent with T.

Now we prove the genericity of A over M.

Recall that M globally κ-covers V. Let f be a function in V from subsets of $\mathcal{B}^M_\kappa$ in M to $\mathcal{B}^M_\kappa$ such that:

If A satisfies some $\psi \in \Phi$ then A satisfies $f(\Phi) \in \Phi$.

Using a wellorder in M we can regard f as a function from some ordinal into M. Apply global κ-covering to get g in M so that $g(\Phi) \subseteq \Phi$ has size $< \kappa$ and $f(\Phi) \in g(\Phi)$ for each Φ.

Consider the following set of formulas T in $\mathcal{B}^M_{(2^{<\kappa})^+}$:

$T = \{(\bigvee \Phi \rightarrow \bigvee g(\Phi)) \mid \Phi \subseteq \mathcal{B}^M_\kappa, \Phi \in M\}$.

Let P be the forcing $(\mathcal{B}^M_\kappa \setminus \mathcal{I}_T)/\mathcal{I}_T$ the set of T-consistent formulas modulo T-provability.

Bukovsky's Theorem

Claim 1. $P = (\mathcal{B}^M_\kappa \setminus \mathcal{I}_T)/\mathcal{I}_T$ is κ-cc.

Proof. Suppose that Φ is a maximal antichain in P. We show that $g(\Phi) = \Phi$ (and therefore Φ has size $< \kappa$). It suffices to show that any $\varphi \in \Phi$ is T-consistent with some element of $g(\Phi)$. Choose any $B \subseteq \kappa$ which satisfies $T \cup \{\varphi\}$ (this is possible because φ is T-consistent). As T includes the formula $\bigvee \Phi \to \bigvee g(\Phi)$ it follows that B also satisfies $\bigvee g(\Phi)$ and therefore ψ for some $\psi \in g(\Phi)$. So φ is T-consistent with $\psi \in g(\Phi)$. $\square$

Claim 2. Let $G(A)$ be $\{[\varphi]_{\mathcal{I}_T} \mid \varphi$ belongs to $\mathcal{B}^M_\kappa$ and A satisfies $\varphi\}$. Then $G(A)$ is P-generic over M.

Proof. Suppose that Φ consists of representatives of a maximal antichain X of equivalence classes in P. Then $T \models \bigvee \Phi$, else the negation of $\bigvee \Phi$ represents an equivalence class violating the maximality of X. As A satisfies the theory T it follows that A satisfies some element of Φ and therefore $G(A)$ meets X. $\square$

Bukovsky's Theorem

It now follows that $M[A]$ is a P-generic extension of M, as $M[A] = M[G(A)]$.

This proves Bukovsky's theorem assuming that $V = M[A]$ for some set of ordinals A.

But the same proof shows that $M[A]$ is a κ-cc generic extension of M for any set of ordinals $A \in V$.
Choose A so that $M[A]$ contains all subsets of $2^{<\kappa}$ in V. Then $M[A]$ must equal all of V:
Otherwise for some set B of ordinals in V, $M[A, B]$ is a nontrivial κ-cc generic extension of $M[A]$ and therefore adds a new subset of $2^{<\kappa}$ to $M[A]$.

Bukovsky's Theorem: Refinements

The above proof shows that for M a definable inner model of V:

V is a κ-cc forcing extension of M iff
M globally κ-covers V

Is there a similar characterisation with "κ-cc" replaced by "size at most κ"?

Bukovsky's Theorem: Refinements

M *κ-decomposes* V iff every subset of M in V is the union of at most κ-many subsets, each of which belongs to M.

Proposition

V is a size at most κ forcing extension of M iff M globally κ^+-covers and κ-decomposes V.

Proof. For the easy direction, suppose that $V = M[G]$ where G is P-generic and P has size at most κ. As P is κ^+-cc it follows that M globally κ^+-covers V. To show that M κ-decomposes V, suppose that $X \in V$ is a subset of M and choose $Y \in M$ that covers X. Let $\dot{X}$ be a name for X and for each $p \in G$ let X_p consist of those $x \in M$ such that p forces $x \in \dot{X}$. Then the X_p's give the desired κ-decomposition of X.

Bukovsky's Theorem: Refinements

Conversely, suppose that M globally κ^+-covers and κ-decomposes V. By Bukovsky's Theorem, V is a P-generic extension of M for some P which is κ^+-cc. We want to argue that P is equivalent to a forcing of size at most κ. We may assume that P is in fact a complete κ^+-cc Boolean algebra which we write as B.

Write V as $M[G]$ where G is B-generic over M. Take a B-name for a κ-decomposition $\dot{G} = \bigcup_{i<\kappa} \dot{G}_i$ of $\dot{G}$, where each $\dot{G}_i$ is forced to belong to M. For each $i < \kappa$ let X_i be a maximal antichain of conditions in B which decide a specific value in M for $\dot{G}_i$. For each p in X_i let $p(\dot{G}_i)$ denote the value of $\dot{G}_i$ forced by p and $b(p)$ the meet of the conditions in $p(\dot{G}_i)$; $b(p)$ is a nonzero Boolean value because if G_p is generic below p then G_p must contain a condition below each element of $p(\dot{G}_i)$. Let D be the set of $b(p)$ for p in the union of the X_i's.

Bukovsky's Theorem: Refinements

Claim. D is dense in B.

If q belongs to P then some r below q forces that q belongs to $\dot{G}_i$ for some i; we can assume that r extends some element p of X_i. But then as p decides a value for $\dot{G}_i$, it also forces that q belongs to $\dot{G}_i$ and therefore q is extended by $b(p) \in D$. $\square$

We have characterised κ-cc generic extensions and size at most κ generic extensions in terms of covering and decomposition properties. As a result, these properties are Π_2 properties of V with a predicate for M.

Question. Is the property "V is a set-forcing extension of M" a strictly Σ_3 property of V with a predicate for M?

Bukovsky's Theorem: Refinements

Class Forcing

I don't know a good version of Laver, Bukovsky for class forcing. Below is a special case.

Morse-Kelley Class Theory MK: Can form new classes by quantifying over classes.

Models of MK (with global choice) correspond to models of:

1. ZFC^- (without Power)
2. There is an inaccessible cardinal κ
3. Every set has cardinality at most κ

Call this theory SetMK.

Bukovsky's Theorem: Refinements

Theorem

Suppose that $M \subseteq V$ are models of SetMK, M is definable in V and κ is the largest cardinal of V. Then every element of V is in a κ-cc set-generic extension of M iff:
$(*)$ For any V-definable function $f : M \to \kappa$ there is an M-definable $g : M \to \kappa$ which dominates f.

In terms of models of MK (with global choice) this says:

Bukovsky's Theorem: Refinements

Theorem

Suppose that $(M, \mathcal{C}^M) \subseteq (V, \mathcal{C}^V)$ *are models of MK with global choice and* $\mathcal{C}^M$ *is definable in* $(V, \mathcal{C}^V)$ *(by a formula which quantifies over classes). Then each class in* $\mathcal{C}^V$ *belongs to a class-generic extension of* $(M, \mathcal{C}^M)$ *via a class forcing whose antichains are sets iff:*
$(*)$ *For any* $(V, \mathcal{C}^V)$*-definable function* f *from* $\mathcal{C}^M$ *to* M *there is an* $(M, \mathcal{C}^M)$*-definable function* g *from* $\mathcal{C}^M$ *to* M *such that* $f(x) \in g(x)$ *for each* $x \in \mathcal{C}^M$*.*

If one goes beyond class theory to hyperclass theory (hyperclasses of classes) then the situation simplifies greatly. In the other direction, working with a weak class theory like Gödel-Bernays looks very difficult.