BACHELOR THESIS

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Ehrenfeucht-Fraïssé Games

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Abstract

The goal of this thesis is to underline the connection between Ehrenfeucht-Fraïssé games and non-definable properties, in particular the connectivity of finite graphs. In the first chapter, I will give a brief introduction about why games are used in logic and I will give the mathematical definition of a game. In the second chapter, I will state what it means to define a property and I will give a proof of the inexpressibility of connectivity over arbitrary graphs. This proof fails over finite graphs, which motivates the definition of Ehrenfeucht-Fraïssé games. I will define the concepts of quantifier rank and rank-k types in order to state and prove the Ehrenfeucht-Fraïssé Theorem. The most interesting result of this thesis is a corollary of that theorem that shows the connection between games and definable properties. In the third chapter, I will prove that the property of a linear order to have even cardinality is not definable. I will use this and the corollary in Chapter 2 to prove that connectivity over finite graphs is not definable.

A note for the reader

In this thesis, I will assume that the reader is comfortable with the following concepts and theorems:

- language, formula, sentence, structure and elementary equivalence;
- theory and model;
- a theory is consistent if and only if it has a model;
- basic notions of graph theory, like graph, node, edge, path.

Furthermore, if not otherwise stated, $\mathcal{L}$ is a finite language.

Contents

1 Logic and Games

1.1 Why games?

The main source for this section is [1].

One could argue that logic and games have always been connected: already Aristotle's writings about syllogism are closely interconnected with his study of debate – and what is a debate, if not some sort of game? Mathematical game theory, however, was only founded at the beginning of the last century. Yet, most mathematicians in the first half of the twentieth century would not have thought of using games in the field of logic. It was only around 1960, when the focus of logical research switched from studying foundations to searching for techniques, that games really came into play. As a matter of fact, it became clear to logicians that they were able to make certain ideas more intuitive if they connected them to a goal: there are many examples of logical games that are centred around (the existence of) a winning strategy for one of the players. More often than not, this strategy - or its existence - is equivalent to something of logical significance, which one could have probably defined without games too. Nonetheless, a definition through a game is more effective, because it provides a concrete motivation: a player wants to win.

1.2 Games in Logic

The main source for this section is [5].

In logic, there are fundamentally three kinds of games: the Semantic Game, the Model Existence Game and the Ehrenfeucht-Fraïssé Game. In the Semantic Game, we are given a sentence and a model; we then want to question the truth of this sentence in this model. In the Model Existence Game, we are only given a sentence; we question the existence of a model for this sentence. Finally, in the Ehrenfeucht-Fraïssé Game, we are given two models; we question whether there exists a sentence that is true in one but false in the other. These games are deeply interconnected: one can "translate" strategies from one game to another. The affinity of these games goes by the name *Strategic Balance of Games in Logic* (see [6]).

But what *is* a game? Let M be any set, $n \in \mathbb{N}$ a natural number and $W \subseteq M^{2n}$. Fix two players, Alice (player a) and Bob (player b). We are going to give the game the name $\mathcal{G}_n(M, W)$.

Definition 1.1. A sequence $\bar{a} = (a_1, \dots, a_n)$, where $a_i \in M$, is called a *play of one of the players*. Furthermore, a sequence $(\bar{a}; \bar{b}) = (a_1, b_1, \dots, a_n, b_n)$, where $a_i, b_i \in M$ is called a *play of* $\mathcal{G}_n(M, W)$.

Definition 1.2. We define a play $(\bar{a}; \bar{b})$ to be a *win for* Bob, if $(a_1, b_1, \dots, a_n, b_n) \in W$, otherwise it's a win for Alice.

Definition 1.3. Next, we call a sequence $f = (f_1, \dots, f_n)$ of functions $f_i : M^i \to M$ a *strategy* of Alice in the game $\mathcal{G}_n(M, W)$.
Likewise, we call a sequence $g = (g_1, \dots, g_n)$ of functions $g_i : M^{i+1} \to M$ a strategy of Bob.
We say that Alice has *used the strategy f in the play* $(\bar{a}; \bar{b})$, if, for all $i = 1, \dots, n$: $a_1 = f_1$ and $a_i = f_i(b_1, \dots, b_n)$.
Similarly, we say that Bob has *used the strategy g in the play* $(\bar{a}; \bar{b})$, if, for all $i = 1, \dots, n$: $b_1 = g_1$ and $b_i = g_i(a_1, \dots, a_i)$.
It is worth noticing, though expected, that strategies only depend on the opponent's moves up until that point.

Definition 1.4. A strategy is called a *winning strategy* for player a, if a wins every game where she uses that strategy, no matter how player b plays. If one player has a winning strategy on a game, we call that a *determined game*.

Example 1.1. Consider the following game: given a set of integers M, Alice chooses an integer $a \in M$, then Bob chooses an integer $b \in M$. Bob wins if $a + b \in M$. We are going to express this game in the form $\mathcal{G}_2(M, W)$. Here, $M \subseteq \mathbb{Z}$ and $W = \{(a, b) \in M^2 : a + b \in M\}$. This is a determined game, because whatever M is, one player has a winning strategy: as a matter of fact, if Bob does not have a winning strategy, that means that for all $b \in M$ there is an $a \in M$ such that $a + b \notin M$. Then Alice has a winning strategy: she plays a. But when does Bob have a winning strategy? Let $\mathcal{L} = \{\mathbf{W}\}$ be a language, where $\mathbf{W}$ is a binary relation symbol. Consider then an $\mathcal{L}$-structure

$\mathcal{M}$ with universe $M \in \mathbb{Z}$ and $\mathbf{W}^{\mathcal{M}} = W$. Then Bob has a winning strategy if and only if

$$\mathcal{M} \models \forall v \exists u \mathbf{W}(v, u).$$

2 Ehrenfeucht-Fraïssé Games

The main source for this chapter is, unless otherwise stated, [2].

As mentioned before, Ehrenfeucht-Fraïssé games are, essentially, games where one is given two structures and tries to find sentences that are true in one and false in the other one. Nonetheless, they have many interesting applications. One of them relates to the impossibility of expressing some properties.

2.1 Defining properties

In this section, the source for Definition 2.1 is [3] and the source for Remark 2.1 is [5].

What does "expressing some properties" even mean? Let's start with a basic definition.

Definition 2.1. Let $\mathcal{L}$ be a language and $\mathcal{S} = (S, \mathcal{C}, \mathcal{F}, \mathcal{R})$ an $\mathcal{L}$-structure. A set $A \subseteq S^n$ is called *definable* if there exists an $\mathcal{L}$-formula $\varphi = \varphi(v_1, \ldots, v_n)$, such that

$$A = \{(a_1, \ldots, a_n) : \mathcal{S} \models \varphi[a_1, \ldots, a_n]\}.$$

Remark 2.1. In the above definition, $\mathcal{S} \models \varphi[a_1, \ldots, a_n]$ means that the formula φ is true in $\mathcal{S}$ for the assignment $(a_1, \ldots, a_n)$ of the free variables $(v_1, \ldots, v_n)$. But what does it mean for a formula to be *true*? The truth of a formula can be interpreted as the existence of a winning strategy for Bob in a type of Semantic Game, the Evaluation Game. Let us have a look at this game for quantifier-free formulas. We have two players, Alice and Bob. Bob wants to show that φ is true in $\mathcal{S}$ for the assignment $(a_1, \ldots, a_n)$ and Alice wants to show the opposite. At the start, Bob holds the pair $(\varphi, \bar{a})$, where $\bar{a} = (a_1, \ldots, a_n)$ is an assignment of the free variables of φ. During the first round of the game, the two players exchange pairs of formulas and assignments follows:

(i). If φ is an atomic formula and it is satisfied by $\bar{a}$ in $\mathcal{S}$, then Bob wins - otherwise Alice wins;

(ii). if $\varphi \equiv \neg\psi$, then Bob gives $(\psi, \bar{a})$ to Alice;

(iii). if $\varphi \equiv \psi_1 \wedge \psi_2$, then Bob switches to hold $(\psi_1, \bar{a})$ or $(\psi_2, \bar{a})$ and Alice chooses which one;

(iv). if $\varphi \equiv \psi_1 \vee \psi_2$, then Bob switches to hold $(\psi_1, \bar{a})$ or $(\psi_2, \bar{a})$ and he chooses which one.

Now either Alice or Bob hold a pair. If Bob holds it, the next round is going to be like the first one. If Alice holds it, the next round is going to look like this:

(i). If φ is an atomic formula and it is not satisfied by $\bar{a}$ in $\mathcal{S}$, then Alice wins - otherwise Bob wins;

(ii). if $\varphi \equiv \neg\psi$, then Alice gives $(\psi, \bar{a})$ to Bob;

(iii). if $\varphi \equiv \psi_1 \wedge \psi_2$, then Alice switches to hold $(\psi_1, \bar{a})$ or $(\psi_2, \bar{a})$ and Bob chooses which one;

(iv). if $\varphi \equiv \psi_1 \vee \psi_2$, then Alice switches to hold $(\psi_1, \bar{a})$ or $(\psi_2, \bar{a})$ and she chooses which one.

Note that $\mathcal{S} \models \varphi[a_1, \ldots, a_n]$ if and only if Bob has a winning strategy in the above game. As a matter of fact, assuming that $\mathcal{S} \models \varphi[a_1, \ldots, a_n]$, Bob can always play in a way that, by the end, he will hold the pair $(\varphi, \bar{a})$. Conversely, by induction, if Bob has a winning strategy, then $\mathcal{S} \models \varphi[a_1, \ldots, a_n]$ must hold.

Definition 2.2. Let $\mathcal{L}$ be a language and $\mathrm{STR}(\mathcal{L})$ the class of all $\mathcal{L}$-structures. A *property* $\mathfrak{P}$ is a mapping

$$\mathfrak{P} : \mathrm{STR}(\mathcal{L}) \to \{\mathit{true}, \mathit{false}\}$$

from the class of all $\mathcal{L}$-structures to the set $\{\mathit{true}, \mathit{false}\}$. Since the codomain of $\mathfrak{P}$ only has two elements, a property can be identified with a collection of structures like so:

$$\mathbf{P} = \{\mathcal{S} \in \mathrm{STR}(\mathcal{L}) \; : \; \mathfrak{P}(\mathcal{S}) = \mathit{true}\}.$$

If $\mathcal{S} \in \mathbf{P}$, we say that $\mathcal{S}$ *has the property* $\mathbf{P}$.
A property is called *definable* if there exist a $\mathcal{L}$-sentence φ such that $\mathcal{S} \in \mathbf{P}$ if and only if $\mathcal{S} \models \varphi$.

2.2 One inexpressibility proof

In this section, the source for Lemma 2.1 is [3].

In this section, we are going to have a look at a property that is not definable. First, we will start with a useful lemma.

Lemma 2.1. *A theory T has a model if and only if every finite subset of T has a model.*

Proof. [$\Rightarrow$] Assume there exists a finite subset $T' \subseteq T$ that has no model. Then there is a sentence $\varphi \in T$ such that $\mathcal{S} \not\models \varphi$ for all $\mathcal{L}$-structures $\mathcal{S}$. But since $T' \subseteq T$, then $\varphi \in T$ and therefore T has no model either. Contradiction!
[$\Leftarrow$] Assume that every finite subset of T has a model, but T itself has no model. We know that a theory has a model if and only if it is consistent, therefore T is not consistent. That means that there exist sentences $\varphi_1, \ldots, \varphi_n \in T$ such that $\{\varphi_1, \ldots, \varphi_n\}$, a finite subset of T, is not consistent. But then $\{\varphi_1, \ldots, \varphi_n\}$ has no model. Contradiction! $\square$

It is possible to show that some properties cannot be expressed without Ehrenfeucht-Fraïssé games as well. Here is an example of how this can be done using fairly easy tools. It will be about connectivity of graphs. As a reminder: a graph is called *connected* if any two nodes are connected by a path.

Theorem 2.2. *The property of an* arbitrary *graph to be connected is not definable.*

Proof. Let $\mathcal{L} = \{E, a, b\}$ be a language, where E is a relational symbol and a, b are constant symbols. Assume that it is possible to define connectivity by a sentence φ over $\mathcal{L}$. For each natural number n, define the sentence ψ_n as follows:

$$\neg(\exists v_1, \ldots, \exists v_n (E(a, v_1) \wedge E(v_1, v_2) \wedge \ldots \wedge E(v_n, b))).$$

This sentence says that there is no path from a to b of length $n + 1$. Furthermore, define the theory

$$\tau = \{\varphi\} \cup \{\neg(a = b)\} \cup \{\neg E(a, b)\} \cup \{\psi_n \forall n \in \mathbb{N} : n > 0\}.$$

Our claim is that the theory τ is consistent. To prove that, we are going to show that every finite subset τ' of τ is consistent. That is equivalent to proving that every finite subset of τ has a model. So, let $\tau' \subseteq \tau$ be finite. Then there exists an $m \in \mathbb{N}$ such that for all ψ_n in τ, $n < m$. Then τ' has a model: a connected graph, in which the shortest path from a to b has length $m + 1$. This shows that τ has a model $\mathcal{G}$ as well. Then $\mathcal{G}$ is connected, since $\mathcal{G} \models \varphi$. However, since $\mathcal{G} \models \psi_n$ for all $n \in \mathbb{N}$, we know that, for all natural numbers n, there is no path from a to b of length n. Contradiction! $\square$

This proof is nice, but it only tells us that connectivity is not definable for arbitrary graphs. So there is still the possibility that connectivity is definable for *finite* graphs: there might be a sentence that successfully checks whether a finite graph is connected or not.

We could try modifying the proof above. There, we used the argument that a theory τ has a model if and only if every finite subset of τ has a model. Is that true for finite models as well? Does a theory τ have a *finite* model if every finite subset of τ has a *finite* model? Unfortunately, this is not true.

Theorem 2.3. *There exists a theory τ that has no finite models, even though every finite subset of τ has a finite model.*

Proof. Define the sentence ψ_n as follows:

$$\exists v_1, \ldots, \exists v_n \bigwedge_{\neg(k=l)} \neg(v_k = v_l).$$

The sentence ψ_n states that there is a minimum of n distinct elements in the universe. Let $\tau := \{\psi_n : n \in \mathbb{N}\}$, it is clear that the theory τ does not have a finite model. However, take any finite subset $\tau' = \{\psi_{n_k} : k = 1, \ldots, l\}$ of τ. Any set A with $|A| > n_k \; \forall k$ is a finite model for τ'. $\square$

It looks like we need to find a new, more powerful tool for proving inexpressibility over finite models.

2.3 Ehrenfeucht-Fraïssé Games

All the examples and pictures in this section are from [4].

Before we start, we are going to have a look at some interesting objects that will be useful later.

2.3.1 Quantifier Rank and Rank-k Types

Definition 2.3. Given a formula ψ, its *quantifier rank* $\mathbf{qr}(\psi)$ is defined as follows:

- if ψ is atomic, then $\mathbf{qr}(\psi) = 0$;
- $\mathbf{qr}(\neg\psi) = \mathbf{qr}(\psi)$
- $\mathbf{qr}(\psi_1 \wedge \psi_2) = \mathbf{qr}(\psi_1 \vee \psi_2) = \max(\mathbf{qr}(\psi_1), \mathbf{qr}(\psi_2))$;
- $\mathbf{qr}(\forall x\psi) = \mathbf{qr}(\exists x\psi) = \mathbf{qr}(\psi) + 1$.

To denote all formulas of quantifier rank up to k, we write FO[k].

Remark 2.2. It is worth noticing that the quantifier rank of a formula is, in general, not equal to the number of quantifiers that appear in the formula. Consider the following example.
Define

$$\varphi \equiv \forall u(\forall u(v < u) \vee \exists u(u < v)).$$

The formula φ contains three quantifiers, however $\mathbf{qr}(\varphi) = 2$:

$$\begin{aligned}\mathbf{qr}(\varphi) &= 1 + \mathbf{qr}(\forall u(v < u) \vee \exists u(u < v)) \\ &= 1 + \max(\mathbf{qr}(\forall u(v < u)), \mathbf{qr}(\exists u(u < v))) \\ &= 1 + \max(1 + 0, 1 + 0) \\ &= 2.\end{aligned}$$

Definition 2.4. Let $\mathcal{L}$ be a relational language, $\mathcal{S}$ a $\mathcal{L}$-structure and $\bar{a} = (a_1, \ldots, a_n)$ a n-tuple over S. The *rank-k n-type* of $\bar{a}$ over $\mathcal{S}$ is defined like so:

$$\mathbf{tp}_k(\mathcal{S}, \bar{a}) = \{\psi \in \mathrm{FO}[k] : \mathcal{S} \models \psi[\bar{a}]\}.$$

Remark 2.3. If $n = 0$, we speak of the rank-k type of $\mathcal{S}$, $\mathbf{tp}_k(\mathcal{S})$. This is the set of sentences of sentences of quantifier rank up to k that are true in $\mathcal{S}$.

Theorem 2.4. *Let T be a rank-k n-type. Then there exists a* FO$[k]$ *formula α_T such that, for every structure $\mathcal{S}$ and $\bar{a} \in S^n$, we have*

$$\mathcal{S} \models \alpha_T[\bar{a}] \Leftrightarrow \mathbf{tp}_k(\mathcal{S}, \bar{a}) = T.$$

Proof. **Step 1.** We are going to prove by induction that rank-k types are finite objects. It is enough to show that, up to logical equivalence, for every k, FO$[k]$ only contains finitely many formulas with n free variables.

- $k = 0$: FO$[0]$ simply contains all Boolean combinations of atomic formulas. Since there is only a finite amount of atomic formulas and thus a finite amount of Boolean combinations of these, the conclusion follows;

· $k-1 \to k$: assume that there are only finitely many formulas with n free variables in $\mathrm{FO}[k-1]$. Note that each formula with n free variables in $\mathrm{FO}[k]$ is a Boolean combination of $\exists a_n \varphi(a_1, \ldots, a_{n-1}, a_n)$, where $\varphi \in \mathrm{FO}[k-1]$. Hence, we can conclude that there are only finitely many formulas with n free variables in $\mathrm{FO}[k]$.

Step 2. Let $T \subseteq \mathrm{FO}[k]$ be a rank-k n-type. Since $\mathrm{FO}[k]$ is finite, we have:

$$T \subseteq \mathrm{FO}[k] = \{\psi_1, \ldots, \psi_N\},$$

where $\psi_1, \ldots, \psi_N$ are all the nonequivalent formulas with free variables $\bar{a} = (a_1, \ldots, a_n)$ in $\mathrm{FO}[k]$. Then there is one unique set $A \subseteq \{1, \ldots, N\}$ that specifies which ones of the ψ_j's belong to T. This way, A uniquely determines T. Define the formula

$$\alpha_T(\bar{a}) \equiv \bigwedge_{j \in A} \psi_j \wedge \bigwedge_{i \notin A} \neg \psi_i. \tag{2.1}$$

It is easy to see that, for all $\mathcal{L}$-structures $\mathcal{S}$ and for all $j \in A, i \notin A$, we have:

$$\mathcal{S} \models \alpha_T[\bar{a}] \Leftrightarrow \mathcal{S} \models \psi_j[\bar{a}] \wedge \mathcal{S} \not\models \psi_i[\bar{a}].$$

This is equivalent to the statement we wanted to prove. Furthermore, note that we did not introduce any new quantifiers in $\alpha_T(\bar{x})$, hence it is a $\mathrm{FO}[k]$ formula. □

Remark 2.4. We say that α_T *defines* T.

2.3.2 Graphs in logical formalism

All the examples in the following sections are going to be finite graphs. What is a finite graph in logic?

Definition 2.5. A finite graph is a finite structure in the language $\mathcal{L} = \{E\}$, where E is a binary relation symbol.

2.3.3 Definition of the Ehrenfeucht-Fraïssé game

Let us now have a look at the fundamental idea of the Ehrenfeucht-Fraïssé game. There are two players, usually called the *duplicator* (female player) and the *spoiler* (male player), and two structures, $\mathcal{S}_1$ and $\mathcal{S}_2$. The goal of the duplicator is to show that $\mathcal{S}_1$ and $\mathcal{S}_2$ are the same, whereas the goal of the spoiler is to show that they are different. This can be done in a number of rounds, each round consisting of these three steps:

(i). The spoiler chooses one structure, $\mathcal{S}_1$ or $\mathcal{S}_2$;

(ii). The spoiler chooses one element in that structure, $s_1 \in \mathcal{S}_1$ or $s_2 \in \mathcal{S}_2$;

(iii). The duplicator chooses one element in the other structure.

Figure 1: Graphs in Example 2.1

Example 2.1. Assume the two structures are graphs, like in Figure 1. The spoiler starts the game by choosing one graph and then one node in that graph. So he can choose either a_1, a_2 or b_1. If he chooses a_1 or a_2, then the only option for the duplicator is to choose b_1. If he chooses b_1, then the duplicator must choose either a_1 or a_2. This ends the first round. We can represent this game as a tree, like in Figure 2.

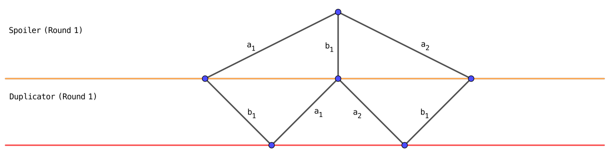

Figure 2: Tree

But what does it mean for two structures to be the *same*? In order to define the winning condition more precisely, we need the following definition.

Definition 2.6. Let $\mathcal{S}_1$, $\mathcal{S}_2$ be $\mathcal{L}$-structures, where $\mathcal{L}$ is a relational language. Furthermore, let $\bar{S_1} = (s_1^1, \ldots, s_n^1) \subseteq \mathcal{S}_1$ and $\bar{S_2} = (s_1^2, \ldots, s_n^2) \subseteq \mathcal{S}_2$. If the three following conditions hold, then $(\bar{S_1}, \bar{S_2})$ defines a *partial embedding*:

(i). for each i,$j \leq n$:

$$s_i^1 = s_j^1 \Leftrightarrow s_i^2 = s_j^2;$$

(ii). for each constant symbol c from $\mathcal{L}$ and for each $i \leq n$:

$$s_i^1 = c^{\mathcal{S}_1} \Leftrightarrow s_i^2 = c^{\mathcal{S}_2};$$

(iii). for each k-ary relation symbol R from $\mathcal{L}$ and for each k-tuple $(i_1, \ldots, i_k)$ of (not necessarily distinct) numbers from $\{1, \ldots, n\}$:

$$R^{\mathcal{S}_1}(s_{i_1}^1, \ldots, s_{i_k}^1) \Leftrightarrow R^{\mathcal{S}_2}(s_{i_1}^2, \ldots, s_{i_k}^2).$$

Remark 2.5. Define $\mathcal{S'}_1$ and $\mathcal{S'}_2$ to be the substructures of $\mathcal{S}_1$ and $\mathcal{S}_2$ generated by $\{s_1^1, \ldots, s_n^1\}$ and $\{s_1^2, \ldots, s_n^2\}$, respectively. Note that, if $\mathcal{L}$ has no constant symbols, like in the case of graphs, the above definition of a partial embedding just says that

$$f : \mathcal{S'}_1 \to \mathcal{S'}_2,\ s_i^1 \mapsto s_i^2,\ i \leq n,$$

is an isomorphism.

So the duplicator wins if she plays in a way that her moves define a partial embedding between the structures. Otherwise, the spoiler wins. Intuitively, the duplicator wins as long as the spoiler hasn't won - as long as she can maintain a partial embedding between the structures.

Definition 2.7. The duplicator has an *n-round winning strategy in the Ehrenfeucht-Fraïssé game on $\mathcal{S}_1$ and $\mathcal{S}_2$*, if she has a strategy that can always guarantee her a winning position after n rounds, independently from the moves of the spoiler, and we write

$$\mathcal{S}_1 \equiv_n \mathcal{S}_2.$$

(see [2])

Example 2.2. With this new notion, we can finish the game we started in Example 2.1. The second round looks like the first round, and now we can have a look at who the winners are. Let's observe the tree in Figure 3. One

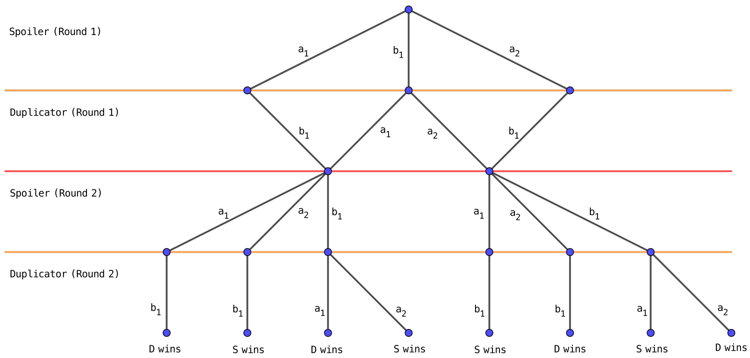

Figure 3: Tree

can notice that the spoiler has a winning strategy in the 2-rounds Ehrenfeucht-Fraïssé Game played on these two graphs: he chooses a_1 in the first round, forcing the duplicator to choose b_1; then he chooses a_2 in the second round, forcing the duplicator to choose b_1 again. Using the notation introduced in Definition 2.4, we can say that $(\bar{a}, \bar{b})$, where $\bar{a} = (a_1, a_2)$ and $\bar{b} = (b_1, b_1)$ is not a partial embedding, because condition (i) does not hold.
Since the spoiler has a winning strategy, the duplicator cannot have one.

Example 2.3. Let us have a look at another game where the duplicator has a winning strategy in the 2-rounds Ehrenfeucht-Fraïssé Game. Consider two graphs like in Figure 4. In the first round, the spoiler either chooses a node x from A or from B; if he chooses from A, the duplicator will pick b_1, if he chooses from B, the duplicator will pick a_1. In the second round, no matter which node the spoiler chooses, the duplicator will always be able to mirror that choice and define a partial embedding.

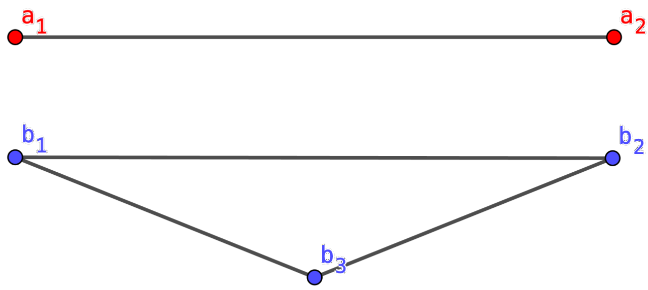

Figure 4: Graphs in Example 2.3

- Assume the duplicator had chosen b_1 in the first round. If, in the second round, the spoiler chooses a node from A, then that node is either x (and the duplicator will choose b_1 again) or it is the other one, which is connected to x (and the duplicator will choose either b_2 or b_3). If, in the second round, the spoiler chooses a node from B, then that node is either b_1 (and the duplicator will choose x) or another one, which is connected to b_1 (and the duplicator will choose the node in A that is connected to x).

- Assume the duplicator had chosen a_1 in the first round. If, in the second round, the spoiler chooses a node from B, then that node is either x (and the duplicator will choose a_1 again) or it is another one, which is connected to x (and the duplicator will choose a_2). If, in the second round, the spoiler chooses a node from A, then that node is either a_1 (and the duplicator will choose x) or a_2, which is connected to a_1 (and the duplicator will choose any node in B since they're all connected to x).

2.3.4 The Ehrenfeucht-Fraïssé Theorem

Definition 2.8. Let $\mathcal{L}$ be a language, $\mathcal{S}$ a $\mathcal{L}$-structure and $\bar{a} = (a_1, \ldots, a_n)$. Then $(\mathcal{S}, \bar{a})$ is a $(\mathcal{L} \cup \{c_1, \ldots, c_n\})$-structure, where $c_1, \ldots, c_n$ are new constant symbols and

$$c_i^{(\mathcal{S},\bar{a})} = a_i \text{ for each } i \in \{1, \ldots, n\}.$$

We now have all the ingredients we need to formulate and prove the main theorem of this section. As we will see later, this theorem is very useful when it comes to proving inexpressibility results.

Theorem 2.5. *Let $\mathcal{S}_1$, $\mathcal{S}_2$ be $\mathcal{L}$-structures in a relational language and $k \in \mathbb{N}$. Then the following are equivalent:*

(i). for all FO-formulas φ of quantifier rank up to k, $\mathcal{S}_1 \models \varphi \Leftrightarrow \mathcal{S}_2 \models \varphi$;

(ii). $\mathcal{S}_1 \equiv_k \mathcal{S}_2$.

Proof. **Step 1.** We will analyse the relation $\equiv_0$. How can the duplicator already have a winning strategy before the game even starts? By definition, that is the case if and only if $(\emptyset, \emptyset)$ defines a partial embedding between $\mathcal{S}_1$ and $\mathcal{S}_2$. Again, by definition, that is the case if and only if:

· for each couple of constant symbols c_i, c_j from $\mathcal{L}$ we have that:

$$c_i^{\mathcal{S}_1} = c_j^{\mathcal{S}_1} \Leftrightarrow c_i^{\mathcal{S}_2} = c_j^{\mathcal{S}_2}$$

· for each k-ary relation symbol R from $\mathcal{L}$ and for each k-tuple $(i_1, \dots, i_k)$ of (not necessarily distinct) numbers from $\{1, \dots, n\}$:

$$R^{\mathcal{S}_1}(c_{i_1}^{\mathcal{S}_1}, \dots, c_{i_k}^{\mathcal{S}_1}) \Leftrightarrow R^{\mathcal{S}_2}(c_{i_1}^{\mathcal{S}_2}, \dots, c_{i_k}^{\mathcal{S}_2}).$$

In conclusion, we found that $\mathcal{S}_1 \equiv_0 \mathcal{S}_2$ if and only if the two structures satisfy the same atomic sentences.

Step 2. For each $k \in \mathbb{N}$, we will define inductively the relation $\simeq_k$. Let $\mathcal{S}_1$, $\mathcal{S}_2$ be $\mathcal{L}$-structures. Then we define:

· $\mathcal{S}_1 \simeq_0 \mathcal{S}_2$ if and only if $\mathcal{S}_1 \equiv_0 \mathcal{S}_2$;

· $\mathcal{S}_1 \simeq_k \mathcal{S}_2$ if and only if the following conditions hold:

- **forth:** for every $s_1 \in \mathcal{S}_1$, there exists a $s_2 \in \mathcal{S}_2$ such that $(\mathcal{S}_1, s_1) \simeq_{k-1} (\mathcal{S}_2, s_2)$;

- **back:** for every $s_2 \in \mathcal{S}_2$, there exists a $s_1 \in \mathcal{S}_1$ such that $(\mathcal{S}_1, s_1) \simeq_{k-1} (\mathcal{S}_2, s_2)$.

Step 3. We will prove the following extension of the Ehrenfeucht-Fraïssé Theorem:

Theorem 2.6. *Let $\mathcal{S}_1$, $\mathcal{S}_2$ be $\mathcal{L}$-structures in a relational language and $k \in \mathbb{N}$. Then the following are equivalent:*

(i). for all FO-formulas φ of quantifier rank up to k, $\mathcal{S}_1 \models \varphi \Leftrightarrow \mathcal{S}_2 \models \varphi$;

(ii). $\mathcal{S}_1 \equiv_k \mathcal{S}_2$;

(iii). $\mathcal{S}_1 \simeq_k \mathcal{S}_2$.

Proof. We will prove the equivalence of the three statements by induction. For the base case $k = 0$, the equivalence of the three statements follows directly by **Step 1** and by definition.

[$(iii) \Rightarrow (ii)$] Going from $k-1$ to k. We are assuming $\mathcal{S}_1 \simeq_k \mathcal{S}_2$ and we need to show that $\mathcal{S}_1 \equiv_k \mathcal{S}_2$, which means that the duplicator wins the k-move EF-game. Indeed, assume that the spoiler plays $s_1 \in \mathcal{S}_1$ for his first move. By the **forth** condition, the duplicator can find $s_2 \in \mathcal{S}_2$ such that $(\mathcal{S}_1, s_1) \simeq_{k-1} (\mathcal{S}_2, s_2)$. Then, by the induction hypothesis, $(\mathcal{S}_1, s_1) \equiv_{k-1} (\mathcal{S}_2, s_2)$ must hold. Hence, the duplicator can continue playing for $k-1$ moves, thus winning the k-move game. In case the spoiler plays $s_2 \in \mathcal{S}_2$ for his first move, the proof is identical, one simply uses the **back** condition.

[$(ii) \Rightarrow (iii)$] Going from $k-1$ to k. We are assuming $\mathcal{S}_1 \equiv_k \mathcal{S}_2$ and we need to show that $\mathcal{S}_1 \simeq_k \mathcal{S}_2$. We know that $\mathcal{S}_1 \equiv_{k-1} \mathcal{S}_2$, which, by the induction hypothesis, yields $\mathcal{S}_1 \simeq_{k-1} \mathcal{S}_2$. However, assuming that $\mathcal{S}_1 \neg \simeq_k \mathcal{S}_2$, then the conditions **forth** and **back** would not hold, and the duplicator would not win the k-move EF-game, a contradiction.

[$(i) \Rightarrow (iii)$] Going from $k-1$ to k. We are assuming that $\mathcal{S}_1$ and $\mathcal{S}_2$ agree on all FO[k] formulas, and we need to show that $\mathcal{S}_1 \simeq_k \mathcal{S}_2$. First, we prove the **forth** condition. Choose $s_1 \in \mathcal{S}_1$ and let α be the sentence that defines its rank-$(k-1)$ 1-type. Then $\mathcal{S}_1 \models \varphi$, where $\varphi \equiv \exists x \alpha(x)$. We know that $\mathbf{qr}(\alpha) = k-1$, hence $\mathbf{qr}(\varphi) = k$. Therefore, by assumption, $\mathcal{S}_2 \models \varphi$. Let s_2 be the witness for $\exists$ in φ. That is, $\mathbf{tp}_{k-1}(\mathcal{S}_1, s_1) = \mathbf{tp}_{k-1}(\mathcal{S}_2, s_2)$. Hence, we have that $(\mathcal{S}_1, s_1)$ and $(\mathcal{S}_2, s_2)$ agree on all FO[k-1] formulas. By the induction hypothesis, this yields that $(\mathcal{S}_1, s_1) \simeq_{k-1} (\mathcal{S}_2, s_2)$. The **back** condition is similar.

[$(iii) \Rightarrow (i)$] Going from $k-1$ to k. We are assuming that $\mathcal{S}_1 \simeq_k \mathcal{S}_2$ and we need to show that $\mathcal{S}_1$ and $\mathcal{S}_2$ agree on all formulas of quantifier rank up to k. First we notice that it is enough to prove the statement for formulas ψ of the form $\exists v \varphi(v)$, where φ is a formula of quantifier rank up to $k-1$. Indeed, any formula of quantifier rank up to k is a Boolean combination of formulas of the form ψ. So assume that $\mathcal{S}_1 \models \exists v \varphi(v)$, which means that, for some $s_1 \in S_1$, it holds $\mathcal{S}_1 \models \varphi(s_1)$. By the **forth** condition, we can find a $s_2 \in S_2$ such that $(\mathcal{S}_1, s_1) \simeq_{k-1} (\mathcal{S}_2, s_2)$. Thus, by the induction hypothesis, $\mathcal{S}_1$ and $\mathcal{S}_2$ agree on all formulas of quantifier rank up to $k-1$. Therefore, $\mathcal{S}_2 \models \varphi(s_2)$ and finally $\mathcal{S}_2 \models \exists v \varphi(v)$, which is what we wanted to show. Conversely, one can show with an identical argument that $\mathcal{S}_2 \models \exists v \varphi(v)$ implies that $\mathcal{S}_1 \models \exists v \varphi(v)$. With this, the proof is complete.

$\square$

$\square$

Remark 2.6. In light of Theorem 2.5, one can notice that k-equivalence is the finite counterpart of elementary equivalence, since:

$$\mathcal{S}_1 \equiv_k \mathcal{S}_2 \ \forall k \in \mathbb{N} \Rightarrow \mathcal{S}_1 \equiv \mathcal{S}_2.$$

Example 2.4. Looking at the oriented graphs in Figure 5, one can notice all the nodes in the left graph (graph A) have an outgoing edge, whereas the node b_4 in the right graph (graph B) has no outgoing edges. Now consider the following sentence, where the relational symbol E expresses the oriented edge relation:

$$\varphi \equiv \exists x \forall y \neg E(x, y),$$

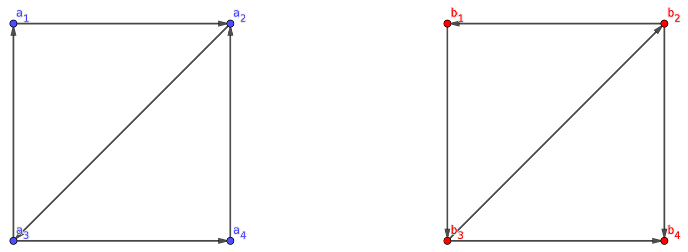

Figure 5: Oriented graphs in Example 2.4

saying that there is a node x that has no outgoing edges. This sentence is true for B but not for A. Furthermore, $\mathbf{qr}(\varphi) = 2$. In light of Theorem 2.5, this means that the spoiler has a strategy to win the Ehrenfeucht-Fraïssé game on A and B in two rounds. Indeed, he can play as follows:

(i). **Round 1.**

(a) The spoiler picks graph B;

(b) the spoiler picks node b_4;

(c) the duplicator picks a node a_i in A.

(ii). **Round 2.**

(a) The spoiler picks graph A;

(b) the spoiler picks a node a_j such that there is an edge from a_j to a_i;

(c) the duplicator loses, because she is unable to find a node b_k in B such that there is an edge from b_k to b_4.

2.3.5 A corollary about inexpressibility

A very useful consequence of Theorem 2.5 is the following corollary.

Corollary 2.7. *A property* $\mathbf{P}$ *of finite* $\mathcal{L}$*-structures is* expressible *if and only if there exists a number* $k \in \mathbb{N}$ *such that, for any two finite* $\mathcal{L}$*-structures* $\mathcal{S}_1$ *and* $\mathcal{S}_2$*, we have the following:*

$$\mathcal{S}_1 \equiv_k \mathcal{S}_2 \ \wedge \ \mathcal{S}_1 \text{ has the property } \mathbf{P} \ \Rightarrow \ \mathcal{S}_2 \text{ has the property } \mathbf{P}. \tag{2.2}$$

(see [2])

Proof. [$\Rightarrow$] By contradiction. Assume that the property $\mathbf{P}$ is definable by a sentence φ and let $k = \mathbf{qr}(\varphi)$. If $\mathcal{S}_1$ has the property $\mathbf{P}$, then, by definition, $\mathcal{S}_1 \models \varphi$. Therefore, since $\mathcal{S}_1 \equiv_k \mathcal{S}_2$, and by Theorem 2.5, we have that $\mathcal{S}_2 \models \varphi$. Thus, $\mathcal{S}_2$ has the property $\mathbf{P}$ too.
[$\Leftarrow$] Assume that $\mathcal{S}_1 \equiv_k \mathcal{S}_2$. This means, in light of Theorem 2.5, that $\mathcal{S}_1$ and $\mathcal{S}_2$ agree on the same formulas of quantifier rank up to k. Therefore, $\mathcal{S}_1$ and $\mathcal{S}_2$ have the same rank-k type. In addition, if (2.2) holds, we have that $\mathcal{S}_1$ and $\mathcal{S}_2$ agree on $\mathbf{P}$. Therefore, $\mathbf{P}$ is a union of types. By Theorem 2.4, $\mathbf{P}$ is definable. □

This is equivalent to the following.

Corollary 2.8. *A property* $\mathbf{P}$ *of finite* $\mathcal{L}$*-structures is* not expressible *in* FO *if and only if, for every* $k \in \mathbb{N}$*, there exist two finite* $\mathcal{L}$*-structures,* $\mathcal{S}_1$ *and* $\mathcal{S}_2$ *such that:*

- $\mathcal{S}_1 \equiv_k \mathcal{S}_2$*, and*
- $\mathcal{S}_1$ *has the property* $\mathbf{P}$ *and* $\mathcal{S}_2$ *does not.*

(see [2])

3 Connectivity of finite graphs

The main source for this chapter is [2].

We have seen in Section 2.2 that connectivity is not definable over arbitrary graphs. Now we have the tools to prove that it is not definable over finite graphs either.

3.1 Games on Linear Orders

In this section, we will prove that the property even is not definable over linear orders.

Definition 3.1. A *linear order* is a structure of the form $(\{1, \ldots, n\}, <)$, where $<$ is a binary relation symbol that satisfies antisymmetry, transitivity and connexity ($<$ is also called a *linear order*).

Lemma 3.1. *Let $n \in \mathbb{N}$ and let O_1, O_2 be linear orders, the cardinality of whose universe is at least 2^n. Then $O_1 \equiv_n O_2$.*

Proof. We are going to prove the lemma by induction on k, the number of rounds. Furthermore, the induction hypothesis is going to be stronger than the mere partial embedding claim - that is because, otherwise, the induction step would not work. So let O_1 have the universe $\{1, \ldots, x\}$ and O_2 the universe $\{1, \ldots, y\}$, where $x, y > 2^n$. Moreover, let us expand the language with two constant symbols M and m, the interpretation of which is, respectively, the maximum and minimum element of the linear orders.

Induction hypothesis. Let $o_1^{-1} := m^{O_1}$ and $o_2^{-1} := m^{O_2}$ be the minimum elements of O_1, O_2 respectively, and $o_1^0 := M^{O_1}$, $o_2^0 := M^{O_2}$ be the maximum elements of O_1, O_2. Furthermore, let $o_1^1, \ldots, o_1^k$ be the k moves in O_1 and $o_2^1, \ldots, o_2^k$ be the k moves in O_2. Now consider the tuples $\bar{o_1} := (o_1^{-1}, o_1^0, o_1^1, \ldots, o_1^k)$ and $\bar{o_2} := (o_2^{-1}, o_2^0, o_2^1, \ldots, o_2^k)$. The induction hypothesis is that the duplicator can play so that, after the k-th round, the following holds:

(i). if $|o_1^i - o_1^j| < 2^{n-k}$, then $|o_2^i - o_2^j| = |o_1^i - o_1^j|$;

(ii). if $|o_1^i - o_1^j| \geq 2^{n-k}$, then $|o_2^i - o_2^j| \geq 2^{n-k}$;

(iii). $o_1^i \leq o_1^j \Leftrightarrow o_2^i \leq o_2^j$;

for all $-1 \leq i, j \leq k$. Notice that the condition (iii) is enough to guarantee a partial embedding.

Base case. The case $k = 0$ is trivial, since we assumed that $|o_1^{-1} - o_1^0| \geq 2^n$ and $|o_2^{-1} - o_2^0| \geq 2^n$.

Induction step. Going from k to $k+1$. We can assume, without loss of generality, that the spoiler makes his $(k+1)$st move in O_1. If the spoiler plays an element that has already been played, that is, if he plays one of o_1^i, $i \leq k$, then the duplicator's response will be o_2^i. The three conditions are preserved. If not, since we assumed the universe is large enough, the spoiler's choice o_1^{i+1} will fall into an interval $[o_1^r, o_1^s]$ such that none of the elements in $\bar{o}_1$ are in that interval. Therefore, by condition (iii), the interval $[o_2^r, o_2^s]$ contains no elements of $\bar{o}_2$. Now we have two cases:

- $|o_1^r - o_1^s| < 2^{n-k}$. In this case, by the induction hypothesis, we have that $|o_1^r - o_1^s| = |o_2^r - o_2^s|$; so, $[o_1^r, o_1^s]$ and $[o_2^r, o_2^s]$ are isomorphic. Then the duplicator can find an element o_2^{i+1} such that $|o_2^r - o_2^{i+1}| = |o_1^r - o_1^{i+1}|$ and $|o_2^{i+1} - o_2^s| = |o_1^{i+1} - o_1^s|$. In this way, all the conditions above are preserved.

- $|o_1^r - o_1^s| \geq 2^{n-k}$. In this case, by the induction hypothesis, we have that $|o_2^r - o_2^s| \geq 2^{n-k}$. There are four cases:

  - $|o_1^r - o_1^{i+1}| < 2^{n-(i+1)}$. Then $|o_1^{i+1} - o_1^s| \geq 2^{n-(i+1)}$ and the duplicator can find an element o_2^{i+1} such that $|o_2^r - o_2^{i+1}| = |o_1^r - o_1^{i+1}|$ and $|o_2^{i+1} - o_2^s| \geq 2^{n-(i+1)}$.
  - $|o_1^{i+1} - o_1^s| < 2^{n-(i+1)}$. This case is similar to the one above.
  - $|o_1^r - o_1^{i+1}| \geq 2^{n-(i+1)}$. In this case, the duplicator chooses o_2^{i+1} to be the middle of $[o_2^r, o_2^s]$. This way, since $|o_2^r - o_2^s| \geq 2^{n-k}$, we ensure that $|o_2^{i+1} - o_2^s| \geq 2^{n-(i+1)}$.

- $|o_1^{i+1} - o_1^s| \geq 2^{n-(i+1)}$. This case is similar to the one above.

In any case, the three conditions will hold. This completes the proof. $\square$

Theorem 3.2. *Even cardinality is not definable over linear orders.*

Proof. We will take for every $k \in \mathbb{N}$ two linear orders O_1, the cardinality of whose universe is 2^k, and O_2, the cardinality of whose universe is $2^k + 1$. In light of Lemma 3.1, we have that $O_1 \equiv_k O_2$. However, O_1 has the property even and O_2 does not. Corollary 2.8 yields the desired result. $\square$

3.2 Connectivity of finite graphs

Theorem 3.3. *The property of a* finite *graph to be connected is not definable.*

Proof. Let $\mathcal{L} = \{E\}$ be a language. Assume that connectivity of finite graphs is definable by a $\mathcal{L}$-sentence φ. The idea of the proof is the following: we will start from a linear order and, from its elements, we will construct a graph (**Step 1** and **Step 2**); this graph will be connected if and only if the cardinality of the universe of the underlying linear order is odd, which will lead to a contradiction (**Step 3**).
Step 1. Starting from a linear order $<$, we will define the successor relation:

$$\mathbf{S}(v, w) \Leftrightarrow (v < w) \wedge \forall u((u \leq v) \vee (u \geq w)).$$

Furthermore, we will define a formula $\psi(v, w)$ that is true if and only if one of these conditions hold:

- $\exists u(\mathbf{S}(v, u) \wedge \mathbf{S}(u, w))$, that is, w is the successor of the successor of v;
- $(\forall z(w \leq z)) \vee (\exists u(\mathbf{S}(v, u) \wedge \forall z(z \leq u)))$, that is, w is the first element and v is the second to last element;
- $(\forall z(z \leq v)) \vee (\exists u(\mathbf{S}(u, w) \wedge \forall z(u \leq z)))$, that is, v is the last element and w is the second element;

Step 2. Let $\{v_1, \ldots, v_n\}$ be the universe of the linear order. On these elements, we will define a graph like so:

$$E(v_i, v_j) \Leftrightarrow \psi(v_i, v_j),$$

meaning that there is an edge between v_i, v_j if one of the three conditions above is satisfied. This construction is illustrated in Figure 6 and Figure 7.

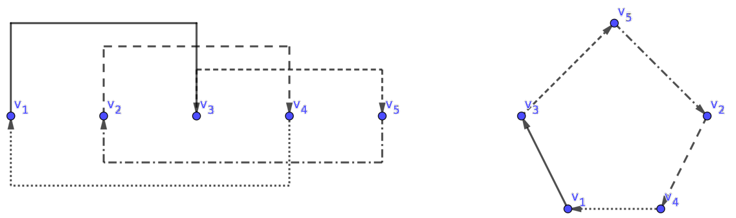

Figure 6: The odd case (see [2])

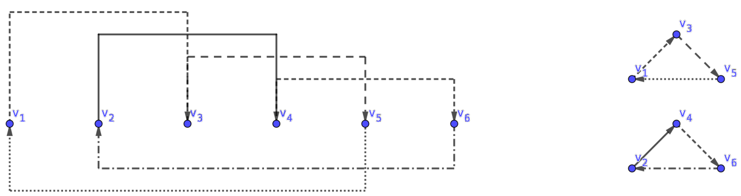

Figure 7: The even case (see [2])

Step 3. Clearly, the constructed graph is connected if and only if n is odd. Since we assumed that connectivity is definable by the sentence φ, we obtain that even is definable on linear orders by the sentence $\neg\varphi$. Contradiction!

$\square$

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