PROSEMINAR AXIOMATIC SET THEORY I (S2018): 23.03.2018

Exercise 1. (Mostowski's Collapsing Theorem)
Let R be a well-founded and set-like relation on a class A.

(1) Prove that $\text{mos}_{A,R}$ is an injection iff R is extensional on A.
Hint: Show inductively that $a \neq b \to \text{mos}_{A,R}(a) \neq \text{mos}_{A,R}(b)$ in case R is extensional.

(2) Show that in this case $\text{mos}_{A,R}$ provides an isomorphism from (A, R) onto $(\text{mos}\,{}''A, \in)$.

Exercise 2. Assume that $\in$ is well-founded and extensional on a class A. Let $T \subseteq A$ be a subclass of A which is transitive. Show that elements of T are collapsed to themselves, i.e. $\text{mos}_{A,\in}(y) = y$ for all $y \in T$.

Exercise 3. Let R be a well-founded, set-like, and transitive relation on a class A. Then $\text{mos}_{A,R}(a) = \text{rank}_{A,R}(a)$ for all $a \in A$.

Exercise 4. Let κ, λ be cardinals and let $+$ and $\cdot$ denote cardinal addition and multiplication.

(1) Show that $\kappa + \lambda = |X \cup Y|$ for any two disjoint sets X and Y with $|X| = \kappa$ and $|Y| = \lambda$.

(2) Show that $\kappa + \lambda \leq \kappa \cdot \lambda$ for $\kappa, \lambda \geq 2$.

(3) If A is a set, we write $[A]^\kappa$ for $\{x \subseteq A : |x| = \kappa\}$. Assume that κ and λ are infinite with $\lambda \leq \kappa$ and show that $|[\kappa]^\lambda| = \kappa^\lambda$.
Hint: You may use the Schröder-Bernstein Theorem (see below) and the fact that $\kappa \cdot \lambda = \kappa$ without proof.

Bonus Exercise.

(1) (Schröder-Bernstein Theorem) Let X and Y be sets such that there is an injection $f\colon X \to Y$ and an injection $g\colon Y \to X$. Prove that this implies that there is a bijection $h\colon X \to Y$.
Hint: For $x \in X$ consider the X-orbit of x given by $g^{-1}(x), f^{-1}(g^{-1}(x)), g^{-1}(f^{-1}(g^{-1}(x))), \ldots$ and similarly for $y \in Y$ the Y-orbit of y. Distinguish whether the maximal length of such an orbit is infinite, finite even, or finite odd and use this to define the bijection h.

(2) Show that $|\mathbb{R}| = |\mathcal{P}(\mathbb{N})| = 2^{\aleph_0} = |{}^\omega\omega|$, where ${}^\omega\omega$ is the set of all functions $f\colon \omega \to \omega$.

E-mail address: mueller.sandra@logic.univie.ac.at