Introduction to Mathematical Logic
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Part 1

Set Theory

CHAPTER 1

The Axiomatic system of Zermelo-Fraenkel

1. ZFC

In the following, we will formulate the axiomatic system of Zermelo-Fraenkel. For this we work in the language of set theory, which has only one non-logical symbol, the binary relation, membership! The language of set theory is denoted $\mathscr{L}_{\in}$. The Axioms:

- Axiom 1 (Extensionality)
$$\forall z(z \in x \leftrightarrow z \in y) \rightarrow x = y$$
- Axiom 2 (Foundation)
$$\exists y(y \in x) \rightarrow \exists y(y \in x \wedge \neg \exists z(z \in x \wedge z \in y))$$
- Axiom 3 (Comprehension Scheme) For each formula φ without y free:
$$\exists y \forall x(x \in y \leftrightarrow x \in v \wedge \varphi(x))$$
- Axiom 4 (Pairing)
$$\exists z(x \in z \wedge y \in z)$$
- Axiom 5 (Union)
$$\exists A \forall Y \forall x(x \in Y \wedge Y \in \mathscr{F} \rightarrow x \in A)$$
- Axiom 6 (Replacement Scheme) For each formula φ in which B is not a free variable
$$\forall x \in A \exists ! y \varphi(x,y) \rightarrow \exists B \forall x \in A \exists y \in B \varphi(x,y)$$

REMARK 1.1. To formulate the last three axioms, we need some defined notions, namely the notions of a subset, emptyset, successor of a set, intersection and singleton:

- $x \subseteq y$ iff $\forall z(z \in x \rightarrow z \in y)$
- $x = \emptyset$ iff $\forall z(z \notin x)$
- $y = S(x)$ iff $\forall z(z \in y \leftrightarrow z \in x \vee z = x)$
- $y = v \cap w$ iff $\forall x(x \in y \leftrightarrow x \in v \wedge x \in w)$
- $\text{Sing}(y)$ iff $\exists y \in x \forall z \in x(z = y)$.

Note that

- Thus $S(x) = x \cup \{x\}$ and $\text{Sing}(y) = \{y\}$.
- The ordered pair (x,y) is the set $\{\{x\},\{x,y\}\}$.

We continue with the axioms.

- Axiom 7 (Infinity)
$$\exists x(\emptyset \in x \wedge \forall y \in x(S(y) \in x))$$

- Axiom 8 (Power Set)

$$\exists y \forall z (z \subseteq x \to z \in y)$$

- Axiom 9 (Axiom of Choice)

$$\emptyset \notin F \wedge \forall x \in F \forall y \in F (x \neq y \to x \cap y = \emptyset) \to \exists C \forall x \in F (\mathrm{Sing}(C \cap x))$$

We refer to the above system of Axioms as ZFC. Note that ZFC is an infinite set of Axioms, because Axioms 3 (Comprehension) and 6 (Replacement) are in fact axiom schemes (one axiom for each formula). Moreover ZFC is not finitely axiomatizable.

2. Relations and Functions

DEFINITION 2.1. Binary relation A set R is said to be a binary relation iff R is a set of ordered pairs, i.e. for each $u \in R$ there are x, y such that $u = (x,y) = \{\{x\}, \{x,y\}\}$.

REMARK 2.2. Recall the following notions associated to a binary relation R:
- R is a pre-order on A if R is reflexive and transitive on A.
- R partially orders A non-strictly if R is a pre-order on A and satisfies $\neg \exists x, y \in A (xRy \wedge yRx \wedge x \neq y)$.
- R is a total-order on A if R is irreflexive, transitive and satisfies trichotomy, i.e. for any $a, b \in A$ either aRb, or bRa or $a = b$.

DEFINITION 2.3. A binary relation R is a function if
- for every x there is at most one y such that $(x,y) \in R$.

If there is y such that xRy then $R(x)$ denotes that unique y.

DEFINITION 2.4. For any set A, $\mathrm{id}_A = \{(x,x) : x \in A\}$ is the identity function of A.

PROOF. (Justification of existence) Note that we can justify the existence of id_A as follows: $\mathrm{id}_A = \{(x,x) \in \mathscr{P}(\mathscr{P}(A)) : x \in A\}$. □

REMARK 2.5.
- Note $(x,x) = \{\{x\}, \{x,x\}\} = \{\{x\}, \{x\}\} = \{\{x\}\}$ and
- whenever $x \in A$ and $x \in B$, then

$$(x,y) = \{\{x\}, \{x,y\}\} \in \mathscr{P}(\mathscr{P}(A \cup B)).$$

DEFINITION 2.6. $A \times B = \{(x,y) : x \in A \wedge y \in B\}$

PROOF. (Justification of existence) The existence of $A \times B$ follows from the Axioms of Power Set and Comprehension, since

$$A \times B = \{(x,y) = \{\{x\}, \{x,y\}\} \in \mathscr{P}(\mathscr{P}(A \cup B)) : x \in A \wedge y \in B\}.$$

□

REMARK 2.7. Justification, yet once again: $A \times B$ is a set Alternatively, one can use the Axioms of Replacement and Union:
- By Replacement for each $y \in B$,

$$A \times \{y\} = \{(x,y) : x \in A\}$$

  is a set. Again by Replacement $S = \{A \times \{y\} : y \in B\}$ is a set.
- Now, by the Union Axiom $\bigcup S$ is a set.
- Thus, we can define $A \times B = \bigcup S$.

DEFINITION 2.8. (Domain and Range) For every set R define

- $\operatorname{dom}(R) = \{x : \exists y((x,y) \in R)\}$,
- $\operatorname{ran}(R) = \{y : \exists x((x,y) \in R)\}$.

PROOF. (Justification of existence: Using Union and Comprehension) If $\{\{x\},\{x,y\}\} \in R$, then

- $\{x\},\{x,y\}$ belong to $\bigcup R$ and so
- $x, y \in \bigcup\bigcup R$.

Thus,

- $\operatorname{dom}(R) = \{x \in \bigcup\bigcup R : \exists y((x,y) \in R)\}$, and
- $\operatorname{ran}(R) = \{y \in \bigcup\bigcup R : \exists x((x,y) \in R)\}$.

$\square$

Note that alternatively, one can use Replacement.

DEFINITION 2.9. (Restriction) $R \restriction A = \{(x,y) \in R : x \in A\}$

PROOF. (Justification of existence) By the Axiom of Comprehension. $\square$

REMARK 2.10. The notions of a function, injection, bijection, surjection, can be defined in a similar way.

LEMMA 2.11. Assume $\forall x \in A \exists! y \varphi(x,y)$ and assume the Axiom of Replacement. Then there is a function f such that $\operatorname{dom}(f) = A$ and such that $\forall x \in A$, $f(x)$ is the unique y such that $\varphi(x,y)$.

DEFINITION 2.12. (A set of functions) Given sets A, B let

$$B^A = {}^A B = \{f \mid f : A \to B\}.$$

PROOF. (Justification of existence: Power set and Comprehension) If f is a function from A to B, then $f \subseteq A \times B$. Therefore

$${}^A B \subseteq \mathscr{P}(A \times B).$$

$\square$

DEFINITION 2.13. Let A be a set and let R be a relation on A. Then, we say that

(1) R totally orders A strictly if R is transitive, irreflexive, satisfies trichotomy on A.
(2) R well-orders A iff R totally orders A and R is well-founded on A, i.e. every $B \subseteq A$ has an R-minimal element.

LEMMA 2.14. If R is a well-order on a set A and $X \subseteq A$, then R is a well-order on X.

PROOF. Clearly R is a total order on X. Moreover, every subset of X has an R-minimal element. $\square$

CHAPTER 2

Ordinal Arithmetic

1. Ordinals

DEFINITION 1.1. A set z is an ordinal if z is transitive, i.e. $\forall x(x \in z \to x \subseteq z)$ and the membership relation $\in$ is well-founded on z.

EXAMPLE 1.2.

- $\emptyset$,
- $\{\emptyset\}$,
- $\{\emptyset, \{\emptyset\}\}$,
- $\{\emptyset, \{\emptyset\}, \{\emptyset, \{\emptyset\}\}\}$
- $\cdots$

REMARK 1.3. Every natural number is an ordinal.

NOTATION. $\mathbb{ON}$ denotes the collection of all ordinals. Greek letters are used to denote ordinals.

LEMMA 1.4. Suppose α is an ordinal, $z \subseteq \alpha$. Then z is also an ordinal.

PROOF. By transitivity of α, $z \subseteq \alpha$. Thus $\in$ is well-founded on z. We need to check if z is transitive. Let $x \in z$ and $y \in x$. Then $x \in \alpha$. But α is transitive and so $x \subseteq \alpha$. Thus $y \in \alpha$. Therefore x, y, z are elements of α. But $\in$ is transitive on α and so we have $y \in x \wedge x \in z \to y \in z$. Thus $y \in z$. That is $x \subseteq z$, i.e. z is transitive. □

LEMMA 1.5. Let α, β be ordinals. Then $\alpha \cap \beta$ is an ordinal.

PROOF.
- Since $\alpha \cap \beta \subseteq \alpha$, the $\in$ is well-founded on $\alpha \cap \beta$.
- Is $\alpha \cap \beta$ transitive? Let $x \in \alpha \cap \beta$ and $y \in x$. Then $x \subseteq \alpha \cap \beta$ and so $y \in \alpha \cap \beta$. Thus $x \subseteq \alpha \cap \beta$, i.e. $\alpha \cap \beta$ is a transitive set.

□

LEMMA 1.6. Let α, β be ordinals. Then $\alpha \subseteq \beta$ if and only if $\alpha \in \beta \vee \alpha = \beta$.

PROOF. ($\Leftarrow$) If $\alpha \in \beta$, then by transitivity of β, we have $\alpha \subseteq \beta$. Therefore $\alpha \in \beta \vee \alpha = \beta$ implies that $\alpha \subseteq \beta$.

($\Rightarrow$) If $\alpha = \beta$, then clearly we are done. So, suppose $\alpha \neq \beta$. Thus $X = \beta \backslash \alpha \neq \emptyset$ and so there is $\xi = \min \beta \backslash \alpha$. Then

$$\xi \in \beta \text{ and } \xi \notin \alpha.$$

We will show that $\xi = \alpha$. First we will show that $\xi \subseteq \alpha$:

- Let $\mu \in \xi$. Then by transitivity of β, we have $\xi \subseteq \beta$ and so $\mu \in \beta$.
- If $\mu \notin \alpha$, we get a contradiction to the minimality of ξ.

Thus $\mu \in \alpha$ and so $\xi \subseteq \alpha$.

Now, suppose $\xi \subseteq \alpha$, but $\xi \neq \alpha$! Then take any pick $\mu \in \alpha \backslash \xi$. Then $\mu \in \beta$ (because $\alpha \subseteq \beta$ by hypothesis) and $\xi \in \beta$, since $\xi = \min \beta \backslash \alpha$. Thus, by the trichotomy of $\in$ on β we get $\mu = \xi \vee \mu \in \xi \vee \xi \in \mu$.

- However $\mu \in \alpha$, but $\xi \notin \alpha$. Thus $\mu \neq \xi$.
- By the choice of μ, $\mu \notin \xi$.
- Thus $\xi \in \mu$.

Since $\mu \in \alpha$ and α is transitive, $\xi \in \alpha$, which is a contradiction to the choice of ξ! Thus $\xi = \alpha$. □ □

THEOREM 1.7. *(The collection of all ordinals "behaves" like an ordinal)*

(1) (Transitivity) For all α, β and γ ordinals, if $\alpha \in \beta \wedge \beta \in \gamma$ then $\alpha \in \gamma$.
(2) (Irreflexivity) for every ordinal α, $\neg(\alpha \in \alpha)$.
(3) (Trichotomy) for all α, β ordinals: $\alpha \in \beta \vee \beta \in \alpha \vee \alpha = \beta$.
(4) (Well-foundedness) If $X \neq \emptyset$ is a set of ordinals, then X has an $\in$-least element.

PROOF. (1) Since γ is a transitive set, $\beta \subseteq \gamma$ and so $\alpha \in \gamma$.

(2) Suppose $\alpha \in \alpha$. That is α is an element of α. But $\in$ is irreflexive on α and so $\neg(\alpha \in \alpha)$. This is a contradiction. Therefore $\alpha \notin \alpha$.

(3) Let $\delta = \alpha \cap \beta$. Then $\delta \subseteq \alpha$, $\delta \subseteq \beta$. But then by a previous Lemma we have:

$$\delta \in \alpha \vee \delta = \alpha \text{ and } \delta \in \beta \vee \delta = \beta.$$

- If $\delta = \alpha$, then $\alpha \subseteq \beta$ and so $\alpha \in \beta \vee \alpha = \beta$.
- If $\delta = \beta$, then $\beta \subseteq \alpha$ and so $\beta \in \alpha \vee \beta = \alpha$.
- Thus suppose $\delta \neq \alpha$, $\delta \neq \beta$. Therefore $\delta \in \alpha$ and $\delta \in \beta$, i.e. $\delta \in \alpha \cap \beta = \delta$, which is a contradiction to (2).

(4) Let $X \neq \emptyset$ and X be a set of ordinals. Let $\alpha \in X$. If $\alpha = \min X$, then we are done. Otherwise

$$X_0 = \{\xi : \xi \in X \wedge \xi \in \alpha\} \neq \emptyset.$$

Then $\mu = \min X_0$ exists, because $X_0 \subseteq \alpha$. Thus $\mu = \min X \cap \alpha$. Note that $\mu = \min X$. Consider any $\delta \in X$ and suppose $\delta \in \mu$. Then $\delta \in \alpha$ (since $\mu \subseteq \alpha$), which is a contradiction to $\mu = \min X \cap \alpha$. □

REMARK 1.8. The above theorem shows that the collection of all ordinals, "behaves" like an ordinal.

- But is the collection of all ordinals a set?
- In fact, is there a set containing all ordinals?

THEOREM 1.9. *(Bourali-Forty Paradox) There is no set containing all ordinals.*

PROOF. Suppose not and let X be a set containing all ordinals. Then let

$$Y = \{y \in X : y \text{ is an ordinal}\}.$$

By the Axiom of Comprehension Y is a set. By the previous theorem $\in$ is well-founded on Y and Y is a transitive set. Thus, Y is an ordinal. But then $Y \in Y$, contradiction to (2) of the previous theorem. Thus, there is no such X. □ □

NOTATION. (1) With $\mathbb{ON}$ we denote the class of all ordinals.
(2) Let α, β be ordinals. Then $\alpha < \beta$ denotes $\alpha \in \beta$ and $\alpha \leq \beta$ denotes $\alpha \in \beta \vee \alpha = \beta$.

LEMMA 1.10. Let α, β be ordinals. Then

$$\alpha \cap \beta = \min\{\alpha, \beta\} \text{ and } \alpha \cup \beta = \max\{\alpha, \beta\}.$$

LEMMA 1.11. If $A \neq \emptyset$ is a set of ordinals, then

(1) $\bigcap A = \min A$,
(2) $\bigcup A \in \mathbb{ON}$
(3) If $\forall \alpha \in A \exists \beta \in A(\alpha < \beta)$, then $\bigcup A$ is the smallest ordinal that exceeds all ordinals in A. Thus, we denote $\bigcup A$ also $\sup A$.

PROOF. (2) We need to show that $\bigcup A$ is a transitive set and $\in$ is well-founded on $\bigcup A$. Let $\alpha \in \bigcup A$. Thus there is $\beta \in A$ such that $\alpha \in \beta$. But β is transitive and so $\alpha \subseteq \beta$. Therefore $\alpha \subseteq \bigcup A$.

To show well-foundedness of $\in$, let $X \subseteq \bigcup A$. Thus $\forall x \in X$ there is $\alpha_x \in A$ such that $x \in \alpha_x$. Now $\{\alpha_x : x \in X\}$ is a set of ordinals and so by well-foundedness of the membership relation on $\mathbb{ON}$, there is $\alpha_0 = \min\{\alpha_x : x \in X\}$. Then either $\alpha_0 = \min X$ or $\alpha_0 \cap X \neq \emptyset$, in which case $\min(\alpha_0 \cap X)$ is as desired.

(3) Let $\delta = \bigcup A$. Then $\delta = \{\alpha : \exists \beta \in A(\alpha \in \beta)\}$. Since for every $\alpha \in A$ there is $\beta \in A$ such that $\alpha < \beta$, we get that every $\alpha \in A$ is an element of δ. On the other hand, if $\alpha < \delta$, then $\alpha \in \delta$ and so there is $\beta \in A$ such that $\alpha \in \beta$. But, then $\beta \notin \alpha$ and so α does not exceed all elements of A. □

LEMMA 1.12. Let α be an ordinal. Then
- $S(\alpha) = \alpha \cup \{\alpha\}$ is an ordinal,
- $\alpha < S(\alpha)$ and
- for all ordinals γ,
$$\gamma < S(\alpha) \text{ iff } \gamma \leq \alpha.$$

PROOF. The membership relation is well-founded on $S(\alpha)$ and clearly $S(\alpha)$ is a transitive set. The rest is straightforward. □

DEFINITION 1.13. (Successor and Limit Ordinals) An ordinal β is
(1) a successor iff there is an ordinal α such that $\beta = S(\alpha) = \alpha \cup \{\alpha\}$,
(2) a limit ordinal iff $\beta \neq 0$ and β is not a successor ordinal,
(3) a finite ordinal or a natural number if and only if $\forall \alpha \leq \beta(\alpha = 0 \vee \alpha$ is a successor).

REMARK 1.14. If n is a natural number, then $S(n)$ is a natural number and every element of n is a natural number.

THEOREM 1.15. *Principle of ordinary induction If $\emptyset \in X$ and for all $y \in X(S(y) \in X)$, then every natural number is in X.*

PROOF. Suppose not and let $n \in \mathbb{N} \backslash X$. Consider $Y = S(n) \backslash X$. Then $n \in Y$ and so $Y \neq \emptyset$. Let $k = \min Y$. Thus $k \leq n$. Therefore $k = \emptyset$ or k is a successor. However $\emptyset \notin Y$, because $\emptyset \in X$ and so $k = S(i)$ for some i. By minimality of k, we must have $i \in X$. But then also $k = S(i) \in X$, which is a contradiction. □

REMARK 1.16.
- Recall the Axiom of Infinity: $\exists x(\emptyset \in X \wedge \forall y \in x(S(y) \in x))$.
- Thus if X is a set which contains all natural numbers, then $\{n \in X : n \text{ is a natural number}\}$ is a set.

LEMMA 1.17. Let X be a set of ordinals, which is an initial segment of $\mathbb{ON}$. That is $\forall \beta \in X \forall \alpha < \beta(\alpha \in X))$. Then X is an ordinal itself.

PROOF. Note that $\in$ is a well-order on X. Since X is an initial segment of the ordinals, X is also a transitive set. Thus X is an ordinal. □

REMARK 1.18. So in particular, every transitive set of ordinals is an ordinal.

DEFINITION 1.19. Let ω denote the set of all natural numbers.

REMARK 1.20. Note that ω is an initial segment of $\mathbb{ON}$ and so ω is an ordinal. Moreover ω is not a successor ordinal and ω is not finite. Thus, ω is the first limit ordinal.

DEFINITION 1.21. Assume the Axiom of Infinity and for each $n \in \mathbb{N}$ let

$$B^n = {}^nB = \{F \mid F : n \to B\}.$$

Then let

$$B^{<\omega} = {}^{<\omega}B := \bigcup\{B^n : n \in \omega\}.$$

PROOF. (Justification of existence) Use the Power Set Axiom or the Axiom of Replacement. □

REMARK 1.22. Let $\mathscr{L} = (\mathscr{C}, \mathscr{F}, \mathscr{R})$ be a first order language and let B be the set of all logical and non-logical symbols of $\mathscr{L}$. Then the set of formulas of $\mathscr{L}$ is a subset of $B^{<\omega}$.

LEMMA 1.23. Let α, β be ordinals and suppose that $f : (\alpha, \in) \to (\beta, \in)$ is an order preserving bijection (i.e. an isomorphism). Then $\alpha = \beta$ and $f = \mathrm{id}$.

PROOF. Let $\xi \in \alpha$. Then $f(\xi) \in \beta$. Furthermore:

$$f(\xi) := \{\nu \in \beta : \nu \in f(\xi)\} = \{f(\mu) : \mu \in \alpha \wedge \mu < \xi\}.$$

That is $f(\xi) = \{f(\mu) : \mu < \xi\}$. Suppose $X_0 = \{\xi \in \alpha : f(\xi) \neq \xi\} \neq \emptyset$. Then X_0 has a minimal element μ. Thus for all $\xi < \mu$, $f(\xi) = \xi$ and so

$$f(\mu) = \{f(\xi) : \xi < \mu\} = \{\xi : \xi < \mu\} = \mu,$$

which is a contradiction. Therefore $X_0 = \emptyset$ and so f is the identity. □

THEOREM 1.24. *Let A be a set and let R be a well-order on A. Then there is a unique ordinal α such that* $(A, R) \cong (\alpha, \in)$.

REMARK 1.25. Uniqueness follows from the previous statement.

PROOF. (Existence) For $a \in A$ let $a\downarrow := \{x \in A : xRa\}$ and let

$$G = \{a \in A : \exists \xi_a \in \mathbb{ON}((a\downarrow, R) \cong (\xi_a, \in))\}.$$

Since A is a set, by the Axiom of Comprehension G is also a set. Since $\forall a \in G\ \exists \xi_a$ as above, by Replacement there is a set $X \subseteq \mathbb{ON}$ and a function $f : G \to X$ such that for all $a \in G$, $f(a) = \xi_a$. Then $\in$ is a well-order on $\mathrm{range}(f) \subseteq X$. Moreover $\mathrm{range}(f)$ is a transitive and so it is an ordinal, say α. Then $f : (G, R) \cong (\alpha, \in)$. Note that:

- if $G = A$, then we are done.
- if $G \subseteq A$ and $G \neq A$, let $e = \min_R(A \backslash G)$. Then $e\downarrow = G$ and $f : (e\downarrow, R) \cong (\alpha, \in)$. That is $\xi_e = \alpha$. But, this implies that $e \in G$, which is a contradiction. Thus $G = A$.

□

DEFINITION 1.26. (Order Type) Let R be a well-order on A. Then $\mathrm{type}(A, R)$ is the unique ordinal α such that $(A, R) \cong (\alpha, \in)$. We denote this ordinal by $\mathrm{type}(A, R)$.

2. Ordinal Arithmetic

DEFINITION 2.1. Let α, β be ordinals. Then

(1) The ordinal multiplication of α and β, denoted $\alpha \cdot \beta$, is the ordinal
$$\text{type}(\beta \times \alpha, <_{lex}).$$

(2) The ordinal addition of α and β, denoted $\alpha + \beta$, is the ordinal
$$\text{type}(\{0\} \times \alpha \cup \{1\} \times \beta, <_{lex}).$$

LEMMA 2.2. If R well-orders A and $X \subseteq A$, then R well-orders X and
$$\text{type}(X,R) \leq \text{type}(A,R).$$

PROOF. We can assume that $(A,R) = (\alpha, \in)$. Thus, in particular, X and A are sets of ordinals. Let $\delta = \text{type}(X,R)$ and let $f : (X,R) \cong (\delta, \in)$. Suppose $X_0 = \{\xi \in X : f(\xi) > \xi\} \neq \emptyset$ and let $\mu = \min X_0$. Then $f(\mu) > \mu$ and $\forall \xi \in X \cap \mu (f(\xi) \leq \xi)$. Since f is an isomorphism
$$f(\mu) = \{f(\xi) : \xi < \mu\} \leq \mu,$$
which is a contradiction. Therefore for all $\xi \in X$, $f(\xi) \leq \xi$. Then
$$\delta = \{f(\xi) : \xi \in X\} \subseteq \alpha \text{ and so } \delta \subseteq \alpha.$$
□

EXAMPLE 2.3.
- $\omega + \omega$
$$0, 1, \cdots, n, n+1, \cdots, \omega = \omega + 0, \omega + 1, \omega + 2, \cdots, \omega + n, \cdots$$
- $\omega \cdot 2 = \text{type}(\{0,1\} \times \omega, <_{\text{lex}})$
$$(0,0), (0,1), \cdots, (0,n), \cdots, (1,0), (1,1), \cdots, (1,n), \cdots$$
Thus $\omega + \omega = \omega \cdot 2$ (because the order type is unique!).
- However $1 + \omega = \omega$, while $\omega < \omega + 1$. Thus $1 + \omega \neq \omega + 1$.
- Also $2 \cdot \omega = \text{type}(\omega \times \{0,1\}, <_{\text{lex}}) = \omega$, while $\omega \cdot 2 = \omega + \omega > \omega$.
- More precisely, what is $2 \cdot \omega$?
$$(0,0), (0,1), (1,0), (1,1), (2,0), (2,1), \cdots, (n,0), (n,1), \cdots$$
- In particular $2 \cdot \omega \neq \omega \cdot 2$.
- Both, ordinal multiplication and ordinal addition are associative, but not commutative.

THEOREM 2.4. *(Transfinite Induction on $\mathbb{ON}$) Let $\psi(\alpha)$ be a formula. If there is an ordinal α such that $\psi(\alpha)$, then there is a least ordinal ξ such that $\psi(\xi)$.*

PROOF. Fix α such that $\psi(\alpha)$. If α is least, then we are done. Otherwise, $X = \{\xi \in \alpha : \psi(\alpha)\} \neq \emptyset$ and so $\xi = \min X$ is as desired. □

REMARK 2.5. (Ordinal Exponentiation) Note that induction is a method for giving proofs, while recursion is a method for giving definitions. Recursively, one can define ordinal exponentiation as follows:
$$\alpha^0 = 1,\ \alpha^{S(\beta)} = \alpha^\beta \cdot \alpha,\ \alpha^\gamma = \sup_{\beta < \gamma} \alpha^\beta \text{ for } \gamma \text{ limit}.$$

THEOREM 2.6. *(Primitive Recursion on $\mathbb{ON}$) Suppose for all s there is a unique y such that $\varphi(s,y)$ and define $G(s)$ to be this unique y. Then there is a formula ψ for which the following two properties are provable:*

(1) $\forall x \exists ! y \psi(x,y)$. *Thus, ψ defines a function F, where $F(x)$ is such that $\psi(x, F(x))$.*
(2) $\forall \xi \in \mathbb{ON}(F(\xi) = G(F(\xi)))$.

PROOF. *δ-approximations to F:* Let $\delta \in \mathbb{ON}$ and let $\mathrm{App}(\delta, h)$ abbreviate

$$h \text{ is a function}, \mathrm{dom}(h) = \delta, \forall \xi \in \delta h(\xi) = G(h \upharpoonright \xi).$$

Uniqueness: We will show that

$$\delta \leq \delta' \wedge \mathrm{App}(\delta, h) \wedge \mathrm{App}(\delta', h') \to h = h' \upharpoonright \delta.$$

In particular, the case $\delta = \delta'$ gives the uniqueness of h.

Fix δ, δ', h, h' as above. Suppose $h \neq h' \upharpoonright \delta$. Then

$$X = \{\xi < \delta : h(\xi) \neq h'(\xi)\} \neq \emptyset$$

and so there is $\mu = \min X$. Then for all $\xi < \mu$ $h(\xi) = h'(\xi)$. That is $h \upharpoonright \mu = h' \upharpoonright \mu$. But then $h(\xi) = G(h \upharpoonright \xi) = G(h' \upharpoonright \xi) = h'(\xi)$, which is a contradiction. Therefore $X = \emptyset$ and $h = h' \upharpoonright \delta$.

Existence: By transfinite induction on $\mathbb{ON}$ show that $\forall \delta \exists h \mathrm{App}(\delta, h)$. Suppose not and let $\delta \in \mathbb{ON}$ be least such that $\neg \exists h \mathrm{App}(\delta, h)$. Thus in particular $\forall \xi < \delta \exists h_\xi$ such that $\mathrm{App}(\xi, h_\xi)$.

Case 1: If $\delta = \beta + 1$ let $f = h_\beta \cup \{\langle \beta, G(h_\beta) \rangle\}$. Then $\mathrm{App}(\delta, f)$ which contradicts our hypothesis.

Case 2: $\delta = \emptyset$ - impossible, since $\mathrm{App}(0, \emptyset)$.

Case 3: δ is a limit ordinal. Let $f = \bigcup \{h_\xi : \xi < \delta\}$. Then uniqueness implies that f is a function and furthermore $\mathrm{App}(\delta, f)$, which is a contradiction to the choice of δ.

Thus $\forall \delta \in \mathbb{ON} \exists ! h \mathrm{App}(\delta, h)$. Let $\psi(x,y)$ be the following formula:

$$(x \notin \mathbb{ON} \wedge y = 0) \vee (x \in \mathbb{ON} \wedge \exists \delta > x \exists h (\mathrm{App}(\delta, h) \wedge h(x) = y)).$$

The uniqueness and existence of h imply that $\forall x \exists ! y \psi(x,y)$ and so $\psi(x,y)$ defines a function F. Now, let $\xi \in \mathbb{ON}$. Then pick any $\delta > \xi$ and h such that $\mathrm{App}(\delta, h)$. Then

$$F(\xi) = h(\xi) = G(h \upharpoonright \xi) = G(F \upharpoonright \xi)$$

as desired. □

REMARK 2.7. One can define ordinal addition and exponentiation by transfinite recursion on the ordinals as follows:

Ordinal addition Let $\alpha \in \mathbb{ON}$. Recursively over $\beta \in \mathbb{ON}$ define $\alpha + \beta$ as follows:

(1) $\alpha + 0 = \alpha$,
(2) $\alpha + \beta = S(\alpha + \beta)$ if $\beta = S(\gamma)$.
(3) $\alpha + \beta = \bigcup_{\gamma \in \beta} (\alpha + \gamma)$ if β is a limit > 0.

Ordinal multiplication Let $\alpha \in \mathbb{ON}$. By recursion over $\beta \in \mathbb{ON}$ define the ordinal $\alpha \cdot \beta$ as follows:

(1) $\alpha \cdot 0 = 0$,
(2) $\alpha \cdot \beta = (\alpha \cdot \gamma) + \alpha$, if $\beta = S(\gamma)$,
(3) $\alpha \cdot \beta = \bigcup_{\gamma \in \beta} (\alpha \cdot \gamma)$, if β is a limit > 0.

EXERCISE 1. The latter two definitions are equivalent to the definitions of ordinal addition and ordinal multiplication respectively, which we gave earlier in the lecture.

CHAPTER 3

Cardinal Arithmetic

1. Comparing infinities

DEFINITION 1.1. Let X, Y be sets.

(1) $X \preceq Y$ iff there is an injective function $f : X \to Y$;
(2) $X \approx Y$ iff there is a bijection $f : X \to Y$.

REMARK 1.2. Note that

- $\preceq$ is transitive and reflexive, and that
- $\approx$ is an equivalence relations.

So, we can think of different infinite sizes as equivalence classes, consisting of sets any two of which are in bijective correspondence.

LEMMA 1.3. If $B \subseteq A$ and there is an injective $f : A \to B$ then $A \approx B$.

PROOF. Using the fact that $f(A) \subseteq B \subseteq A$ obtain:

$$A \supseteq B \supseteq f(A) \supseteq f(B) \supseteq f^2(A) \supseteq f^2(B) \supseteq f^3(A) \supseteq ...$$

Let $f^0 = \text{id}$ and for each $n \in \mathbb{N}$ let

$$H_n = f^n(A) \setminus f^n(B),\ K_n = f^n(B) \setminus f^{n+1}(A).$$

We will show that for each n, the functions

$$f \restriction H_n : H_n \to H_{n+1} \text{ and } f \restriction K_n : K_n \to K_{n+1}$$

are bijections.

CLAIM 1.4. $f \restriction H_n : H_n \to H_{n+1}$ is a bijection, where $H_n = f^n(A) \setminus f^n(B)$.

PROOF. Let $g = f \restriction H_n$. Clearly since f is injective, then also g is injective. We need to show that g is onto.

- Let $x \in H_{n+1}$. Thus $x \in f^{n+1}(A) \setminus f^{n+1}(B)$. So clearly, there is $y \in f^n(A)$ such that $x = f(y)$.
- We need to show that $y \notin f^n(B)$. However, if $y \in f^n(B)$ then $f(y) = x \in f^{n+1}(B)$ which is a contradiction.

Thus, $x = f(y)$ for some $y \in H_n = f^n(A) \setminus f^n(B)$, i.e. g is a bijection. □

Consider the set $P = \bigcap_{n \in \omega} f^n(A) = \bigcap_{n \in \omega} f^n(B)$. Then

$$A = P \cup H_0 \cup H_1 \cup H_2 \cup \cdots \cup K_0 \cup K_1 \cup \cdots$$

$$B = P \cup H_1 \cup H_2 \cup H_3 \cup \cdots \cup K_0 \cup K_1 \cup \cdots$$

are partitions of A, B. Then the function $k : A \to B$ defined by

- $k \restriction H_n = f \restriction H_n$ for each n,
- $k \restriction P = \text{id}$ and

- $k \restriction K_n = \mathrm{id}$ for each n,

is a bijection from A to B. □

THEOREM 1.5. *(Schröder-Bernstein) $A \approx B$ iff $A \preceq B$ and $B \preceq A$.*

PROOF. ($\Rightarrow$) If $f : A \to B$ is a bijection, then f witnesses $A \preceq B$ and f^{-1} witnesses $B \preceq A$.

($\Leftarrow$) Suppose $f : A \to B$ and $h : B \to A$ are injective. Let $\hat{B} = h(B)$. Then $\hat{B} \subseteq A$ and $h : B \to \hat{B}$ is a bijection. Thus, by definition $B \approx \hat{B}$. On the other hand $\hat{B} \subseteq A$ and so $h \circ f : A \to \hat{B}$ witnesses $A \preceq \hat{B}$. Thus, by the previous Lemma $A \approx \hat{B}$. Since $B \approx \hat{B}$ we obtain $A \approx B$. □

DEFINITION 1.6. $X \prec Y$ iff $X \preceq Y$ and it is not the case that $Y \preceq X$.

REMARK 1.7. By the theorem of Schröder-Bernstein, $X \prec Y$ means that X can be mapped injectively into Y, but there is no bijection between X and Y.

LEMMA 1.8. (Cantor's Diagonal Element) If F is a function, $\mathrm{dom}(f) = A$ and $\mathbb{D} = \{x \in A : x \notin f(x)\}$ then $\mathbb{D} \notin \mathrm{ran}(f)$.

PROOF. Suppose $\mathbb{D} \in \mathrm{ran}(f)$. Then there is $x \in A$ such that $\mathbb{D} = f(x)$. There are two possibilities:

If $x \in f(x)$, then $x \in \mathbb{D}$ (since $f(x) = \mathbb{D}$) and so x the defining characteristic of $\mathbb{D}$, i.e. x is an element of A such that $x \notin f(x)$. This is a contradiction.

If $x \notin f(x)$, then since $x \in A$ we have that x satisfies the defining characteristic of $\mathbb{D}$ and so we must have that $x \in \mathbb{D}$, i.e. $x \in f(x)$. Again we reach a contradiction.

Therefore $\mathbb{D} \notin \mathrm{ran}(f)$. □

THEOREM 1.9. $A \prec \mathscr{P}(A)$.

PROOF. Clearly $A \preceq \mathscr{P}(A)$ witnessed by the mapping $x \mapsto \{x\}$ for each $x \in A$. We claim that $\mathscr{P}(A) \not\preceq A$. Well, suppose to the contrary that $\mathscr{P}(A) \preceq A$. Then by Schröder-Bernstein $\mathscr{P}(A) \approx A$ and so there is a bijection

$$f : A \to \mathscr{P}(A).$$

Then since $\mathbb{D} = \{x \in A : x \notin f(x)\} \in \mathscr{P}(A)$ and f is onto we must have

$$\mathbb{D} = \{x \in A : x \notin f(x)\} \in \mathrm{ran}(f)$$

contradicting Cantor's Diagonal Element Lemma. □

COROLLARY 1.10. $\mathbb{N} \prec \mathscr{P}(\mathbb{N})$.

REMARK 1.11. Characteristic Functions Let A be a set and let $B \subseteq A$. Then we refer to $\chi_B : A \to 2 = \{0,1\}$ defined by

$$\chi_B(a) = \begin{cases} 1 & \text{if } a \in B \\ 0 & \text{otherwise} \end{cases}$$

as the characteristic function of B.

The mapping $B \mapsto \chi_B$ where $B \in \mathscr{P}(A)$ is a bijection between ${}^A2$ and $\mathscr{P}(A)$. Thus

$${}^A2 \approx \mathscr{P}(A).$$

In particular ${}^{\mathbb{N}}2 = 2^{\mathbb{N}} \approx \mathscr{P}(\mathbb{N})$.

Note that

(1) If $A \prec B$ and $C \prec D$, then ${}^AC \preceq {}^BD$.

(2) If $2 \preceq C$, then $A \prec \mathscr{P}(A) \preceq {}^{A}C$, simply because $\mathscr{P}(A) \approx {}^{A}2 \prec {}^{A}C$.

LEMMA 1.12. (1) ${}^{C}({}^{B}A) \approx {}^{C\times B}A$
(2) ${}^{(B\cup C)}A \approx {}^{B}A \times {}^{C}A$, where B and C are disjoint.

PROOF. (1) Consider the mapping $\Phi : {}^{C}({}^{B}A) \to {}^{C\times B}A$ defined by

$$\Phi(f)(c,b) = (f(c))(b).$$

(2) Consider the mapping $\Psi : {}^{B\cup C}A \to {}^{B}A \times {}^{C}A$ given by

$$\Psi(f) = (f \restriction B, f \restriction C).$$

□

DEFINITION 1.13. (Finite, countable and uncountable sizes)

(1) A set A is said to be countable, if $A \preceq \omega$.
(2) A set A is said to be finite if $A \preceq n$ for some $n \in \omega$.
(3) Infinite means not finite. Uncountable means not countable.
(4) A countably infinite set is a countable set which is infinite.

2. Cardinal Numbers

FACT 1.

(1) If $B \subseteq \alpha$ then type$(B, \in) \leq \alpha$.
(2) If $B \preceq \alpha$, then $B \approx \delta$ for some $\delta \leq \alpha$.
(3) If $\alpha \leq \beta \leq \gamma$ and $\alpha \approx \gamma$ then $\alpha \approx \beta \approx \gamma$.

PROOF. (2) If $B \preceq \alpha$, then $B \approx \delta$ for some $\delta \leq \alpha$ (identify B with a subset of α and apply part (1)).
(3) Since $\alpha \subseteq \beta$ and $\beta \preceq \alpha$ imply that $\alpha \approx \beta$. □

Thus, the ordinals come in blocks of the same size. Informally, the first ordinal in a block is called a cardinal.

DEFINITION 2.1. A cardinal is an ordinal α such that $\xi \prec \alpha$ for all $\xi \in \alpha$.

REMARK 2.2. Thus, an ordinal α fails to be a cardinal iff there is $\xi < \alpha$ such that $\xi \approx \alpha$. We denote by $\mathbb{CD}$ the collection of all cardinals.

THEOREM 2.3. *(1) If $\alpha \geq \omega$ is a cardinal, then α is a limit ordinal.*
(2) Every natural number is a cardinal.
(3) If A is a set of cardinals, then sup$A$ *is a cardinal.*
(4) ω is a cardinal.

PROOF. (1) Let $\alpha \geq \omega$ be an infinite cardinal. Suppose α is a successor ordinal. Thus $\alpha = \delta + 1 = \delta \cup \{\delta\}$. Then $f : \delta \cup \{\delta\} \to \delta$ defined by $f(\delta) = 0$, $f(n) = n+1$ for all $n \in \omega$ and $f(\xi) = \xi$ for all ξ such that $\omega \leq \xi < \delta$ is a bijection. Thus $\delta \in \alpha$, but $\delta \not\prec \alpha$, which is a contradiction to α being a cardinal.

(2) Proceed by induction. Now, 0 is trivially a cardinal. Suppose n is a cardinal and suppose $S(n) = n+1$ is not a cardinal. Then $\exists \xi (\xi < S(n))$ such that $\xi \approx S(n)$. Thus there is a bijection $f : \xi \to S(n) = n \cup \{n\}$. Clearly $\xi \neq 0$ and so $\xi = S(m)$ for some $m < n$. But, then

$$f : m \cup \{m\} \to n \cup \{n\}$$

is a bijection. Thus $\xi = S(m)$ for some $m < n$ and $f : m \cup \{m\} \to n \cup \{n\}$ is a bijection. We have the following options:

If $f(m) = n$, then $f \restriction m : m \to n$ is a bijection, contradiction to the assumption that n is a cardinal.

Otherwise $f(m) = j \in n$. Now $n \in \mathrm{ran}(f)$ and so there is $i \in m$ such that $f(i) = n$. Consider the mapping

$$g : m \to n$$

defined by $g(i) = j$ and $g \restriction m \backslash \{i\} = f$. Then g is a bijection, again a contradiction to the assumption that n is a cardinal.

(3) Suppose, by way of contradiction that $\sup A = \bigcup A$ is not a cardinal. Thus there is $\xi < \sup A$ such that $\xi \approx \sup A$. Recall that $\sup A$ is the least ordinal, which is greater or equal each element of A. Thus there is $\alpha \in A$ such that $\xi < \alpha$. Then by one of the earlier Lemmas

$$\xi \approx \alpha,$$

which is a contradiction to α being a cardinal.

(4) Note that

$$\omega = \sup_{n \in \mathbb{N}} n = \bigcup_{n \in \mathbb{N}} n$$

and so the claim follows from items (2) and (3) above. □

DEFINITION 2.4.

(1) We say that a set A is well-orderable, if there is a relation R on A such that (A, R) is a well-order.
(2) If A is well-orderable, then the cardinality of A, denoted $|A|$, is the least ordinal α such that $A \approx \alpha$.

REMARK 2.5.

- Note that the cardinality of a set is always a cardinal number.
- Under the Axiom of Choice every set can be well-ordered and so under the AC every set is characterised by its cardinality.

LEMMA 2.6.

(1) If A is a set, which can be well-ordered and $f : A \to B$ is an onto mapping, then B can be well-ordered and $|B| \leq |A|$.
(2) Let κ be a cardinal and $B \neq \emptyset$. Then $B \preceq \kappa$ if and only if there is an onto mapping $f : \kappa \to B$.

COROLLARY 2.7. (A) set $B \neq \emptyset$ is countable if and only if there is an onto function $f : \omega \to B$.

THEOREM 2.8. *(Hartogs, 1915) Let A be a set. Then there is a cardinal κ such that $\kappa \not\preceq A$.*

PROOF. Fix A and let $W = \{(X, R) : X \subseteq A \wedge R \text{ well-orders } X\}$. Then if α is an ordinal, we have that

$$\alpha \preceq A \text{ iff } \exists (X, R) \in W \text{ s.t. } \alpha = \mathrm{type}(X, R).$$

By the Axiom of Replacement $Z = \{\mathrm{type}(X, R) + 1 : (X, R) \in W\}$ is a set. But then $\beta = \sup Z$ is an ordinal. Moreover, for each $\alpha \preceq A$, we have that $\beta > \alpha$. Thus, $\beta \not\preceq A$. Take $\kappa = |\beta|$. Then $\kappa \approx \beta$ and $\kappa \not\preceq A$. □

DEFINITION 2.9. Let A be a set. Then $\aleph(A)$ denotes the least cardinal κ such that $\kappa \not\preceq A$. For ordinals α define $\alpha^+ = \aleph(\alpha)$.

DEFINITION 2.10. By transfinite recursion on $\mathbb{ON}$, define the cardinal numbers $\aleph_\xi$ as follows:

(1) $\aleph_0 = \omega_0 = \omega$
(2) $\aleph_{\xi+1} = \omega_{\xi+1} = (\aleph_\xi)^+$

(3) $\aleph_\eta = \omega_\eta = \sup\{\aleph_\xi : \xi < \eta\}$ whenever η is a limit ordinal.

REMARK 2.11. (The class of all cardinals) The collection of all cardinals is a proper class.

$$\aleph_0 = |\mathbb{N}| < \aleph_1 < \aleph_2 < \dots < \aleph_n \dots < \aleph_\omega < \aleph_{\omega+1} < \dots$$

DISCUSSION 2.12. The cardinality of the real line How large is $\mathbb{R}$? What is $|\mathbb{R}|$? Note that $|\mathbb{R}| = |\mathscr{P}(\mathbb{N})|$ and $|\mathscr{P}(\mathbb{N}) = 2^{\aleph_0}$ where $2^{\aleph_0}$ is cardinal exponentiation (to be defined shortly) and is the cardinality of the set of functions from $\mathbb{N}$ to 2.

THEOREM 2.13. *Suppose $\alpha \geq \omega$ is an ordinal. Then $|\alpha \times \alpha| = |\alpha|$. Thus in particular, if $\kappa \geq \omega$ is a cardinal, then $|\kappa \times \kappa| = \kappa$.*

REMARK 2.14. Observe that it is sufficient to prove the claim for cardinal numbers. Indeed. Suppose α is an infinite ordinal and we have proved that $||\alpha| \times |\alpha|| = |\alpha|$. Now $\alpha \approx |\alpha|$, which induces a bijection witnessing

$$|\alpha| \times |\alpha| \approx \alpha \times \alpha,$$

and so $||\alpha| \times |\alpha|| = |\alpha|$.

PROOF. Define a relation $\lhd$ on $\mathbb{ON} \times \mathbb{ON}$ as follows: $(\xi_1, \xi_2) \lhd (\eta_1, \eta_2)$ iff

- either $\max\{\xi_1, \xi_2\} < \max\{\eta_1, \eta_2\}$,
- or $\max\{\xi_1, \xi_2\} = \max\{\eta_1, \eta_2\}$ and $(\xi_1, \xi_2) <_{\text{lex}} (\eta_1, \eta_2)$.

Note that $\lhd$ is a well-order. It is sufficient to show that

CLAIM 2.15. For each infinite cardinal κ, $\text{type}(\kappa \times \kappa, \lhd) = \kappa$.

PROOF. Proceed by transfinite induction on κ. Let κ be the least infinite cardinal such that $\text{type}(\kappa \times \kappa, \lhd) \neq \kappa$. Now, let $\delta = \text{type}(\kappa \times \kappa, \lhd)$ and let $F : (\delta, <) \to (\kappa \times \kappa, \lhd)$ be an order preserving bijection.

Suppose $\delta > \kappa$. Then $F(\kappa)$ is defined and so $\exists (\xi_1, \xi_2) \in \kappa \times \kappa$ such that $F(\kappa) = (\xi_1, \xi_2)$. Let $\alpha = \max\{\xi_1, \xi_2\} + 1$. Then since κ is a limit ordinal, $\alpha < \kappa$. Moreover since F is order preserving, $F''\kappa \subseteq \alpha \times \alpha$. Therefore $\kappa \preceq \alpha \times \alpha \prec \kappa$, which is clearly a contradiction.

Now, suppose $\delta < \kappa$. Then $\kappa \preceq \kappa \times \kappa \approx \delta$, which is a contradiction, since κ is a cardinal.

Thus $\kappa = \delta$, which is a contradiction to the choice of κ.

Therefore there is no such κ, i.e. for each infinite cardinal κ, $|\kappa \times \kappa| = \kappa$. This proves the claim and the theorem. □

□

3. Cardinal Arithmetic

DEFINITION 3.1. (Cardinal addition, multiplication and exponentiation) Let κ and λ be cardinals. Then:

(1) $\kappa + \lambda$ is defined to be the cardinality of the set $\{0\} \times \kappa \cup \{1\} \times \lambda$.
(2) $\kappa \times \lambda$ is defiend to be the cardinality of the set $\kappa \times \lambda$.
(3) κ^λ is the cardinality of the set ${}^\kappa\lambda := \{f \mid f : \kappa \to \lambda\}$.

LEMMA 3.2. (Monotonicity) Let $\kappa, \kappa', \lambda, \lambda'$ be cardinals such that $\kappa \leq \kappa'$, $\lambda \leq \lambda'$. Then:

(1) $\kappa + \lambda \leq \kappa' + \lambda'$,
(2) $\kappa \cdot \lambda \leq \kappa' \cdot \lambda'$,
(3) $\kappa^\lambda \leq (\kappa')^{\lambda'}$.

PROOF. (1) Note that $\{0\}\times\kappa\cup\{1\}\times\lambda\subseteq\{0\}\times\kappa'\cup\{1\}\times\lambda'$. Thus

$$\mathrm{id}:\kappa+\lambda\preceq\kappa'+\lambda'$$

and so $\kappa+\lambda\le\kappa'+\lambda'$.

(2) Similarly $\kappa\times\lambda\subseteq\kappa'\times\lambda'$ and so $\mathrm{id}:\kappa\cdot\lambda\preceq\kappa'\cdot\lambda'$. Therefore $\kappa\cdot\lambda\le\kappa'\cdot\lambda'$.

(3) Consider the mapping $\varphi:{}^{\lambda}\kappa\to{}^{(\lambda')}(\kappa')$ defined by

- $\varphi(f)\restriction\lambda=f$ and
- $\varphi(f)(\xi)=0$ for all $\lambda\le\xi<\lambda'$.

When $\kappa=\kappa'=0$, note that

$$0^0=|{}^00|=|\{\emptyset\}|=1$$

and for $\lambda>0$,

$$0^\lambda=|{}^\lambda0|=|\emptyset|=0.$$

□

LEMMA 3.3. Let κ, λ, θ be cardinals. The following properties refer to cardinal arithmetic:

(1) $\kappa+\lambda=\lambda+\kappa$,
(2) $\kappa\cdot\lambda=\lambda\cdot\kappa$,
(3) $(\kappa+\lambda)\cdot\theta=\kappa\cdot\theta+\lambda\cdot\theta$,
(4) $\kappa^{(\lambda\cdot\theta)}=(\kappa^\lambda)^\theta$,
(5) $\kappa^{(\lambda+\theta)}=\kappa^\lambda\cdot\kappa^\theta$.

PROOF. To see (1) note that $A\cup B=B\cup A$. To see (2) note that $A\times B=B\times A$. To see (3) observe that $(A\cup B)\times C=A\times C\cup B\times C$. To see (4) note that ${}^C({}^BA)\approx{}^{C\times B}A$. To see (5) observe that ${}^{(B\cup C)}A\approx{}^BA\times{}^CA$ provided that B,C are disjoint. □

EXAMPLE 3.4.

(1) ω, $\omega\cdot\omega$, $\omega+\omega$ are three different ordinals, all of the same cardinality.
(2) ω^ω as ordinal exponentiation is equal to $\sup_{n\in\omega}\omega^n$, which is a countable set.
(3) However, ω^ω as cardinal exponentiation is uncountable: $|{}^\omega\omega|=|\mathscr{P}(\omega)|=\aleph_0^{\aleph_0}=2^{\aleph_0}$ (to be proven shortly).

LEMMA 3.5. Let κ, λ be cardinals and suppose at least one of them is infinite.

- Then the cardinal sum of κ and λ is equal to $\max\{\kappa,\lambda\}$.
- If none of them is 0, then the cardinal product of κ and λ is equal to $\max(\kappa,\lambda)$.

PROOF. Let $\kappa\le\lambda$. Thus λ is infinite. But then

$$\lambda\preceq\kappa+\lambda\preceq\lambda\times\lambda.$$

However we proved that $\lambda\times\lambda\approx\lambda$. Therefore $\lambda\preceq\kappa+\lambda$ and $\kappa+\lambda\preceq\lambda$. Therefore $\kappa+\lambda=\max\{\kappa,\lambda\}=\lambda$.

To see the second claim assume that $\kappa\le\lambda$. Thus λ is infinite. Then

$$\lambda\preceq\kappa\times\lambda\preceq\lambda\times\lambda\approx\lambda$$

and so $\kappa\times\lambda\approx\lambda$. □

LEMMA 3.6. If $2\le\kappa\le2^\lambda$ and λ is infinite, then $\kappa^\lambda=2^\lambda$. All exponentiation here is cardinal exponentiation.

PROOF. $2^\lambda\preceq\kappa^\lambda\preceq(2^\lambda)^\lambda\preceq2^{\lambda\times\lambda}=2^{\lambda\cdot\lambda}=2^\lambda$. □ □

COROLLARY 3.7. $2^\omega = \omega^\omega$.

REMARK 3.8. (CH and GCH)

(1) For every ordinal α, $2^{\aleph_\alpha} \geq \aleph_{\alpha+1}$.
(2) The Continuum Hypothesis(abbreviated CH) is the statement that
$$2^{\aleph_0} = \aleph_1.$$
(3) The Generalized Continuum Hypothesis (abbreviated GCH) is the statement
$$2^{\aleph_\alpha} = \aleph_{\alpha+1}$$
for all $\alpha \in \mathbb{ON}$.

REMARK 3.9. Thus CH is the statement that the cardinality of the real line is the first uncountable cardinal, i.e. $|\mathbb{R}| = \aleph_1$. If CH holds, then there are no infinite sizes between $|\mathbb{N}|$ and $|\mathbb{R}|$.

CHAPTER 4

Cofinality and Lemma of König

1. Cofinality

DEFINITION 1.1. (Cofinality)

(1) If γ is a limit ordinal, then the cofinality of γ is defined as follows:

$$\mathrm{cf}(\gamma) = \min\{\mathrm{type}(X) : X \subseteq \gamma \wedge \sup(X) = \gamma\}.$$

(2) We say that γ is a regular ordinal, if $\mathrm{cf}(\gamma) = \gamma$.

REMARK 1.2. Note that $\mathrm{cf}(\gamma) \leq \gamma$.

EXAMPLE 1.3.

$$\aleph_0 < \aleph_1 < \ldots < \aleph_n < \ldots < \aleph_\omega < \ldots$$

LEMMA 1.4. Let γ be a limit ordinal. Then:

(1) If $A \subseteq \gamma$ and $\sup(A) = \gamma$, then $\mathrm{cf}(\gamma) = \mathrm{cf}(\mathrm{type}(A))$.
(2) $\mathrm{cf}(\mathrm{cf}(\gamma)) = \mathrm{cf}(\gamma)$. Thus $\mathrm{cf}(\gamma)$ is a regular ordinal.
(3) $\omega \leq \mathrm{cf}(\gamma) \leq |\gamma| \leq \gamma$.
(4) If γ is a regular ordinal, then γ is a cardinal.

PROOF. (1) Let $\alpha = \mathrm{type}(A)$. Since γ is limit and A is unbounded in γ, α must be limit as well. Let $f : (\alpha, \in) \to (A, \in)$ be an isomorphism.

<u>$\mathrm{cf}(\gamma) \leq \mathrm{cf}(\alpha)$</u>: If $Y \subseteq \alpha$ is unbounded in α, then $f''(Y)$ is unbounded in γ and $\mathrm{type}(f''(Y)) = \mathrm{type}(Y)$. Now, take $Y \subseteq \alpha$ such that $\mathrm{type}(Y) = \mathrm{cf}(\alpha)$. Then $Y \subseteq \gamma$ is unbounded in γ, $\mathrm{type}(Y) = \mathrm{cf}(\alpha)$. Thus $\mathrm{cf}(\gamma) \leq \mathrm{cf}(\alpha)$.

<u>$\mathrm{cf}(\alpha) \leq \mathrm{cf}(\gamma)$</u>: Let $X \subseteq \gamma$ be unbounded and let $\mathrm{type}(X) = \mathrm{cf}(\gamma)$ and consider the mapping $h : X \to A (\subseteq \gamma)$ given by:

$$h(\zeta) = \min\{\eta : \eta \in A \wedge \eta \geq \zeta\}.$$

Then h is non-decreasing. Consider the set

$$X' = \{\eta \in X : \forall \xi \in X \cap \eta\, (h(\xi) < h(\eta))\}.$$

Therefore $h \restriction X' : X' \to A$ is order preserving and so injective. Thus $h(X')$ is unbounded in A. However the set A was chosen to be of order type α. Therefore

$$\mathrm{cf}(\alpha) \leq \mathrm{type}(X') \leq \mathrm{type}(X) = \mathrm{cf}(\gamma).$$

(2) Let $A \subseteq \gamma$ be an unbounded subset of γ of order type $\mathrm{cf}(\gamma)$. Then by part (1) of this Lemma, $\mathrm{cf}(\gamma) = \mathrm{cf}(\mathrm{type}(A)) = \mathrm{cf}(\mathrm{cf}(\gamma))$.

(3) By definition $\omega \leq \mathrm{cf}(\gamma)$ and $|\gamma| \leq \gamma$. So, we need to show that $\mathrm{cf}(\gamma) \leq |\gamma|$. For this purpose, let $\kappa := |\gamma|$ and fix an onto function $f : \kappa \to \gamma$. Recursively, define the following function $g : \kappa \to \mathbb{ON}$:

$$g(\eta) := \max\{f(\eta), \sup\{g(\xi)+1 : \xi < \eta\}\}.$$

What can we say about g?

(1) $\mathrm{dom}(g) = \mathrm{dom}(f) = \kappa$,
(2) $g(\eta) \geq f(\eta)$ for all $\eta \in \kappa$,
(3) if $\xi < \eta$ then $g(\xi) < g(\eta)$, because $g(\eta) \geq g(\xi)+1 > g(\xi)$,
(4) If $\eta = \zeta + 1$, then

$$g(\zeta+1) = \max\{f(\zeta+1), \sup\{g(\xi) : \xi \leq \zeta\}\} = \max\{f(\zeta+1), g(\zeta)+1\}.$$

In particular we have that $g : \kappa \cong \mathrm{ran}(g)$ and so $\mathrm{type}(\mathrm{ran}(g)) = \kappa$.

If $\mathrm{ran}(g) \subseteq \gamma$, then since $g(\eta) \geq f(\eta)$ and $\mathrm{ran}(f) = \gamma$, we have $\mathrm{ran}(g)$ is unbounded in γ. Therefore $\mathrm{cf}(\gamma) \leq \kappa = |\gamma|$. Done!

If $\mathrm{ran}(g) \not\subseteq \gamma$, we can find $\eta \in \kappa$ least such that $g(\eta) \geq \gamma$.

Suppose $\eta = \xi + 1$. Then

$$g(\eta) = g(\xi+1) = \max\{g(\xi)+1, f(\eta)\}.$$

However $g(\eta) \geq \gamma$ and $f(\eta) < \gamma$. Thus $g(\eta) = g(\xi)+1$. By minimality of η, $g(\xi) < \gamma$ and so $g(\xi)+1 \leq \gamma$. Therefore $g(\eta) = g(\xi)+1 \leq \gamma \leq g(\eta)$. But then $\gamma = g(\xi)+1$ is a successor, which is a contradiction!

Therefore η is a limit ordinal and $g''\eta$ is unbounded in γ. Moreover $g \restriction \eta : \eta \approx g''\eta$. In particular $\mathrm{type}(g''\eta) \leq \eta$.

Then $\mathrm{cf}(\gamma) \leq \mathrm{type}(g''\eta) \leq \eta < \kappa = |\gamma|$. Done!

(4) This is a direct corollary to (3). Indeed, suppose γ is regular. Then $\gamma = \mathrm{cf}(\gamma)$. But, by item (3)

$$\mathrm{cf}(\gamma) \leq |\gamma| \leq \gamma.$$

Thus $\gamma \leq |\gamma| \leq \gamma$ and so $\gamma = |\gamma|$ is a cardinal. □

DEFINITION 1.5. (Regular and Singular Cardinals) Let γ be an infinite cardinal.

(1) If $\gamma = \mathrm{cf}(\gamma)$, we say that γ is regular.
(2) If $\mathrm{cf}(\gamma) < \gamma$, we say that γ is singular.

REMARK 1.6. By the previous Lemma, part (1), we have that $\mathrm{cf}(\alpha+\beta) = \mathrm{cf}(\beta)$. Indeed, the set

$$A = \{\alpha + \xi : \xi < \beta\}$$

is unbounded in $\alpha+\beta$. Thus, for every limit ordinal $\gamma < \omega_1$,

$$\mathrm{cf}(\gamma) = \omega.$$

For every limit ordinal γ such that $\gamma < \omega_2$,

$$\text{either } \mathrm{cf}(\gamma) = \omega \text{ or } \mathrm{cf}(\gamma) = \omega_1.$$

LEMMA 1.7. Let γ be a limit ordinal.

(1) Suppose $\gamma = \aleph_\alpha$, where $\alpha = 0$ or $\alpha = \beta+1$ is a successor ordinal. Then γ is regular.
(2) If $\gamma = \aleph_\alpha$ for a limit ordinal α, then $\mathrm{cf}(\gamma) = \mathrm{cf}(\alpha)$.

PROOF. (1) If $\alpha = 0$, then $\aleph_\alpha = \aleph_0 = \omega$ and $\omega \leq \text{cf}(\omega) \leq |\omega| \leq \omega$ is regular. Thus, suppose $\gamma = \aleph_{\beta+1}$. Consider any $A \subseteq \aleph_{\beta+1}$ such that

$$\text{type}(A) < \aleph_{\beta+1}.$$

It is sufficient to show that A is not unbounded in $\aleph_{\beta+1}$, since then $\aleph_{\beta+1} \leq \text{cf}(\gamma)$. But $\text{cf}(\gamma) \leq |\gamma| = \aleph_{\beta+1}$ and so $\text{cf}(\aleph_{\beta+1}) = \aleph_{\beta+1}$.

To show that A is not unbounded in γ, consider $\sup A = \bigcup A$. Note that $|A| \leq \aleph_\beta$, because $|A| \leq \text{type}(A) < \aleph_{\beta+1}$.

- Moreover, every element of A is of cardinality at most $\aleph_\beta$. Therefore we can view A as a collection of $\leq \aleph_\beta$-many sets, each of cardinality at most $\aleph_\beta$.
- Then, by the Axiom of Choice we obtain that $|\sup A| = |\bigcup A| \leq \aleph_\beta$ (see Lemma A).

Thus $\sup A < \aleph_{\beta+1}$ (otherwise contradiction to the notion of a cardinal!) Thus A can not be unbounded in $\aleph_{\beta+1}$.

(2) Let $A = \{\aleph_\xi : \xi < \alpha\}$. Then $A \subseteq \aleph_\alpha$ and $\sup A = \aleph_\alpha$. By a previous Lemma

$$\text{cf}(\aleph_\alpha) = \text{cf}(\text{type}(A)).$$

However $\text{cf}(\text{type}(A)) = \text{cf}(\alpha)$. Thus $\text{cf}(\aleph_\alpha) = \text{cf}(\alpha)$. □

EXAMPLE 1.8.

- $\text{cf}(\aleph_n) = \aleph_n$ for each $n \in \omega$, and
- $\text{cf}(\aleph_\omega) = \omega$.

2. König's Lemma

LEMMA 2.1. (AC) Let A, B be sets such that $A \neq \emptyset$. Then there is an injective function $g : B \to A$ if and only if there is an onto function $f : A \to B$.

PROOF. ($\Leftarrow$) Suppose there is an onto mapping $f : A \to B$. Then $\{f^{-1}(b)\}_{b \in B}$ is a non-empty family of non-empty sets and so for each $b \in B$ we can chose $a_b \in f^{-1}(b)$ such that $f(a_b) = b$. Since $f^{-1}(b_1) \cap f^{-1}(b_2) = \emptyset$ whenever $b_1 \neq b_2$, we must have $a_{b_1} \neq a_{b_2}$. Therefore the mapping $g : B \to A$ defined by $g(b) = a_b$ is injective.

($\Rightarrow$) Let $g : B \to A$ be an injective mapping. For each $a \in \text{ran}(g)$, we have $a = g(b)$ for some $b \in B$ (note that b is unique by the injectivity of g). For such a's define $f(a) = b$, i.e. define $f : A \to B$ so that $f \restriction \text{ran}(g) = g^{-1}$. It remains to define $f \restriction (A \setminus \text{ran}(g))$. To do this, fix an arbitrary $b^* \in B$ and for each $a \in A \setminus \text{ran}(g)$ define $f(a) = b^*$. Thus $f = g^{-1} \cup ((A \setminus \text{ran}(g)) \times \{b^*\})$ is an onto mapping from A to B. □ □

LEMMA 2.2. (AC) Let κ be an infinite cardinal. If $\mathscr{F}$ is a family of sets with $|\mathscr{F}| \leq \kappa$ and $|X| \leq \kappa$ for each $X \in \mathscr{F}$, then $|\bigcup \mathscr{F}| \leq \kappa$.

PROOF. Assume $\mathscr{F} \neq \emptyset$ and $\emptyset \notin \mathscr{F}$. Then there is an onto function $f : \kappa \to \mathscr{F}$. Similarly, for each $B \in \mathscr{F}$ fix an onto function

$$g_B : \kappa \to f(\alpha).$$

This defines an onto mapping $h : \kappa \times \kappa \to \bigcup \mathscr{F}$ given by

$$h(\alpha, \beta) = g_{f(\alpha)}(\beta).$$

Since $|\kappa \times \kappa| = \kappa$, we obtain an onto mapping from κ onto $\bigcup \mathscr{F}$. □

THEOREM 2.3. *(AC) Let θ be a cardinal.*

(1) Suppose θ is regular and $\mathscr{F}$ is a family of sets, such that $|\mathscr{F}| < \theta$ and moreover $|S| < \theta$ for all $S \in \mathscr{F}$. Then $|\bigcup \mathscr{F}| < \theta$.

(2) *Suppose* $\mathrm{cf}(\theta)=\lambda<\theta$. *Then there is a family* $\mathscr{F}$ *of subsets of* θ *with* $|\mathscr{F}|=\lambda$ *and* $|\bigcup\mathscr{F}|=\theta$ *such that* $|S|<\theta$ *for all* $S\in\mathscr{F}$.

PROOF. (1) Let $X=\{|S|:S\in\mathscr{F}\}$. Then $X\subseteq\theta$, $|X|<\theta$ and so $\mathrm{type}(X)<\theta$. Since θ is regular, $\mathrm{type}(X)<\mathrm{cf}(\theta)$ and so X is not unbounded in θ. Thus $\sup(X)<\theta$. Consider $\kappa:=\max\{\sup(X),|\mathscr{F}|\}$. Then $\kappa<\theta$. If κ is infinite, then by Lemma A $|\bigcup\mathscr{F}|\leq\kappa$. If κ is finite, then $\bigcup\mathscr{F}$ is finite. In either of those two cases $|\bigcup\mathscr{F}|<\theta$.

(2) Just take $\mathscr{F}$ to be a subset of θ such that $\mathrm{type}(\mathscr{F})=\lambda$ and $\sup(\mathscr{F})=\bigcup\mathscr{F}=\theta$. □

THEOREM 2.4. *(König) Let* $\kappa\geq 2$ *and* λ *be infinite. Then* $\mathrm{cf}(\kappa^\lambda)>\lambda$.

PROOF. Let $\theta=\kappa^\lambda$. Note that θ is infinite and $\theta^\lambda=\kappa^{\lambda\cdot\lambda}=\kappa^\lambda=\theta$. Thus, we can enumerate ${}^\lambda\theta$ is order type θ, i.e. ${}^\lambda\theta=\{f_\alpha:\alpha\in\theta\}$. There are two options. Either $\mathrm{cf}(\kappa^\lambda)\leq\lambda$ or $\mathrm{cf}(\kappa^\lambda)>\lambda$.

If $\mathrm{cf}(\kappa^\lambda)\leq\lambda<2^\lambda\leq\kappa^\lambda$, then by Lemma B we have $\theta=\bigcup_{\xi<\lambda}S_\xi$, where each $|S_\xi|<\theta$. Let $g:\lambda\to\theta$ be the function $g(\xi)=\min(\theta\setminus\{f_\alpha(\xi):\alpha\in S_\xi\})$. Then $g\in{}^\lambda\theta$ and so there is $\alpha\in\theta$ such that $g=f_\alpha$. Take $\xi<\lambda$ such that $\alpha\in S_\xi$. Then $g(\xi)\neq f_\alpha(\xi)$, contradiction.

Therefore $\mathrm{cf}(\kappa^\lambda)>\lambda$. □

EXAMPLE 2.5.

(1) $\mathrm{cf}(2^{\aleph_0})>\aleph_0=\omega$ and so $2^{\aleph_0}$ can not be $\aleph_\omega$.
(2) Consistently (using the method of forcing) $2^{\aleph_0}$ is any cardinal of uncountable cofinality, e.g. $\aleph_{2020}$, $\aleph_{\omega+1}$, $\aleph_{\omega_1}$, etc.

THEOREM 2.6. *Assume GCH. Let* κ, λ *be cardinals such that* $\max\{\kappa,\lambda\}\geq\omega$.

(1) *Suppose* $2\leq\kappa\leq\lambda^+$. *Then* $\kappa^\lambda=\lambda^+$.
(2) *Suppose* $1\leq\lambda\leq\kappa$. *Then* $\kappa^\lambda=\kappa$ *provided that* $\lambda<\mathrm{cf}(\kappa)$ *and* $\kappa^\lambda=\kappa^+$ *provided that* $\lambda\geq\mathrm{cf}(\kappa)$.

PROOF. (1) Since we have GCH, $2^\lambda=\lambda^+$. Then $2\leq\kappa\leq 2^\lambda$. But then

$$2^\lambda\leq\kappa^\lambda\leq(2^\lambda)^\lambda=2^{\lambda\cdot\lambda}=2^\lambda$$

and so $\kappa^\lambda=2^\lambda$. Thus by GCH we obtain $\kappa^\lambda=\lambda^+$.

(2) Since $1\leq\lambda\leq\kappa$ we have that $\kappa\leq\kappa^\lambda\leq\kappa^\kappa=2^\kappa=\kappa^+$ (the latter equality by GCH). Therefore either $\kappa^\lambda=\kappa$ or $\kappa^\lambda=\kappa^+$. By König's Lemma $\mathrm{cf}(\kappa^\lambda)>\lambda$. Thus:

- If $\mathrm{cf}(\kappa)\leq\lambda$, then $\kappa^\lambda\neq\kappa$. Therefore $\kappa^\lambda=\kappa^+$. Done!
- If $\lambda<\mathrm{cf}(\kappa)$, then every $f:\lambda\to\kappa$ is bounded. Thus for all $f\in{}^\lambda\kappa$ there is $\alpha_f<\kappa$ such that $f\in{}^\lambda\alpha_f$ and so ${}^\lambda\kappa=\bigcup_{\alpha<\kappa}{}^\lambda\alpha$. Now ${}^\lambda\alpha\subseteq\mathscr{P}(\lambda\times\alpha)$ and for $\alpha<\kappa$, $|\lambda\times\alpha|<\kappa$. Therefore $|{}^\lambda\alpha|\leq\kappa$ by GCH. Then by Lemma 2.2 we have also $|{}^\lambda\kappa|\leq\kappa$ and so $\kappa^\lambda=\kappa$. Done!

□

DEFINITION 2.7. (The beth function) By recursion on the ordinals define $\beth_\zeta$ as follows:

(1) $\beth_0=\aleph_0=\omega$,
(2) $\beth_{\zeta+1}=2^{\beth_\zeta}$,
(3) $\beth_\eta=\sup\{\beth_\zeta:\zeta<\eta\}$ for η limit ordinal.

REMARK 2.8. CH is equivalent to the statement that $\beth_1=\aleph_1$ and GCH is equivalent to the statement that $\beth_\xi=\aleph_\xi$ for all $\xi\in\mathbb{ON}$.

DEFINITION 2.9.

- A cardinal κ is said to be weakly inaccessible if $\kappa>\omega$, κ is regular and $\kappa>\lambda^+$ for all $\lambda<\kappa$.

- A cardinal κ is strongly inaccessible if $\kappa > \omega$ is regular and $\kappa > 2^{\lambda}$ for all $\lambda < \kappa$.

REMARK 2.10. If κ is strong inaccessible, then κ is weakly inaccessible. The existence of a strong inaccessible cardinal is not provable in ZFC.

Part 2

Structures and Theories

CHAPTER 5

Structures and Embeddings

1. Structures, Substructures, Expansions

DEFINITION 1.1. ($\mathscr{L}$-structure) Let $\mathscr{L}$ be a language. An $\mathscr{L}$-structure is a pair $\mathfrak{A} = (A, (Z^{\mathfrak{A}}))_{Z\in\mathscr{L}}$ where:

(1) A is a non-empty set, referred to as the *domain* or *universe* of $\mathfrak{A}$,
(2) $Z^{\mathfrak{A}} \in A$ whenever Z is a constant symbol,
(3) $Z^{\mathfrak{A}} : A^n \to A$ whenever Z is an n-ary function symbol,
(4) $Z^{\mathfrak{A}} \subseteq A^n$ if Z is an n-ary relation symbol.

By the cardinality of a structure $\mathfrak{A}$, we understand the cardinality of its universe.

DEFINITION 1.2. (Homomorphic Structures) Let $\mathfrak{A}$ and $\mathfrak{B}$ be $\mathscr{L}$-structures.

(1) A map $h : A \to B$ is called a *homomorphism* if:
  - for all $c \in \mathscr{C}_{\mathscr{L}}$
$$h(c^{\mathfrak{A}}) = c^{\mathfrak{B}},$$
  - for all $a_1, \cdots, a_n \in A$ and all $f \in \mathscr{F}_{\mathscr{C}}$
$$h(f^{\mathfrak{A}}(a_1, \cdots, a_n)) = f^{\mathfrak{B}}(h(a_1), \cdots, h(a_n)),$$
  - for all $a_1, \cdots, a_n \in A$ and all $R \in \mathscr{R}_{\mathscr{L}}$
$$\text{if } R^{\mathfrak{A}}(a_1, \cdots, a_n) \text{ then } R^{\mathfrak{B}}(h(a_1), \cdots, h(a_n)).$$

DEFINITION 1.3. (Embedding) An injective homomorphism is called an embedding.

DEFINITION 1.4. (Isomorphic Structures) Let $\mathfrak{A}$ and $\mathfrak{B}$ be $\mathscr{L}$-structures.

(1) If $h : \mathfrak{A} \to \mathfrak{B}$ is an injective homomorphism and
$$R^{\mathfrak{A}}(a_1, \cdots, a_n) \text{ iff } R^{\mathfrak{B}}(h(a_1), \cdots, h(a_n)),$$
then h is called an *(isomorphic) embedding*.
(2) An *isomorphism* is a surjective embedding.
(3) Two structures are said to be *isomorphic* if there is an isomorphism between them. We use the notation $\mathfrak{A} \cong \mathfrak{B}$.

DEFINITION 1.5. (Automorphism) An *automorphism* of a structure $\mathfrak{A}$ is an isomorphism of $\mathfrak{A}$ with itself. The set of automorphisms of a structure $\mathfrak{A}$ is denoted $\mathrm{Aut}(\mathfrak{A})$.

EXAMPLE 1.6. Show that $\mathrm{Aut}(\mathfrak{A})$ is a group under the operation of composition.

DEFINITION 1.7. (Substructure)

(1) Let $\mathfrak{A}, \mathfrak{B}$ be structures. We say that $\mathfrak{A}$ is a substructure of $\mathfrak{B}$, denoted

$$\mathfrak{A} \subseteq \mathfrak{B},$$

if the universe A of $\mathfrak{A}$ is a subset of the universe B of $\mathfrak{B}$ and the identity mapping is an embedding.

(2) If $\mathfrak{A}$ is a substructure of $\mathfrak{B}$, we also say that $\mathfrak{B}$ is an extension of $\mathfrak{A}$.

EXAMPLE 1.8. Let $\mathscr{L} = (\mathscr{C}, \mathscr{F}, \mathscr{R})$ be a language and let $\mathfrak{B}$ be a $\mathscr{L}$-structure.

(1) Suppose $A \neq \emptyset$ and $A \subseteq B$, where B is the universe of $\mathfrak{B}$. Show that A is the universe of a uniquely determined substructure $\mathfrak{A}$ of $\mathfrak{B}$ if and only if the set A is closed under all functions $f^{\mathfrak{B}}$, where $f \in \mathscr{F}_{\mathscr{L}}$. That is, for each n-ary f we must have

$$f^{\mathfrak{B}} \restriction A^n : A^n \to A.$$

(2) If $\mathscr{F}_{\mathscr{L}} = \mathscr{C}_{\mathscr{L}} = \emptyset$, then any non-empty subset C of B is the universe of a substructure $\mathfrak{C}$ of $\mathfrak{B}$.

REMARK 1.9. Let $\mathfrak{B}$ be a $\mathscr{L}$-structure. Let $\{\mathfrak{A}_i\}_{i\in I}$ be an enumeration of all substructures of $\mathfrak{B}$. Note that:

(1) $\{c^{\mathfrak{B}} : c \in \mathscr{C}_{\mathscr{L}}\} \subseteq \bigcap_{i\in I} A_i$ (indeed $c^{\mathfrak{B}} = c^{\mathfrak{A}_i} \in A_i$ for each i),

(2) $\bigcap_{i\in I} A_i$ is closed under $f^{\mathfrak{B}}$ for all $f \in \mathscr{F}_{\mathscr{L}}$ (justify!).

Thus, if $\mathscr{C}_{\mathscr{L}} \neq \emptyset$ then

$$\bigcap_{i\in I} A_i \neq \emptyset$$

is the universe of a substructure of $\mathfrak{B}$. Moreover, this is the universe of the smallest substructure of $\mathfrak{B}$ (Why? Explain!), denoted $\langle\emptyset\rangle^{\mathfrak{B}}$. If $\mathscr{C}_{\mathscr{L}} = \emptyset$, then we set $\langle\emptyset\rangle^{\mathfrak{B}} = \emptyset$.

EXAMPLE 1.10. (Generated Substructures) Let $\mathfrak{B}$ be a structure and let $S \neq \emptyset$ be a subset of the universe of $\mathfrak{B}$. Let $\mathrm{St}(S) = \{\mathfrak{A} \subseteq \mathfrak{B} : S \subseteq A\}$. Show that:

(1) $\bigcap\{A : \mathfrak{A} \in \mathrm{St}(S)\}$ is the domain of a structure $\mathfrak{M}$ such that S is contained in its universe;

(2) $\mathfrak{M}$ is the smallest substructure of $\mathfrak{B}$ containing S in its universe.[1]

DEFINITION 1.11. We refer to $\mathfrak{M}$ as the structure generated by S and denote it $\langle S\rangle^{\mathfrak{B}}$. If S is finite, then we say that $\mathfrak{M} = \langle S\rangle^{\mathfrak{B}}$ is finitely generated.

LEMMA 1.12. *Let $\mathfrak{A}$ be a structure generated by the set S. Then, every homomorphism $h : \mathfrak{A} \to \mathfrak{B}$ is determined by its values on the set S.*

PROOF. Suppose $h, h' : \mathfrak{A} \to \mathfrak{B}$ are homomorphisms from $\mathfrak{A}$ to $\mathfrak{B}$ such that for each $s \in S$ we have that $h(s) = h(s')$. Show that $h = h'$. □

LEMMA 1.13. *Let $\mathfrak{A} \cong \mathfrak{A}'$ be isomorphic structures witnessed by an isomorphism h. Whenever $\mathfrak{B}$ is an extension of $\mathfrak{A}$ (i.e. $\mathfrak{A} \subseteq \mathfrak{B}$), then there is an extension $\mathfrak{B}'$ of $\mathfrak{A}$ and an isomorphism $g : \mathfrak{B} \cong \mathfrak{B}'$ which extends h, i.e. $g \restriction A = h$.*

PROOF. Extend the bijection $h : A \to A'$ to a bijection $g : B \to B'$ and use g to define an $\mathscr{L}$-structure on B'. □

DEFINITION 1.14. (Directed System of Structures, Chains) Let $\langle I, \leq\rangle$ be a directed partial order. This means that for all $i, j \in I$ there exists $k \in I$ such that $i \leq k$ and $j \leq k$.

(1) A family $\{\mathfrak{A}_i\}_{i\in I}$ of $\mathscr{L}$-structures is said to be *directed* if whenever $i \leq j$, then $\mathfrak{A}_i \subseteq \mathfrak{A}_j$.

(2) If the set I is linearly ordered, we say that the family $\{\mathfrak{A}_i\}_{i\in I}$ is a *chain*.

---

[1] What is the partial order on the collection of all substructures of $\mathfrak{B}$ that we are referring to here?

LEMMA 1.15. Let $\{\mathfrak{A}_i\}_{i\in I}$ be a directed family of $\mathscr{L}$-structures. Then $A=\bigcup_{i\in I}A_i$ is the universe of a uniquely determined $\mathscr{L}$-structure, denoted $\bigcup_{i\in I}\mathfrak{A}_i$, which is an extension of each $\mathfrak{A}_i$.

PROOF. Let $A=\bigcup_{i\in I}A_i$. We will define a structure $\mathfrak{A}$ with universe A as desired. Let R be an n-ary relation symbol and $a_1,\cdots,a_n$ elements of A. Find $k\in I$ such that $\{a_i\}_{i=1}^n\subseteq A_k$ and define $R^{\mathfrak{A}}(a_1,\cdots,a_n)$ iff $R^{\mathfrak{A}_k}(a_1,\cdots,a_k)$. Note that to claim the existence of k we have to use fact that $\{\mathfrak{A}_i\}_{i\in I}$ is directed. Constant and function symbols are treated similarly. □

DEFINITION 1.16. (Reduct, Expansion) Let $K\subseteq L$ be a sublanguage, i.e. $\mathscr{C}_{\mathscr{K}}\subseteq\mathscr{C}_{\mathscr{L}}$, $\mathscr{R}_{\mathscr{K}}\subseteq\mathscr{R}_{\mathscr{L}}$ and $\mathscr{F}_{\mathscr{K}}\subseteq\mathscr{F}_{\mathscr{L}}$.

(1) To every $\mathscr{L}$-structure, we associate a $\mathscr{K}$-structure, called the reduct to $\mathscr{K}$, by forgetting the interpretation symbols from $\mathscr{L}\backslash\mathscr{K}$.
(2) The reduct is usually denoted $\mathfrak{A}\restriction K=(A,(Z^{\mathfrak{A}})_{Z\in\mathscr{K}})$.
(3) Conversely, $\mathfrak{A}$ is said to be an expansion of $\mathfrak{A}\restriction\mathscr{K}$.

EXAMPLE 1.17.

(1) Let R be an n-ary relation on A. Introduce a new relation symbol $\tilde{R}$ and denote by $(\mathfrak{A},R)$ the expansion $\mathfrak{B}$ of $\mathfrak{A}$ to an $\mathscr{L}\cup\{\tilde{R}\}$-structure in which $\tilde{R}$ is interpreted by R, i.e. $\tilde{R}^{\mathfrak{B}}=R$.
(2) For given elements $a_1,\cdots,a_n$ of $\mathfrak{A}$ introduce new constants $\underline{a}_1,\underline{a}_2,\cdots,\underline{a}_n$ and consider the $\mathscr{L}\cup\{\underline{a}_1,\cdots,\underline{a}_n\}$-structure
$$\mathfrak{B}=(\mathfrak{A},\{a_i\}_{i=1}^n)$$
where $\underline{a}_i^{\mathfrak{B}}=a_i$, and $\mathfrak{B}$ has the same universe as $\mathfrak{A}$.

EXAMPLE 1.18.

(1) Let $B\subseteq A$, where A is the universe of the structure $\mathfrak{A}$. Then, for every element b of the set B we can introduce a new constant symbol $\underline{b}$. Thus, we expand the language $\mathscr{L}$ to a new language $\mathscr{L}(B)=\mathscr{L}\cup\{\underline{b}: b\in B\}$ and the structure $\mathfrak{A}$ to a $\mathscr{L}(B)$-structure
$$\mathfrak{A}_B=(\mathfrak{A},b)_{b\in B}.$$
(2) The group of automorphisms of $\mathfrak{A}_B$ consists of exactly those automorphisms f of $\mathfrak{A}$ which are the identity on B, i.e. for all $b\in B(f(b)=b)$.
(3) In general, whenever we expand a language $\mathscr{L}$ by a set of new constant symbols C, we denote the new language with $\mathscr{L}(C)$.

Recall the notion of a term:

DEFINITION 1.19. (Term)

(1) Every variable v_i and every constant c is an $\mathscr{L}$-term.
(2) If f is an n-ary function symbol and $t_1,\cdots,t_n$ are $\mathscr{L}$-terms, then $ft_1\cdots t_n$ is also an $\mathscr{L}$-term.
(3) The number of occurrences of a function symbols in a term is called its complexity.

DEFINITION 1.20. (Term evaluation) Let $\mathfrak{A}$ be a $\mathscr{L}$-structure.

(1) A mapping which assigns to every variable v_i a value $b_i\in A$, where $i\in\mathbb{N}$, is said to be an assignment. We denote such assignments with $\vec{b}$.
(2) For an $\mathscr{L}$-term t and an assignment $\vec{b}$, we define the interpretation $t^{\mathfrak{A}}[\vec{b}]$ by
    - $v_i^{\mathfrak{A}}[\vec{b}]=b_i$,
    - $c^{\mathfrak{A}}[\vec{b}]=c^{\mathfrak{A}}$,

- $ft_1 \cdots t_n^{\mathfrak{A}}[\vec{b}] = f^{\mathfrak{A}}(t_1^{\mathfrak{A}}[\vec{b}], \cdots, t_n^{\mathfrak{A}}[\vec{b}])$.

EXAMPLE 1.21. (Term evaluation) Suppose $\mathfrak{A}_A$ is a structure associated to an expanded language $\mathscr{L}(A)$, where A is the universe of a $\mathscr{L}$-structure $\mathfrak{A}$. If $t(x_1, \cdots, x_n)$ is an $\mathscr{L}$-term, then $t(a_1, \cdots, a_n)$ is a term in the expanded language $\mathscr{L}(A)$ and

$$t(a_1, \cdots, a_n)^{\mathfrak{A}_A} = t^{\mathfrak{A}}[a_1, \cdots, a_n].$$

PROOF. Induction on the complexity of t. □

REMARK 1.22. Let $\mathfrak{A}$ be a $\mathscr{L}$-structure and suppose $S \subseteq A$, where A is the universe of $\mathfrak{A}$. Then the structure $\langle S \rangle^{\mathfrak{A}}$ generated by S is closed under the evaluation of terms. In fact:

(1) $\langle S \rangle^{\mathfrak{A}} = \{t^{\mathfrak{A}}[a_1, \cdots, a_n] : t(x_1, \cdots, x_n) \text{ is an } \mathscr{L}\text{-term}, a_1, \cdots, a_n \in A\}$.
(2) $\langle \emptyset \rangle^{\mathfrak{A}} = \{t^{\mathfrak{A}} : t \text{ is a term with no variables}\}$.

Recall the definition:

DEFINITION 1.23. ($\mathscr{L}$-formulas)

(1) $t_1 \doteq t_2$ where t_1, t_2 are $\mathscr{L}$-terms;
(2) $Rt_1 \cdots t_n$ where R is an n-ary relation symbol from $\mathscr{L}$ and $t_1, \cdots, t_n$ are $\mathscr{L}$-terms;
(3) $\neg\psi$ where ψ is an $\mathscr{L}$-formula;
(4) $(\psi_1 \wedge \psi_2)$ where ψ_1 and ψ_2 are $\mathscr{L}$-formulas;
(5) $(\exists x \psi)$ where ψ is an $\mathscr{L}$-formula, x is a variable.

The formulas corresponding to items $(1)-(2)$ above are called atomic formulas. The number of occurrences of $\neg, \exists, \wedge$ in a formula is referred to as the complexity of that formula.

DEFINITION 1.24. Let $\mathfrak{A}$ be an $\mathscr{L}$-structure. For an $\mathscr{L}$-formula ψ and all assignments $\vec{b}$ define the relation $\mathfrak{A} \models \varphi[\vec{b}]$ recursively over the complexity of φ:

(1) $\mathfrak{A} \models t_1 \doteq t_2[\vec{b}]$ iff $t_1^{\mathfrak{A}}[\vec{b}] = t_2^{\mathfrak{A}}[\vec{b}]$,
(2) $Rt_1 \cdots t_n[\vec{b}]$ iff $R^{\mathfrak{A}}(t_1^{\mathfrak{A}}[\vec{b}], \cdots, t_n^{\mathfrak{A}}[\vec{b}])$,
(3) $\mathfrak{A} \models \neg\psi[\vec{b}]$ iff $\mathfrak{A} \not\models \psi[\vec{b}]$.
(4) $\mathfrak{A} \models (\psi_1 \wedge \psi_2)[\vec{b}]$ iff $\mathfrak{A} \models \psi_1[\vec{b}]$ and $\mathfrak{A} \models \psi_2[\vec{b}]$,
(5) $\exists x \psi[\vec{b}]$ iff $\exists a \in A$ such that $\mathfrak{A} \models \psi[\vec{b}\frac{a}{x}]$, where $\vec{b}\frac{a}{x}$ is the assignment which maps each v_i (except x) to b_i and x to a[2]

If $\mathfrak{A} \models \varphi[\vec{b}]$ holds, then we say that φ holds in the structure $\mathfrak{A}$ for the assignment $\vec{b}$. Alternatively, we say that $\vec{b}$ satisfies φ in $\mathfrak{A}$.

DEFINITION 1.25. (Free variables) Let φ be a formula and x a variable. We say that the variable x occurs *free* in φ if it occurs in a place in the formula φ which is not on the scope of a quantifier. If x occurs in φ and is not free, then we say that x is*bound* in φ. A recursive definition of the concept on the complexity of terms is here:

(1) x is free in $t_1 \doteq t_2$ iff x occurs in t_1 and in t_2,
(2) x is free in $Rt_1 \cdots t_n$ iff x occurs in one of t_i,
(3) x is free in $\neg\psi$ iff x is free in ψ,
(4) x is free in $(\psi_1 \wedge \psi_2)$ iff x is free in ψ_1 or x is free in ψ_2,
(5) x is free in $\exists y \psi$ iff $x \neq y$ and x is free in ψ.

DEFINITION 1.26. (Definable subsets) Let $\mathfrak{A}$ be a $\mathscr{L}$-structure.

[2]Remember that x is one of the variables in the list $\{v_i\}_{i \in \mathbb{N}}$.

(1) A $\mathscr{L}$-formula φ defines an n-ary relation

$$\varphi(\mathfrak{A}) = \{\vec{a} : \mathfrak{A} \models \varphi[\vec{a}]\}$$

on the set A, referred to as the *realisation set* of φ (in $\mathfrak{A}$).

(2) Let $B \subseteq A$ and let φ be a $\mathscr{L}(B)$-formula. Then the set $\varphi(\mathfrak{A}_B)$ is said to be a B-definable subset of $\mathfrak{A}$. Thus, a definable subset of $\mathfrak{A}$ is simply a set definable over the empty set.

(3) Two formulas are said to be equivalent if in every structure they define the same set.

LEMMA 1.27. (Substitution Lemma) $\mathfrak{A} \models \varphi(t_1, \cdots, t_n)[\vec{b}]$ if and only if $\mathfrak{A} \models \varphi[t_1^{\mathfrak{A}}[\vec{b}], \cdots, t_n^{\mathfrak{A}}[\vec{b}]$.

PROOF. Induction on the complexity of φ. □

REMARK 1.28. Note that $\mathfrak{A}_A \models \varphi(a_1, \cdots, a_n)$ if and only if $\mathfrak{A} \models \varphi[a_1, \cdots, a_n]$.

Recall the definitions:

DEFINITION. We refer to formulas without free variables as *sentences*. Atomic formulas and their negations are called *basic formulas*. Formulas without quantifiers are Boolean combinations of basic fromulas, i.e. built from basic formulas by successively applying $\neg, \wedge$. Recall that $\top$ denotes a formula which is always true, $\perp$ a formula which is always false.

DEFINITION 1.29. (Basic formulas, Negation normal form)

(1) Atomic formulas and their negations are called basic.
(2) A formula is in a negation normal form if it is build from basic formulas using $\wedge, \vee, \exists, \forall$.

REMARK 1.30. Every formula can be transformed into an equivalent formula which is in a negation normal form (i.e. it is logically equivalent to a formula in negation normal form).

LEMMA 1.31. Let $h : \mathfrak{A} \to \mathfrak{B}$ be an embedding (we write $h : \mathfrak{A} \prec \mathfrak{B}$).

(1) Let $\varphi(x_1, ..., x_n)$ be an existential formula. Let $a_1, ..., a_n$ be elements of A. If $\mathfrak{A} \models \varphi[a_1, ..., a_n]$ then $\mathfrak{B} \models \varphi[h(a_1), ..., h(a_n)]$.
(2) Let ψ be universal. If $\mathfrak{B} \models \psi[h(a_1), ..., h(a_n)]$ then $\mathfrak{A} \models \psi[a_1, ..., a_n]$.

PROOF. By induction on the complexity of the formulas. Suppose $\varphi(\bar{x})$ is $\exists y \psi(\bar{x}, y)$. If

$$\mathfrak{A} \models \varphi[\bar{a}]$$

then $\exists a \in A$ such that $\mathfrak{A} \models \psi[\bar{a}, a]$. However, by inductive hypothesis

$$\mathfrak{B} \models \psi[h(\bar{a}), h(a)]$$

and so $\mathfrak{B} \models \varphi[h(\bar{a})]$. □

DEFINITION 1.32. (Atomic diagram) Let $\mathfrak{A}$ be an $\mathscr{L}$-structure. The atomic diagram of $\mathfrak{A}$ is the set of all basic $\mathscr{L}(A)$-sentences such that $\mathfrak{A}_A \models \varphi$.

REMARK 1.33. Given a structure $\mathfrak{A}$ we denote by $\mathrm{Diag}(\mathfrak{A})$ the atomic diagram of $\mathfrak{A}$.

LEMMA 1.34. Let $\mathfrak{A}$, $\mathfrak{B}$ be $\mathscr{L}$-structures.

(1) Let $h : \mathfrak{A} \to \mathfrak{B}$ be an embedding. Then $(\mathfrak{B}, h(a))_{a \in A} \models \mathrm{Diag}(\mathfrak{A})$.
(2) Let $h : A \to B$ and $(\mathfrak{B}, h(a))_{a \in A} \models \mathrm{Diag}(\mathfrak{A})$. Then $h : \mathfrak{A} \to \mathfrak{B}$ is an embedding.

PROOF. (1): Let $\varphi \in \text{Diag}(\mathfrak{A})$. Then $\mathfrak{A} \vDash \varphi[a_1, ...a_n]$. By the previous Lemma $\mathfrak{B} \vDash \varphi[h(a_1), ..., h(s_n)]$. Thus $(\mathfrak{B}, h(a))_{a\in A} \vDash \text{Diag}(\mathfrak{A})$.

(2): First we will show that h is injective. Consider the formula $\varphi_1(x_1, x_2) : \neg(x_1 \doteq x_2)$. Then for each $a_1 \neq a_2$ from A, $\varphi(a_1, a_2) \in \text{Diag}(\mathfrak{A})$. Thus $(\mathfrak{B}, h(a))_{a\in A} \vDash \varphi(a_1, a_2)$. More precisely, $\underline{a}_1^{\mathfrak{B}} \neq \underline{a}_2^{\mathfrak{B}}$, i.e. $h(a_1) \neq h(a_2)$. Therefore h is injective.

Now, we will show that h respects the interpretation of constants. Let $c \in \mathscr{C}_{\mathscr{L}}$ and let $a = c^{\mathfrak{A}} \in A$. The formula $\varphi(x_1) : x_1 \doteq c$ is atomic and $\varphi(\underline{a}) \in \text{Diag}(\mathfrak{A})$. Then $(\mathfrak{B}, h(a))_{a\in A} \vDash \varphi(\underline{a})$ and so $c^{\mathfrak{B}} = \underline{a}^{\mathfrak{B}}$, i.e. $c^{\mathfrak{B}} = h(a) = h(c^{\mathfrak{A}})$. Thus h respects the interpretation of all constants.

Next we show that h respects interpretation of function symbols. Let $f \in \mathscr{F}_{\mathscr{L}}$. Consider the formula $\varphi(x_0, ...x_n) : x_0 \doteq f(x_1, ..., x_n)$. For each $(a_0, ..., a_n)$ such that $f^{\mathfrak{A}}(a_1, ..., a_n) = a_0$, $\varphi(\underline{a}_0, ..., \underline{a}_n) \in \text{Diag}(\mathfrak{A})$. Thus $(\mathfrak{B}, h(a))_{a\in A} \vDash \varphi(\underline{a}_0, ..., \underline{a}_n)$ and so $\underline{a}_0^{\mathfrak{B}} = f^{\mathfrak{B}}(\underline{a}_1^{\mathfrak{B}}, ..., \underline{a}_n^{\mathfrak{B}})$. That is $h(a_0) = f^{\mathfrak{B}}(h(a_1), ..., h(a_n))$. Now, since $a_0 = f^{\mathfrak{A}}(a_1, ..., a_n)$ we also get $h(a_0) = h(f^{\mathfrak{A}}(a_1, ..., a_n))$. Thus $h(f^{\mathfrak{A}}(a_1, ..., a_n)) = f^{\mathfrak{B}}(h(a_1), ..., h(a_n))$ and so h respects f.

Finally, we show that h respects the interpretation of relation symbols. The fact that if $(a_1, ..., a_n) \in R^{\mathfrak{A}}$ then $(h(a_1), ..., h(a_n)) \in R^{\mathfrak{B}}$ is shown similarly. Indeed, given $R \in \mathscr{L}_{\mathscr{R}}$ for each tuple $(a_1, ..., a_n) \in R^{\mathfrak{A}}$ one can use the formula $R(\underline{a}_1, ..., \underline{a}_n)$. □

2. Theories

DEFINITION 2.1. Definition: Theory A theory is a set of $\mathscr{L}$-sentences.

DEFINITION 2.2. (Consistency)

(1) A theory T is consistent, if it has a model.
(2) A set of $\mathscr{L}$-formulas Φ is consistent if there is an $\mathscr{L}$-structure and an assignment $\vec{b}$ such that

$$\mathfrak{A} \vDash \varphi[\vec{b}]$$

for all $\varphi \in \Phi$.
(3) A set of formulas Φ is consistent with a theory T if $T \cup \Phi$ is consistent.

LEMMA 2.3. Let T be an $\mathscr{L}$-theory, $\mathscr{L}'$ an expansion of $\mathscr{L}$. Then T is consistent as an $\mathscr{L}$-theory iff T is consistent as an $\mathscr{L}'$-theory.

PROOF. Every $\mathscr{L}$-structure is expandable to an $\mathscr{L}'$-structure. □ □

DEFINITION 2.4. (Valid formulas)

(1) If a sentence φ holds in all models of T, then we say that φ follows from T and write $T \vdash \varphi$.
(2) If $\emptyset \vdash \varphi$, we say φ is valid.

REMARK 2.5. Because of completeness of first order logic the above definitions coincides with the notion of validity given in the first lecture.

LEMMA 2.6.

(1) If $T \vdash \varphi$ and $T \vdash (\varphi \to \psi)$ then $T \vdash \psi$.
(2) If $T \vdash \varphi(c_1, ..., c_n)$ and the constants $c_1, ..., c_n$ occur neither in T, nor in $\varphi(x_1, ..., x_n)$ then $T \vdash \forall x_1 ... x_n \varphi(x_1, ..., x_n)$.

PROOF. To see (1) consider any $\mathfrak{A} \vDash T$. Then $\mathfrak{A} \vDash \varphi$ and $\mathfrak{A} \vDash \neg\varphi \vee \psi$. Clearly $\mathfrak{A} \vDash \psi$. Thus $T \vdash \psi$. To see item (2) consider $\mathscr{L}' = \mathscr{L} \backslash \{c_1, ..., c_n\}$. Let $\mathfrak{A}$ be a $\mathscr{L}'$-structure such that $\mathfrak{A} \vDash T$. Then for each $a_1, ..., a_n$ in $\mathfrak{A}$ we have

$$(\mathfrak{A}, a_1, ..., a_n) \vDash \varphi(c_1, \cdots, c_n).$$

Thus $\mathfrak{A} \models \forall x_1...x_n \varphi(x_1,...,x_n)$. Then $T \vdash \forall x_1...x_n \varphi(x_1,...,x_n)$. □

DEFINITION 2.7. Let T, S be theories.

(1) We say that $T \models S$ iff every model of T is also a model of S.
(2) $T \equiv S$ iff they have the same models. The theories are said to be elementarily equivalent.

DEFINITION 2.8. (Completeness) A consistent $\mathscr{L}$-theory T is said to be complete iff for every $\mathscr{L}$-sentence φ either $T \vdash \varphi$ or $T \vdash \neg\varphi$.

EXAMPLE 2.9. Let $\mathfrak{A}$ be a $\mathscr{L}$-structure. Then

$$\mathrm{Th}(\mathfrak{A}) = \{\varphi : \mathfrak{A} \models \varphi, \varphi \text{ is a } \mathscr{L}\text{-sentence}\}$$

is a complete theory.

LEMMA 2.10. Let T be a consistent $\mathscr{L}$-theory. Then T is complete iff T is maximal consistent, i.e. T is elementarily equivalent to every consistent $\mathscr{L}$-theory T' such that $T \subseteq T'$.

PROOF. ($\Leftarrow$): Suppose T is maximal consistent, but not complete. Then there is a $\mathscr{L}$-sentence φ such that neither $T \vdash \varphi$, nor $T \vdash \neg\varphi$. Thus:

- there is $\mathfrak{A}$ such that $\mathfrak{A} \models T$, bit $\mathfrak{A} \not\models \varphi$, and
- there is $\mathfrak{B}$ such that $\mathfrak{B} \models T$ but $\mathfrak{B} \not\models \neg\varphi$.

Then in particular $\mathfrak{A} \models \neg\varphi$, $\mathfrak{B} \models \varphi$. Now $T \cup \{\neg\varphi\}$ is a consistent extension of T (with model $\mathfrak{A}$), but $T \cup \{\neg\varphi\} \not\equiv T$. Indeed, $\mathfrak{B} \models T$, but $\mathfrak{B} \not\models T \cup \{\neg\varphi\}$. Thus, T is not maximal consistent, which is a contradiction.

($\Rightarrow$): Suppose T is complete, but not maximal consistent. Then there is a $\mathscr{L}$-sentence φ such that $T \cup \{\varphi\}$ is consistent, but $T \cup \{\varphi\} \not\equiv T$.

- Thus there is $\mathfrak{A} \models T$ such that $\mathfrak{A} \not\models \varphi$, i.e. $\mathfrak{A} \models \neg\varphi$.
- Moreover, since $T \cup \{\varphi\}$ is consistent, it has a model $\mathfrak{B}$.
- But, then $T \not\vdash \varphi$ (because of $\mathfrak{A}$) and $T \not\vdash \neg\varphi$ (because of $\mathfrak{B}$).

Therefore T is not complete, which is a contradiction. □

DEFINITION 2.11. (Elementary equivalence) Two $\mathscr{L}$-structures $\mathfrak{A}$, $\mathfrak{B}$ are said to be elementarily equivalent iff they satisfy the same sentences, i.e. $\mathrm{Th}(\mathfrak{A}) = \mathrm{Th}(\mathfrak{B})$. We write $\mathfrak{A} \equiv \mathfrak{B}$.

EXERCISE 2.

(1) If $\mathfrak{A} \cong \mathfrak{B}$ then $\mathfrak{A} \equiv \mathfrak{B}$.
(2) Give an example of $\mathfrak{A}, \mathfrak{B}$ such that $\mathfrak{A} \equiv \mathfrak{B}$, but $\mathfrak{A} \ncong \mathfrak{B}$.

LEMMA 2.12. The following are equivalent:

(1) T is complete.
(2) All models of T are elementarily equivalent.
(3) There is a structure $\mathfrak{A}$ such that $T \equiv \mathrm{Th}(\mathfrak{A})$.

PROOF. $(1) \Rightarrow (3)$: Let $\mathfrak{A} \models T$. Take $\varphi \in \mathrm{Th}(\mathfrak{A})$. Then $\mathfrak{A} \models \varphi$. Since T is complete $T \vdash \varphi$. Then $\mathrm{Th}(\mathfrak{A})$ and T have the same models.

$(3) \Rightarrow (1)$: Straightforward.

$(3) \Rightarrow (2)$: Let $\mathfrak{B} \models T$. Then $\mathfrak{B} \models \mathrm{Th}(\mathfrak{A})$ and so $\mathfrak{B} \equiv \mathfrak{A}$.

$(2) \Rightarrow (1)$: Let $\mathfrak{A} \models T$. If $\varphi \in \mathrm{Th}(\mathfrak{A})$ then $T \models \varphi$ (otherwise there is $\mathfrak{B}$ such that $\mathfrak{B} \models T$ and $\mathfrak{B} \models \neg\varphi$, which is a contradiction to (2)). □

DEFINITION 2.13. (Elementary Class) Let T be a $\mathscr{L}$-theory. The class of all $\mathscr{L}$-structures $\mathfrak{A}$ such that $\mathfrak{A} \vDash T$ is called an elementary class.

3. Elementary Extensions

DEFINITION 3.1. (Elementary embedding) Let $\mathfrak{A}$ and $\mathfrak{B}$ be two $\mathscr{L}$-structures. A map $h : A \to B$ is said to be elementary if for every formula $\varphi(x_1, \cdots, x_n)$ and all $a_1, \cdots, a_n$ in A we have:

$$\mathfrak{A} \vDash \varphi[a_1, \cdots, a_n] \leftrightarrow \mathfrak{B} \vDash \varphi[h(a_1), \cdots, h(a_n)].$$

REMARK 3.2. Since the function h from the above definition preserves quantifier free formulas, h is an embedding. We use the notation $h : \mathfrak{A} \preceq \mathfrak{B}$.

LEMMA 3.3. Let $\mathfrak{A}$ and $\mathfrak{B}$ be $\mathscr{L}$-structres.

(1) If $h : \mathfrak{A} \preceq \mathfrak{B}$ then $(\mathfrak{B}, h(a))_{a \in A} \vDash \mathrm{Th}(\mathfrak{A}_A)$.
(2) If $h : A \to B$ where B is the universe of a structure $\mathfrak{B}$ such that $(B, h(a))_{a \in A} \vDash \mathrm{Th}(\mathfrak{A}_A)$ then $h : \mathfrak{A} \preceq \mathfrak{B}$.

REMARK 3.4. The proof is almost identical to an earlier proof.

DEFINITION 3.5.

(1) $\mathrm{Th}(\mathfrak{A}_A)$ is called the elementary diagram of $\mathfrak{A}$.
(2) Let $\mathfrak{A} \subseteq \mathfrak{B}$. Then $\mathfrak{A}$ is said to be an elementary substructure of $\mathfrak{B}$ if $\mathrm{id} : \mathfrak{A} \preceq \mathfrak{B}$ is an elementary embedding. We say also that $\mathfrak{B}$ is an elementary extension of $\mathfrak{A}$.

REMARK 3.6. Recall that $\mathrm{id} : \mathfrak{A} \preceq \mathfrak{B}$ is just saying that for every formula $\varphi(x_1, \cdots, x_n)$ and all $a_1, \cdots, a_n$ in A we have:

$$\mathfrak{A} \vDash \varphi[a_1, \cdots, a_n] \leftrightarrow \mathfrak{B} \vDash \varphi[a_1, \cdots, a_n].$$

THEOREM 3.7. *(Tarski's Test) Let $\mathfrak{B}$ be an $\mathscr{L}$-structure and let $A \subseteq B$. Then A is the universe of a structure $\mathfrak{A}$ such that $\mathfrak{A} \preceq \mathfrak{B}$ if and only if every $\mathscr{L}(A)$-formula $\varphi(x)$ which is satisfiable in $\mathfrak{B}$, is also satisfiable by an element of A.*

PROOF. ($\Rightarrow$): Suppose $\mathfrak{A} \preceq \mathfrak{B}$. Then if $\mathfrak{B} \vDash \exists x \varphi(x)$ we must have $\mathfrak{A} \vDash \exists x \varphi(x)$.

($\Leftarrow$): We have to show that there is a structure $\mathfrak{A}$ with universe the set A such that $\mathfrak{A} \preceq \mathfrak{B}$. First of all consider the $\mathscr{L}(A)$-formula $x \doteq x$. Now $\mathfrak{B} \vDash x \doteq x$ and so for some $b \in B$, $\mathfrak{B} \vDash (x \doteq x)(b)$. By hypothesis, $x \doteq x$ is satisfiable by an element of $\mathfrak{A}$ and so $A \neq \emptyset$. To show that A is the universe of a substructure of $\mathfrak{B}$, it is sufficient to show that A is closed wrt the interpretation of constant and function symbols. Fix $f \in \mathscr{L}_{\mathscr{L}}$, n-ary, where $n \geq 0$. Let $\varphi(x)$ be the formula $f(a_1, \cdots, a_n) \doteq x$ for fixed $a_1, \cdots, a_n$ in A. Since it is satisfiable in $\mathfrak{B}$ there must be $a \in A$ such that $f^{\mathfrak{B}}(a_1, \cdots, a_n) = a$. Thus $f^{\mathfrak{B}} \restriction A^n : A^n \to A$. Thus A is the universe of a substructure $\mathfrak{A}$ of $\mathfrak{B}$.

Next, we will show that $\mathfrak{A}$ is an elementary substructure of $\mathfrak{B}$. Thus, we need to show that for every $\mathscr{L}(A)$-sentence ψ, we have $\mathfrak{A} \vDash \psi \leftrightarrow \mathfrak{B} \vDash \psi$. If ψ is atomic, or of the form $\neg\varphi$, $(\psi_1 \wedge \psi_2)$ this is straightforward. Suppose $\psi = \exists x \varphi(x)$. Clearly if $\mathfrak{A} \vDash \psi$ then $\mathfrak{B} \vDash \psi$. Suppose $\mathfrak{B} \vDash \exists x \varphi(x)$. Then by hypothesis on the theorem there is $a \in A$ such that $\mathfrak{B} \vDash \varphi(a)$. Now by induction hypothesis $\mathfrak{A} \vDash \varphi(a)$ and so $\mathfrak{A} \vDash \exists x \varphi(x)$. □

COROLLARY 3.8. Let $\mathfrak{B}$ be a $\mathscr{L}$-structure, $S \subseteq B$. Then there is $\mathfrak{A} \preceq B$ such that S is contained in the universe $\mathfrak{A}$ and $|\mathfrak{A}| \leq \max\{|S|, |\mathscr{L}|, \aleph_0\}$.

PROOF. Construct an increasing chain of sets $\{S_i\}_{i \in \mathbb{N}}$ where $S_0 = S$ as follows. Suppose S_i is defined. Let

$$F_i : \{\varphi(x) \mid \varphi \text{ is } \mathscr{L}(S_i)\text{-formula s.t. } \mathfrak{B} \vDash \varphi(x)\} \to B,$$

where

$$F(\varphi(x)) = a_\varphi \text{ and } \mathfrak{B} \vDash \varphi(a_\varphi).$$

Take $S_{i+1} = S_i \cup \text{ran}(F)$. Then $A = \bigcup_{i\in\omega} S_i$ is the universe of an elementary substructure. Note also that the number of $\mathscr{L}$-formulas does not exceed $\max\{|\mathscr{L}|, \aleph_0\}$. □

THEOREM 3.9. *(Löwenheim-Skolem Downwards) Let $\mathfrak{B}$ be a $\mathscr{L}$-structure, S a subset of $\mathfrak{B}$ and κ an infinite cardinal. Suppose*

$$\max\{|S|, |\mathscr{L}|\} \le \kappa \le |\mathfrak{B}|.$$

Then $\mathfrak{B}$ has an elementary substructure of cardinality κ containing S.

PROOF. Take $S' \subseteq B$, such that $S \subseteq S'$ and $|S'| = \kappa$. Apply the previous Corollary. □

DEFINITION 3.10. A directed family $\{\mathfrak{A}_i\}_{i\in I}$ is elementary if $\mathfrak{A}_i \prec \mathfrak{A}_j$ for all $i \le j$.

LEMMA 3.11. (Tarski's Chain Lemma) Let $\{\mathfrak{A}_i\}_{i\in I}$ be an elementary directed family. Then $\mathfrak{A} = \bigcup_{i\in I} \mathfrak{A}_i$ is an elementary extension of all $\mathfrak{A}_i$'s.

PROOF. Let $\mathfrak{A} = \bigcup_{i\in I} \mathfrak{A}_i$. We will prove by induction on $\varphi(\bar{x})$ that for all $i \in I$ and all tuples $\bar{a}$ in $\mathfrak{A}_i$,

$$\mathfrak{A}_i \vDash \varphi(\bar{a}) \leftrightarrow \mathfrak{A} \vDash \varphi(\bar{a}).$$

Fix i. If φ is atomic, or φ is negation or conjunction of formulas for which the claim has been proved, the argument is straightforward.

Thus, suppose $\varphi(\bar{x}) = \exists y \psi(\bar{x}, y)$. Fix $\bar{a}$ in $\mathfrak{A}_i$. Note that

$$\mathfrak{A} \vDash \varphi(\bar{a}) \text{ iff } \exists b \in A \text{ s.t. } \mathfrak{A} \vDash \psi(\bar{a}, b).$$

Then $\exists j \ge i$ such that $b \in A_j$. By Inductive Hypothesis

$$\mathfrak{A} \vDash \psi(\bar{a}, b) \leftrightarrow \mathfrak{A}_j \vDash \psi(\bar{a}, b).$$

However $\mathfrak{A}_i \prec \mathfrak{A}_j$ and so there is $b' \in A_i$ such that $\mathfrak{A}_i \vDash \psi(\bar{a}, b')$. Thus $\mathfrak{A}_i \vDash \exists y \psi(\bar{a}, y)$. □

CHAPTER 6

Theorem of Compactness

1. Theorem of Compactness

THEOREM 1.1. *(Compactness) If T is finitely satisfiable, i.e. every finite subset of T is consistent, then T is satisfiable.*

DEFINITION 1.2. (Henkin theory) Let $\mathscr{L}$ be a language, C a set of new constants. A $\mathscr{L}(C)$-theory T' is called a Henkin theory if for every $\mathscr{L}(C)$-formula $\varphi(x)$ there is $c \in \mathscr{C}$ such that

$$\exists x\varphi(x) \to \varphi(c) \in T'.$$

REMARK 1.3. The elements of $\mathscr{C}$ are called Henkin constants.

Until the end of the section, we will be occupied with the proof of the Theorem of Compacntess. For the purposes of the proof we will work with the following notion.

DEFINITION. A $\mathscr{L}$-theory T is said to be finitely complete, if it is finitely satisfiable and if every $\mathscr{L}$-sentence φ satisfies

$$\varphi \in T \text{ or } \neg\varphi \in T.$$

We will make use of the following Lemma.

LEMMA 1.4. Every finitely satisfiable $\mathscr{L}$-theory T can be extended to a finitely complete Henkin theory T^*.

PROOF. Inductively we will define an increasing sequence $\emptyset = C_0 \subseteq C_1 \subseteq \cdots$ of new constants by assigning to every $\mathscr{L}(C_i)$-formula $\varphi(x)$ a constant $c_{\varphi(x)}$ and defining

$$C_{i+1} = \{c_{\varphi(x)} : \varphi(x) \text{ is an } \mathscr{L}(C_i)\text{-formula}\}.$$

Take $C = \bigcup_{i\in\mathbb{N}} C_i$,

$$T^H = \{\exists x\varphi(x) \to \varphi(c_{\varphi(x)}) : \varphi(x) \text{ is an } \mathscr{L}(C)\text{-formula}\}.$$

Suppose $\mathfrak{A}$ is a $\mathscr{L}$-structure and $\mathfrak{A} \vDash \exists x\varphi(x)$. Then let $c^{\mathfrak{A}}_{\varphi(x)} = a$ where $\mathfrak{A} \vDash \varphi(a)$. Thus $\mathfrak{A}$ can be extended to a $\mathscr{L}(C)$-structure $\mathfrak{A}'$ such that $\mathfrak{A}' \vDash T^H$.

Note that, the above shows that every $\mathscr{L}$-structure can be extended to a $\mathscr{L}(C)$-structure satisfying T^H. Therefore $T \cup T^H$ is a finitely satisfiable Henkin theory in $\mathscr{L}(C)$. Extend $T \cup T^H$ to a maximal finitely satisfiable $\mathscr{L}(C)$-theory T^*.

CLAIM. T^* is finitely complete.

PROOF. Suppose not. Thus, there is a $\mathscr{L}(C)$-sentence φ such that neither φ, nor $\neg\varphi$ is in T^*. Then neither $T^* \cup \{\varphi\}$, nor $T^* \cup \{\neg\varphi\}$ is finitely satisfiable, by maximality of T^*. Thus in particular there are $\Delta_1, \Delta_2 \in [T^*]^{<\omega}$ such that neither $\Delta_1 \cup \{\varphi\}$ nor $\Delta_2 \cup \{\neg\varphi\}$ is satisfiable. Therefore $\Delta = \Delta_1 \cup \Delta_2$ is a finite subset of T^* and $\Delta \cup \{\varphi\}$ as well as $\Delta \cup \{\neg\varphi\}$ is not satisfiable. Thus Δ is not satisfiable, which is a contradiction. □

LEMMA. Suppose T^* is a finitely complete Henkin theory. Then T^* has a model, the universe of which consists of constants. This model is unique up to isomorphism.

REMARK. Note that every sentence which follows from a finite subset of T^* belongs to T^*. Indeed. Let φ be a $\mathscr{L}$-formula. Then either $\varphi \in T^*$ or $\neg \in T^*$, because T^* is finitely complete. Suppose $\Delta \in [T^*]^{<\omega}$ and $\Delta \vdash \varphi$ for some φ. If $\neg\varphi \in T^*$ then $\Delta \cup \{\neg\varphi\}$ is not satisfiable, contradicting T^* being finitely complete. Thus $\varphi \in T^*$.

Now, define for $c, d \in C$: $c \cong d \leftrightarrow c \doteq d \in T^*$. Then $\cong$ is an equivalence relation. Define $a_c := [c]_{\cong}$. Take $A = \{a_c : c \in C\}$ and define a $\mathscr{L}$-structure $\mathfrak{A}$ with universe A as follows. For each relation symbol R and each n-ary function symbol f, where $n \geq 0$, let

$$R^{\mathfrak{A}}(a_{c_1}, \cdots, a_{c_n}) \leftrightarrow R(c_1, \cdots, c_n) \in T^*$$
$$f^{\mathfrak{A}}(a_{c_1}, \cdots, a_{c_n}) = a_{c_0} \leftrightarrow f(c_1, \cdots, c_n) \doteq c_0 \in T^*.$$

EXERCISE. Check that the above is well defined.

First we will show that the definition of $R^{\mathfrak{A}}$ for $R \in \mathscr{R}_{\mathscr{L}}$ does not depend on the representatives of the equivalence classes. Thus, suppose $\wedge_{i=1}^n c_i \cong d_i$ and $R \in \mathscr{R}_{\mathscr{L}}$. We need to show that

$$R(c_1, \cdots, c_n) \in T^* \text{ iff } R(d_1, \cdots, d_n) \in T^*.$$

Suppose $R(c_1, \cdots, c_n) \in T^*$. By hypothesis also $\{c_i \doteq d_i\}_{i=1}^n \subseteq T^*$. However

$$\wedge_{i=1}^n (c_i \doteq d_i) \to (R(c_1, \cdots, c_n) \leftrightarrow R(d_1, \cdots, d_n))$$

is a valid formula and so $T^* \vdash R(d_1, \cdots, d_n)$. Thus, by our Remark $R(d_1, \cdots, d_n) \in T^*$

EXERCISE. Which is the finite set of formulas referred to in the last item above?

Next, we will deal with the interpretation of function symbols. Thus, fix $f \in \mathscr{F}_{\mathscr{L}}$. We need to show that $f^{\mathfrak{A}}(a_{c_1}, \cdots, a_{c_n})$ (provided it is defined) does not depend on the representatives, argue as above. To show that $f^{\mathfrak{A}}$ is a function, take any $\{a_{c_i}\}_{i=1}^n \subseteq A$. We want to show that $f^{\mathfrak{A}}(a_{c_1}, \cdots, a_{c_n})$ is defined. Pick representatives $c_i \in a_{c_i}$. Since $f \in \mathscr{L}_{\mathscr{F}}$,

$$\exists x (f(c_1, \cdots, c_n) \doteq x)$$

is a valid formula and so $\exists x (f(c_1, \cdots, c_n) \doteq x) \in T^*$. However T^* is Henkin and so there is $c_0 \in C$ such that

$$\exists x f(c_1, \cdots, c_n) \doteq x \to f(c_1, \cdots, c_n) \doteq c_0 \in T^*.$$

Therefore $f(c_1, \cdots c_n) \doteq c_0 \in T^*$ and so $f^{\mathfrak{A}}(a_{c_1}, \cdots, a_{c_n})$ is defined.

Expand $\mathfrak{A}$ to the $\mathscr{L}(C)$-structure $\mathfrak{A}^* = (\mathfrak{A}, a_c)_{c \in C}$.

EXERCISE. Show by induction on the complexity of φ that for every $\mathscr{L}(C)$-sentence φ

$$\mathfrak{A}^* \models \varphi \leftrightarrow \varphi \in T^*.$$

□

COROLLARY 1.5. $T \vdash \varphi$ iff there is a finite $\Delta \subseteq T$ such that $\Delta \vdash \varphi$.

PROOF. By the Compactness theorem, φ follows from T iff $T \cup \{\neg\varphi\}$ is inconsistent iff $T \cup \{\neg\varphi\}$ is not finitely satisfiable iff there is a finite $\Delta \subseteq T$ such that $\Delta \cup \{\neg\varphi\}$ is not satisfiable iff $\Delta \vdash \varphi$. □

COROLLARY 1.6. A set of formulas $\Sigma(x_1, \cdots, x_n)$ is consistent with T iff every finite subset of Σ is consistent with T.

PROOF. Let $c_1, \cdots, c_n$ be new constants. Then Σ is consistent with T if and only if $T \cup \Sigma(c_1, \cdots, c_n)$ is consistent if and only if every finite subset is consistent. □

2. Theorem of Löwenheim-Skolem Upwards

THEOREM 2.1. *(Löwenheim-Skolem Upwards) Let $\mathfrak{B}$ be a $\mathscr{L}$-structure, S a subset of the universeof $\mathfrak{B}$ and let κ be an infinite cardinal. Suppose $\mathfrak{B}$ is infinite and* $\max\{|\mathfrak{B}|,|\mathscr{L}|\} \leq \kappa$*. Then $\mathfrak{B}$ has an elementary extension of cardinality κ.*

PROOF. Let C be a set of new constant symbols of cardinality κ. Since the universe of $\mathfrak{B}$ is an infinite set, the theory $\mathrm{Th}(\mathfrak{B}_B) \cup \{\neg c \doteq d : c, d \in C, c \neq q\}$ is finitely satisfiable and so by the theorem of Compactness it has a model $\mathfrak{A}$. Thus in particular $\mathfrak{A} \models \mathrm{Th}(\mathfrak{B}_B)$ and so $\mathfrak{B} \preceq \mathfrak{A}$. Moreover, for each $c \neq d$ in C, we have $c^{\mathfrak{A}} \neq d^{\mathfrak{A}}$ and so the universe of $\mathfrak{A}$ is of cardinality at least κ. It remains to observe that by the Downwards Löwenheim-Skolem Theorem $\mathfrak{A}$ has an elementary submodel containing B, which is of cardinality κ. □

COROLLARY 2.2. Let T be a $\mathscr{L}$-theory. Suppose T has an infinite model. Then T has a model of every cardinality $\kappa \geq \max\{|\mathscr{C}|, \aleph_0\}$.

DEFINITION 2.3. (Categoricity) Let κ be an infinite cardinal. A theory T is called κ-categorical if all models of T of cardinality κ are isomorphic.

3. The Separation Theorem

THEOREM 3.1. *(The Separation Theorem) Let T_1 and T_2 be two theories and let $\mathscr{H}$ be a set of sentences which is closed under $\wedge, \vee$ contains $\top$, $\bot$. The following are equivalent:*

(1) There is $\varphi \in \mathscr{H}$ such that $T_1 \vdash \varphi$ and $T_2 \vdash \neg\varphi$.
(2) For each pair of models $\mathfrak{A}_1$ and $\mathfrak{A}_2$ such that $\mathfrak{A}_1 \models T_1$ and $\mathfrak{A}_2 \models T_2$ there is a formula $\varphi \in \mathscr{H}$ such that

$$\mathfrak{A}_1 \models \varphi \text{ and } \mathfrak{A}_2 \models \neg\varphi.$$

REMARK 3.2. We say that φ separates $\mathfrak{A}_1$ and $\mathfrak{A}_2$.

PROOF. The implication $(1) \Rightarrow (2)$ is straightforward. We need to show $(2) \Rightarrow (1)$. We can assume that the theories are consistent as otherwise the statement if vacuously true. For each model $\mathfrak{A}$ of T_1 define

$$\mathscr{H}_{\mathfrak{A}} = \{\varphi \in \mathscr{H} : \mathfrak{A} \models \varphi\}.$$

By the hypothesis of (2) for each $\mathfrak{A} \models T_1$, $\mathscr{H}_{\mathfrak{A}} \neq \emptyset$.

CLAIM 3.3. Let $\mathfrak{A} \models T_1$. Then $T_2 \cup \mathscr{H}_{\mathfrak{A}}$ is not consistent.

PROOF. Assume $\mathfrak{B} \models T_2 \cup \mathscr{H}_{\mathfrak{A}}$. Now, by (2) there is $\varphi \in \mathscr{H}$ such that $\mathfrak{A} \models \varphi$ and $\mathfrak{B} \models \neg\varphi$. Then $\varphi \in \mathscr{H}_{\mathfrak{A}}$ and so $\mathfrak{B} \models \varphi \wedge \neg\varphi$, contradiction. □

CLAIM 3.4. Let $\mathfrak{A} \models T_1$. Then there is $\varphi_{\mathfrak{A}} \in \mathscr{H}$ such that $\mathfrak{A} \models \varphi_{\mathfrak{A}}$ and $T_2 \vdash \neg\varphi$.

PROOF. Since $T_2 \cup \mathscr{H}_{\mathfrak{A}}$ is not consistent, there is a finite $\{\varphi_j\}_{j=1}^n \subseteq \mathscr{H}_{\mathfrak{A}}$ such that $T_2 \cup \{\varphi_j\}_{j=1}^n$ is inconsistent. Thus $T_2 \cup \{\wedge_{j=1}^n \varphi_j\}$ is also inconsistent. Let $\varphi_{\mathfrak{A}} = \wedge_{j=1}^n \varphi_j$. Then $T_2 \cup \{\varphi_{\mathfrak{A}}\}$ is inconsistent and so $T_2 \vdash \neg\varphi_{\mathfrak{A}}$. Since $\mathscr{H}$ is closed under conjunctions , $\varphi_{\mathfrak{A}} \in \mathscr{H}$. □

CLAIM 3.5. $T_1 \cup \{\neg\varphi_{\mathfrak{A}} : \mathfrak{A} \models T_1\}$ is inconsistent.

PROOF. If $\mathfrak{A}^* \models T_1$ then $\mathfrak{A}^* \models \varphi_{\mathfrak{A}^*}$ and so $\mathfrak{A}^* \not\models \neg\varphi_{\mathfrak{A}^*}$ □

By the Theorem of Compactness there are finitely many models $\mathfrak{A}_1, \cdots, \mathfrak{A}_n$ of T_1 such that $T_1 \cup \{\neg\varphi_{\mathfrak{A}_i}\}_{i=1}^n$ is inconsistent. Thus $T_1 \cup \{\wedge_{i=1}^n \neg\varphi_{\mathfrak{A}_i}\}$ is inconsistent and so

$$T_1 \vdash \neg \wedge_{i=1}^n \neg\varphi_{\mathfrak{A}_i}.$$

That is $T_1 \vdash \vee_{i=1}^n \varphi_{\mathfrak{A}_i}$. We claim that $\varphi = \vee_{i=1}^n \varphi_{\mathfrak{A}_i} \in \mathscr{H}$ separates T_1 and T_2.

Well, $\varphi \in \mathscr{H}$, because $\mathscr{H}$ is closed under $\vee$. Thus, it remains to show that $T_2 \vdash \neg\varphi$. By the choice of $\varphi_{\mathfrak{A}_i}$ for each i, $T_2 \vdash \neg\varphi_{\mathfrak{A}_i}$ and so $T_2 \vdash \wedge_{i=1}^n \neg\varphi_{\mathfrak{A}_i}$, which is equivalent to $T_2 \models \neg\varphi$. $\square$

CHAPTER 7

Preservation Theorems

1. Δ-elementary mappings

DEFINITION 1.1. Let $\mathscr{L}$ be a language, $\mathfrak{A},\mathfrak{B}$ structures, Δ a set of $\mathscr{L}$-formulas.

(1) Let $f: A \to B$. We write $f: \mathfrak{A} \to_\Delta \mathfrak{B}$ if f preserves the formulas in Δ. That is, if for each $\varphi \in \Delta$ and each $a_1,\cdots,a_n \in A$,
$$\text{if } \mathfrak{A} \models \varphi(a_1,\cdots,a_n) \text{ then } \mathfrak{B} \models \varphi(f(a_1),\cdots,f(a_n)).$$
(2) $\mathfrak{A} \Rightarrow_\Delta \mathfrak{B}$ denotes the fact that $\text{Th}(\mathfrak{A}) \cap \Delta \subseteq \text{Th}(\mathfrak{B}) \cap \Delta$.

QUESTION 1.2. In the above definition, item (1): If $\mathfrak{B} \models \varphi(f(a_1),\cdots,f(a_n))$ is it necessarily the case that $\mathfrak{A} \models \varphi(a_1,\cdots,a_n)$?

REMARK 1.3. Well, if $\mathfrak{A} \not\models \varphi(a_1,\cdots,a_n)$ then $\mathfrak{A} \models \neg\varphi(a_1,\cdots,a_n)$ and so $\mathfrak{B} \models \neg\varphi(f(a_1),\cdots,f(a_n))$, contradiction. Thus $f: \mathfrak{A} \to_\Delta \mathfrak{B}$ means that $\varphi \in \Delta$ and each $a_1,\cdots,a_n \in A$,
$$\mathfrak{A} \models \varphi(a_1,\cdots,a_n) \text{ iff } \mathfrak{B} \models \varphi(f(a_1),\cdots,f(a_n)).$$

DISCUSSION 1.4. Can you formulate item (1) from the above Definition in a different way? How about the following:

Let $\mathscr{L}$ be a language, $\mathfrak{A}$, $\mathfrak{B}$ structures and Δ a set of $\mathscr{L}$-formulas. Let
$$\Delta(A) = \{\delta(\bar{a}) : \delta(\bar{x}) \in \Delta \text{ and } \bar{a} \text{ is a tuple in } A\}.$$
Let $f: A \to B$. Then, say $f: \mathfrak{A} \to_\Delta \mathfrak{B}$ if
$$\text{Th}(\mathscr{A}_A) \cap \Delta(A) \subseteq \text{Th}((\mathfrak{B}, f(a))_{a\in A}).$$
The latter two theories are theories in the expanded language $\mathscr{L}(A)$.

REMARK 1.5. (A special case) Now, if Δ is the set of all $\mathscr{L}$-formulas, then $f: \mathfrak{A} \to_\Delta \mathfrak{B}$ states that
$$\text{Th}(\mathfrak{A}_A) \subseteq \text{Th}((\mathfrak{B}, f(a))_{a\in A}).$$
That is, $(\mathfrak{B}, f(a))_{a\in A} \models \text{Th}(\mathfrak{A}_A)$ and so by our earlier characterisation, we get that f is an elementary embedding of $\mathfrak{A}$ into $\mathfrak{B}$, i.e.
$$f: \mathfrak{A} \preceq \mathfrak{B}.$$

THEOREM 1.6. *Let T be a theory, $\mathfrak{A}$ a structure, Δ a set of $\mathscr{L}$-formulas which is closed under existential quantification, conjunction and substitution of variables. The following are equivalent:*

(1) $(Th(\mathfrak{A}) \cap \Delta) \cup T$ is consistent.
(2) there is a model $\mathfrak{B} \models T$ and there is a map $f: \mathfrak{A} \to_\Delta \mathfrak{B}$.

PROOF. $((2) \Rightarrow (1))$: Let $\mathfrak{B} \models T$ and $f : \mathfrak{A} \to_\Delta \mathfrak{B}$. Consider any $\varphi \in \mathrm{Th}(\mathfrak{A}) \cap \Delta$. Since $f : \mathfrak{A} \to_\Delta \mathfrak{B}$ we must have $\mathfrak{B} \models \varphi$. Thus $\mathfrak{B} \models (\mathrm{Th}(\mathfrak{A}) \cap \Delta) \cup T$.

$((1) \Rightarrow (2))$: Let $\mathrm{Th}_\Delta(\mathfrak{A}_A) = \{\Delta(\bar{a}) : \delta(\bar{x}) \in \Delta, \mathfrak{A}_A \models \delta(\bar{a})\}$. Note that

- If $\mathfrak{B} \models \mathrm{Th}_\Delta(\mathfrak{A}_A)$ and $f(a) = a^{\mathfrak{B}}$ for each $a \in A$, then $f : \mathfrak{A} \to_\Delta \mathfrak{B}$.
- Also if $f : \mathfrak{A} \to_\Delta \mathfrak{B}$, then $(\mathfrak{B}, f(a))_{a \in A} \models \mathrm{Th}_\Delta(\mathfrak{A}_A)$.

Thus, there is a one-to-one correspondence between the models $\mathfrak{B}$ of $\mathrm{Th}_\Delta(\mathfrak{A}_A)$ and the maps $f : \mathfrak{A} \to_\Delta \mathfrak{B}$.

$((1) \Rightarrow (2))$: It is sufficient to find a model of $T \cup \mathrm{Th}_\Delta(\mathfrak{A}_A)$. By Compactness, it is sufficient to find a model of $T \cup D$ for each $D \in [\mathrm{Th}_\Delta(\mathfrak{A}_A)]^{<\omega}$. Fix such a finite set D and let $\delta(\bar{a})$ be the conjunction of all elements in D. Then $\mathfrak{A} \models \exists \bar{x} \delta(\bar{x})$. By hypothesis of (1), there is a model $\mathfrak{M}$ of $T \cup \{\exists x \delta(\bar{x})\}$ and so for some finite $\vec{b}$ in M, $\mathfrak{M} \models \delta(\vec{b})$, i.e $(\mathfrak{M}, \bar{b}) \models \delta(\bar{a})$. Thus $(\mathfrak{M}, \vec{b}) \models T \cup D$ and so by the theorem of Compactness, $T \cup \mathrm{Th}_\Delta(\mathfrak{A}_A)$ has model $\mathfrak{B}$. Then $\mathfrak{B} \models T$ and if $f(a) = a^{\mathfrak{B}}$ then $f : \mathfrak{A} \to_\Delta \mathfrak{B}$. □

QUESTION 1.7. Where in the above proof did we use the fact that Δ is closed with respect to substitution of variables? How about existential quantification and conjunction?

COROLLARY 1.8. Let $\mathfrak{A}$, $\mathfrak{B}$ be $\mathscr{L}$-structures, $T = \mathrm{Th}(\mathfrak{B})$ and Δ a set of formulas, which is closed under existential quantification, conjunction and substitution of variables. Then the following are equivalent:

(1) $\mathrm{Th}(\mathfrak{A}) \cap \Delta$ is consistent with $T = \mathrm{Th}(\mathfrak{B})$.
(2) There is a model $\mathfrak{B}' \models T$ and $f : \mathfrak{A} \to_\Delta \mathfrak{B}'$.

REMARK 1.9. Note that $\mathfrak{B}' \models T$ is equivalent to $\mathfrak{B}' \equiv \mathfrak{B}$. Thus, item (1) is equivalent to the existence of $\mathfrak{B}' \equiv \mathfrak{B}$ such that $f : \mathfrak{A} \to_\Delta \mathfrak{B}'$ for some f.

COROLLARY 1.10. Let $\mathfrak{A}, \mathfrak{B}$ be $\mathscr{L}$-structures and Δ a set of $\mathscr{L}$-formulas which is closed under existential quantification, conjunction and substitution of variables. The following are equivalent:

(1) $\mathfrak{A} \Rightarrow_\Delta \mathfrak{B}$
(2) $\exists f : \mathfrak{A} \to_\Delta \mathfrak{B}'$ where $\mathfrak{B}' \equiv \mathfrak{B}$.

PROOF. $((1) \Rightarrow (2))$: By hypothesis $\mathrm{Th}(\mathfrak{A}) \cap \Delta \subseteq \mathrm{Th}(\mathfrak{B}) \cap \Delta$. Thus $\mathrm{Th}(\mathfrak{B}) \cup (\mathrm{Th}(\mathfrak{A}) \cap \Delta)$ is consistent (well, this set is in fact just $\mathrm{Th}(\mathfrak{B})$) and so by the Preservation Theorem applied to $T = \mathrm{Th}(\mathfrak{B})$, there is a model $\mathfrak{B}' \models T$ and $f : \mathfrak{A} \to_\Delta \mathfrak{B}'$. Since $\mathfrak{B}' \models T$ we get $\mathfrak{B}' \equiv \mathfrak{B}$.

$((2) \Rightarrow (1))$: By the hypothesis (2) if $\varphi \in \mathrm{Th}(\mathfrak{A}) \cap \Delta$, then $\varphi \in \mathrm{Th}(\mathfrak{B}') = \mathrm{Th}(\mathfrak{B})$. Thus $\mathrm{Th}(\mathfrak{A}) \cap \Delta \subseteq \mathrm{Th}(\mathfrak{B}) \cap \Delta$. □

REMARK 1.11. If $\mathfrak{A}_1 \subseteq \mathfrak{A}_2$ and $\mathfrak{A}_1$, $\mathfrak{A}_2$ can not be separated by a universal sentence, then in particular for every formula $\exists x \varphi(x)$ we have:

$$\text{if } \mathfrak{A}_2 \models \exists x \varphi(x) \text{ then } \mathfrak{A}_1 \models \exists x \varphi(x).$$

Note that this is not sufficient to conclude that $\mathfrak{A}_1 \prec \mathfrak{A}_2$. Can you see why? For this to be the case, we need the above property for all $\mathscr{L}(A_1)$ formulas, not only the $\mathscr{L}$-ones.

THEOREM 1.12. *(Universal Separation) Let T_1, T_2 be theories. The following are equivalent:*

(1) There is a universal sentence separating T_1 from T_2.
(2) No model of T_2 is a substructure of a model of T_1.

PROOF. $((1) \Rightarrow (2))$ Let φ be an universal sentence such that

$$T_1 \vdash \varphi \text{ and } T_2 \vdash \neg\varphi.$$

Suppose $\mathfrak{A}_1 \models T_1$ and $\mathfrak{A}_2 \models T_2$ such that $\mathfrak{A}_2 \subseteq \mathfrak{A}_1$. Since φ is universal, $\mathfrak{A}_2 \models \varphi$ (by downwards absoluteness of universal formulas) and so

$$\mathfrak{A}_2 \models \varphi \wedge \neg\varphi,$$

which is a contradiction.

$((2) \Rightarrow (1))$ We will show $\neg(1) \Rightarrow \neg(2)$. Thus, suppose there is no universal sentence which separates T_1 from T_2. Then by the Separation Theorem, there are models $\mathfrak{A}_1 \models T_1$ and $\mathfrak{A}_2 \models T_2$ which can not be separated by an universal sentence. Then in particular,

$$\text{if } \mathfrak{A}_2 \models \exists\bar{x}\varphi(\bar{x}) \text{ then } \mathfrak{A}_1 \models \exists\bar{x}\varphi(\bar{x})$$

(we write $\mathfrak{A}_2 \Rightarrow_\exists \mathfrak{A}_1$). By Corollary B there is $\mathfrak{A}_1' \equiv \mathfrak{A}_1$ and $f : \mathfrak{A}_2 \to_\exists \mathfrak{A}_1'$. But, then for some $\mathfrak{A}_2'$ we have that

$$\mathfrak{A}_2' \equiv \mathfrak{A}_1' \text{ and } \mathfrak{A}_2 \subseteq \mathfrak{A}_2'.$$

Thus a model of T_1 is a substructure of a model of T_2, which is what we wanted to prove. □

QUESTION 1.13. The use of the subscript $\exists$ in the above proof (for example in $\mathfrak{A}_2 \Rightarrow_\exists \mathfrak{A}_1$) is just an abbreviation for which set $\Delta_\exists$ of $\mathscr{L}$-formulas?

DEFINITION 1.14. Let T be a $\mathscr{L}$-theory. The formulas $\varphi(\bar{x})$ and $\psi(\bar{x})$ are said to be equivalent under T if $T \vdash \forall\bar{x}(\varphi(\bar{x}) \leftrightarrow \psi(\bar{x}))$.

COROLLARY 1.15. Let T be a theory and $\varphi(\bar{x})$ a formula. The following are equivalent:

(1) There is an universal $\psi(\bar{x})$ such that $T \vdash \forall\bar{x}(\varphi(\bar{x}) \leftrightarrow \psi(\bar{x}))$.
(2) If $\mathfrak{A} \subseteq \mathfrak{B}$ are models of T and $\bar{a} = (a_1, \cdots, a_n)$ is an n-tuple in A, then:

$$\text{if } \mathfrak{B} \models \varphi(\bar{a}) \text{ then } \mathfrak{A} \models \varphi(\bar{a}).$$

PROOF. $((2) \Rightarrow (1))$: Extend $\mathscr{L}$ by adding new constants $\{c_i\}_{i=1}^n$ and let $\bar{c} = (c_1, \cdots, c_n)$. Let $T_1 = T \cup \{\varphi(\bar{c})\}$, $T_2 = T \cup \{\neg\varphi(\bar{c})\}$. Now, if $\mathfrak{A} \models T_1$ and $\mathfrak{B} \subseteq \mathfrak{A}$, then $\mathfrak{B} \not\models T_2$. By the Universal Separation Theorem the theories T_1 and T_2 can be separated by a universal $\mathscr{L}(\mathscr{C})$-sentence $\psi(\bar{c})$.

Thus $T \vdash \varphi(\bar{c}) \to \psi(\bar{c})$ and so $T \vdash \forall\bar{x}(\varphi(\bar{x}) \to \psi(\bar{x}))$. Similarly, $T \vdash \neg\psi(\bar{c}) \to \neg\varphi(\bar{c})$ and so $T \models \forall\bar{x}(\neg\varphi(\bar{x}) \to \neg\psi(\bar{x}))$. Thus $\varphi(\bar{x})$ is modulo T equivalent to the universal formula $\psi(\bar{x})$, i.e.

$$T \vdash \forall\bar{x}(\varphi(\bar{x}) \leftrightarrow \psi(\bar{x})).$$

$((1) \Rightarrow (2))$: Straightforward. □

COROLLARY 1.16. A theory T is equivalent to an universal theory (i.e. a theory consisting of universal sentences) if and only if all substructures of models of T are again models of T.

PROOF. $(\Rightarrow)$: If T is equivalent to an universal theory, then by downwards absoluteness of universal formulas, every substructure of a model of T is again a model of T.

$(\Leftarrow)$: Suppose T is a theory with the property that all substructures of a model of T are again models of T. Fix $\varphi \in T$. Consider the theories $T_1 = T$ and $T_2 = \{\neg\varphi\}$. If $\mathfrak{B} \models T_1$ and $\mathfrak{A} \subseteq \mathfrak{B}$, then by hypothesis $\mathfrak{A} \models T_1$. Thus in particular, $\mathfrak{A} \not\models T_2$. Thus, no model of T_2 is a substructure of a model of T_1. Therefore there is an universal sentence ψ which separates T_1 and T_2, i.e. $T_1 \vdash \psi$ and $\neg\varphi \vdash \neg\psi$. Note that

$$\neg\varphi \vdash \neg\psi \text{ if and only if } \psi \vdash \varphi.$$

Thus for every formula $\varphi \in T$ there is an universal formula ψ_φ such that

$$T \vdash \psi_\varphi \text{ and } \psi_\varphi \vdash \varphi.$$

Thus every sentence of T follows from

$$T_\forall = \{\psi : T \vdash \psi, \psi \text{ is universal}\},$$

and so $T_\forall \equiv T$. □

2. Inductive Theories

DEFINITION 2.1. ($\forall\exists$-formulas) A formula is said to be a $\forall\exists$-formula if it is of the form $\forall\bar{x}\psi(\bar{x})$ where $\psi(\bar{x})$ is existential.

LEMMA 2.2. Suppose φ is a $\forall\exists$-sentence, $\{\mathfrak{A}_i\}_{i\in I}$ a directed family of models for φ and $\mathfrak{B} = \bigcup_{i\in I}\mathfrak{A}_i$. Then $\mathfrak{B} \models \varphi$.

PROOF. We can write φ in the form $\forall\bar{x}\psi(\bar{x})$, where $\psi(\bar{x})$ is existential. Now, pick any tuple $\bar{b}$ of elements in $\mathfrak{B}$. Then we can find $i \in I$ such that $\bar{b}$ is a tuple of elements in $\mathfrak{A}_i$. Since $\mathfrak{A}_i \models \forall\bar{x}\psi(\bar{x})$, we have

$$\mathfrak{A}_i \models \psi(\bar{b}).$$

But $\psi(\bar{b})$ is existential and since existential formulas are upwards absolute, we must have

$$\mathfrak{B} \models \psi(\bar{b}).$$

Thus $\mathfrak{B} \models \forall\bar{x}\psi(\bar{x})$, i.e. $\mathfrak{B} \models \varphi$. □

DEFINITION 2.3. (Inductive theories) A theory T is called inductive, if the union of any directed family of models of T is again a model of T.

THEOREM 2.4. *Let T_1 and T_2 be two theories. The following are equivalent:*

(1) There is a $\forall\exists$-sentence which separates T_1 from T_2.
(2) No model of T_2 is the union of a chain (or of a directed family) of models of T_1.

PROOF. ((1)⇒(2)): Suppose φ is a $\forall\exists$-sentence, $T_1 \vdash \varphi$ and $T_2 \vdash \neg\varphi$. Let $\{\mathfrak{A}_i\}_{i\in I}$ be a directed family of models of T_1, $\mathfrak{B} = \bigcup_{i\in I}\mathfrak{A}_i$. Since $\forall\exists$-formulas are inductive, $\mathfrak{B} \models \varphi$ and so $\mathfrak{B} \not\models T_2$.

((2)⇒(1)) Suppose $\neg$(1). Then T_1 and T_2 have models $\mathfrak{A} \models T_1$ and $\mathfrak{B}^0 \models T_2$ which cannot be separated by a $\forall\exists$-sentence. Thus in particular, $\mathfrak{B}^0 \Rightarrow_\forall \mathfrak{A}$. By Corollary B there is $\mathfrak{A}^0 \equiv \mathfrak{A}$ and $f : \mathfrak{B}^0 \to_\forall \mathfrak{A}^0$. We can assume that $\mathfrak{B}^0 \subseteq \mathfrak{A}^0$ and that f is the identity mapping. Let B be the universe of $\mathfrak{B}^0$. Then $\mathfrak{B}^0_B \Rightarrow_\forall \mathfrak{A}^0_B$ and so $\mathfrak{A}^0_B \Rightarrow_\exists \mathfrak{B}^0_B$. Apply Corollary B to obtain a model $\mathfrak{B}^1_B \equiv \mathfrak{B}^0_B$ and $f : \mathfrak{A}^0_B \to_\exists \mathfrak{B}^1_B$. Without loss of generality $\mathfrak{A}^0_B \subseteq \mathfrak{B}^1_B$ and so $\mathfrak{B}^0_B \subseteq \mathfrak{B}^1_B$. Note that since $\mathfrak{B}^1_B \equiv \mathfrak{B}^0_B$, in particular $\mathfrak{B}^1_B \models \mathrm{Th}(\mathfrak{B}^0_B)$. Thus, $\mathfrak{B}^0 \prec \mathfrak{B}^1$ and so $\mathfrak{B}^0 \subseteq \mathfrak{A}^0 \subseteq \mathfrak{B}^1$ and $\mathfrak{B}^0 \prec \mathfrak{B}^1$.

Consider $\mathfrak{A}$ and $\mathfrak{B}^1$ and suppose they can be separated by a $\forall\exists$-formula ψ. Say, $\psi = \forall\bar{x}\rho(\bar{x})$ where ρ is existential. Thus $\mathfrak{A} \models \psi$ and $\mathfrak{B}^1 \models \neg\psi$, i.e. $\mathfrak{B}^1 \models \exists\bar{x}\neg\rho(\bar{x})$. However, $\mathfrak{B}^0 \prec \mathfrak{B}^1$, which implies that for some tuple $\bar{b}$ in the universe B of $\mathfrak{B}^0$, $\mathfrak{B}^1 \models \neg\rho(\bar{b})$. By downwards absoluteness of universal formulas: $\mathfrak{B}^0 \models \neg\rho(\bar{b})$. Thus, $\mathfrak{B}^0 \models \exists\bar{x}\neg\rho(\bar{x})$, i.e. $\mathfrak{B}^0 \models \neg\psi$. That is $\mathfrak{A}$ and $\mathfrak{B}^0$ can be separated by a $\forall\exists$-formula, which is a contradiction to their choice. Thus, $\mathfrak{A}$ and $\mathfrak{B}^1$ can not be separated by a $\forall\exists$-formula. Now, apply the same argument to $\mathfrak{A}$ and $\mathfrak{B}^1$ to obtain an extension $\mathfrak{A}^1$ of $\mathfrak{B}^1$ such that $\mathfrak{A}^1 \equiv \mathfrak{A}$ and an extension $\mathfrak{B}^2$ of $\mathfrak{A}^1$ such that $\mathfrak{B}^1 \prec \mathfrak{B}^2$.

Proceeding inductively we can obtain an infinite chain

$$\mathfrak{B}^0 \subseteq \mathfrak{A}^0 \subseteq \mathfrak{B}^1 \subseteq \mathfrak{A}^1 \subseteq \mathfrak{B}^2 \subseteq \cdots,$$

where for each $i \in \mathbb{N}$

$$\mathfrak{B}^i \prec \mathfrak{B}^{i+1} \text{ and } \mathfrak{A} \equiv \mathfrak{A}^i.$$

Let $\mathfrak{B} = \bigcup_{i\in\mathbb{N}} \mathfrak{A}^i$. But then also $\mathfrak{B} = \bigcup_{i\in\mathbb{N}} \mathfrak{B}^i$ and so $\mathfrak{B}$ is an elementary extension of $\mathfrak{B}^0$, which implies that $\mathfrak{B} \models T_2$. Since each $\mathfrak{A}^i \models T_1$, we obtain a model of T_2 which is the union of a chain of models of T_1, i.e. we established $\neg(2)$. □

COROLLARY 2.5. Let T be a theory. For each sentence φ the following are equivalent:

(1) φ is equivalent modulo T to an $\forall\exists$-sentence.
(2) If $\{\mathfrak{A}_i\}_{i\in\mathbb{N}}$ is a chain of models of a theory T and $\mathfrak{B} = \bigcup_{i\in\mathbb{N}} \mathfrak{A}_i$ is also a model of T, then $\mathfrak{B} \models \varphi$ if $\mathfrak{A}_i \models \varphi$ for each $i \in \mathbb{N}$.

PROOF. ((1)⇒(2)) Let $\mathfrak{B} = \bigcup_{i\in\mathbb{N}} \mathfrak{A}_i$ and suppose $\mathfrak{A}_i \models T \cup \{\varphi\}$ for each $i \in \mathbb{N}$. Moreover, suppose $\mathfrak{B} \models T$. By hypothesis there is a $\forall\exists$-sentence ψ which is equivalent modulo T to φ. Then in particular for each i, $\mathfrak{A}_i \models \psi$ and since $\forall\exists$-sentences are inductive, we obtain $\mathfrak{B} \models \psi$.

((2)⇒(1)) Consider the theories $T_1 = T \cup \{\varphi\}$ and $T_2 = \{\neg\varphi\}$. By hypothesis (2) no model of T_2 is the union of a chain of models of T_1. By the $\forall\exists$-separation Theorem, the theories T_1 and T_2 can be separated by a $\forall\exists$-sentence ψ. Thus $T_1 \vdash \psi$ and $T_2 \vdash \neg\psi$. That is

$$T \cup \{\varphi\} \vdash \psi \text{ and } \{\neg\varphi\} \vdash \neg\psi,$$

which implies $T \vdash \varphi \to \psi$ and $\vdash \neg\varphi \to \neg\psi$. That is $T \vdash \varphi \leftrightarrow \psi$. □

COROLLARY 2.6. A theory T is inductive if and only if $T \equiv T_{\forall\exists}$ where

$$T_{\forall\exists} = \{\psi : \psi \text{ is a } \forall\exists\text{-sentence such that } T \vdash \psi\}.$$

PROOF. (⇐) Suppose $T \equiv T_{\forall\exists}$ and let $\{\mathfrak{A}_i\}_{i\in\mathbb{N}}$ be an increasing chain of models of T. Then $\{\mathfrak{A}_i\}_{i\in\mathbb{N}}$ is also an increasing chain of models of $T_{\forall\exists}$. However $\forall\exists$-sentences are preserved by increasing chains of models and so $\mathfrak{B} = \bigcup_{i\in\mathbb{N}} \mathfrak{A}_i \models T_{\forall\exists}$. Since $T_{\forall\exists} \equiv T$ we obtain $\mathfrak{B} \models T$.

(⇒) Suppose T is inductive and let $\varphi \in T$. Let $\mathfrak{B}$ be the increasing union of the chain $\{\mathfrak{A}_i\}_{i\in\mathbb{N}}$. Then $\mathfrak{B} \models \varphi$ and so $\mathfrak{B} \not\models \neg\varphi$. Thus, no increasing chain of models of T is a model of $\{\neg\varphi\}$. Therefore by the $\forall\exists$-Separation Theorem, the theories T and $\{\neg\varphi\}$ are separated by a $\forall\exists$-sentence ψ. Thus

$$T \vdash \psi \text{ and } \{\neg\varphi\} \vdash \neg\psi.$$

Therefore $T \vdash \varphi \leftrightarrow \psi$. Since φ was arbitrary in T we get $T \equiv T_{\forall\exists}$. □

DISCUSSION 2.7. Consider the language $\mathscr{L}$ consisting of a single binary relation symbol $<$. Let $\mathfrak{A}_0$ be the $\mathscr{L}$-structure with universe $A_0 = \{0, 1, \cdots\}$ and the natural interpretation of $<$. Moreover for each $n \in \mathbb{N}$ let $\mathfrak{A}_n$ be the $\mathscr{L}$-structure with universe $A_n := \{-n, \cdots, -1, 0, 1, 2, \cdots\}$ again with the natural interpretation of $<$. Note that $\{\mathfrak{A}_i\}_{i\in\mathbb{N}}$ forms an increasing chain of $\mathscr{L}$-structures and let $\mathfrak{B} = \bigcup_{i\in\mathbb{N}} \mathfrak{A}_i$.

(1) Show that $\mathrm{Th}(\mathfrak{A}_0) = \mathrm{Th}(\mathfrak{A}_i)$ for each $i \in \mathbb{N}$.
(2) Find a $\forall\exists$-sentence ψ such that $\mathfrak{B} \models \psi$, but $\mathfrak{A}_0 \not\models \psi$.
(3) Is $\mathrm{Th}(\mathfrak{A}_0)$ inductive?

3. Quantifier Elimination

DEFINITION 3.1. A theory T has *quantifier elimination* if every $\mathscr{L}$-formula $\varphi(x_1, \cdots, x_n)$ in T is equivalent modulo T to a quantifier free formula $\rho(x_1, \cdots, x_n)$.

EXAMPLE 3.2. If T has quantifier elimination, then every sentence is equivalent to a quantifier free sentence.

EXAMPLE 3.3. Let $\mathscr{L}$ be a language and T an $\mathscr{L}$-theory. Expand the language $\mathscr{L}$ by adjoining new relation symbols R_φ for each $\mathscr{L}$-formula φ. In the expanded language the theory

$$T \cup \{\forall x_1 \cdots x_n (R_\varphi(x_1, \cdots, x_n) \leftrightarrow \varphi(x_1, \cdots, x_n)) : \varphi \text{ is an } \mathscr{L} - \text{formula}\}$$

has quantifier elimination.

Recall that atomic formulas and their negations are called *basic formulas*, as well as the following fact.

FACT 2.

(1) Every quantifier free formula is equivalent to a formula in the form $\wedge_{i<m} \vee_{j<m_i} \pi_{i,j}$ and to a formula in the form $\vee_{i<m} \wedge j < m_i \pi_{i,j}$, where in both formulas each $\pi_{i,j}$ is basic. The former is referred to as a *conjunctive normal form*, the latter as a *disjunctive normal form*.
(2) Every formula is equivalent to a formula in *prenex normal form*, i.e. to a formula in the form $Q_1 x_1 \cdots Q_n x_n \varphi$, where $Q_i \in \{\exists, \forall\}$ for each i and φ is a quantifier free formula.

DEFINITION 3.4.

(1) A *simple existential* formula is a formula in the form $\exists y \varphi$, where φ is a quantifier-free formula.
(2) A *primitive existential* formulas is a formula in the form $\exists y \varphi$, where φ is a conjunction of basic formulas.

LEMMA 3.5. A theory T has quantifier elimination if and only if every primitive existential formula is equivalent modulo T to a quantifier free formula.

PROOF. ($\Leftarrow$): Let φ be simple existential. That is $\varphi = \exists y \rho$ for some quantifier free formula ρ. Write ρ in disjunctive normal form. Then $\varphi = \exists y \vee_{i<n} \rho_i$ where each $\rho_i = \wedge_{j<m_i} \pi_{ij}$, π_{ij} basic. Then φ is equivalent to $\vee_{i<n} \exists y \rho_i$. Thus φ is equivalent to a disjunction of primitive existential formulas. Therefore every simple existential formula is equivalent modulo T to a quantifier free formula.

($\Leftarrow$): Now, consider an arbitrary formula φ. Then φ can be written in prenex normal form $Q_1 x_1 \cdots Q_n x_n \rho$ where each Q_i is a quantifier (i.e. $\exists$ or $\forall$) and ρ is quantifier free.

<u>Suppose $Q_n = \exists$</u>. Then $\exists x_n \rho$ is a simple existential formula and so there is a quantifier free formula ρ_0 equivalent modulo T to $\exists x_n \rho$. Continue by considering the formula $Q_1 x_1 \cdots Q_{n-1} x_{n-1} \rho_0$.

<u>If $Q_n = \forall$</u> then $\forall x_n \rho$ is equivalent to $\neg\neg\forall x_n \rho$. Note that $\neg\forall x_n \rho$ is equivalent to $\exists x_n \neg\rho$. Moreover $\exists x_n \neg\rho$ is simple existential and thus it is equivalent modulo T to a quantifier free formula ρ_1. Then $\neg\rho_1$ is still quantifier free and equivalent to $\forall x_n \rho$. Proceed with $Q_1 x_1 \cdots Q_{n-1} x_{n-1} \neg\rho_1$.

($\Rightarrow$): Straightforward. □

THEOREM 3.6. *Let T be a theory. Then the following are equivalent:*

(1) T has quantifier elimination.
(2) Whenever $\mathfrak{M}^1$, $\mathfrak{M}^2$ are models of T with a common $\mathscr{L}$-substructure $\mathfrak{A}$, then

$$\mathfrak{M}^1_A \equiv \mathfrak{M}^2_A.$$

(3) Whenever $\mathfrak{M}^1$, $\mathfrak{M}^2$ are models of T with common substructure $\mathfrak{A}$, then for all primitive existential formulas $\varphi(x_1, \cdots, x_n)$ and parameters $a_1, \cdots, a_n$ from A:

$$\mathfrak{M}^1 \models \varphi(a_1, \cdots, a_n) \Rightarrow \mathfrak{M}^2 \models \varphi(a_1, \cdots, a_n).$$

PROOF. $(1) \Rightarrow (2))$: Fix $\mathfrak{M}^1$, $\mathfrak{M}^2$ with a common substructure $\mathfrak{A}$. Consider the expanded language $\mathscr{L}(A)$ and a $\mathscr{L}(A)$-sentence $\varphi(\bar{a})$ such that $\mathfrak{M}^1 \vDash \varphi(\bar{a})$. Since T has quantifier elimination there is a quantifier free formula $\rho(\bar{x})$ which is equivalent modulo T to $\varphi(\bar{x})$. Then $\mathfrak{M}^1 \vDash \rho(\bar{a})$ and so $\mathfrak{A} \vDash \rho(\bar{a})$, which implies that $\mathfrak{M}^2 \vDash \rho(\bar{a})$ and finally $\mathfrak{M}^2 \vDash \varphi(\bar{a})$.

$((2) \Rightarrow (3))$ Straightforward.

$((3) \Rightarrow (1))$ It is sufficient to show that every primitive existential formula is equivalent modulo T to a quantifier free formula. Fix a primitive existential formula $\varphi(\bar{x})$. Consider the expanded language $\mathscr{L}(C)$, where $C = \{c_i\}_{i=1}^n$ is a set of new constants. Let $\bar{c} = (c_1, \cdots, c_n)$. It is sufficient to show that $T_1 = T \cup \{\varphi(\bar{c})\}$ and $T_2 = T \cup \{\neg\varphi(\bar{c})\}$ can be separated by a quantifier free sentence $\rho(\bar{c})$. Consider two $\mathscr{L}(\bar{c})$-structures, $(\mathfrak{M}^1, \bar{a}^1)$ and $(\mathfrak{M}^2, \bar{a}^2)$ satisfying T_1 and T_2 respectively. Here $\bar{a}^1 = (a_1^1, \cdots, a_n^1)$ and $\bar{a}^2 = (a_1^2, \cdots, a_n^2)$ are designated n-tuples in M^1 and M^2 respectively.

Suppose $(\mathfrak{M}^1, \bar{a}^1)$ and $(\mathfrak{M}^2, \bar{a}^2)$ can not be separated by a quantifier free sentence in $\mathscr{L}(C)$.

EXERCISE. Show that the generated substructure $\mathfrak{A}^1 = \langle \{a_i^1\}_{i=1}^n \rangle^{\mathfrak{M}^1}$ is isomorphic to the generated substructure $\mathfrak{A}^2 = \langle \{a_i^2\}_{i=1}^n \rangle^{\mathfrak{M}^2}$.

Thus, without loss of generality $(\mathfrak{M}^1, \bar{a}^1)$ and $(\mathfrak{M}^2, \bar{a}^2)$ have a common substructure $\mathfrak{A}$. Then by the hypothesis of (3) we obtain:

$$\mathfrak{M}^1 \vDash \varphi(a_1, \cdots, a_n) \Rightarrow \mathfrak{M}^2 \vDash \varphi(a_1, \cdots, a_n),$$

which is a contradiction. Then, T_1 and T_2 can be separated via a quantifier free sentence and so $\varphi(\bar{x})$ is equivalent to a quantifier free sentence. □

4. Model Completeness

DEFINITION 4.1. A theory T is said to be *model complete* if for all models $\mathfrak{M}^1$ and $\mathfrak{M}^2$ of T, if $\mathfrak{M}^1 \subseteq \mathfrak{M}^2$ then $\mathfrak{M}^1 \prec \mathfrak{M}^2$.

COROLLARY 4.2.

(1) If T has quantifier elimination, then T is model complete.
(2) A theory T is model complete if and only if for every model $\mathfrak{M}$ of T, the theory $T \cup \mathrm{Diag}(\mathfrak{M})$ is complete.

REMARK 4.3. Recall that

$$\mathrm{Diag}(\mathfrak{M}) = \{\varphi : \varphi \text{ is a basic } \mathscr{L}(M)\text{-sentence such that } \mathfrak{M}_M \vDash \varphi\}.$$

COROLLARY 4.4. (Robinson Test) Let T be a theory. The following are equivalent:

(1) T is model complete.
(2) Whenever $\mathfrak{M}^1 \subseteq \mathfrak{M}^2$ are models of T and φ is an $\mathscr{L}(M^1)$-existential sentence and $\mathfrak{M}^2 \vDash \varphi$, then $\mathfrak{M}^1 \vDash \varphi$.
(3) Each formula is modulo T equivalent to a universal formula.

PROOF. Item (1) implies item (2) by definition of model completeness; (1) is equivalent to (3) is a restatement of an earlier Corollary. To see that (2) implies (3) note that by (2) and same Corollary, every existential formula is equivalent modulo T to a universal formula.

Take an arbitrary formula φ and write it in prenex normal form $Q_1x_1 \cdots Q_nx_n\rho$ where ρ is quantifier free and each Q_i is a quantifier, $\exists$ or $\forall$. Proceed inductively. Here are the interesting cases:

If $Q_n = \exists$ then $Q_nx_n\rho$ is existential and by the above observation $Q_nx_n\rho$ is equivalent modulo T to a universal formula ψ. Consider $Q_{n-1}x_{n-1}\psi$ and suppose $Q_{n-1} = \exists$. Let $\chi = \exists x_{n-1}\psi$. Then χ is equivalent to $\neg\forall x_{n-1}\neg\psi$. However $\neg\psi$ is equivalent to an existential formula φ_1 and φ_1 is equivalent modulo T to a universal formula ψ_1

(by the above observation). Thus χ is equivalent modulo T to the formula $\neg\forall x_{n-1}\psi_1$. However the negation of a universal formula is equivalent to an existential formula and so in particular, $\neg\forall x_{n-1}\psi_1$ is equivalent to an existential formula φ_2. Again by the above observation φ_2 is equivalent modulo T to a universal formula ψ_2. Thus, χ is modulo T equivalent to the universal formula ψ_2. Following the same argument, in finitely many steps one can show that φ (the formula we started with) is equivalent modulo T to a universal formula. $\square$

EXERCISE 3. Let T be a model complete theory and let $\mathfrak{M}$ be a model of T which embeds into every model of T. Show that T is complete.

REMARK 4.5. Note that there are theories which are model complete, but do not have quantifier elimination.

Part 3

Countable Models

CHAPTER 8

ω-saturatedness

1. The Omitting Type Theorem

DEFINITION 1.1. Let T be an $\mathscr{L}$-theory and let $\Sigma(x)$ be a set of $\mathscr{L}$-formulas.

(1) A model of T which does not realize $\Sigma(x)$ is said to omit $\Sigma(x)$.
(2) A formula $\varphi(x)$ is said to isolate $\Sigma(x)$ in T if $T \cup \{\varphi(x)\}$ is consistent and for each $\sigma(x) \in \Sigma(x)$ we have $T \vdash \forall x(\varphi(x) \to \sigma(x))$.

THEOREM 1.2. *(Omitting type) Suppose T is a countable consistent theory and $\Sigma(x)$ is a set of formulas which is not isolated in T. Then T has a model which omits $\Sigma(x)$.*

REMARK 1.3. If T is complete and $\varphi(x)$ isolates $\Sigma(x)$ in T, then $\Sigma(x)$ is realised in every model of T. Moreover every element which realizes $\varphi(x)$ will realize $\Sigma(x)$.

PROOF. Let $\mathscr{C} = \{c_i\}_{i<\omega}$ be a countable set of new constants. Let $\{\psi_i(x)\}_{i<\omega}$ enumerate all $\mathscr{L}(\mathscr{C})$-formulas. Inductively construct an increasing chain $\{T_i\}_{i\in\mathbb{N}}$ of consistent extensions of T as follows. Let $T_0 = T$. Assume T_{2i} has been defined. Pick $c \in \mathscr{C}$ such that c does not occur in $T_{2i} \cup \{\psi_i(x)\}$ and take $T_{2i+1} = T_{2i} \cup \{\exists x \psi_i(x) \to \psi_i(c)\}$. Since T_{2i} is consistent, then so is T_{2i+1}. Without loss of generality, $T_{2i+1} = T \cup \{\delta(c_i, \bar{c})\}$, where $\bar{c}$ is a finite tuple of new constants and c_i does not occur in $\bar{c}$. Now, consider the formula $\exists \bar{y} \delta(x, \bar{y})$. Then by hypothesis $\exists \bar{y} \delta(x, \bar{y})$ does not isolate $\Sigma(x)$ and so there is $\sigma(x) \in \Sigma(x)$ such that

$$T \not\vdash \forall x(\exists \bar{y} \delta(x, \bar{y}) \to \sigma(x)).$$

Thus, $T \cup \{\exists \bar{y} \delta(x, \bar{y}) \wedge \neg\sigma(x)\}$ is consistent and so $T_{2i+1} \cup \{\neg\sigma(c_i)\}$ is also consistent. Take $T_{2i+2} = T_{2i+1} \cup \{\neg\sigma(c_i)\}$. Finally, let $T' = \bigcup T_i$. Then:

(1) T' is Henkin. Indeed, for every $\mathscr{L}(\mathscr{C})$-formula $\psi(x)$ there is a constant $c \in \mathscr{C}$ such that the formula $\exists x \psi(x) \to \psi(c)$ is in T'.
(2) for every constant $c \in \mathscr{C}$ there is a formula $\sigma(x) \in \Sigma(x)$ such that $\neg\sigma(c) \in T'$.

Then T' is consistent with a model say $(\mathfrak{A}, a_c)_{c\in\mathscr{C}}$. Since T' is Henkin, by the Tarski-Vaught criterion there is an elementary submodel $\mathfrak{A}' \prec \mathfrak{A}$ with universe $\{a_c\}_{c\in\mathscr{C}}$. Then for every $c \in \mathscr{C}$, $\mathfrak{A}' \models \neg\sigma(c)$ which gives us a model of T which does not realize $\Sigma(x)$. □

COROLLARY 1.4. For each i let $\Sigma(x_1, \cdots, x_{n_i})$ be a set of formulas with free variables in $\{x_i\}_{i=1}^{n_i}$. Suppose T is a countable consistent theory and none of the sets $\Sigma(x_1, \cdots, x_{n_i})$ is isolated in T. Then T has a model omitting all of these partial types.

PROOF. Generalize the proof of the Omitting type theorem. □

2. The Space of Types

DEFINITION 2.1. Let T be a theory.

(1) An n-type is a maximal set of formulas $p(x_1, \cdots, x_n)$ which is consistent with T.
(2) $S_n(T)$ denotes the set of all n-types.
(3) $S_1(T)$ is often denoted $S(T)$.
(4) $S_0(T)$ is the set of all maximal consistent extensions of T (note that if $p \in S_0(T)$ then p consists of closed formulas, i.e. sentences).

The following is a revision, however it might be helpful.

DEFINITION 2.2. Let $\mathfrak{A}$ be an $\mathscr{L}$-structure, $B \subseteq A$, $a \in A$, $\Sigma(x)$ a set of $\mathscr{L}(B)$ formulas with at most x as a free variable.

- Then $a \in A$ is said to realize $\Sigma(x)$, if $\mathfrak{A}_B \models \sigma(a)$ for all $\sigma \in \Sigma(x)$.
- We write $\mathfrak{A}_B \models \Sigma(a)$ or simply $\mathfrak{A} \models \Sigma(a)$.

Note that by the Compactness Theorem, the set $\Sigma(x)$ is finitely satisfiable in $\mathfrak{A}$ if and only if there is an elementary extension of $\mathfrak{A}$ which realizes $\Sigma(x)$.

DEFINITION 2.3. Let $\mathfrak{A}$ be a $\mathscr{L}$-structure, $B \subseteq A$.

(1) A set $p(x)$ of $\mathscr{L}(B)$-formulas is said to be a type over B if $p(x)$ is maximal finitely satisfiable in $\mathfrak{A}$. The set B is called the domain of p.
(2) $S(B) = S^{\mathfrak{A}}(B)$ denotes the set of types over B.

REMARK 2.4. Let $\mathfrak{A}$ be an $\mathscr{L}$-structure and let $a \in A$. Then the set

$$\text{tp}(a/B) = \text{tp}^{\mathfrak{A}}(a/B) = \{\varphi(x) : \mathfrak{A} \models \varphi(a), \varphi \text{ is a } \mathscr{L}(B)\text{-formula}\}.$$

Note that $a \in A$ realizes the type $p \in S(B)$ if and only if $p = \text{tp}(a/B)$.

EXERCISE 4. Show that if $\mathfrak{A} \prec \mathfrak{A}'$, $B \subseteq A$, $a \in A$ then:

(1) $S^{\mathfrak{A}}(B) = S^{\mathfrak{A}'}(B)$
(2) $\text{tp}^{\mathfrak{A}'}(a/B) = \text{tp}^{\mathfrak{A}}(a/B)$

DEFINITION 2.5. Let $\mathfrak{A}$ be a $\mathscr{L}$-structure, $B \subseteq A$.

(1) Maximal finitely satisfiable sets of $\mathscr{L}(B)$ formulas in the variables $x_1, \cdots, x_n$ are called n-types and $S_n(B) = S_n^{\mathfrak{A}}(B)$ denotes the set of n-types over B.
(2) Given an n-tuple $\bar{a}$ from $\mathfrak{A}$, denote

$$\text{tp}^{\mathfrak{A}}(\bar{a}/B) = \{\varphi(\bar{x}) : \mathfrak{A} \models \varphi(\bar{a}), \varphi(\bar{a}) \text{ is } \mathscr{L}(B)\text{-formula}\}.$$

Note that $\text{tp}^{\mathfrak{A}}(\bar{a}/B) \in S_n^{\mathfrak{A}}(B)$.
(3) Let C be an arbitrary set. Then

$$\text{tp}(C/B) = \{\varphi(x_{c_1}, \cdots, x_{c_n}) : \mathfrak{A} \models \varphi(c_1, \cdots, c_n), \varphi \text{ is an } \mathscr{L}(B) - \text{fromula}\}.$$

Recall that every structure $\mathfrak{A}$ has an elementary extension in which all types over A are realised. This concludes the revision.

REMARK 2.6. Let $\mathfrak{A}$ be a $\mathscr{L}$-structure and $B \subseteq A$. Then $S_n^{\mathfrak{A}}(B) = S_n(\text{Th}(\mathfrak{A}_B))$. Thus if T is a complete theory and $\mathfrak{A} \models T$ then $S^{\mathfrak{A}}(\emptyset) = S(T)$.

DEFINITION 2.7. Let $\varphi(\bar{x})$ be an $\mathscr{L}$-formula. Then $[\varphi]$ is the set of all types containing φ.

LEMMA 2.8.

(1) $[\varphi] = [\psi]$ if and only if $T \models \forall \bar{x}(\varphi(\bar{x}) \leftrightarrow \psi(\bar{x}))$.
(2) $[\varphi] \cap [\psi] = [\varphi \wedge \psi]$, $[\varphi] \cup [\psi] = [\varphi \vee \psi]$, $S_n(T) \setminus [\varphi] = [\neg\varphi]$, $S_n(T) = [\top]$, $\emptyset = [\bot]$.

PROOF. Straightforward. □

COROLLARY 2.9. The set $\mathscr{B} = \{[\varphi] : \varphi(x_1, \cdots, x_n)$ is a $\mathscr{L}$-formula$\}$ is the base for a topology $\mathscr{O}$ on $S_n(T)$.

PROOF. Why? Because $\mathscr{B}$ is closed under finite intersections and $S_n(T) = \bigcup \mathscr{B}$. □

LEMMA 2.10. $(S_n(T), \mathscr{O})$ is a topological space with the properties:
(1) $S_n(T)$ has a basis consisting of clopen sets.
(2) $S_n(T)$ is Hausdorff.
(3) $S_n(T)$ is compact.

PROOF. Note that $\mathscr{O}$ contains $S_n(T) = [\top]$ and $\emptyset = [\bot]$.

(1) The set $S_n(T) \setminus [\varphi] = [\neg\varphi]$ and so each set of the form $[\varphi]$ is clopen.

(2) Let $p, q \in S_n(T)$ distinct. Then there is φ such that $\varphi \in p$ and $\varphi \notin q$. Thus $p \in [\varphi]$ and since $q \cup T$ is maximal consistent set $\neg\varphi \in q$, i.e. $q \in [\neg\varphi]$. Thus $[\varphi]$ and $[\neg\varphi]$ are disjoint open sets containing p and q respectively.

(3) Consider an arbitrary family $\{[\varphi_i] : i \in I\}$ with the finite intersection property. Thus, for each $J \in [I]^{<\omega}$ the set $\{\wedge_{j \in J} \varphi_j\} \cup T$ is consistent and so for all $J \in [I]^{<\omega}$ the set $\{\varphi_j\}_{j \in J} \cup T$ is consistent. Therefore by compactness $\{\varphi_j\}_{j \in I} \cup T$ is consistent and so we can extend it to a maximal consistent set, i.e. a type p such that $p \in [\varphi_j]$ for each $j \in I$. Thus $\bigcap_{j \in I} [\varphi_j] \neq \emptyset$, which completes the proof. □

LEMMA 2.11. Let U be a clopen subset of $(S_n(T), \mathscr{O})$. Then $U = [\varphi]$ for some formula φ.

PROOF. Note that $U = \bigcup \{[\varphi] : [\varphi] \subseteq U\}$. Since U is also a closed set, its complement $W = S_n(T) \setminus U$ is open and so it is covered by all basic open sets contained in it. That is $W = \bigcup \{[\psi] : [\psi] \subseteq S_n(T) \setminus U\}$. Since $(S_n(T), \mathscr{O})$ is a compact topological space we can find a finite sub-cover. Thus,

$$S_n(T) = \bigcup_{i=1}^{n} [\varphi_i] \cup \bigcup_{j=1}^{k} [\psi_j] = [\vee_{i=1}^{n} \varphi_i] \cup [\wedge_{j=1}^{k} \psi_j].$$

Thus, $U = [\vee_{i=1}^{n} \varphi_i]$. □

DEFINITION 2.12. Let $\mathfrak{A}$, $\mathfrak{B}$ be $\mathscr{L}$-structures and let $A_0 \subseteq A$. A mapping $f : A_0 \to B$ is said to be elementary if for every $\mathscr{L}$-formula $\varphi(x_1, \cdots, x_n)$ and every n-tuple $\bar{a}$ from A_0:

$$\mathfrak{A} \models \varphi(\bar{a}) \Rightarrow \mathfrak{B} \models \varphi(f(\bar{a})).$$

REMARK 2.13. The special case of $A_0 = A$ gives the notion of an elementary embedding of $\mathfrak{A}$ into $\mathfrak{B}$. The case $A_0 = \emptyset$ is eqiuvalent to saying that $\mathfrak{A} \equiv \mathfrak{B}$.

LEMMA 2.14. Let $\mathfrak{A}$ and $\mathfrak{B}$ be $\mathscr{L}$-structures and let $A_0 \subseteq A$, $B_0 \subseteq B$. Let $f : A_0 \to B_0$ be elementary. Then the mapping $S(f) : S_n(B_0) \to S_n(A_0)$ defined by

$$S(f)(q) = \{\varphi(\bar{x}, \bar{a}) : \bar{a} \text{ in } A_0, \varphi(\bar{x}, f(\bar{a})) \in q\}$$

is continuous and surjective.

PROOF. (*Surjectivity*) Let $p \in S_n(A_0)$. Since f is elementary $\{\varphi(\bar{x}, f(\bar{a})) : \varphi(\bar{x}, \bar{a}) \in p\}$ is finitely satisfiable and so it can be extended to a type $q \in S_n(B_0)$. But then $S(f)(q) = p$.

(*Continuity*) It is sufficient to show that the preimage of a basic open set is open. However $(S(f))^{-1}(\varphi[\bar{x}, \bar{a}]]) = [\varphi(\bar{x}, f(\bar{a}))]$. □

REMARK 2.15.

(1) Suppose $f : A_0 \to B_0$ is an elementary bijection. Then the mapping $S_n(A_0) \to S_n(B_0)$ defined by $p \mapsto f(p)$ is a homeomorphism.

(2) The special case $\mathfrak{A} = \mathfrak{B}$ and $A_0 \subseteq B_0$ gives rise to the notion of a restriction of a type: for each $q \in S_n(B_0)$, define the restriction of q to A_0, denoted $q \upharpoonright A_0$, as $S(\mathrm{id})(q)$.

LEMMA 2.16. Let $p \in S_n(T)$. Then φ isolates p if and only if $[\varphi] = \{p\}$.

PROOF. ($\Rightarrow$) Suppose φ isolates p. That is for every $\psi(\bar{x}) \in p$, we have $T \vdash \forall\bar{x}(\varphi(\bar{x}) \to \psi(\bar{x}))$. Take any $q \in S_n(T)$ such that $\psi \in q$. But then $p \subseteq q$ and by maximality of p we get $p = q$. Thus $[\varphi] = \{p\}$ is an isolated point in $(S_n(T), \mathscr{O})$.

($\Leftarrow$) Suppose $[\varphi] = \{p\}$ for some $p \in S_n(T)$. Consider any $\psi(\bar{x}) \in p$ and suppose $T \nvdash \forall\bar{x}(\varphi(\bar{x}) \to \psi(\bar{x}))$. Thus $T \cup \{\varphi(\bar{x}), \neg\psi(\bar{x})\}$ is consistent and so we can extend the latter to a type q such that $q \in [\varphi]$ and $q \neq p$, which is a contradiction to $[\varphi] = \{p\}$. □

DEFINITION 2.17. A formula $\varphi(x)$ is complete if $\{\psi(\bar{x}) : \forall\bar{x}(\varphi(\bar{x}) \to \psi(\bar{x}))\}$ is a type.

Thus, a formula isolates a type if and only if it is complete.

QUESTION 2.18. If $p \in S_n(T)$ is not an isolated point (in topological sense), is there a model of T which omits p? What if p is isolated?

3. Saturated models

DISCUSSION 3.1. A consistent set of formulas in the free variables $x_1, \cdots, x_n$ is often referred to as a *partial type*. If T is a theory and $\Sigma(\bar{x})$ is a partial type consistent with T, then $\Sigma(\bar{x})$ can be extended to a maximal (under inclusion) set of formulas $\Sigma^*(\bar{x})$ which is consistent with T. Thus $\Sigma^*(\bar{x}) \in S_n(T)$. Note that if Δ is a maximal consistent set of n-formulas, then Δ is complete in a natural sense: for each n-formula φ, either $\varphi \in \Delta$ or $\neg\varphi \in \Delta$.

LEMMA 3.2. Let Γ be a theory and t a partial n-type.

(1) Let $\mathscr{C} = \{c_i\}_{i=1}^n$ be a set of new constant symbols. Then $t \cup \Gamma$ is consistent if and only if $t(c_1, \cdots, c_n) \cup \Gamma$ is consistent.
(2) $t \cup \Gamma$ is consistent (i.e. is a partial type) if and only if for all m and all $\varphi_1, \cdots, \varphi_m$ in t the set $\Gamma \cup \{\exists\bar{x}(\wedge_{j=1}^m \varphi_j)\}$ is consistent.
(3) If Γ is complete, then $t \cup \Gamma$ is consistent if and only if for all m and all $\varphi_1, \cdots, \varphi_m$ in t, $\Gamma \vdash \exists\bar{x}(\wedge_{j=1}^m \varphi_j)$.
(4) If t is complete, then t is consistent with Γ if and only if $\Gamma \subseteq t$.

PROOF. (1) $t \cup \Gamma$ is inconsistent if and only if there is a finite $\Phi_0 = \{\varphi_1, \cdots, \varphi_k\} \subseteq t$ and a finite $\Gamma_0 = \{\gamma_1, \cdots, \gamma_k\} \subseteq \Gamma$ such that $\Phi_0 \cup \Gamma_0$ is inconsistent. Now, $\Phi_0 \cup \Gamma_0$ is inconsistent if and only if $\{\bar{\varphi}, \bar{\gamma}\}$ is inconsistent (where $\bar{\varphi} = \wedge_{j=1}^k \varphi_j$, $\bar{\gamma} = \wedge_{j=1}^k \gamma_j$) if and only if $\bar{\gamma} \vdash \neg\bar{\varphi}$ if and only if $\bar{\gamma} \vdash \neg\bar{\varphi}(c_1, \cdots, c_m)$ if and only if $\{\bar{\gamma}, \varphi(c_1, \cdots, c_n)\}$ is inconsistent.

(2) $t \cup \Gamma$ is consistent if and only if for every finite subset t_0 of Γ the set $t_0 \cup \Gamma$ is consistent. Now, note that $\Gamma \cup \{\varphi_1, \cdots, \varphi_m\}$ is inconsistent if and only if $\Gamma \cup \{\wedge_{j=1}^m \varphi_j\}$ is inconsistent if and only if $\Gamma \vdash \neg \wedge_{j=1}^m \varphi_j$ if and only if $\Gamma \vdash \forall\bar{x}\neg \wedge_{j=1}^m \varphi_j$ if and only if $\Gamma \vdash \neg\exists\bar{x} \wedge_{j=1}^m \varphi_j$ if and only if $\Gamma \cup \{\exists\bar{x} \wedge_{j=1}^m \varphi_j\}$ is inconsistent.

(3) For any Γ we have $\Gamma \cup \{\exists\bar{x} \wedge_{j=1}^m \varphi_j\}$ is consistent if and only if $\Gamma \nvdash \neg\exists\bar{x} \wedge_{j=1}^m \varphi_j$. For Γ complete this is equivalent to $\Gamma \vdash \exists\bar{x} \wedge_{j=1}^m \varphi_j$.

(4) If $\Gamma \subseteq t$, then clearly $\Gamma \cup t = t$ is consistent. Conversely, since t is complete, for all $\varphi \in \Gamma$ either $\varphi \in t$ or $\neg\varphi \in t$. However $\neg\varphi \in t \Rightarrow t \cup \Gamma$ is inconsistent. Thus $\varphi \in t$ and so $\Gamma \subseteq t$. □

DEFINITION 3.3. A model $\mathfrak{M}$ is ω-saturated if and only if for every finite $A \subseteq M$ and every $t \in S_1^{\mathfrak{M}}(A)$, $\mathfrak{M}_A$ realizes t.

LEMMA 3.4. Let $\mathfrak{M}$ be a saturated model, $A \subseteq M$ finite, $t \in S_n^{\mathfrak{M}}(A)$. Then $\mathfrak{M}_A$ realizes t.

REMARK 3.5. Note the definition gives the above property only for $n = 1$.

PROOF. We proceed by induction on n simultaneously for all $A \in [M]^{<\omega}$. If $n = 1$ the statement is true by definition of saturatedness. Suppose $n = k+1$ and the claim holds for all k-types and all finite $A \subseteq M$.

Now, let $t \in S_{k+1}^{\mathfrak{M}}(A)$. Consider $t' = \{\exists x_{k+1}\varphi : t \vdash \varphi\}$. Note that $\vdash \varphi \to \exists x_{k+1}\varphi$ is a logic axiom and so for all $\varphi \in t$, $t \vdash \exists x_{k+1}\varphi$. Thus t' is contained in the deductive closure of t. Therefore $t' \in S_k^{\mathfrak{M}}(A)$. By Inductive Hypothesis there is a finite set $B = \{m_i\}_{i=1}^k \subseteq M$ which realizes t' in $\mathfrak{M}_A$. Now, consider

$$t'' = \{\varphi(m_1, \cdots, m_k, x_{k+1}) : t \vdash \varphi\}.$$

Then $t'' \in S_1^{\mathfrak{M}}(A \cup B)$ and so by inductive hypothesis there is $m_{k+1} \in M$ such that $\mathfrak{M}_{A\cup B} \models t''(m_{k+1})$. Then $m_1, \cdots, m_{k+1}$ realise t in $\mathfrak{M}_A$. □

LEMMA 3.6. Let $\mathscr{L}' = \mathscr{L}(\mathscr{C})$ where $\mathscr{C}$ is a finite set of new constants. Then an $\mathscr{L}'$-structure $\mathfrak{M}$ is saturated for $\mathscr{L}'$ if and only if $\mathfrak{M} \restriction \mathscr{L}$ is saturated for $\mathscr{L}$.

THEOREM 3.7. *(Saturated models are universal) Let $\mathfrak{N}$ be a saturated model, Th$(\mathfrak{N}) = \Gamma$. If $\mathfrak{M}$ is countable and $\mathfrak{M} \models \Gamma$ then there is an elementary embedding of $\mathfrak{M}$ into $\mathfrak{N}$.*

PROOF. Let $M = \{m_k\}_{k\in\mathbb{N}}$. Consider $\mathrm{tp}^{\mathfrak{M}}(m_1)$. Clearly $\mathrm{tp}^{\mathfrak{M}}(m_1) \cup \Gamma$ is consistent and since $\mathfrak{N}$ is saturated there is $n_1 \in N$ such that $\mathfrak{N} \models (\mathrm{tp}_{\mathfrak{M}}(m_1))(n_1)$. Thus $\mathrm{tp}^{\mathfrak{M}}(m_1) \subseteq \mathrm{tp}^{\mathfrak{N}}(n_1)$. However, these are complete types and so they must be equal. Suppose we have chosen $\{n_j\}_{j=1}^k$ such that

$$t = \mathrm{tp}^{\mathfrak{M}}(m_1, \cdots, m_k) = \mathrm{tp}^{\mathfrak{N}}(n_1, \cdots n_k).$$

Let $\mathscr{L}' = \mathscr{L} \cup \{c_j\}_{j=1}^k$, where the c_j's are new constant symbols. Let $\mathfrak{M}'$, $\mathfrak{N}'$ be expansions of $\mathfrak{M}$, $\mathfrak{N}$ defined as follows: for each j, let $c_j^{\mathfrak{M}'} = m_j$ and $c_j^{\mathfrak{N}'} = n_j$. Then

$$\mathfrak{M}' \models t(c_1, \cdots, c_k) \text{ and } \mathfrak{N}' \models t(c_1, \cdots, c_k).$$

Since $t(\bar{c})$ is a complete theory, we must have $\mathrm{Th}(\mathfrak{M}') = \mathrm{Th}(\mathfrak{N}')$.

Now, consider $\mathrm{tp}^{\mathfrak{M}}(m_{k+1})$. Since $\mathrm{Th}(\mathfrak{M}') \cup \mathrm{tp}^{\mathfrak{M}'}(m_{k+1})$ is consistent, we must have $\mathrm{Th}(\mathfrak{N}') \cup \mathrm{tp}^{\mathfrak{M}'}(m_{k+1})$ is consistent. However $\mathfrak{N}'$ is saturated (by Lemma 3.6) and so there is n_{k+1} such that

$$\mathfrak{N}' \models (\mathrm{tp}^{\mathfrak{M}'}(m_{k+1}))(n_{k+1})$$

which implies that $\mathrm{tp}^{\mathfrak{M}'}(m_{k+1}) \subseteq \mathrm{tp}^{\mathfrak{N}'}(n_{k+1})$ and again since these are complete types, we must have equality. However, this implies

$$\mathrm{tp}^{\mathfrak{M}}(m_1, \cdots, m_{k+1}) = \mathrm{tp}^{\mathfrak{N}}(n_1, \cdots, n_{k+1}).$$

Thus, inductively we can construct a sequence $\{n_k\}_{k\in\omega}$ such that for all k,

$$\mathrm{tp}^{\mathfrak{M}}(m_1, \cdots, m_k) = \mathrm{tp}^{\mathfrak{N}}(n_1, \cdots, n_k).$$

Now, define a mapping $g : \{m_k\}_{k\in\omega} \to \{n_k\}_{k\in\omega}$ by $g(m_k) = n_k$. We can define a structure $\mathfrak{N}^*$ on $N^* = \{n_k\}_{k\in\omega}$ such that $g : \mathfrak{M} \cong \mathfrak{N}^*$. Then for an arbitrary k-formula φ, a tuple $\bar{n} = (n_1, \cdots, n_k)$ and $\bar{m} = (m_1, \cdots, m_k)$ we have

$$\mathfrak{N}^* \models \varphi(\bar{n}) \text{ iff } \mathfrak{M} \models \varphi(\bar{m}) \text{ iff } \varphi \in \mathrm{tp}^{\mathfrak{M}}(\bar{m}) \text{ iff } \varphi \in \mathrm{tp}_{\mathfrak{N}}(\bar{n}) \text{ iff } \mathfrak{N} \models \varphi(\bar{n}).$$

Thus, $\mathfrak{N}^* \cong \mathfrak{M}$ and $\mathfrak{N}^* \prec \mathfrak{N}$. That is $\mathfrak{M}$ elementary embeds in $\mathfrak{N}$. □

THEOREM 3.8. *Any two countable saturated models of a complete theory* Γ *are isomorphic.*

PROOF. Let $\mathfrak{A}$ and $\mathfrak{B}$ be two countable saturated models of a complete theory Γ. Thus in particular $\mathrm{Th}(\mathfrak{A}) = \mathrm{Th}(\mathfrak{B}) = \Gamma$. Fix enumerations $A = \{a_n\}_{n\in\omega}$ and $B = \{b_n\}_{n\in\omega}$ of their universes. Inductively, we will construct sequences $\{A_i\}_{i\in\omega}$ and $\{B_i\}_{i\in\omega}$ of finite subsets of A and B respectively, and elementary bijections $f_i : A_i \to B_i$ such that $A = \bigcup_{i\in\omega} A_i$, $B = \bigcup_{i\in\omega} B_i$ and $f = \bigcup f_i$ will be the desired isomorphism.

Since $\mathfrak{A} \equiv \mathfrak{B}$, the empty map $f_0 = \emptyset$ is elementary. Thus, let $A_0 = B_0 = \emptyset$. Now, suppose $f_i : A_i \to B_i$ has been defined. We have two cases to consider: if i is even and if i is odd.

If $i = 2n$ for some n (in particular if $i = 0$, i.e. $n = 0$), we extend f_i to $A_{i+1} = A_i \cup \{a_n\}$. Since f_i is an elementary mapping and $p = \mathrm{tp}(a_n/A_i) \in S_1^{\mathfrak{A}}(A_i)$, the image $f_i(p) \in S_1^{\mathfrak{B}}(B_i)$. However, $\mathfrak{B}$ is saturated and by definition $f_i(p)$ is realized in $\mathfrak{B}_{B_i}$ by an element b'. Now, extend f_i to a mapping $f_{i+1} : A_{i+1} \to B_{i+1}$ where $B_{i+1} = B_i \cup \{b'\}$ and $f_{i+1}(a_n) = b'$. We need to show that f_i is elementary. Well, consider an arbitrary tuple of parameters $\bar{a}$ in A_i and a $\mathscr{L}$-formula φ such that $\mathfrak{A} \models \varphi(a_n, \bar{a})$. Then $\varphi \in p$ and so by our choice of b', $\mathfrak{B} \models \varphi(b', f_i(\bar{a}))$ and so f_{i+1} is indeed elementary.

If $i = 2n+1$ for some n, consider the set $B_{i+1} = B_i \cup \{b_n\}$ and let $q = \mathrm{tp}(b_n/B_i) \in S_1^{\mathfrak{B}}(B_i)$. Then since f_i is an elementary bijection $p = f_i^{-1}(q) \in S_1^{\mathfrak{A}}(A_i)$ and so since $\mathfrak{A}$ is saturated, p is realized in $\mathfrak{A}$ by an element a'. Then define $A_{i+1} = A_i \cup \{a'\}$ and extend f_i to $f_{i+1} = f_i \cup \{(a', b_n)\}$. Consider an arbitrary $\mathscr{L}$-formula φ, a tuple $\bar{a}$ in A_i and suppose $\mathfrak{A} \models \varphi(a', \bar{a})$. Then $\varphi \in \mathrm{tp}^{\mathfrak{A}}(a'/A_i) = f_i^{-1}(q)$. By our choice of a', we get $\mathfrak{B} \models \varphi(b_n, f_i(\bar{a}))$. Thus, f_{i+1} is elementary. □

REMARK 3.9. Note that if $\mathfrak{M} \prec \mathfrak{N}$ and $A \subseteq M$, then $\mathfrak{M}_A \prec \mathfrak{N}_A$.

THEOREM 3.10. *(Characterization of theories with Saturated Models) Let T be a countable complete theory. Then the following are equivalent:*

(1) T has a countable saturated model.
(2) For all n, there are at most countably many n-types t extending T. Stated shortly, $|S_n(T)| \leq \aleph_0$ for all $n \in \mathbb{N}$.
(3) For every model $\mathfrak{M}$ of T and every $A \subseteq M$ finite, there are at most countably many types over A extending $Th(\mathfrak{M}_A)$. Shortly, we can formulate this as follows: $|S_n^{\mathfrak{M}}(A)| \leq \aleph_0$ for all $n \in \mathbb{N}$ and all finite $A \subseteq M$.

PROOF. $((1) \Rightarrow (2))$ Let $\mathfrak{M} \models T$ be a countable saturated model. Then $T = \mathrm{Th}(\mathfrak{M})$ and if $t \in S_n(T)$ then $\mathfrak{M}$ realizes t. However if $t_1 \neq t_2$ are in $S_n(T)$, then t_1 and t_2 are complete and so can not be realized by the same tuple. Thus the mapping $i : S_n(T) \to M^n$ defined by $i(t) = \bar{m}$ if and only if $\bar{m}$ realizes t in $\mathfrak{M}$ is injective. Since M^n is countable, $|S_n(T)| \leq \aleph_0$.

$((2) \Rightarrow (3))$ Let $\mathfrak{M} \models T$, $A = \{a_1, \cdots, a_k\} \subseteq M$. If θ is an n-formula in $\mathscr{L}(A)$ then there is an $(n+k)$-formula $\bar{\theta}$ in $\mathscr{L}$ such that:

$$\mathfrak{M} \models (\theta(x_1, \cdots, x_n) \leftrightarrow \bar{\theta}(x_1, \cdots, x_n, a_1, \cdots, a_k)).$$

If $t \in S_n^{\mathfrak{M}}(A)$, then $\bar{t} = \{\bar{\theta} : \theta \in t\} \in S_{n+k}^{\mathfrak{M}}(\emptyset) = S_{n+k}(T)$. If $t_1 \neq t_2$ then $\bar{t}_1 \neq \bar{t}_2$. However $S_{n+k}(T)$ is at most countable (by our hypothesis (2)) and so $S_n^{\mathfrak{M}}(A)$ is also countable.

$((3) \Rightarrow (1))$ Note that if $\mathfrak{M}$ is a countable model of T and $A \subseteq M$ is finite, $t \in S_1^{\mathfrak{M}}(A)$, then there is a countable model $\mathfrak{M}'$ realizing t such that $\mathfrak{M} \prec \mathfrak{M}'$. Indeed, let c be a new constant and let $T' = \mathrm{Th}(\mathfrak{M}_M) \cup t(c)$. Take $\mathfrak{M}' \models T'$. Then $\mathfrak{M} \prec \mathfrak{M}'$ and $\mathfrak{M}'$ realizes t. This leads us to the following construction:

CLAIM. Let $\mathfrak{M}$ be a countable model of T, $\{A_n\}_{n\in\omega}$ a sequence of finite subsets of M and $\{t_n\}_{n\in\omega}$ a sequence of types where $t_n \in S_1^{\mathfrak{M}}(A_n)$ for each n. Then there is a model $\mathfrak{M}^*$ such that $\mathfrak{M} \prec \mathfrak{M}^*$ and $\mathfrak{M}^*$ realizes all types t_n.

PROOF. Let $\mathfrak{M}^1$ be an elementary extension of $\mathfrak{M}$ such that which realizes t_1. Then $\mathfrak{M}_{A_1} \prec \mathfrak{M}^1_{A_1}$ and so we can find an elementary extension $\mathfrak{M}^2$ of $\mathfrak{M}^1$ realizing t_2. Proceed inductively, to construct an elementary chain $\{\mathfrak{M}^n\}_{n\in\omega}$ such that $\mathfrak{M}^n$ realizes t_n. Then $\mathfrak{M}^* = \bigcup_{n\in\omega} \mathfrak{M}^n$ is desired. □

CLAIM. If $\mathfrak{M}$ is a countable model of T, then there is a countable saturated model $\mathfrak{N}$ of T such that $\mathfrak{M} \prec \mathfrak{N}$.

PROOF. Let $\{(t_n, A_n)\}_{n\in\omega}$ enumerate all pairs (t,A) where $A \in [M]^{<\omega}$, $t \in S_1^{\mathfrak{M}}(A)$. Then there is a model $\mathfrak{M}^1$ such that $\mathfrak{M} \prec \mathfrak{M}^1$ and $\mathfrak{M}^1$ realizes all t_n's. Proceed inductively. Let $\{(t_n^1, A_n^1)\}_{n\in\omega}$ enumerate all pairs (t,A) where $A \in [M^1]^{<\omega}$ and $t \in S_1^{\mathfrak{M}^1}(A)$. Then, we can find a model $\mathfrak{M}^2$ of T such that $\mathfrak{M}^1 \prec \mathfrak{M}^2$ and $\mathfrak{M}^2$ realises all t_n^1's. Obtain an elementary chain $\{\mathfrak{M}^n\}_{n\in\mathbb{N}}$ such that $\mathfrak{M}_{n+1}$ realizes types in $S_1^{\mathfrak{M}^n}(A)$ for all finite $A \subseteq M^n$. Then $\mathfrak{N} = \bigcup_{n\in\omega} \mathfrak{M}^n$ is a countable saturated model of T. □

□

COROLLARY 3.11. If a countable complete theory T has only countably many countable models, then T has a saturated model.

PROOF. Every maximal consistent set of n-formulas which is consistent with T is realized in a countable model of T. On the other hand, each countable model can realize only countably many types. Thus, there are only countably many types consistent with T. □

CHAPTER 9

$\aleph_0$-categorical theories

1. Atomic Models

Recall that, an n-formula φ is said to be n-complete over a theory T if and only if $T \cup \{\varphi\}$ is consistent and for all n-formulas ψ either $T \cup \{\varphi\} \vdash \psi$ or $T \cup \{\varphi\} \vdash \neg\psi$. An n-type $t \in S_n(T)$ is atomic over T if it contains an n-complete over T formula. Thus, as we saw earlier, atomic types are isolated points in $S_n(T)$.

LEMMA 1.1. Let t be a partial n-type such that $t \supseteq T$ for some theory T. If t contains an n-complete over T formula φ, then t is complete (i.e. $t \in S_n(T)$).

PROOF. Let ψ be an arbitrary n-formula. Then $T \cup \{\varphi\} \vdash \psi$, or $T \cup \{\varphi\} \vdash \neg\psi$. Since $T \cup \{\varphi\} \subseteq t$, we obtain that $t \vdash \psi$, or $t \vdash \neg\psi$. □

DEFINITION 1.2. Let $\mathfrak{M}$ be a model of a theory T. Then,

(1) $\mathfrak{M}$ is atomic if and only if for every $k \in \omega$ and every k-tuple $\bar{a} = (a_1, \cdots a_k)$ in M, $\text{tp}^{\mathfrak{M}}(\bar{a})$ is atomic.
(2) $\mathfrak{M}$ is prime if and only if for every $\mathfrak{N} \models T$, $\mathfrak{M}$ elementary embeds in $\mathfrak{N}$.

THEOREM 1.3. *Let T be a complete theory (in a countable language). If $\mathfrak{M} \models T$ and $\mathfrak{M}$ is countable and atomic, then $\mathfrak{M}$ is prime.*

PROOF. Fix an arbitrary model $\mathfrak{N}$ of T. We will show that $\mathfrak{M}$ elementary embeds into $\mathfrak{N}$. Let $M = \{m_k\}_{k\in\mathbb{N}}$. For each k let $\varphi_k \in \text{tp}^{\mathfrak{M}}(m_1, \cdots, m_k)$ be a k-complete over T formula. Inductively define a sequence $\{n_k\}_{k\in\mathbb{N}} \subseteq N$ such that for each k, $\varphi_k \in \text{tp}^{\mathfrak{N}}(n_1, \cdots, n_k)$ as follows: Consider φ_1, m_1. Then $\mathfrak{M} \models \varphi_1(m_1)$ and so $\mathfrak{M}_1 \models \exists x_1 \varphi_1(x_1)$. Therefore $T \cup \{\exists x_1 \varphi_1\}$ is consistent and so $T \not\vdash \neg\exists x_1 \varphi_1$. Since T is complete, we must have $T \vdash \exists x_1 \varphi_1$. Now, $\mathfrak{N} \models T$ and so there is $n_1 \in N$ such that $\mathfrak{N} \models \varphi_1(n_1)$. Thus, $\varphi_1 \in \text{tp}^{\mathfrak{N}}(n_1)$. Suppose we have defined $n_1, \cdots, n_k$. Note that

$$\mathfrak{M} \models \varphi_k(m_1, \cdots, m_k) \wedge \exists x_{k+1} \varphi_{k+1}(m_1, \cdots, m_k, x_{k+1})$$

and so $T \not\models \forall x_1 \cdots x_k(\varphi_k(x_1, \cdots, x_k) \to \neg\varphi/x_1, \cdots, x_{k+1})$. However φ_k is a k-complete formula over T and so

$$T \vdash \forall x_1 \cdots x_k(\varphi_k(x_1, \cdots, x_k) \to \exists x_{k+1} \varphi_{k+1}(x_1, \cdots, x_{k+1})).$$

Thus

$$\mathfrak{N} \models \forall x_1 \cdots x_k(\varphi(x_1, \cdots, x_k) \to \exists x_{k+1} \varphi(x_1, \cdots, x_{k+1}))$$

and so there is n_{k+1} such that $\mathfrak{N} \models \varphi_{k+1}(n_1, \cdots, n_{k+1})$.

Let $N^* = \{n_k\}_{k\in\omega}$ and let $g : M \to N^*$ be defined by $g(m_k) = n_k$. Then g extends to an isomorphism and so $\mathfrak{M}$ elementary embeds into $\mathfrak{N}$. □

COROLLARY 1.4. Any two countable atomic models of a complete theory T are isomorphic.

PROOF. A back-and-forth argument. □

2. Characterisation of complete theories with atomic models

We will make use of the following theorem, which for now will consider without a proof.

THEOREM 2.1. *A complete theory T has an atomic model if and only if for every n, every n-formula consistent with T is contained in some atomic type.*

THEOREM 2.2. *If T has a saturated countable model, then T has an atomic model.*

PROOF. We will make use of the following notion: *Let ψ be an n-formula and φ an n-formula which is complete over T such that $T \vdash \varphi \to \psi$. Then φ is called the completion of ψ. An n-formula is said to be "incompletable" if it has no completion.* Note that a formula ψ is incompletable if and only if for every atomic $t \in S_n(T)$, $\psi \notin t$.

Now, suppose T has no atomic model. Then there is an n-formula ψ which is not contained in any atomic type over T and $\{\psi\} \cup T$ is consistent. Thus, $\{\psi\} \cup T$ is consistent, but ψ is incompletable. In particular, ψ is not complete itself , since otherwise it would be its own completion. Therefore there is ψ_0 such that $T \cup \{\psi\} \not\vdash \psi_0$ and $T \cup \{\psi\} \not\vdash \neg\psi_0$. Let $\psi_1 := \neg\psi_0$. Then $\{\psi, \psi_0\}$, $\{\psi, \neg\psi_0\} = \{\psi, \psi_1\}$ are partial n-types consistent with T which are mutually inconsistent. Then $\psi \wedge \psi_0$ is also incomplete and so there is an n-formula ψ_{00} such that $T \cup \{\psi \wedge \psi_0\} \not\vdash \psi_{00}$, $T \cup \{\psi \wedge \psi_0\} \not\vdash \neg\psi_{00}$. Let $\psi_{01} := \neg\psi_{00}$. Similarly, there is a formula ψ_{10} such that $T \cup \{\psi \wedge \psi_1\} \not\vdash \psi_{10}$, $T \cup \{\psi \wedge \psi_1\} \not\vdash \neg\psi_{10}$. Let $\psi_{11} = \neg\psi_{10}$. Proceed by induction.

If $A \subseteq \mathbb{N}$, then identify A with its characteristic function $\chi_A : \mathbb{N} \to \{0,1\}$, where $\chi_A(n) = 1$ if and only if $n \in A$. For each $A \subseteq \mathbb{N}$ let $t_A = \{\psi_{\chi_A \restriction n}\}_{n \in \mathbb{N}}$. Since each finite subset of t_A is consistent with T, by compactness $t_A \cup T$ is consistent. Therefore there are uncountably many n-types consistent with T and so T can not have a saturated model, which is a contradiction. $\square$

COROLLARY 2.3. Let T be a complete theory, $\mathfrak{M} \models T$, $\mathfrak{M}$ countable. Then $\mathfrak{M}$ is both saturated and atomic if and only if for every countable model $\mathfrak{N}$ of T, $\mathfrak{M} \cong \mathfrak{N}$.

3. $\aleph_0$-categorical theories

DEFINITION 3.1. A theory T is said to be $\aleph_0$ categorical if up to isomorphism it has a unique countable model.

REMARK 3.2. Thus by the above Corollary, a countable complete theory T is $\aleph_0$-categorical if and only if it has a model $\mathfrak{M}$ which is both atomic and saturated.

THEOREM 3.3. *Let T be a complete theory. Then the following are equivalent:*

(1) T is $\aleph_0$-categorical.
(2) All countable models are atomic.
(3) All types over T are atomic.
(4) $|S_n(T)| < \aleph_0$ for each $n \in \omega$
(5) for each n there is a finite list of n-formulas such that every n-formula is modulo T equivalent to a formula from the list.

PROOF. Clearly items (1), (2) and (3) are equivalent.
((4) $\Rightarrow$ (5)) Fix n and let $\{t_i\}_{i=1}^k = S_n(T)$. For each n-formula φ, let

$$A_\varphi := \{t \in S_n(T) : \varphi \in t\}.$$

CLAIM 3.4. If $A_\varphi \subseteq A_\psi$ then $T \vdash \varphi \to \psi$.

PROOF. If $T \not\vdash \varphi \to \psi$ then $T \cup \{\varphi, \neg\psi\}$ is consistent and so it can be extended to some $t \in S_n(T)$. Thus, there is $i \in \{1, \cdots, k\}$ such that $T \cup \{\varphi, \neg\psi\} \subseteq t_i$. Then $t_i \in A_\varphi$, $t_i \notin A_\psi$. Thus $A_\varphi \not\subseteq A_\psi$. $\square$

Therefore if $A_\varphi = A_\psi$ then $T \vdash \varphi \leftrightarrow \psi$. Now, consider the map

$$\chi : \{\varphi \mid \varphi \text{ is an } n\text{-formula}\} \to \mathscr{P}(\{t_i\}_{i=1}^k),$$

where $\chi(\varphi) = A_\varphi$. By the above Claim, if $\chi(\varphi_1) = \chi(\varphi_2)$ then $T \vdash \varphi_1 \leftrightarrow \varphi_2$. However $|\mathscr{P}(\{t_1, \cdots, t_k\})| = 2^k$ and so there are no more than 2^k many formulas such that every formulas is equivalent to one of them.

$((5) \Rightarrow (3))$ Let $t \in S_n(T)$ and let $\{\varphi_1, \cdots, \varphi_k\}$ be a list of formulas given by (5). For each $i \leq k$ define

$$\psi_i := \begin{cases} \varphi_i, & \text{if } \varphi_i \in t \\ \neg\varphi_i, & \text{if } \varphi_i \notin t \end{cases}$$

Let $\psi^* = \psi_1 \wedge \cdots \wedge \psi_k$. Then $\psi^* \in t$. We will show that ψ^* is n-complete over T. Consider an arbitrary n-formula φ. Since t is a complete n-type, either $\varphi \in t$ or $\neg\varphi \in t$. Assume without loss of generality that $\varphi \in t$. Then by (5) there is a $j \in \{1, \cdots, k\}$ such that $T \vdash \varphi_j \leftrightarrow \varphi$ and so $\varphi_j \in t$. Therefore $\psi_j = \varphi_j$ and since $\psi^* \to \psi_j$ is a tautology, we obtain $T \cup \{\psi^*\} \vdash \varphi_j$. Thus $T \cup \{\psi^*\} \vdash \varphi_j$, $T \cup \{\psi^*\} \vdash \varphi_j \to \varphi$, which gives $T \cup \{\psi^*\} \vdash \varphi$. The case $\neg\varphi \in t$ is dealt with analogously.

$((3) \Rightarrow (4))$ Assume for all $n \in \mathbb{N}$, all $t \in S_n(T)$ are atomic. Fix n. Assume towards a contradiction that $|S_n(T)| \geq \aleph_0$ (i.e.assume that $S_n(T)$ is infinite). Fix an enumeration $\{t_i\}_{i=1}^\infty$ of $S_n(T)$. Now, for each i let $\varphi_i \in t_i$ be an n-complete formula. Thus $i \neq j$, $T \vdash \varphi_i \to \neg\varphi_j$ and so $\neg\varphi_j \in t_i$. Consider $t^* = \{\neg\varphi_i\}_{i=1}^\infty$. For each k, $\{\neg\varphi_i\}_{i=1}^k \subseteq t_{k+1}$ and so by compactness t^* is consistent. Thus it can be extended to $\bar{t}^* \in S_n(T)$. However $\bar{t}^* \neq t_i$ for each i, which is a contradiction. □

4. Vaught's Never Two Theorem

THEOREM 4.1. *(Vaught) A countable complete theory can not have exactly two non-isomorphic countable models.*

PROOF. Suppose T is a countable complete theory with exactly two countable non-isomorphic models. Thus T has an ω-saturated model $\mathfrak{N}$. However, every theory with a saturated model has also an atomic model $\mathfrak{M}$. Since T has two non-isomorphic models, T is not $\aleph_0$-categorical. Then in particular $\mathfrak{N}$ is not atomic and $\mathfrak{M}$ is not saturated. Since $\mathfrak{N}$ is not atomic, there are $n \in \mathbb{N}$ and $a_1, \cdots a_n$ in N (the universe of $\mathfrak{N}$) such that $\mathrm{tp}^{\mathfrak{N}}(a_1, \cdots, a_n)$ is not atomic. Consider the language $\bar{\mathscr{L}} = \mathscr{L} \cup \{a_1, \cdots, a_n\}$ and let $\bar{\mathfrak{N}}$ be an expansion of $\mathfrak{N}$ to $\bar{\mathscr{L}}$. Then $\bar{\mathfrak{N}}$ is saturated and so $\bar{T} = \mathrm{Th}(\bar{\mathfrak{N}})$ has an atomic model $\bar{\mathfrak{M}}$.

CLAIM. $\bar{T}$ is not $\aleph_0$-categorical.

PROOF. Since T is not $\aleph_0$-categorical, there infinitely many $\mathscr{L}$-formulas $\{\varphi_i\}_{i\in\omega}$ such that for all $i \neq j$, $T \nvdash \varphi_i \leftrightarrow \varphi_j$. But then $\bar{T} \nvdash \varphi_i \leftrightarrow \varphi_j$. □

Therefore $\bar{\mathfrak{M}}$ is not saturated and so $\bar{\mathfrak{M}} \upharpoonright \mathscr{L}$ is also not saturated. Note that $\bar{\mathfrak{M}} \upharpoonright \mathscr{L} \models T$. Thus $\bar{\mathfrak{M}} \upharpoonright \mathscr{L} \not\cong \mathfrak{N}$. It remains to show that $\bar{\mathfrak{M}} \upharpoonright \mathscr{L}$ is not atomic and so $\bar{\mathfrak{M}} \upharpoonright \mathscr{L} \not\cong \mathfrak{M}$, which will be a contradiction.

Since atomic models are prime, there is an elementary embedding $f : \bar{\mathfrak{M}} \to \bar{\mathfrak{N}}$ such that $f(a_j^{\bar{\mathfrak{M}}}) = a_j^{\bar{\mathfrak{N}}} = a_j$. Let φ be an n-formula in $\mathscr{L}$. Then:

$$\bar{\mathfrak{M}} \upharpoonright \mathscr{L} \models \varphi(a_1^{\bar{\mathfrak{M}}}, \cdots, a_n^{\bar{\mathfrak{M}}}) \text{ iff } \mathfrak{N} \models \varphi(a_1, \cdots, a_n).$$

Thus,

$$\mathrm{tp}^{\bar{\mathfrak{M}} \upharpoonright \mathscr{L}}(a_1^{\bar{\mathfrak{M}}}, \cdots, a_n^{\bar{\mathfrak{M}}}) = \mathrm{tp}^{\mathfrak{N}}(a_1, \cdots, a_n)$$

which is not atomic over T. Thus $\bar{\mathfrak{M}} \upharpoonright \mathscr{L}$ is not atomic. □

Part 4

Lachland and Morley Downwards

CHAPTER 10

Indiscernibles

1. The Theorem of Ramsey

We will start with a little detour into combinatorial set theory. Recall the following notation and terminology:

(1) Given a set A and a natural number n, $[A]^n$ denotes the family of all subsets B of A such that $|B| = n$.
(2) Similarly, $[A]^\omega = \{B : B \subseteq A,\ |B| = \aleph_0\}$
(3) A partition of a set X is a family $\mathscr{C}$ of pairwise disjoint non-empty subsets of X which cover the set, i.e. such that $X = \bigcup \mathscr{C}$.

THEOREM 1.1. *(Ramsey) Let A be an infinite set and let $k \in \omega$. For every natural number $n \geq 1$ and every partition $\mathscr{C} = \{C_i\}_{i=1}^k$ of $[A]^n$, there are $B \in [A]^\omega$ and $i \in \{1, \cdots, k\}$ such that $[B]^n \subseteq C_i$.*

PROOF. By induction on n. For $n = 1$, this is the pigeonhole principle. Assume the theorem is true for n. We have to prove it for $n+1$. Thus, let $f : [A]^{n+1} \to \{1, \cdots, k\}$ be arbitrary. Now pick an arbitrary $a_0 \in A$ and consider the mapping

$$f_{a_0} : [A \backslash \{a_0\}]^n \to \{1, \cdots, k\}$$

defined by $f_{a_0}(x) = f(x \cup \{a_0\})$. Let $A_1 = A \backslash \{a_0\}$. By induction hypothesis there are a set $B_1 \in [A_1]^\omega$ and an integer $i_0 \in \{1, \cdots, k\}$ such that $f''_{a_0}[B_1]^n = i_0$. Now pick $a_1 \in B_1$ and consider the mapping

$$f_{a_1} : [B_1 \backslash \{a_1\}]^n \to \{1, \cdots, k\}$$

such that $f_{a_1}(x) = f(x \cup \{a_1\})$. Let $A_2 = B_1 \backslash \{a_1\}$. By the inductive hypothesis there is a set $B_2 \in [A_1]^\omega$ and $i_1 \in \{1, \cdots, k\}$ such that $f''_{a_1}[B_2]^n = i_1$. Proceed inductively to obtain a decreasing sequence of sets

$$A = B_0 \supseteq B_1 \supseteq B_2 \supseteq \cdots,$$

elements $a_l \in B_l \backslash B_{l+1}$ and colors $i_l \in \{1, \cdots, k\}$ such that for each $l \in \omega$

$$f''_{a_l}[B_{l+1}]^n = i_l.$$

Now consider the set $\{i_l\}_{l \in \omega}$. By the pigeonhole principle there is an infinite set $I \subseteq \omega$ and an integer $i^* \in \{1, \cdots, k\}$ such that for each $l \in I$, $i_l = i^*$. Then $B = \{a_l\}_{l \in I}$ is as desired. Indeed, consider any $B^* \in [B]^{n+1}$. Then we can list B^* as $B^* = \{a_{l_j}\}_{j=0}^n$ and we can assume that $l_0 < l_1 < \cdots < l_n$. Then $B^* \backslash \{a_{l_0}\} \subseteq B_{l_0}$ and so $f(B^*) = f_{a_{l_0}}(B^* \backslash \{a_{l_0}\}) = i^*$. □

QUESTION 1.2. What is the connection between the function f in the proof and the partitions of $[A]^{n+1}$? How exactly did we use the pigeonhole principle to claim the existence of i^* in the paragraph above?

2. Indiscernibles

DEFINITION 2.1. Let $(I, <)$ be a linear order, $\mathfrak{A}$ an $\mathscr{L}$-structure. A family $(a_i)_{i \in I}$ of elements of A is called *a sequence of order indiscernibles* if for all $\mathscr{L}$-formulas φ and all $i_1 < \cdots < i_n$, $j_1 < \cdots < j_n$ from I

$$\mathfrak{A} \models \varphi(a_{i_1}, \cdots, a_{i_n}) \text{ if and only if } \mathfrak{A} \models \varphi(a_{j_1}, \cdots, a_{j_n}).$$

REMARK 2.2.

(1) Note that if there are $i \neq j$ in I such that $a_i = a_j$, then all the a_i's above coincide. Thus, we usually assume that the elements $(a_i)_{i\in I}$ are all distinct. Which formula φ can we use to prove this claim?
(2) In the above defintion we can require not only that the a_i's are elements of A, but even more - that they are k-tuples of elements of A for some fixed $k \geq 1$.

DEFINITION 2.3. Let $\mathfrak{M}$ be an $\mathscr{L}$-structure, $(I,<)$ an infinite linear order and $\mathscr{I} = (a_i)_{i\in I}$ a sequence of elements of M. Let $A \subseteq M$. The Ehrenfeucht-Mostowski type of $\mathscr{I}$ over A is the set $\mathrm{EM}(\mathscr{I}/A) = \{\varphi(x_1,\cdots,x_n) : \varphi(x_1,\cdots,x_n)$ is an $\mathscr{L}(A)$-formula such that $\mathfrak{M} \models \varphi(a_{i_1},\cdots,a_{i_n})$ for all $i_1 < \cdots < i_n$, where $n \in \omega\}$.

LEMMA 2.4. (Standard Lemma) Let $\mathscr{I} = (I,<)$ and $\mathscr{J} = (J,<)$ be two infinite linear orders, $A = (a_i)_{i\in I}$ a sequence of elements of a structure $\mathfrak{M}$. Then there is a structure $\mathfrak{N}$ such that $\mathfrak{M} \equiv \mathfrak{N}$ and $\mathfrak{N}$ has a sequence of indiscernibles $(b_j)_{j\in J}$ which realizes $\mathrm{EM}(\mathscr{I})$.

PROOF. Consider the Ehrenfeucht-Mostowski type $\mathrm{EM}(\mathscr{I})$ of $\mathscr{I}$. That is consider the set $\mathrm{EM}(\mathscr{I}) = \{\varphi(\bar{x}) : \varphi(\bar{x})$ is a n-$\mathscr{L}$-formula, $\mathfrak{M} \models \varphi(a_{i_1},\cdots,a_{i_n})$ for all $i_1 < \cdots < i_n, n \in \omega\}$. Let $\mathscr{C} = \{c_j\}_{j\in\mathscr{J}}$ be a set of new constants supplied with the linear order $c_{j_1} < c_{j_2}$ if and only if $j_1 < j_2$. Consider the theories:

- $T' = \{\varphi(\bar{c}) : \varphi(\bar{x}) \in \mathrm{EM}(\mathscr{I})\}$, where $\bar{c}$ is an increasing tuple of new constants;
- $T'' = \{\varphi(\bar{c}) \leftrightarrow \varphi(\bar{d}) : \varphi$ is a $\mathscr{L}$-formula, $\bar{c},\bar{d}$ are increasing tuples of new constants of the same length$\}$.

Now, let $T = \mathrm{Th}(\mathfrak{M})$ and let $T^* = T \cup T' \cup T''$. If this theory is consistent and $\mathfrak{M}^* \models T^*$, then $\mathfrak{N} = \mathfrak{M}^* \restriction \mathscr{L}$, where $\mathscr{L}$ is the original language, will be as desired. Indeed, $\mathfrak{N} \models T$ and so $\mathfrak{N} \equiv \mathfrak{M}$. Since $\mathfrak{M}^* \models T'$, the sequence $(b_j)_{j\in J}$ where $b_j = c_j^{\mathfrak{M}^*}$ realizes $\mathrm{EM}(\mathscr{I})$ in $\mathfrak{N}$ and since $\mathfrak{M}^* \models T''$, the sequence $(b_j)_{j\in J}$ is a sequence of indiscernibles in $\mathfrak{N}$. Thus, it is sufficient to show that T^* is consistent.

By Compactness, it is sufficient to show that whenever $\mathscr{C}_0$ is a finite set of new constants and Δ is a finite set of $\mathscr{L}$-formulas, the theory $T_{\mathscr{C}_0,\Delta} = T \cup \Delta' \cup \Delta''$ is consistent, where

- $\Delta' = \{\varphi(\bar{c}) \in T' : \bar{c}$ increasing tuple in $\mathscr{C}_0\}$,
- $\Delta'' = \{\varphi(\bar{c}) \leftrightarrow \varphi(\bar{d}) : \varphi(\bar{x}) \in \Delta, \bar{c},\bar{d}$ increasing tuples of the same length in $\mathscr{C}_0\}$.

Thus, Δ' is just a finite fragment of T' and Δ'' is a finite fragment of T''.

Consider the set $A = \{a_i\}_{i\in I}$ with the linear order inherited from $(I,<)$. Define an equivalence relation $\sim_\Delta$ on $[A]^n$ as follows. Let $\bar{a} \sim_\Delta \bar{b}$ if and only if

$$\mathfrak{M} \models \varphi(\bar{a}) \leftrightarrow \varphi(\bar{b}) \text{ for all } \varphi(\bar{x}) \in \Delta \text{ and all tuples } \bar{a},\bar{b} \text{ in } A \text{ taken in increasing order.}$$

Enumerate the power set $\mathscr{P}(\Delta)$ as $\{D_i\}_{i=0}^{2^{|\Delta|}-1}$ and define $f : [A]^n \to 2^{|\Delta|}$ as follows:

$$f(\bar{a}) = i \text{ if and only if } \Big(\forall\varphi \in D_i(\mathfrak{M} \models \varphi(\bar{a})) \text{ and } \forall\varphi \in \Delta\backslash D_i(\mathfrak{M} \not\models \varphi(\bar{a}))\Big).$$

Note that f is well-defined. Why? Well, take an arbitrary tuple $\bar{a}$. If it does not realize in $\mathfrak{M}$ any formulas from Δ, then take an index i such that $D_i = \emptyset$. If $\bar{a}$ realizes a formula from the set Δ, take $D = \{\varphi \in \Delta : \mathfrak{M} \models \varphi(\bar{a})\}$ and find an index i such that $D = D_i$. Then clearly, for every $\psi \in \Delta\backslash D$, $\mathfrak{M} \not\models \psi(\bar{a})$. Thus, f is indeed well-defined.

By the Theorem of Ramsey, there is $B \in [A]^\omega$ such that $f \restriction [B]^n$ is the constant i for some $i \in 2^{|\Delta|}$. By definition of the function f, we obtain that $\bar{a} \sim_\Delta \bar{b}$ for each $\bar{a},\bar{b}$ in $[B]^n$. To complete the proof, suppose $\mathscr{C}_0 = \{c_l\}_{l=1}^k$ for some k. Take any $B_0 \in [B]^k$, enumerate it as $\{b_l\}_{l=1}^k$ and for each l define $b_l = c_l^{\mathfrak{M}}$. Then

$$\mathfrak{M}' = (\mathfrak{M}, b_c)_{c\in\mathscr{C}_0} \models T_{\mathscr{C}_0,\Delta}$$

where $b_c = b_l$ for $c = c_l$ is as desired. □

QUESTION 2.5. Is it clear that sets of the form $T_{\mathscr{C}_0,\Delta}$ cover indeed all finite subsets of T^* in the proof above? Why?

EXERCISE 5. Prove that indeed $\mathfrak{M}' = (\mathfrak{M}, b_c)_{c\in\mathscr{C}_0} \models T_{\mathscr{C}_0,\Delta}$.
Hint: Use the fact that $\mathfrak{M}' \restriction \mathscr{L} = \mathfrak{M}$ and that $\bar{a} \sim_\Delta \bar{b}$ for each $\bar{a}, \bar{b}$ in $[B]^n$.

COROLLARY 2.6. Let T be a theory with an infinite model. Then for every infinite linear order I, the theory T has a model with a sequence $(b_i)_{i\in I}$ of (distinct) indiscernibles.

PROOF. By the upwards Löwenheim-Skolem theorem, find a model $\mathfrak{M} \models T$ such that $|M| \geq |I|$ and fix $(a_i)_{i\in I}$ a sequence of distinct elements of M. By the Standard Lemma there is a model $\mathfrak{M}'$ such that $\mathfrak{M}' \cong \mathfrak{M}$ with a sequence $(b_i)_{i\in I}$ of (distinct) indiscernibles. Thus $\mathfrak{M}'$ is as desired. □

Recall that a well-order is a linear order with the property that every non-empty subset has a least element.

LEMMA 2.7. Assume $\mathscr{L}$ is countable. If the $\mathscr{L}$-structure $\mathfrak{M}$ is generated by a well-ordered sequence $(a_i)_{i\in I}$ of indiscernibles, then $\mathfrak{M}$ realises only countably many types over every countable subset of M.

PROOF. Let $A = \{a_i\}_{i\in I}$. Since $\mathfrak{M}$ is generated by A, we have that

$$M = \{t^{\mathfrak{M}}(\bar{a}) : t \text{ is an } \mathscr{L}\text{-term, } \bar{a} \text{ is a finite tuple in } A\}.$$

Fix any countable subset S of M. Since, every element of M is of the form $t^{\mathfrak{M}}(\bar{a})$ for some $\mathscr{L}$-term and a tuple $\bar{a}$ from A, in particular every element from S is in this form. Moreover, since S is countable, we can find a countable $A_0 \subseteq A$ such that only elements from A_0 occur in "describing" S. Thus $A_0 = \{a_i : i \in I_0\}$ for some countable subset I_0 of I.

We have to find out, how many types over S does $\mathfrak{M}$ realise. Recall

$$\text{tp}(\bar{a}/S) = \{\varphi(\bar{x}) : \mathfrak{M} \models \varphi(\bar{a}), \text{ where } \varphi \text{ is a } \mathscr{L}(S)\text{-formula}\}.$$

Thus, having in mind the description of S, we obtain that

$$\text{tp}(\bar{a}/S) = \{\varphi(\bar{x}) : \mathfrak{M} \models \varphi(\bar{a}), \text{ where } \varphi \text{ is a } \mathscr{L}(A_0)\text{-formula}\}.$$

Thus,

$$\text{tp}(\bar{a}/S) = \text{tp}(\bar{a}/A_0).$$

Note that the type $\text{tp}(\bar{a}/A_0)$ depends only on the quantifier free type $\text{tp}_{qf}(\bar{i}/I_0)$ in the structure $(I,<)$ where $\bar{i} = (i_0,\cdots,i_n)$ is such that $\bar{a} = (a_{i_0},\cdots,a_{i_n})$.[1] Moreover $\text{tp}_{qf}(\bar{i}/I_0)$ depends only on $\text{tp}_{qf}(\bar{i})$ and on the quantifier free types $\text{tp}_{qf}(i/I_0)$ of the elements i of $\bar{i}$ in the structure $(I,<)$. For the former, there are finitely many possibilities, while for the latter - countably many. Indeed, there are exactly three options for the latter type:

(1) either for each $i_0 \in I_0 (i_0 < i)$, or
(2) there is $i_0 \in I_0$ such that $i_0 = i$, or
(3) there is $i_0 \in I_0$ such that $i < i_0$ and for each $j \in I_0$ such that $j < i_0$ we have $j < i$. Do we use well-foundedness of the linear order $\mathscr{I} = (I,<)$? How do we know that if we are not in case (1) or in case (2) above, exactly case (3) occurs? Well, suppose neither (1), nor (2) above holds. Then there is $i^* \in I_0$ such that $i < i^*$. Thus, the set $I^* = \{j \in I_0 : i < j\}$ is non-empty. Now, since we have a well-order we can take $i_0 = \min I^*$. Then i_0 has exactly the properties from (3).

[1]Note that $\bar{i}$ is not necessarily $<$-increasing.

There is only one type in the first case, while the latter two are determined by i_0. Since I_0 is countable, this gives rise to only countably many possibilities.

Now, to count all types over S which $\mathfrak{M}$ realises, it is sufficient to count the number of types of the form $\operatorname{tp}(\bar{b}/S)$ for $\bar{b}$ a tuple in M. This number is determined by:

(1) the number of $\mathscr{L}$-terms $t(\bar{x})$ such that $\bar{b} = t(\bar{a})$ for some finite tuple $\bar{a}$ in A, and
(2) the number of types of the form $\operatorname{tp}(\bar{i}/I_0)$ in the structure $(I, <)$.

The first item above gives rise to countably many possibilities, as the language $\mathscr{L}$ is countable. By our observations above, the second item also gives rise to only countably many possibilities. Thus, there are only countably many types over S that $\mathfrak{M}$ realizes. $\square$

REMARK 2.8. Why is it sufficient to consider the types of the form $\operatorname{tp}(\bar{b}/S)$ in the proof of Theorem 10? If $t \in S_n^{\mathfrak{M}}(S)$ and $\mathfrak{M}$ realizes t, then $t = \operatorname{tp}^{\mathfrak{M}}(\bar{b}/S)$ for some tuple $\bar{b}$ in M.

CHAPTER 11

ω-stable theories

1. Skolem Theories

THEOREM 1.1. *Let $\mathscr{L}$ be a first order language. Then there is an expansion $\mathscr{L}_{Skolem}$ of $\mathscr{L}$ and a theory in this expansion, denoted Skolem($\mathscr{L}$), which has quantifier elimination, is universal and such that:*

(1) every $\mathscr{L}$-structure can be extended to a model of Skolem($\mathscr{L}$),
(2) $|\mathscr{L}_{Skolem}| \leq \max(|\mathscr{L}|, \aleph_0)$.

PROOF. We will define an increasing sequence of languages $\{\mathscr{L}_i\}_{i\in\omega}$ such that $\mathscr{L}_0 = \mathscr{L}$ as follows. Suppose $\mathscr{L}_i$ is defined. For every quantifier free $(n+1)$-ary formula $\varphi(\bar{x}, y)$ in the language $\mathscr{L}_i$ take a new n-ary function symbol f_φ.

REMARK 1.2. The intended interpretation of f_φ is to be the Skolem function associated to φ. That is, for every $\mathscr{L}_i$ structure $\mathfrak{M}$ if $\mathfrak{M} \models \exists y \varphi(\bar{x}, y)$ and $\mathfrak{M} \models \varphi(\bar{a}, b)$, then $f_\varphi^{\mathfrak{M}}(\bar{a}) = b$.

Define $\mathscr{L}_{i+1} = \mathscr{L}_i \cup \{f_\varphi : \varphi \text{ is a quantifier free } \mathscr{L}_i\text{-formula}\}$ and let $\mathscr{L}_{\text{Skolem}} = \bigcup_{i\in\omega} \mathscr{L}_i$. Finally, take Skolem($\mathscr{L}$) to be the set

$$\{\forall \bar{x}(\exists y \varphi(\bar{x}, y) \to \varphi(\bar{x}, f_\varphi(\bar{x}))) \mid \varphi(\bar{x}, y) \text{ is a q.f. } \mathscr{L}_{\text{Skolem}}\text{-formula}\}.$$

□

COROLLARY 1.3. Let T be a countable theory, $\mathfrak{N} \models T$ where $|N| \geq \aleph_0$. Let κ be an infinite cardinal. Then there is a model $\mathfrak{M} \models T$ such that $|\mathfrak{M}| = \kappa$ and for every $S \subseteq M$, $|S| \leq \aleph_0$, $\mathfrak{M}$ realises only countably many types over S.

PROOF. Expand the language to $\mathscr{L}_{\text{Skolem}}$ and take $T^* = T \cup \text{Skolem}(\mathscr{L})$. Now, T^* is a countable theory and if $\mathfrak{N}^*$ is the natural expansion of $\mathfrak{N}$ to $\mathscr{L}_{\text{Skolem}}$, then $\mathfrak{N}^* \models T^*$. Thus T^* has an infinite model. Note that T^* is equivalent to a universal theory, since for every $\varphi \in T^*$ there is a q.f. $\varphi^* \in \text{Skolem}(\mathscr{L})$ such that φ and φ^* are equivalent modulo Skolem($\mathscr{L}$). Thus, T^* is equivalent to the universal theory

$$\{\varphi^* : \varphi \in T\} \cup \text{Skolem}(\mathscr{L}).$$

Now, take a well-order I of cardinality κ. By a previous theorem, there is a model $\mathfrak{N}^* \models T^*$ with indiscernibles $(a_i)_{i\in I}$. Take $\mathfrak{M}^*$ to be the substructure of $\mathfrak{N}^*$ generated by the set of indiscernibles i.e. take $\mathfrak{M}^* = \langle\{a_i\}_{i\in I\rangle}\rangle$. Since universal sentences are downwards absolute, we get $\mathfrak{M}^* \models T^*$. On the other hand, T^* has quantifier elimination and so $\mathfrak{M}^* \prec \mathfrak{N}^*$, which implies that $(a_i)_{i\in I}$ is a sequence of indiscernibles for $\mathfrak{M}^*$. Thus over every countable subset of its universe, $\mathfrak{M}^*$ realizes only countable many types. Clealry, the same holds for $\mathfrak{M} = \mathfrak{M}^* \restriction \mathscr{L}$. □

DISCUSSION 1.4.

(1) Once, again: In which language is the theory T^* from the theorem above?
(2) Why is the cardinality of $\mathfrak{M}^*$ exactly κ? How about the cardinality of $\mathfrak{M}$?

2. ω-stable theories

Until the end of the section, unless explicitly otherwise stated, T is a complete theory with an infinite model. First of all recall the following definition:

DEFINITION 2.1. Let κ be an infinite cardinal. A complete theory T of cardinality at most κ is said to be *κ-categorical* if it has up to isomorphism a unique model of cardinality κ.

DEFINITION 2.2. A theory T is said to be *κ-stable* if for every model $\mathfrak{M}$ of T, every $A \subseteq M$ such that $|A| \leq \kappa$ and every $n \in \omega$, $|S_n(A)| \leq \kappa$.

FACT 3.

(1) Note that if T is κ-stable, then up to logical equivalence $|T| \leq \kappa$.
(2) T is κ-stable if and only if T is κ-stable for 1-types, i.e. if and only if $|S(A)| \leq \kappa$ whenever $|A| \leq \kappa$.

THEOREM 2.3. *A countable theory T which is κ-categorical for some $\kappa > \aleph_0$ is ω-stable. Note that since T is κ-cateogrial, then T is in particular complete.*

PROOF. Proceed by contradiction. Suppose $\mathfrak{N} \models T$, $A \subseteq \mathfrak{N}$, $|A| \leq \aleph_0$ and $|S(A)| > \aleph_0$. Thus, we can find a sequence $(b_i)_{i\in I}$ of $\aleph_1$ distinct elements of $\mathfrak{N}$ realising $\aleph_1$-many distinct types in $S(A)$. By Löwenheim-Skolem there is $\mathfrak{M}_0$ such that

$$A \cup \{b_i\}_{i\in I} \subseteq \mathfrak{M}_0, \mathfrak{M}_0 \prec \mathfrak{N} \text{ and } |\mathfrak{M}_0| = \aleph_1.$$

Now, find $\mathfrak{M}_1$ such that $\mathfrak{M}_0 \prec \mathfrak{M}_1$ and $|\mathfrak{M}_1| = \kappa$. Thus, in particular $\mathfrak{M}_1$ realises $\aleph_1$-many types over the countable set A.

On the other hand, by Corollary 1.3, we can find a model $\mathfrak{M}_2 \models T$ such that $|\mathfrak{M}_2| = |\mathfrak{M}_1|$, $\mathfrak{M}_2$ realised only countably many types over A. Thus $\mathfrak{M}_1 \ncong \mathfrak{M}_2$, which means that T is not κ-categorical. □

Recall that ${}^{<\omega}2$ denotes the set $\bigcup_{n\in\omega} {}^n2$.

DEFINITION 2.4. For each $s \in {}^{<\omega}2$ let φ_s be a formula. We say that $\{\varphi_s\}_{s\in{}^{<\omega}2}$ forms a *binary tree of consistent with T formulas* if

(1) $T \models \forall \bar{x}(\varphi_{s0}(\bar{x}) \vee \varphi_{s1}(\bar{x}) \rightarrow \varphi_s(\bar{x}))$
(2) $T \models \forall \bar{x}(\neg\varphi_{s0}(\bar{x}) \wedge \varphi_{s1}(\bar{x}))$.

FACT 4.

(1) If a theory T has an ω-saturated model, then it has an atomic model (otherwise, we saw that there is a binary tree of consistent with T formulas).
(2) Suppose T is a countable theory. Then,
    - for all $n \in \omega$, $|S_n(T)| \leq \aleph_0$ if and only if T has no binary tree of consistent formulas.
    - for all $n \in \omega$, $|S_n(T)| \leq \aleph_0$ if and only if T has a countable ω-saturated model.

DEFINITION 2.5. A countable theory T is *totally transcendental* if it has no model $\mathfrak{M}$ with a binary tree of consistent $L(M)$-formulas.

THEOREM 2.6.

(1) If T is ω-stable, then T is totally transcendental.
(2) If T is totally transcendental, then T is κ-stable for all $\kappa \geq |T|$.

PROOF. Recall that we always assume that our theories are complete!

(1) Suppose T is not totally transcendental. Then there is $\mathfrak{M} \models T$ with a binary tree $\{\varphi_s\}_{s\in^{<\omega}2}$ of consistent with $\mathscr{L}(M)$-formulas. But, then if A is the set of parameters occurring in $\{\varphi_s\}_{s\in^{<\omega}2}$, we get $|S(A)| > \aleph_0$, which is a contradiction to stability.

(2) This is a counting argument. Suppose T is not κ-stable. Then there is $\mathfrak{M} \models T$ such that $A \subseteq \mathfrak{M}$, $|A| \leq \kappa$ and $|S_n(A)| > \kappa$. Let us say that a $\mathscr{L}(A)$-formula φ is *big*, if $[\varphi] = \{b \in S_n(A) : \varphi \in b\}$ is a set of cardinality $> \kappa$. If φ is not big, then we say that φ is *thin*. Clearly, the true formula $\top$ is big and $[\top] = S_n(A)$. If φ is thin, then $|[\varphi]| \leq \kappa$. Now, $|\mathscr{L}(A)| \leq \kappa$ and so the set

$$\mathrm{Thin}(\mathscr{L}(A)) := \bigcup\{[\varphi] : \varphi \text{ is a thin } \mathscr{L}(A)\text{-n-formula}\}$$

is of cardinality at most κ. However $|S_n(A)| > \kappa$ and so we can take

$$p \in S_n(A) \backslash \mathrm{Thin}(\mathscr{L}(A)).$$

Take any $\varphi \in p$. Then φ is big. Pick $q \in S_n(A) \backslash \mathrm{Thin}(\mathscr{L}(A))$ such that $q \neq p$ and $\varphi \in p \cap q$. Then there is ψ such that $\psi \in p$, $\neg\psi \in q$ and so

$$\varphi \leftrightarrow \varphi \wedge (\psi \vee \neg\psi) \leftrightarrow (\varphi \wedge \psi) \vee (\varphi \wedge \neg\psi).$$

Thus, we can generate a binary tree consisting of big formulas which is consistent with T, contradicting that T is transcendental. □

REMARK 2.7. T is totally transcendental if and only if there is no binary tree of consistent formulas in one variable.

We will work with the following natural generalization of the notion of ω-saturatedness.

DEFINITION 2.8.

(1) Let $\kappa \geq \aleph_0$. An $\mathscr{L}$-structure $\mathfrak{M}$ is κ-*saturated* if in $\mathfrak{M}$ for all $A \subseteq \mathfrak{M}$, $|A| < \kappa$, all types in $S_1(A)$ are realised.

(2) An infinite structure $\mathfrak{M}$ is *saturated*, if it is $|\mathfrak{M}|$-saturated.

REMARK 2.9. Recall the argument showing that if all 1-types are realized, then also all n-types are realised.

LEMMA 2.10. Elementarily equivalent saturated structures of the same cardinality are isomorphic.

PROOF. Let $\mathfrak{A} \equiv \mathfrak{B}$, $|\mathfrak{A}| = |\mathfrak{B}| = \kappa$. Choose an enumeration $A = \{a_\alpha\}_{\alpha<\kappa}$, $B = \{b_\alpha\}_{\alpha<\kappa}$. The construction of the isomorphism between $\mathfrak{A}$ and $\mathfrak{B}$ reproduces the back-and-forth argument we have seen earlier, with the difference that we have to account for all limit ordinals below κ. That is, at every successor stage of the construction we either take the next element if the fixed enumeration of A, or the next element of B, while at limit stages, we just take the union of mappings constructed up to that stage. □

LEMMA 2.11. If T is κ-stable, then for all regular $\lambda \leq \kappa$ there is a model of cardinality κ which is λ-saturated.

PROOF. If T is κ-stable then up to logical equivalence $|T| \leq \kappa$. Let $\mathfrak{M}_0 \models T$, $|\mathfrak{M}_0| = \kappa$. Since T is κ-stable, $|S_1(M_0)| \leq \kappa$. For any $p \in S_1(M_0)$ fix a new constant c_p. Then by compactness

$$\mathrm{Th}(\mathfrak{M}_M) \cup \bigcup_{p \in S_n(M)} p(c_p)$$

is consistent, which implies that there is an elementary extension $\mathfrak{M}_1$ which realises all types in $S(M_0)$. By the Löwenheim-Skolem theorem, we can assume that $\mathfrak{M}_1$ has cardinality κ. Recursively, define a continuous elementary chain of models

$$\mathfrak{M}_0 \prec \mathfrak{M}_1 \prec \cdots \prec \mathfrak{M}_\alpha \prec \cdots (\alpha < \lambda),$$

of models of T with cardinality κ such that all $p \in S(M_\alpha)$ are realized in $\mathfrak{M}_{\alpha+1}$. Recall, that continuous, just means that for limit α, $\mathfrak{M}_\alpha = \bigcup_{\beta<\alpha} \mathfrak{M}_\beta$. Let $\mathfrak{M}$ be the union of this chain. Then $\mathfrak{M}$ is λ-saturated. Indeed. Suppose $|A| < \lambda$ and for each $a \in A$ find $\alpha(a) < \lambda$ such that $a \in M_{\alpha(a)}$. Since λ is regular and $|A| < \lambda$, we obtain that

$$\mu := \sup\{\alpha(a) : a \in A\}$$

is strictly below λ and so $A \subseteq M_\mu$. Thus, all types over A are realised in $\mathfrak{M}_{\mu+1}$ (by construction of the elementary chain). □

THEOREM 2.12. *A countable theory T is κ-categorical if and only if all its models of cardinality κ are saturated.*

PROOF. ($\Leftarrow$) Well, by Lemma 2.10 any two saturated models of cardinality κ are isomorphic. Thus, T is κ-categorical.

($\Rightarrow$) Assume T is κ-categorical. If $\kappa = \aleph_0$, then T has an ω-saturated model and since all countable models of T are isomorphic, any countable model of T is ω-saturated. If $\kappa > \aleph_0$, then by Theorem 2.3 the theory T is ω-stable, which by Theorem 2.6.(1) implies that T is totally transcendental and so by Theorem 2.6.(2) the theory T is κ-stable. Therefore by Lemma 2.11, if $\mathfrak{M} \models T$ and $|\mathfrak{M}| = \kappa$, then $\mathfrak{M}$ is μ^+-saturated for all $\mu < \kappa$. That is, the model $\mathfrak{M}$ is saturated. □

CHAPTER 12

Prime models and prime extensions

1. Prime Models

Unless explicitly stated otherwise, T is a countable complete theory with infinite models. Recall the following definitions:

DEFINITION 1.1. Let T be a countable theory with an infinite model (T not necessarily complete). A model $\mathfrak{M}$ of T is said to be *prime* if $\mathfrak{M}$ elementarily embeds into every model of T.

DEFINITION 1.2. A structure $\mathfrak{M}$ is said to be atomic if all n-tuples $\bar{a}$ in A are atomic, i.e. if $\text{tp}^{\mathfrak{M}}(\bar{a})$ is an isolated point in $S_n^{\mathfrak{M}}(\emptyset)$.

EXAMPLE 1.3. Consider a language $\mathscr{L}$ which has a unary predicate symbol P_s for every $s \in {}^{<\omega}2$. Consider the theory T which is the deductive closure of the following axioms:

- $\forall x P_{\emptyset}(x)$
- $\exists x P_s(x)$
- $\forall x((P_{s0}(x) \vee P_{s1}(x)) \leftrightarrow P_s(x))$
- $\forall x \neg(P_{s0}(x) \vee P_{s1}(x))$

where $s \in {}^{<\omega}2$. Then T is a complete theory which has quantifier elimination. The theory T has no complete formulas and no prime models.

One more definition, which we should recall:

DEFINITION 1.4. Given $\mathscr{L}$-structures $\mathfrak{A}$, $\mathfrak{B}$ and $A_0 \subseteq A$, a mapping $f : A_0 \to B$ is said to be *elementary* if for every formula $\varphi(x_1, \cdots, x_n)$ and every n-tuple $\bar{a}$ in $\mathfrak{A}$

$$\text{if } \mathfrak{A} \vDash \varphi(\bar{a}) \text{ then } \mathfrak{B} \vDash \varphi(f(\bar{a})).$$

We have the following characterisation of the prime models of a given theory:

THEOREM 1.5. *A model of a theory T is prime if and only if it is countable and atomic.*

PROOF. ($\Rightarrow$) Suppose $\mathfrak{M}$ is prime. Since T is a countable complete theory (in a countable language) T has a countable model $\mathfrak{N}$. But $\mathfrak{M}$ being prime can be embedded into $\mathfrak{N}$ and so $\mathfrak{M}$ is countable. By the Omitting Types theorem non-isolated types can be omitted in suitable models of T. This in particular implies that $\mathfrak{M}$ can realize only isolated types! Recall that a type is isolated if and only if it contains a complete formula. Then in particular $\text{tp}^{\mathfrak{M}}(\bar{a})$ is atomic for each tuple $\bar{a}$ in M, i.e. $\mathfrak{M}$ is atomic.

($\Leftarrow$) Now, suppose $\mathfrak{M}_0$ is a countable atomic model of T. Consider an arbitrary model $\mathfrak{M}$ of T. Since T is complete, $\mathfrak{M} \equiv \mathfrak{M}_0$ and so the empty mapping f_0 from $\mathfrak{M}_0$ to $\mathfrak{M}$ is elementary. It is sufficient to show that any elementary mapping $f : A \to M$ on a given finite set A can be extended on a given a (in $M_0 \backslash A$).

Thus, suppose f, A, a are given. Let $p(x) = \text{tp}^{\mathfrak{M}_0}(a/A)$ and let $f(p)$ be the image of p under f. Fix a finite tuple $\bar{a}$ enumerating A and using the fact that $\mathfrak{M}_0$ is atomic fix a $\mathscr{L}$-formula $\varphi(x, \bar{x})$ isolating $\text{tp}^{\mathfrak{M}_0}(a\bar{a})$. Then $\varphi(x, \bar{a})$

isolates p and since f is elementary (by hypothesis), $\varphi(x, f(\bar{a}))$ isolates $f(p)$. If b realizes $\varphi(x, f(\bar{a}))$ in $\mathfrak{M}$ then b realizes also $f(p)$ and so $f \cup \{(a,b)\}$ is the desired extension of f. □

As an immediate corollary we obtain:

COROLLARY 1.6. Any two prime models of a theory T are isomorphic.

In addition, the proof of the above theorem shows that ω-saturated models are ω-homogenous in the following sense:

DEFINITION 1.7. Let $\mathfrak{M}$ be a $\mathscr{L}$-structure. Then, $\mathfrak{M}$ is said to be ω-*homogenous* if for every elementary map $f_0 : A \to M$ where A is a finite subset of M and every $a \in M$ there is $b \in M$ such that $f = f_0 \cup \{\langle a, b \rangle\}$ is elementary.

COROLLARY 1.8. Prime models are ω-homogenous.

DISCUSSION 1.9. Recall the space of types $S_n(T)$. A point x in a topological space $(X, \mathscr{O})$ is an isolated point if the set $\{x\}$ is open. Thus, a type p is said to be isolated if $\{p\}$ is an isolated point, which we proved is equivalent to $[\varphi] = \{p\}$ for some $\mathscr{L}$-formula φ. Then φ isolates p and φ is complete. Now, suppose the isolated types are dense in $S_n(T)$. Then every open set $[\psi]$ contains an isolated type, i.e. every formula ψ consistent with T, belongs to some isolated type. Recalling the definition of a complete formula, we obtain that the set of isolated types is dense in $S_n(T)$ if and only if for every consistent with T formula ψ there is a complete formula φ such that $T \vDash \forall \bar{x}(\varphi(\bar{x}) \to \psi(\bar{x}))$.

2. Prime Extensions

DEFINITION 2.1. Let $\mathfrak{M}$ be a model of T, $A \subseteq M$.

(1) The structure $\mathfrak{M}$ is *prime over A* (also $\mathfrak{M}$ is said to be a *prime extension of A*), if every elementary map $f : A \to N$ (where N is the universe of a structure $\mathfrak{N}$) extends to an elementary map f' from $\mathfrak{M}$ to $\mathfrak{N}$.
(2) A set $B \subseteq M$ is *constructible over A* if $B = \{b_\alpha\}_{\alpha<\lambda}$ where for each $\alpha < \lambda$, if $B_\alpha = \{b_\beta\}_{\beta<\alpha}$ then b_α is atomic over $A \cup B_\alpha$, i.e. $\mathrm{tp}^{\mathfrak{M}}(b_\alpha / A \cup B_\alpha)$ is isolated.

LEMMA 2.2. Let $\mathfrak{M}$ be an $\mathscr{L}$-structure such that M is constructible over A, where $A \subseteq M$. Then $\mathfrak{M}$ is prime over A.

PROOF. We need to show that every elementary map $f : A \to N$, where N is the universe of a structure $\mathfrak{N}$ extends to an elementary embedding from $\mathfrak{M}$ to $\mathfrak{N}$.

Fix an enumeration $\{m_\alpha\}_{\alpha<\lambda}$ of the universe M of $\mathfrak{M}$ so that if for each $\alpha < \lambda$ we have $M_\alpha = \{m_\beta\}_{\beta<\alpha}$, then m_α is atomic over $A \cup M_\alpha$. That is, the type $p_\alpha = \mathrm{tp}^{\mathfrak{M}}(m_\alpha / A \cup M_\alpha)$ is an isolalted point in $S(A \cup M_\alpha)$.

Now, let $f : A \to \mathfrak{N}$ be a given elementary mapping. Recursively, we will define elementary mappings $f_\alpha : A \cup M_\alpha \to \mathfrak{N}$ so that $f_\lambda = \bigcup_{\alpha<\lambda} f_\alpha : M \to \mathfrak{N}$ will be elementary. Suppose

$$f_\alpha : A \cup M_\alpha \to \mathfrak{N}$$

has been defined. Note that f_α induces a homeomorphism

$$S(f_\alpha) : S(A \cup M_\alpha) \to S(f[A \cup M_\alpha]),$$

since $S(f_\alpha)$ is a bijective continuous mapping between Hausdorff, compact spaces (this is a good exercise in general topology; can you give a proof?). Then $S(f_\alpha)(p_\alpha)$ is isolated in $S(f_\alpha[A \cup M_\alpha])$ and so realised by some $b_\alpha \in \mathfrak{N}$. Then take $f_{\alpha+1} = f_\alpha \cup \{(m_\alpha, b_\alpha)\}$ and note that

$$f_{\alpha+1} : A \cup M_{\alpha+1} \to \mathfrak{N}.$$

□

LEMMA 2.3. Suppose T is a totally transcendental theory, $\mathfrak{M} \models T$ and $A \subseteq \mathfrak{M}$. Then the isolated types in $S_n^{\mathfrak{M}}(A)$ are dense.

PROOF. Suppose not. Thus there is a basic open $[\varphi] \subseteq S_n^{\mathfrak{M}}(A)$ which does not contain an isolated type. Then in particular φ is not complete and so we can find mutually inconsistent formulas φ_0, φ_1 such that $\varphi \wedge \varphi_0$ and $\varphi \wedge \varphi_1$ are consistent with T, but none of them is complete. This will give rise to a binary tree of consistent $\mathscr{L}(M)$-formulas, which is a contradiction to T being totally transcendental. □

THEOREM 2.4. *Suppose T is totally transcendental, $\mathfrak{M} \models T$, $A \subseteq \mathfrak{M}$. Then A has a prime, constructible over A extension.*

PROOF. By Lemma 2.2, it suffices to find $\mathfrak{M}_0 \prec \mathfrak{M}$ such that M_0 is constructible over A. We will use the axiom of choice. Consider the collection $\mathbb{P}$ of all families $\{a_\alpha\}_{\alpha<\mu} \subseteq M$ such that for all $\alpha < \mu$, $\mathrm{tp}^{\mathfrak{M}}(a_\alpha/A \cup \{a_\beta\}_{\beta<\alpha})$ is atomic. Then $\mathbb{P}$ is a non-empty partial order (under inclusion) with the property that the union of an increasing chain in the partial order, is an element of $\mathbb{P}$ (i.e. $\mathbb{P}$ is inductive). Thus, by Zorn's Lemma we can find a maximal element. That is a maximal under inclusion family $\{a_\alpha\}_{\alpha<\lambda} \subseteq \mathfrak{M}$ such that for all $\alpha < \lambda$, the type $\mathrm{tp}^{\mathfrak{M}}(a_\alpha/A \cup \{a_\beta\}_{\beta<\alpha})$ is atomic. We claim that if $M_0 = A \cup \{a_\alpha\}_{\alpha<\lambda}$, then M_0 is the universe of an elementary substructure $\mathfrak{M}_0$ of $\mathfrak{M}$. Clearly, $M_0 \subseteq M$ and M_0 is constructible over A.

Suppose $\mathfrak{M} \models \exists x \varphi(x)$ for some $\mathscr{L}(M_0)$-formula $\varphi(x)$. Consider $[\varphi(x)]$. By Lemma 2.3 there is an isolated point $p \in S_1^{\mathfrak{M}}(M_0)$ such that $p \in [\varphi(x)]$, i.e. $\varphi(x) \in p$. But, every isolated type is realised in $\mathfrak{M}$ and so we can find $b \in M$ realising it. Thus $\mathfrak{M} \models \varphi(b)$. But then b is atomic over M_0, i.e. $\mathrm{tp}(b/A \cup \{a_\alpha\}_{\alpha<\lambda})$ is isolated. If $b \notin A \cup \{a_\alpha\}_{\alpha<\lambda}$, we get a contradiction to the maximality of $\{a_\alpha\}_{\alpha<\lambda}$. Therefore $b \in M_0$ and so M_0 is the universe of an elementary substructure $\mathfrak{M}_0$ of $\mathfrak{M}$. □

LEMMA 2.5. Let $\bar{a}$ and $\bar{b}$ be finite tuples of elements of a structure $\mathfrak{M}$. Then $\mathrm{tp}(\bar{a}\bar{b})$ is atomic if and only if $\mathrm{tp}(\bar{a}/\bar{b})$ and $\mathrm{tp}(\bar{b})$ are atomic.

PROOF. ($\Rightarrow$) Suppose $\mathrm{tp}(\bar{a}\bar{b})$ is atomic. Thus, there is a formula $\varphi(\bar{x},\bar{y})$ which isolates $\mathrm{tp}(\bar{a}\bar{b})$. Then clearly $\varphi(\bar{x},\bar{b})$ isolates $\mathrm{tp}(\bar{a}/\bar{b})$ and we will show that $\exists \bar{x}\varphi(\bar{x},\bar{y})$ isolates $\mathrm{tp}(\bar{b}) = p(\bar{y})$.

Note that $\exists \bar{x}\varphi(\bar{x},\bar{y}) \in \mathrm{tp}(\bar{b})$ and if $\sigma(y) \in \mathrm{tp}(\bar{b})$ then

$$\mathfrak{M} \models \forall \bar{x},\bar{y}(\varphi(\bar{x},\bar{y}) \to \sigma(\bar{y})).$$

Therefore $\mathfrak{M} \models \forall \bar{y}(\exists \bar{x}\varphi(\bar{x},\bar{y}) \to \sigma(\bar{y}))$.

($\Leftarrow$) Now, suppose $\mathrm{tp}(\bar{a}/\bar{b})$ and $\mathrm{tp}(\bar{b})$ are isolated by $\rho(\bar{x},\bar{b})$ and $\sigma(\bar{y})$ respectively. We will show that $\rho(\bar{x},\bar{y}) \wedge \sigma(\bar{y})$ isolates $\mathrm{tp}(\bar{a}\bar{b})$. On one hand $\rho(\bar{x},\bar{y}) \wedge \sigma(\bar{y}) \in \mathrm{tp}(\bar{a}\bar{b})$. On the other hand if $\varphi(\bar{x},\bar{y}) \in \mathrm{tp}(\bar{a}\bar{b})$ then $\varphi(\bar{x},\bar{b}) \in \mathrm{tp}(\bar{a}/\bar{b})$ and so

$$\mathfrak{M} \models \forall \bar{x}(\rho(\bar{x},\bar{b}) \to \varphi(\bar{x},\bar{b})).$$

Then $\forall \bar{x}(\rho(\bar{x},\bar{b}) \to \varphi(\bar{x},\bar{b})) \in \mathrm{tp}(\bar{b})$ and so

$$\mathfrak{M} \models \forall \bar{y}(\sigma(\bar{y} \to \forall \bar{x}(\rho(\bar{x},\bar{y}) \to \varphi(\bar{x},\bar{y})).$$

Therefore $\mathfrak{M} \models \forall \bar{x},\bar{y}(\rho(\bar{x},\bar{y}) \wedge \sigma(\bar{y}) \to \varphi(\bar{x},\bar{y}))$. □

LEMMA 2.6. Constructible extensions are atomic.

PROOF. Suppose $\mathfrak{M}$ is a constructible extension of A. Thus, the universe of $\mathfrak{M}$ is of the form $\{a_\alpha\}_{\alpha<\lambda}$ where each b_α is atomic over $A \cup M_\alpha$ for $M_\alpha = \{a_\beta\}_{\beta<\alpha}$. Let $\bar{b}$ be a tuple from $M \backslash A$. We will show that $\mathrm{tp}^{\mathfrak{M}}(\bar{a})$ is atomic. Write $\bar{a} = a_\alpha\bar{b}$ where $\bar{b}$ is a tuple in M_α. By definition of constructibility $\mathrm{tp}(a_\alpha/(M_\alpha \cup A))$ is atomi and so we can fix $\mathscr{L}(M_\alpha)$ complete formula $\varphi(x,\bar{c})$ isolating the type. Then a_α is also atomic over $A \cup \{\bar{b}\bar{c}\}$. By induction $\bar{b}\bar{c}$ is atomic over A. Applying Lemma 2.5 to $\mathfrak{M}_A$, $a_\alpha\bar{b}\bar{c}$ is atomic over A and so $\bar{a} = a_\alpha\bar{b}$ is atomic over A. □

COROLLARY 2.7. Prime extensions of totally transcendental theories are atomic.

PROOF. Fix T totally transcendental. Let $\mathfrak{M} \models T$, $A \subseteq \mathfrak{M}$. Then there is $\mathfrak{M}_0$ such that $\mathfrak{M}_0$ is constructible over A (by Theorem 1.6) and so by Lemma 2.6, $\mathfrak{M}_0$ is atomic. Now, let $\mathfrak{M}_1$ be prime over A and let $f : A \to \mathfrak{M}_0$ be elementary. Then f extends to an elementary mapping $\bar{f} : \mathfrak{M}_1 \to \mathfrak{M}_0$ and since $\mathfrak{M}_1$ has an isomorphic copy in an atomic model, the model $\mathfrak{M}_1$ must be itself atomic. $\square$

CHAPTER 13

Morley Downwards

1. Theorem of Lachland

THEOREM 1.1. *(Lachland) Let T be a countable, complete theory, which is totally transcendental, $\mathfrak{M} \models T$, $|\mathfrak{M}| > \aleph_0$. Then for every $\kappa \geq |\mathfrak{M}|$ there is a model $\mathfrak{N}^*$ such that:*

(1) $\mathfrak{M} \prec \mathfrak{N}^*$, $\mathfrak{N}^* \models T$, $|\mathfrak{N}^*| = \kappa$,

(2) If $\Sigma(x)$ is a countable set of formulas omitted in $\mathfrak{M}$, then $\Sigma(x)$ is omitted in $\mathfrak{N}^*$.

PROOF. We will say that a formula $\varphi(x)$ is *large*, if

$$\varphi(\mathfrak{M}) = \{a \in \mathfrak{M} : \mathfrak{M} \models \varphi(a)\}$$

is uncountable. Since, T is totally transcendental, T does not have a binary tree of consistent big $\mathscr{L}(M)$-formulas. However, there are big formulas as $|\mathfrak{M}| > \aleph_0$ and every tautology is big. Thus, there is a formula $\varphi_0(x)$ which is big and for each $\psi(x)$ either $\varphi_0(x) \wedge \psi(x)$ of $\varphi_0(x) \wedge \neg\psi(x)$ is not big. Take

$$p(x) = \{\psi(x) : \varphi_0(x) \wedge \psi(x) \text{ is big}\}$$

and note that $p(x) \in S(M)$, which consists of big formulas (for a detailed proof on the existence of $p(x)$ see Subsection 3).

CLAIM. $p(x)$ is not realized in $\mathfrak{M}$.

PROOF. Otherwise, pick $a \in M$ such that $\mathfrak{M} \models p(a)$. Since $p(x)$ is maximal, $x \doteq a \in p(x)$, but $x \doteq a$ is not big. Contradiction. □

CLAIM 1.2. If $\Pi(x) \in [p(x)]^{\leq\omega}$, then $\Pi(x)$ is realised in $\mathfrak{M}$.

PROOF. Fix $\Pi(x)$. Each $\psi(x) \in \Pi(x)$ is big and $\varphi_0(x) \wedge \neg\psi(x)$ is not big. Thus in particular there is a countable set $\Delta_\psi \subseteq M$ such that $\Delta_\psi = (\varphi_0 \wedge \neg\psi)(\mathfrak{M})$. That is

- for all $m \in \Delta_\psi$, $\mathfrak{M} \models \varphi_0(m) \wedge \neg\psi(m)$ and
- for all $m \in M \backslash \Delta_\psi$, $\mathfrak{M} \not\models \varphi_0(m) \wedge \neg\psi(m)$.

Now, $\Pi(x)$ is countable and so

$$\Delta = \varphi_0(\mathfrak{M}) \backslash \bigcup\{\Delta_\psi : \psi \in \Pi(x)\}$$

is uncountable and so in particular non-empty! Clearly for any $m_0 \in \Delta$, $\mathfrak{M} \models \Pi(m_0)$. □

Thus in particular $p(x)$ is finitely satisfiable in $\mathfrak{M}$ and so $\mathfrak{M}$ has an elementary extension $\mathfrak{N}$ in which $p(x)$ is realised. Fix $a \in \mathfrak{N}$ such that $\mathfrak{N} \models p(a)$. Then clearly $a \in \mathfrak{N} \backslash \mathfrak{M}$ and so $\mathfrak{M}$ is a proper elementary submodel of $\mathfrak{N}$. By Theorem 13 from the last lecture we can assume that $\mathfrak{N}$ is constructible over $M \cup \{a\}$ and so by Lemma 15 of the same lecture, the model $\mathfrak{N}$ is atomic over $M \cup \{a\}$.

CLAIM. If $\Sigma(x)$ is a countable set of $\mathscr{L}(M)$-formulas, which is realised in $\mathfrak{N}$, then $\Sigma(x)$ is realised in $\mathfrak{M}$.

PROOF. Fix $\Sigma(x)$. Since $\mathfrak{N}$ is atomic, there is $b \in N$ which realises $\Sigma(x)$, i.e.

$$\Sigma(x) \subseteq \mathrm{tp}(b/M).$$

Consider the type $q(y) = \mathrm{tp}(b/M \cup \{a\})$. Since $\mathfrak{N}$ is atomic over $M \cup \{a\}$, there is a complete $\mathscr{L}(M)$-formula $\chi(x,y)$ such that $\chi(a,y)$ isolates $q(y)$. Observe that if $\mathfrak{N} \models \sigma(b)$, where $\sigma(y)$ is an $\mathscr{L}(M)$-formula (in particular if $\sigma(y) \in \Sigma(x)$) then

$$\mathfrak{N} \models \forall y(\chi(a,y) \to \sigma(y)).$$

Thus a realises $\sigma^*(x) = \forall y(\chi(x,y) \to \sigma(y))$. Therefore $\sigma^*(x) \in p(x)$. Analogously, $\mathfrak{N} \models \exists y \chi(a,y)$ (just because $\mathfrak{N} \models \chi(a,b)$). Therefore $\exists y \chi(x,y) \in p(x)$. Now,

$$\Sigma^*(x) = \{\sigma^*(x) : \sigma \in \Sigma\} \cup \{\exists y \chi(x,y)\} \in [p(x)]^{\leq \omega}.$$

By Claim 1.2, $\Sigma^*(x)$ is realized in $\mathfrak{M}$. Thus, there is $a' \in M$ such that $\mathfrak{M} \models \Sigma^*(a')$. Then in particular, $\mathfrak{M} \models \exists y \chi(a',y)$ and we can pick a witness $b' \in M$ such that $\mathfrak{M} \models \chi(a',b')$. Now, $\mathfrak{M} \models \sigma^*(a')$, i.e.

$$\mathfrak{M} \models \forall y(\chi(a',y) \to \sigma(y)).$$

Thus, we must have $\mathfrak{M} \models \chi(a',b') \to \sigma(b')$ and so $\mathfrak{M} \models \sigma(b')$. Therefore $\mathfrak{M} \models \Sigma(b')$. □

REMARK. Can we assume that $|\mathfrak{N}| = |\mathfrak{M}|$ above? If this is not the case, since the language is countable, we can take $\mathfrak{N}' \preceq \mathfrak{N}$ such that $M \cup \{a\} \subseteq N'$ and $|N'| = |M|$.

Using the same analysis, we can construct an increasing continuous chain of proper elementary extensions $\{\mathfrak{N}_\alpha\}_{\alpha<\kappa}$. Then $\mathfrak{N}^* = \bigcup_{\alpha<\kappa} \mathfrak{N}_\alpha$ is as desired. □

QUESTION 1.3. Why is $|\mathfrak{N}^*| \geq \kappa$ in the proof above?

2. Morely Downwards

COROLLARY 2.1. (Morley downwards) A countable complete theory T, which is κ-categorical for some uncountable cardinal κ is $\aleph_1$-categorical.

REMARK. Recall that a countable complete theory T is κ-categorical if and only if all its models of cardinality κ are saturated.

PROOF. (Morley Downwards) Let T be κ-categorical. Assume T is not $\aleph_1$-categorical. Then T has a model $\mathfrak{M}$, $|\mathfrak{M}| = \aleph_1$ such that $\mathfrak{M}$ is not saturated. But then there is a type $p = \Sigma(x)$ over some $A \subseteq \mathfrak{M}$, $|A| \leq \aleph_0$ such that p is not realised in $\mathfrak{M}$ (i.e. p is omitted in $\mathfrak{M}$). However, κ-categorical theories are ω-stable and ω-stable theories are totally transcendental. Therefore T is totally transcendental. By the theorem of Lachland, $\mathfrak{M}$ has an elementary extension $\mathfrak{N}$ of cardinality κ, which realizes the same countable types as $\mathfrak{M}$ and so $\mathfrak{N}$ omits p. Thus $\mathfrak{N}$ is not saturated and so T is not κ-categorical. Contradiction! □

3. A type of big formulas

DISCUSSION 3.1. In the proof of the Theorem of Lachland the existence of a type consisting of big formulas, plays a crucial role! But, why is there such a type? Here is a detailed proof.

Suppose $\mathfrak{M}$ is an uncountable model of a transcendental theory T in a countable language. We say that a formula $\chi(x)$ is big if

$$\chi(\mathfrak{M}) = \{m \in M : \mathfrak{M} \models \chi(m)\}$$

is an uncountable set. The formula $x \doteq x$ is satisfied by every $m \in M$ and so it is big. Thus, there are big formulas.

CLAIM 3.2. Suppose $\varphi(x)$ is a big formula. Then for any formula $\psi(x)$ either $\varphi \wedge \psi$ or $\varphi \wedge \neg\psi$ is big.

PROOF. Take any $\psi(x)$. Then the sets

$$M_{\psi(x)} = \{m \in M : \mathfrak{M} \models \psi(x)\} \text{ and } M_{\neg\psi(x)} = \{m \in M : \mathfrak{M} \models \neg\psi(x)\}.$$

form a partition of M, i.e. they are disjoint and

$$M = M_\psi \cup M_{\neg\psi}.$$

Moreover

$$\varphi(\mathfrak{M}) = M_{\varphi\wedge\psi} \cup M_{\varphi\wedge\neg\psi}.$$

Since $\varphi(\mathfrak{M})$ is uncountable at least one of the sets $M_{\varphi\wedge\psi}$ and $M_{\varphi\wedge\neg\psi}$ is uncountable, i.e. at least one of the formulas $\varphi \wedge \psi$ or $\varphi \wedge \neg\psi$ is big. □

Let φ_0 be an arbitrary big formula. Suppose we can find a formula ψ_0 such that both $\varphi_0 \wedge \psi_0$ and $\varphi_0 \wedge \neg\psi_0$ are big. Call the first φ_{00} and the second φ_{01}. Repeating the argument for the big formulas φ_{00} and φ_{01} we are starting to generate a binary three of big formulas (each branch through which is a set consistent with T, because $\mathfrak{M} \models T$). Proceed inductively. Suppose for each $s \in {}^{<\omega}2$ we can find a formula ψ_s such that both $\varphi_s \wedge \psi_s$ and $\varphi_s \wedge \neg\psi_s$ are big. Then take $\varphi_{s0} = \varphi_s \wedge \psi_s$ and $\varphi_{s1} = \varphi_s \wedge \neg\psi_s$. Thus, we can complete the process and construct a binary tree $\mathscr{X} = \{\varphi_s(x)\}_{s\in{}^{<\omega}2}$ of big formulas.

EXERCISE 6. Let $f \in {}^{\omega}2$. Then $\{\varphi_{f\upharpoonright n} : n \in \omega\}$ is consistent with T.

Therefore, either we can complete the process and construct a binary free $\mathscr{X} = \{\varphi_s(x)\}_{s\in{}^{<\omega}2}$ of big formulas such that for each $f \in {}^{\omega}2$ the set $\{\varphi_{f\upharpoonright n} : n \in \omega\}$ is consistent with T, or we will find a formula for which the process can not be continued.

Well, if we succeed with the construction of $\mathscr{X}$, then we reach a contradiction to the fact that T is transcendental. So, there is a big formula $\varphi^*(x)$ with the property that for each other formula $\psi(x)$ either $\varphi \wedge \psi$ or $\varphi \wedge \neg\psi$ is not big. Then, we define

$$p(x) = \{\psi(x) : \varphi^*(x) \wedge \psi(x) \text{ is big}\}.$$

CLAIM 3.3. $p(x)$ consists of big formulas.

PROOF. For each $\psi \in p(x)$ we have that $(\varphi^* \wedge \psi)(\mathfrak{M})$ is uncountable. However

$$(\varphi^* \wedge \psi)(\mathfrak{M}) \subseteq \psi(\mathfrak{M})$$

and so $\psi(\mathfrak{M})$ is also uncountable. That is $\psi(x)$ is big. □

Thus, we found a type $p(x)$ consisting of big formulas.

Part 5

The Categoricity Theorem

CHAPTER 14

Homogeniety

1. Homogenous Models

Recall the definition:

DEFINITION. Let $\mathscr{L}$ be a countable language. An $\mathscr{L}$-structure is said to be *ω-homogenous* if for every finite $A \subseteq M$, every $a \in M$ and every elementary map $f_0 : A \prec \mathfrak{M}$ there is $b \in M$ such that $f_0 \cup \{(a,b)\}$ is elementary.

REMARK 1.1. In the above definition, note that $f_0 \cup \{(a,b)\}$ is elementary if and only if b realises $f(\text{tp}(a/A))$.

LEMMA 1.2. Let T be a countable complete theory with infinite models.

(1) Let $\mathfrak{M} \models T$, $|\mathfrak{M}| = \aleph_0$. Then there is $\mathfrak{N}$ such that $\mathfrak{M} \prec \mathfrak{N}$, $|\mathfrak{N}| = \aleph_0$ and $\mathfrak{N}$ is ω-homogenous.
(2) Suppose for all n, $\mathfrak{M}_n$ is ω-homogenous, $\mathfrak{M}_n \prec \mathfrak{M}_{n+1}$. Then $\lim_n \mathfrak{M}_n = \bigcup_{n\in\omega} \mathfrak{M}_n$ is ω-homogenous.

PROOF. (1) Let $\mathfrak{M}$ be a countable model of T. For this proof we will say that a triple (A, f, a) is *relevant* for $\mathfrak{M}$ if A is a finite subset of M, $f : A \prec M$ is elementary and $a \in M$. Note that if $\mathfrak{M}$ is countable, then

$$\Delta(\mathfrak{M}) = \{(A, f, a) : (A, f, a) \text{ is relevant for } \mathfrak{M}\}$$

is a countable set. Now, consider $\mathfrak{M}_0$ and let $\{(A_n, f_n, a_n)\}_{n\in\omega}$ be an enumeration of $\Delta(\mathfrak{M}_0)$. Let

$$\Delta_0 = \{q_n(x) : q_n(x) = f_n(\text{tp}(a_n/A_n))\}_{n\in\omega}.$$

Note that since f_n is elementary, $q_n(x)$ is well-defined. Now, fix $\mathscr{C} = \{c_n\}_{n\in\omega}$ a set of countably many new constants.

Since $\text{Th}(\mathfrak{M}_{0_{M_0}}) \cup q_0(x)$ is finitely satisfiable, $\text{Th}(\mathfrak{M}_{0_{M_0}}) \cup q_0(c_0)$ is consistent. Let $\mathfrak{M}_0^1$ be its model. Then $\mathfrak{M}_0 \prec \mathfrak{M}_0^1$ and $\mathfrak{M}_0^1$ realises $q_0(x)$. Suppose we have defined $\{\mathfrak{M}_0^j\}_{j=1}^k$, $\mathfrak{M}_0^j \prec \mathfrak{M}_0^{j+1}$ for each $j < k$ and $\mathfrak{M}_0^0 = \mathfrak{M}_0$.

Now, repeat the argument. Since $\text{Th}(\mathfrak{M}^k_{0_{M_0^k}}) \cup q_k(x)$ is finitely satisfiable, there is a countable $\mathfrak{M}_0^{k+1} \models \text{Th}(\mathfrak{M}^k_{0_{M_0^k}}) \cup q_k(c_{k+1})$. But, then

$$\mathfrak{M}_0^k \prec \mathfrak{M}_0^{k+1} \text{ and } \mathfrak{M}_0^{k+1} \text{ realises } q_k(x).$$

Thus, we can construct a countable elementary chain $\{\mathfrak{M}_0^k\}_{k\in\omega}$ such that for each k, $\mathfrak{M}_0^{k+1}$ realises $q_k(x)$. Take $\mathfrak{M}_1 = \lim_k \mathfrak{M}_0^k$. Then,

$$\mathfrak{M}_0 \prec \mathfrak{M}_1$$

and $\mathfrak{M}_1$ realizes all types in Δ_0. Again, since $\mathfrak{M}_1$ is countable, there are only countably many relevant for $\mathfrak{M}_1$ triples and so

$$\Delta_1 = \{q(x) : q(x) = f(\text{tp}(a/A)), (A, f, a) \in \Delta(\mathfrak{M}_1)\}$$

is countable. Applying the same arguments as above, obtain an increasing elementary chain $\{\mathfrak{M}_1^k\}_{k\in\omega}$ of countable models such that $\mathfrak{M}_2 = \lim_k \mathfrak{M}_1^k$ realises all types in Δ_1. Proceed inductively, to construct an elementary chain $\{\mathfrak{M}_n\}_{n\in\omega}$ of countable models such that for all n, $\mathfrak{M}_{n+1}$ realises the types in Δ_n. Then $\mathfrak{N} = \lim_n \mathfrak{M}_n$ is an elementary extension of $\mathfrak{M}$ which is ω-homogenous.

(2) Suppose $\mathfrak{M} = \lim_{n\in\omega} \mathfrak{M}_n$, where for each n, the structure $\mathfrak{M}_n$ is ω-homogenous. Let $f : A \prec M$, where A is a finite subset of M and let $a \in M$. Then there is $n \in \omega$ such that $A \cup \{a\} \subseteq M_n$. But $\mathfrak{M}_n$ is homogenous and so there is $b \in M_n$ such that $f' = f \cup \{\langle a,b\rangle\}$ is elementary, i.e.

$$f' : A \cup \{a\} \prec \mathfrak{M}_n \prec \mathfrak{M}.$$

□

THEOREM 1.3. *Let T be a countable complete theory in a countable language. Suppose $\mathfrak{M}, \mathfrak{N}$ are countable homogenous models of T and $\mathfrak{M}, \mathfrak{N}$ realise the same types in $S_n(T)$ for $n \geq 1$. Then $\mathfrak{M} \cong \mathfrak{N}$.*

QUESTION 1.4. Is it true that $\mathfrak{M}$ and $\mathfrak{N}$ realise the same types in $S_0(T)$, where $\mathfrak{M}$, $\mathfrak{N}$ are from the above statement?

PROOF. We will build an isomorphism $f : \mathfrak{M} \to \mathfrak{N}$ by a back-and-forth argument. To this this, we will construct an increasing sequence $\{f_n\}_{n\in\omega}$ of partial elementary maps with finite domains. Let $f = \bigcup_{i=0}^{\infty} f_i$, let $\{a_i\}_{i=0}^{\infty} = M$ and $\{b_i\}_{i=0}^{\infty} = N$. To insure that f is a total, onto function we guarantee that $a_i \in \mathrm{dom}(f_{2i+1})$, $b_i \in \mathrm{image}(f_{2i+2})$.

We start with $f_0 = \emptyset$. Because T is a complete theory, f_0 is an elementary mapping. Assume by induction that f_s is elementary, let $\bar{a} = \mathrm{dom}(f_s)$ and $\bar{b} = \mathrm{image}(f_s)$. We differentiate two different cases into the inductive step:

If $s+1 = 2i+1$ Assume by induction hypothesis that $\{a_j\}_{j<i}$ occur in $\bar{a}$. Let $p = \mathrm{tp}^{\mathfrak{M}}(\bar{a}, a_i)$. Since $\mathfrak{M}$ and $\mathfrak{N}$ realise the same types, there are a tuple $\bar{c}$ and an element d in $\mathfrak{N}$ such that $\mathrm{tp}^{\mathfrak{N}}(\bar{c}, d) = p$, $\mathrm{tp}^{\mathfrak{N}}(\bar{c}) = \mathrm{tp}^{\mathfrak{M}}(\bar{a})$. Moreover, since by hypothesis f_s is elementary we have $\mathrm{tp}^{\mathfrak{M}}(\bar{a}) = \mathrm{tp}^{\mathfrak{N}}(\bar{b})$. Thus $\mathrm{tp}^{\mathfrak{N}}(\bar{c}) = \mathrm{tp}^{\mathfrak{N}}(\bar{b})$. Because $\mathfrak{N}$ is homogenous, there is $e \in N$ such that $\mathrm{tp}^{\mathfrak{N}}(\bar{b}, e) = \mathrm{tp}^{\mathfrak{N}}(\bar{c}, d) = p$. Thus, $f_{s+1} = f_s \cup \{(a_i, e)\}$ is partial elementary with a_i in the domain.

If $s+1 = 2i+2$ As in the previous case, we can find $\bar{c}$, d in M such that $\mathrm{tp}^{\mathfrak{M}}(\bar{c}, d) = \mathrm{tp}^{\mathfrak{N}}(\bar{b}, b_i)$. Because $\mathfrak{M}$ is homogenous, there is $e \in M$ such that $\mathrm{tp}^{\mathfrak{M}}(\bar{c}, d) = \mathrm{tp}^{\mathfrak{M}}(\bar{a}, e)$. Then $f_{s+1} = f_s \cup \{(e, b_i)\}$ is as desired. □

COROLLARY 1.5.

(1) The number of countable non-isomorphic homogenous models of a given theory T is at most $2^{2^{\aleph_0}}$.
(2) If T has a countable saturated model, then the number of countable non-isomorphic homogenous models of T is at most $2^{\aleph_0}$.

PROOF. (1) Homogenous models of are determined by the set of types they realise. Note that the number of possible sets of formulas in a countable language does not exceed $2^{\aleph_0}$. Therefore $|S_n(T)| \leq 2^{\aleph_0}$. Thus, the number of possible sets of types which are realised is at most $2^{2^{\aleph_0}}$.

(2) If $\mathfrak{M}$ is a countable saturated model of T, then $|S_n(T)| \leq \aleph_0$ for all $n \in \omega$. Therefore there are at most $2^{\aleph_0}$ many possible sets of types (which can be realised.) □

2. Vaughtian Pairs

Recall that whenever $\mathscr{L}$ is a language, $\mathfrak{M}$ is an $\mathscr{L}$-structure and $\varphi(\bar{x})$ is an n-formula,

$$\varphi(\mathfrak{M}) = \{\bar{x} \in M^n : \mathfrak{M} \models \varphi(\bar{x})\}.$$

DEFINITION 2.1. Let $\kappa > \lambda \geq \aleph_0$. A $\mathscr{L}$-theory T has a (κ, λ)-model if there is a model $\mathfrak{M}$ of T and a $\mathscr{L}$-formula $\varphi(\bar{x})$ such that

$$|\mathfrak{M}| = \kappa \text{ and } |\varphi(\mathfrak{M})| = \lambda.$$

EXERCISE 7. Suppose T is a theory in a countable language $\mathscr{L}$ with an infinite model. Let κ be an infinite cardinal. Then T has a model $\mathfrak{M}$ such that for every $\mathscr{L}$-formula $\varphi(\bar{x})$

$$\text{either } |\varphi(\mathfrak{M})| = \kappa \text{ or } \varphi(\mathfrak{M}) \text{ is finite.}$$

Hint: Formulas for which $|\varphi(\mathfrak{M})| < \aleph_0$ are called algebraic. We will study them more closely in the next lecture. To obtain the claim for formulas with $|\varphi(\mathfrak{M})| \geq \aleph_0$ argue as in the upwards Löwenheim-Skolem theorem.

The goal for this lecture is to obtain the following theorem:

THEOREM 2.2. *(Vaught) If T has a (κ,λ)-model where $\kappa > \lambda \geq \aleph_0$, then T has a $(\aleph_1, \aleph_0)$-model.*

In order to obtain the above, we will work with the notion of a (κ,λ)-model:

DEFINITION 2.3. A pair $(\mathfrak{N},\mathfrak{M})$ is said to be a *Vaughtian pair* of modesl if

$$\mathfrak{M} \prec \mathfrak{N}, \mathfrak{M} \neq \mathfrak{N}$$

and there is a $\mathscr{L}_{\mathfrak{M}}$-formula φ such that $|\varphi(\mathfrak{M})| \geq \aleph_0$ and $\varphi(\mathfrak{M}) = \varphi(\mathfrak{N})$.

LEMMA 2.4. If T has a (κ,λ)-model where $\kappa > \lambda \geq \aleph_0$, then there is a Vaughtian pair $(\mathfrak{N},\mathfrak{M})$ of models of T.

REMARK. Thus the existence of (κ,λ)-models for a theory T is an obstacle to κ-categoricity.

PROOF. Let $\mathfrak{N}$ be a (κ,λ)-model. Suppose that $X = \varphi(\mathfrak{N})$ has cardinality λ. By the Löwenheim-Skolem theorem there is a model $\mathfrak{M} \prec \mathfrak{N}$ such that $X \subseteq M$ and $|M| = \lambda$. Because $X \subseteq M$, the pair $(\mathfrak{N},\mathfrak{M})$ is a Vaughtian pair. $\square$

DISCUSSION 2.5. Suppose $\mathscr{L}$ is a language and $\mathfrak{M} \subseteq \mathfrak{N}$ are $\mathscr{L}$-structures. Introducing a special unary predicate symbol $\mathscr{U}$, we consider the pair $(\mathfrak{N},\mathfrak{M})$ as a $\mathscr{L}^*$-structure, where $\mathscr{L}^* = \mathscr{L}(\mathscr{U}) = \mathscr{L} \cup \{\mathscr{U}\}$, by interpreting $\mathscr{U}$ as M.

DEFINITION 2.6. For each formula $\varphi = \varphi(x_1,\cdots,x_n)$ in $\mathscr{L}$ define a $\mathscr{L}(\mathscr{U})$-formula $\varphi^{\mathscr{U}}$, meant to be the restriction of φ to $\mathscr{U}$, inductively as follows:

(1) if φ is atomic, then $\varphi^{\mathscr{U}}$ is $\wedge_{j=1}^{n} \mathscr{U}(x_j) \wedge \varphi$;
(2) if φ is $\neg\psi$ then $\varphi^{\mathscr{U}}$ is $\neg\psi^{\mathscr{U}}$;
(3) if φ is $\psi \wedge \theta$, then $\varphi^{\mathscr{U}}$ is $\psi^{\mathscr{U}} \wedge \theta^{\mathscr{U}}$;
(4) if φ is $\exists x\psi$ then $\varphi^{\mathscr{U}}$ is $\exists x \mathscr{U}(x) \wedge \psi^{\mathscr{U}}$.

CLAIM 2.7. Let $\mathfrak{N}$ be an $\mathscr{L}^*$-structure, $\mathfrak{M} \subseteq \mathfrak{N}$ such that $\mathfrak{M} = \mathscr{U}^{\mathfrak{N}}$. Then for $\bar{a} \in \mathfrak{M}^k$:

$$\mathfrak{M} \models \varphi(\bar{a}) \text{ iff } (\mathfrak{N},\mathfrak{M}) \models \varphi^{\mathscr{U}}(\bar{a}).$$

LEMMA 2.8. Let $(\mathfrak{N},\mathfrak{M})$ be a Vaughtian pair for T. Then there is a Vaughtian pair $(\mathfrak{N}_0,\mathfrak{M}_0)$ for T where $\mathfrak{N}_0$ is countable.

PROOF. Let $\varphi(x)$ be a formula, witnessing that $(\mathfrak{N},\mathfrak{M})$ is a Vaughtian pair for T. Consider the theory T'_{vp} which is the union of the following theories:

(1) all formulas in T,
(2) $\{\exists x_1 \cdots \exists x_k \big(\wedge_{i<j} x_i \neq x_j \wedge (\wedge_{i=1}^{k} \varphi(x_i))\big)\}_{k\in\omega}$,
(3) $\{\exists x \neg\mathscr{U}(x)\}$,
(4) $\{\forall x(\varphi(x) \to \mathscr{U}(x))\}$.

Note that if $\mathfrak{N} \vDash T'_{vp}$, then $\mathscr{U}^{\mathfrak{N}}$ is a proper subset of the universe of $\mathfrak{N}$, $\varphi(\mathfrak{N}) \subseteq \mathscr{U}^{\mathfrak{N}}$ and $\varphi(\mathfrak{N})$ is infinite. Now, consider the $\mathscr{L}^*$-theory

$$T_{vp} = T'_{vp} \cup \{\forall \bar{x}\Big(\big(\wedge_{i=1}^{k} \mathscr{U}(x_i) \wedge \psi(\bar{x})\big) \to \psi^{\mathscr{U}}(\bar{x})\Big) : \psi \text{ is a k-}\mathscr{L}\text{-formula}, k \in \omega\}.$$

Now $(\mathfrak{N}, \mathfrak{M}) \vDash T_{vp}$ and by the theorem of Löwenheim-Skolem, there is a countable $\mathfrak{N}_0 \prec \mathfrak{N}$ such that $\mathfrak{N}_0 \vDash T_{vp}$. But then $\mathscr{U}^{\mathfrak{N}_0} = \mathfrak{M}_0 \prec \mathfrak{N}_0$ and $(\mathfrak{N}_0, \mathfrak{M}_0)$ is a countable Vaughtian pair for T, witnessed by the same formula. □

LEMMA 2.9. Suppose $\mathfrak{M}_0 \prec \mathfrak{N}_0$ are countable models of $T = T_{vp}$. There are $(\mathfrak{N}, \mathfrak{M})$ countable, homeogenous models such that

$$(\mathfrak{N}_0, \mathfrak{M}_0) \prec (\mathfrak{N}, \mathfrak{M})$$

and $\mathfrak{N}$, $\mathfrak{M}$ realise the same types in $S_n(T)$. Thus in particular $\mathfrak{M} \cong \mathfrak{N}$.

PROOF. We will start with proving the following Claim.

CLAIM 2.10. Let $\bar{a} \in \mathfrak{M}_0$, $p \in S_n(\bar{a})$ which is realised in $\mathfrak{N}_0$. Then there is an elementary extension $(\mathfrak{N}', \mathfrak{M}')$ of $(\mathfrak{N}_0, \mathfrak{M}_0)$ such that p is realised in $\mathfrak{M}'$.

PROOF. Consider the theory

$$\Gamma(\bar{x}) = \{\psi^{\mathscr{U}}(\bar{x}, \bar{a}) : \psi(\bar{x}, \bar{a}) \in p\} \cup \mathrm{Th}^{\mathscr{L}^*}(\mathfrak{N}_{0_{N_0}}).$$

If $\psi_1, \cdots, \psi_n$ are in p, then $\mathfrak{N}_0 \vDash \exists \bar{x} \wedge_{i=1}^{m} \psi_i(\bar{x}, \bar{a})$ and so by elementarity $\mathfrak{M}_0 \vDash \exists \bar{x} \wedge_{i=1}^{m} \psi_i(\bar{x}, \bar{a})$. Thus,

$$(\mathfrak{N}_0, \mathfrak{M}_0) \vDash \exists \bar{x} \wedge_{i=1}^{m} \psi_i^{\mathscr{U}}(\bar{x}, \bar{a}).$$

Therefore $\Gamma(\bar{x})$ is finitely satisfiable and so we can find an elementary extension $(\mathfrak{N}', \mathfrak{M}')$ realising $\Gamma(\bar{x})$. Then in particular, $\mathfrak{M}'$ realises p. □

CLAIM 2.11. If $\bar{b} \in \mathfrak{N}_0$ and $p \in S_n(\bar{b})$, then there is $(\mathfrak{N}_0, \mathfrak{M}_0) \prec (\mathfrak{N}', \mathfrak{M}')$ such that p is realised in $\mathfrak{N}'$.

PROOF. Let $\Gamma(\bar{x}) = p \cup \mathrm{Th}^{\mathscr{L}^*}(\mathfrak{N}_{0_{N_0}})$. Clearly $\Gamma(\bar{x})$ is finitely satisfiable in $\mathfrak{N}_0$. Thus $\Gamma(\bar{x})$ is consistent and we can get an elementary extension $\mathfrak{N}'$ realising $\Gamma(\bar{x})$. Then $(\mathfrak{N}', \mathfrak{M}')$ elementary extends $(\mathfrak{N}_0, \mathfrak{M}_0)$ and $\mathfrak{N}'$ realises p. □

Now, to prove the Lemma, we will build an elementary chain:

$$(\mathfrak{N}_0, \mathfrak{M}_0) \prec (\mathfrak{N}_1, \mathfrak{M}_1) \prec \cdots$$

of countable models $\{(\mathfrak{N}_i, \mathfrak{M}_i)\}_{i \in \omega}$ such that

(1) if $p \in S_n(T)$ is realised in $\mathfrak{N}_{3i}$ then p is realised in $\mathfrak{M}_{3i+1}$. Then $\mathfrak{N} = \lim_{i \in \omega} \mathfrak{N}_i$ and $\mathfrak{M} = \lim_{i \in \omega} \mathfrak{M}_i$ realise the same types.
(2) if $\bar{a}$, $\bar{b}$, c are in $\mathfrak{M}_{3i+1}$ and $\mathrm{tp}^{\mathfrak{M}_{3i+1}}(\bar{a}) = \mathrm{tp}^{\mathfrak{M}}_{3i+1}(\bar{b})$ then there is $d \in \mathfrak{M}_{3i+2}$ such that

$$\mathrm{tp}^{\mathfrak{M}_{3i+2}}(\bar{a}, c) = \mathrm{tp}^{\mathfrak{M}_{3i+2}}(\bar{b}, d).$$

Note that (2) implies that $\mathfrak{M} = \lim_{i \in \omega} \mathfrak{M}_i$ is ω-homogenous.
(3) if $\bar{a}$, $\bar{b}$, c are in $\mathfrak{N}_{3i+2}$ and $\mathrm{tp}^{\mathfrak{N}_{3i+2}}(\bar{a}) = \mathrm{tp}^{\mathfrak{N}_{3i+2}}(\bar{b})$, then there is $d \in \mathfrak{N}_{3i+3}$ such that $\mathrm{tp}^{\mathfrak{N}_{3i+3}}(\bar{a}, c) = \mathrm{tp}^{\mathfrak{N}_{3i+3}}(\bar{b}, d)$. Note that property (3) implies that $\mathfrak{N}$ is ω-homogenous.

Thus $(\mathfrak{N}, \mathfrak{M})$ is a countable Vaughtian pair for T as desired. □

THEOREM 2.12. *(Vaught) If T has a (κ, λ)-model, where $\kappa > \lambda \geq \aleph_0$ then T has a $(\aleph_1, \aleph_0)$-model.*

PROOF. Fix T such that T has a (κ,λ)-model. Then there is a Vaughtian pair $(\mathfrak{N},\mathfrak{M})$ for T such that $\mathfrak{N},\mathfrak{M}$ are countable homogenous and realise the same types. Fix $\varphi(\bar{x})$ witnessing that $\mathfrak{N}$, $\mathfrak{M}$ are a Vaughtian pair for T, i.e. $|\varphi(\mathfrak{M})| = \aleph_0$, $\varphi(\mathfrak{M}) = \varphi(\mathfrak{N})$ and so $\varphi(\mathfrak{N}) \cap N \backslash M = \emptyset$.

Inductively, we will construct a continuous elementary chain $\{\mathfrak{N}_\alpha\}_{\alpha\in\omega_1}$ such that for all $\alpha \in \omega_1$, $\mathfrak{N}_\alpha \cong \mathfrak{N}$ and $N_{\alpha+1} \backslash N$ will contain no elements satisfying φ. Let $\mathfrak{N}_0 = \mathfrak{N}$. If α is a limit, take $\mathfrak{N}_\alpha = \bigcup_{\beta<\alpha} \mathfrak{N}_\beta$. Then $\mathfrak{N}_\alpha$ is ω-homogenous (as the union of an increasing chain of ω-homogenous models) and $\mathfrak{N}_\alpha$ realises the same types as $\mathfrak{N}$. Thus $\mathfrak{N}_\alpha \cong \mathfrak{N}$. If we are at successor stage $\alpha+1$ and $\mathfrak{N}_\alpha$ is defined, proceed as follows: $\mathfrak{N}_\alpha \cong \mathfrak{N}$ and $\mathfrak{N} \cong \mathfrak{M}$. Thus $\mathfrak{M} \cong \mathfrak{N}_\alpha$ and we can fix an isomorphism $f : \mathfrak{M} \to \mathfrak{N}_\alpha$. However $\mathfrak{M} \prec \mathfrak{N}$ and so we can an extension $\mathfrak{N}_{\alpha+1}$ of $\mathfrak{N}_\alpha$ and an isomorphism $\bar{f} : \mathfrak{N} \to \mathfrak{N}_{\alpha+1}$ extending f. Then $\mathfrak{N}_{\alpha+1}$ is as desired.

Finally, take $\mathfrak{N} = \lim_{\alpha<\omega_1} \mathfrak{N}_\alpha$. Then $\mathfrak{N}$ is a model of T of cardinality $\aleph_1$, $\varphi(\bar{\mathfrak{N}}) = \varphi(\mathfrak{M})$ is countable and so $\bar{\mathfrak{N}}$ is an $(\aleph_1, \aleph_0)$-model for T. □

COROLLARY 2.13. Suppose T is κ-categorical for some $\kappa \geq \aleph_1$. Then T does not have a Vaughtian pair.

PROOF. If T has a Vaughtian pair, then T has an $(\aleph_1, \aleph_0)$ model $\mathfrak{M}$ witnessed by some formula φ. Thus in particular, $|\varphi(\mathfrak{M})| = \aleph_0$. But $\mathfrak{M}$ has an elementary extension $\mathfrak{M}^*$ such that $|\mathfrak{M}^*| = \aleph_1$ and for every formula ψ, if $|\psi(\mathfrak{M})| \geq \aleph_0$, then $|\psi(\mathfrak{M}^*)| = \aleph_1$ (recall Exercise 7). However $\mathfrak{M}^* \cong \mathfrak{M}$ since T is $\aleph_1$-categorical (by Morley Downwards) and so $\aleph_0 = |\varphi(\mathfrak{M})| = |\varphi(\mathfrak{M}^*)| = \aleph_1$, which is a contradiction! □

3. Minimal Extensions

DEFINITION 3.1. Let $A \subseteq M$, where M is the universe an a $\mathscr{L}$-structure $\mathfrak{M}$. The structure $\mathfrak{M}$ is said to be minimal extension of A if there is no proper elementary submodel $\mathfrak{N}$ of $\mathfrak{M}$ such that $A \subseteq N$.

REMARK 3.2. Equivalently, we can say that $\mathfrak{M}$ is a minimal extension of A if whenever $\mathfrak{N} \preceq \mathfrak{M}$ and $A \subseteq \mathfrak{N}$, we have $\mathfrak{M} = \mathfrak{N}$.

LEMMA 3.3. Let $\mathfrak{M} \models T$, $A \subseteq \mathfrak{M}$. If A has a prime extension $\mathfrak{M}$ and a minimal extension $\mathfrak{N}$, then they are isomorphic over A, i.e. there is an isomorphism $\mathfrak{M} \cong \mathfrak{N}$ fixing A elementwise.

PROOF. Note that $A \subseteq M$, $A \subseteq N$ and the identity mapping $\mathrm{id} : A \to A$ is elementary. Since $\mathfrak{M}$ is prime, the identity extends to an elementary embedding f from $\mathfrak{M}$ to $\mathfrak{N}$. Since $\mathfrak{N}$ is minimal, the embedding f is surjective and so $\mathfrak{M} \cong \mathfrak{N}$. □

COROLLARY 3.4. Let T be κ-categorical for some uncountable κ, let $\mathfrak{M} \models T$ and let A be an infinite definable subset of $\mathfrak{M}$. Then $\mathfrak{M}$ is the unique up to isomorphism prime extension of A.

PROOF. Let φ be a $\mathscr{L}$-formula such that $A = \varphi(\mathfrak{M})$. If $\mathfrak{M}$ is not minimal over A, then there is a proper elementary submodel $\mathfrak{N}$ of $\mathfrak{M}$ such that $A \subset N$. Then $\varphi(\mathfrak{M}) = \varphi(\mathfrak{N})$ and so $(\mathfrak{M},\mathfrak{N})$ is a Vaughtian pair for T. Now, by the Theorem of Vaught the theory T has an $(\aleph_1, \aleph_0)$ model, which is a contradiction to T being $\aleph_1$-categorical. Thus $\mathfrak{M}$ is minimal over A.

On the other hand, T is ω-stable and so totally transcendental. But then A has a prime extension $\mathfrak{N}$. By the previous Lemma $\mathfrak{M} \cong \mathfrak{N}$ over A. Thus $\mathfrak{M}$ is the unique up to isomorphism prime extension of A. □

CHAPTER 15

Algebraic Formulas and Strongly Minimal Sets

1. Algebraic Formulas

DEFINITION 1.1. Let $\mathfrak{M}$ be a structure, $A \subseteq M$.

(1) A $\mathscr{L}(A)$-formula $\varphi(x)$ is said to be *algebraic* if $|\varphi(\mathfrak{M})| < \aleph_0$.
(2) An element $a \in M$ is *algebraic over* A if there is a $\mathscr{L}(\mathscr{A})$-formula $\varphi(x)$ which is algebraic such that $\mathfrak{M} \models \varphi(a)$.
(3) An element of M is said to be algebraic if it is algebraic over the empty set.
(4) $\mathrm{acl}(A) = \{a \in M : a \text{ is algebraic over } A\}$.
(5) A set A is said to be algebraically closed if $A = \mathrm{acl}(A)$.

LEMMA 1.2. Let $\varphi(x)$ be an algebraic $\mathscr{L}(A)$-formula. Let $\mathfrak{M} \prec \mathfrak{N}$ and $A \subseteq \mathfrak{M}$. Then:

(1) $\varphi(\mathfrak{M}) = \varphi(\mathfrak{N})$,
(2) $\mathfrak{M}$ is algebraically closed.

PROOF. (1) Let $|\varphi(\mathfrak{M})| = n$. Then $\mathfrak{M} \models \exists x_1 \cdots \exists x_n \Big((\wedge_{i<j} x_i \neq x_j) \wedge (\wedge_{i=1}^{n} \varphi(x_i)) \Big)$ and

$$\mathfrak{M} \models \forall x_1 \cdots x_{n+1} \Big((\wedge_{i<j}^{n+1} x_i \neq x_j) \to (\vee_{i=1}^{n+1} \neg\varphi(x_i)) \Big).$$

Suppose there is $a \in \mathfrak{N} \setminus \mathfrak{M}$, $\mathfrak{N} \models \varphi(a)$. But then

$$\mathfrak{N} \models \exists x_1 \cdots \exists x_n \exists x_{n+1} \Big((\wedge_{i<j}^{n+1} x_i \neq x_j) \wedge (\wedge_{i<j}^{n+1} \varphi(x_i)) \Big)$$

and since $\mathfrak{M} \prec \mathfrak{N}$ we obtain

$$\mathfrak{M} \models \exists x_1 \cdots \exists x_n \exists x_{n+1} \Big((\wedge_{i<j}^{n+1} x_i \neq x_j) \wedge (\wedge_{i<j}^{n+1} \varphi(x_i)) \Big)$$

and so

$$\mathfrak{M} \not\models \forall x_1 \cdots \forall x_{n+1} \Big((\wedge_{i<j}^{n+1} x_i \neq x_j) \to (\vee_{i=1}^{n+1} \neg\varphi(x_i)) \Big),$$

which is a contradiction.

(2) To prove that$\mathfrak{M} = \mathrm{acl}(\mathfrak{M})$ note that

$$\mathrm{acl}^{\mathfrak{N}}(M) = \bigcup \{\varphi(\mathfrak{N}) : \exists A \in [M]^{<\omega} \text{ s.t. } \varphi \text{ is } \mathscr{L}(A)\text{-algebraic}\}.$$

But, we just proved that for every $\mathscr{L}(A)$-formula φ which is algebraic (over $\mathfrak{M}$), $\varphi(\mathfrak{M}) = \varphi(\mathfrak{N})$ and so $\varphi(\mathfrak{N}) \subseteq M$. Thus $M \subseteq \mathrm{acl}^{\mathfrak{N}}(\mathfrak{M}) \subseteq M$. and so $\mathfrak{M}$ is algebraically closed. □

REMARK 1.3. In particular, we proved that if $\varphi(x)$ is a $\mathscr{L}(A)$-formula, which is algebraic over $A \subseteq M$, $\mathfrak{M}$ a $\mathscr{L}$-structure, then for every $\mathfrak{N}$ such that $\mathfrak{M} \prec \mathfrak{N}$ we have

$$|\varphi(\mathfrak{N})| = |\varphi(\mathfrak{M})|.$$

FACT 5. Let $a \in \mathrm{acl}(A)$. Let σ be an automorphism of $\mathfrak{M}$ such that $\sigma \restriction A = \mathrm{id}$. Then there are $a_1, \cdots, a_n$ finitely many elements in $\mathrm{acl}(A)$ such that $\sigma(a) \in \{a_1, \cdots, a_n\}$.

PROOF. Let $\varphi(x)$ be algebraic such that $a \in \varphi(\mathfrak{M})$ and let $\sigma \in \mathrm{Aut}(\mathfrak{M})$ such that $\sigma \restriction A = \mathrm{id}$. Since σ is an automorphism

$$\mathfrak{M} \vDash \varphi(a) \text{ iff } \mathfrak{M} \vDash \varphi(\sigma(a))$$

which is equivalent to $\sigma(a) \in \varphi(\mathfrak{M})$. Take $\{a_1, \cdots, a_n\}$ to be an enumeration of $\varphi(\mathfrak{M})$. □

PROOF. Let $\varphi(x)$ be algebraic such that $a \in \varphi(\mathfrak{M})$. Then by definition $\mathfrak{M} \vDash \varphi(a)$ and since σ is an automorphism the latter holds if and only if $\mathfrak{M} \vDash \varphi(\sigma(a))$ which is equivalent to $\sigma(a) \in \varphi(\mathfrak{M})$. Then take $\{a_1, \cdots, a_n\}$ to be an enumeration of $\varphi(\mathfrak{M})$. □

FACT 6. $\mathrm{acl}(\mathrm{acl}(A)) = \mathrm{acl}(A)$.

PROOF. Take $c \in \mathrm{acl}(\mathrm{acl}(A))$. Then there is $\mathscr{L}(\mathrm{acl}(A))$-formula φ such that $c \in \varphi(\mathfrak{M})$, i.e. $\mathfrak{M} \vDash \varphi(c)$. Now, φ is of the form $\varphi(x, b_1, \cdots, b_n)$ for some parameters $b_j \in \mathrm{acl}(A)$. Thus for each $j \in \{1, \cdots, n\}$ there is an algebraic formula $\varphi_j = \varphi_j(y, \bar{a}_j)$ for some finite $\bar{a}_j \subseteq A$ such that $b_j \in \varphi_j(\mathfrak{M})$ (all parameters shown). Suppose $|\varphi(\mathfrak{M})| = k$ and consider the formula

$$\chi(x) = \exists y_1 \cdots \exists y_n \big(\varphi_1(y_1) \wedge \cdots \wedge \varphi(y_n) \wedge \exists^{\leq k} z \varphi(z, y_1, \cdots, y_n) \wedge \varphi(x, y_1, \cdots, y_n)\big).$$

Clearly, $\chi(x)$ has finitely many parameters from A and moreover $\chi(x)$ is algebraic over A. Since $\mathfrak{M} \vDash \chi(c)$, we get $c \in \mathrm{acl}(A)$. □

QUESTION 1.4. It should be clear, but why is $\chi(x)$ algebraic?

DEFINITION 1.5.

(1) A type $p(x) \in S(A)$ is said to be algebraic if and only if p contains an algebraic formula.
(2) Let $p(x) \in S(A)$ be algebraic. Then

$$\deg(p) := \min\{|\varphi(\mathfrak{M})| : \varphi \text{ is algebraic and } \varphi(x) \in p(x)\}.$$

REMARK. Let $p(x)$ be algebraic. Then $p(x)$ is isolated by any algebraic $\varphi \in p(x)$ such that $|\varphi(\mathfrak{M})| = \deg(p)$. Thus if $\varphi \in p(x)$ is algebraic of minimal degree, then

$$\varphi(\mathfrak{M}) \subseteq \bigcap\{\psi(\mathfrak{M}) : \psi \in p(x)\}.$$

LEMMA 1.6. A type $p(x)$ is algebraic over A if and only if $p(x) = \mathrm{tp}(a/A)$ where a is algebraic over A.

PROOF. $(\Rightarrow)$ Suppose $p(x)$ is algebraic over A. Then there is $\varphi(x) \in p(x)$ and a $\mathscr{L}(A)$-formula such that $\varphi(\mathfrak{M})$ is finite, where $A \subseteq M$. Furthermore for all $\psi(x) \in p(x)$

$$T \vdash \forall x(\varphi(x) \to \psi(x))$$

where $T = \mathrm{Th}(\mathfrak{M})$ and so if $\varphi \in \mathrm{tp}(a/A)$ then $p(x) \subseteq \mathrm{tp}(a/A)$. Since these are complete types, we have equality.

$(\Leftarrow)$ If $p(x) = \mathrm{tp}(a/A)$ where a is algebraic, then there is $\varphi(x) \in \mathrm{tp}(a/A)$ such that $\varphi(x)$ is algebraic and $a \in \varphi(\mathfrak{M})$. Thus, by definition $\mathrm{tp}(a/A)$ is algebraic. □

DEFINITION 1.7. If $a \in M$ is algebraic over A, then $\deg(a/A)$ is defined as $\deg(\mathrm{tp}(a/A))$.

LEMMA 1.8. Let $p(x) \in S(A)$ be a non-algebraic type, $A \subseteq B$. Then p has a non-algebraic extension $q \in S(B)$.

PROOF. Let $q_0(x) = p(x) \cup \{\neg\psi(x) : \psi(x) \text{ is an algebraic } \mathscr{L}(B)\text{-formula}\}$. We claim that $q_0(x)$ is a consistent set of formulas. Otherwise, there is a finite $\Gamma_0 \subseteq q(x)$ such that $T \cup \Gamma_0$ is inconsistent, where $T = \mathrm{Th}(\mathfrak{M})$ for $\mathfrak{M}$ a model realising $p(x)$. Thus, there is $\varphi(x) \in p(x)$ and there are $\psi_1, \cdots, \psi_n$ algebraic such that

$$T \cup \{\varphi(x), \neg\psi_1(x), \cdots, \neg\psi_n(x)\}$$

is inconsistent and so $T \cup \{\varphi(x), \wedge_{i=1}^{n} \neg\psi_i(x)\}$ is also inconsistent. However $T \cup \{\varphi(x)\}$ is consistent and so

$$T \cup \{\varphi(x)\} \vdash \neg \wedge_{i=1}^{n} \neg \psi_i(x),$$

i.e. $T \cup \{\varphi(x)\} \vdash \vee_{i=1}^{n} \psi_i(x)$. Therefore $T \vdash \forall x(\varphi(x) \to \vee_{i=1}^{n} \psi_i(x))$.

If $\mathfrak{M} \models T$, then $\varphi(\mathfrak{M}) \subseteq \bigcup_{i=1}^{n} \psi_i(\mathfrak{M})$ and so $\varphi(\mathfrak{M})$ is finite. That is $\varphi(x)$ is algebraic and so $p(x)$ is algebraic, which is a contradiction. Take $q(x)$ to be any maximal consistent set of $\mathscr{L}(B)$-formulas extending $q_0(x)$. In particular, $q(x)$ does not contain algebraic formulas, as otherwise $p(x)$ would be algebraic. □

LEMMA 1.9. $p(x) \in S(A)$ is algebraic if and only if $p(x)$ has only finitely many realisations in all elementary extensions of $\mathfrak{M}$.

PROOF. (⇒) Clear, since an algebraic formula has only finitely many (in fact, exactly the same number of) realisations in every elementary extension of $\mathfrak{M}$.

(⇐) Suppose $p(x) \in S(A)$, where $A \subseteq M$, Th($\mathfrak{M}$) is consistent with $p(x)$ and for every elementary extension $\mathfrak{N}$ of $\mathfrak{M}$, $p(x)$ has only finitely many realisations in $\mathfrak{N}$. Suppose by way of contradiction, that $p(x)$ does not contain an algebraic formula. Then for all n and all $\varphi(x) \in p(x)$,

$$\mathfrak{M} \models \exists x_1 \cdots \exists x_n \Big((\wedge_{i \neq j}^{n} x_i \neq x_j) \wedge (\wedge_{i=1}^{n} \varphi(x_i)) \Big)$$

and so for each n the set

$$\Delta_n = \{(\wedge_{i\neq j}^{n} x_i \neq x_j) \wedge (\wedge_{i=1}^{n} \varphi(x_i))\}_{\varphi \in p(x)} \cup p(x)$$

is a consistent set of n-formulas. Then inductively, we can construct an elementary chain $\{\mathfrak{N}_n\}_{n\in\omega}$ and a set of elements $\{a_n\}_{n=1}^{\infty}$ such that $\mathfrak{N}_n \prec \mathfrak{N}_{n+1}$ and $(a_1, \cdots, a_n)$ is an n-tuple of in $\mathfrak{N}_n$ realising $\Delta_n(x_1, \cdots, x_n)$. Take $\mathfrak{N}^* = \lim_{n\in\omega} \mathfrak{N}_n$. Then $\mathfrak{N}^*$ is an elementary extension of $\mathfrak{M}$ and $\mathfrak{N}^*$ has infinitely many realisations of $p(x)$, which is a contradiction. □

2. Strongly Minimal Sets

Unless otherwise specified T is a complete theory with infinite models.

DEFINITION 2.1. Let $\mathfrak{M}$ be a model of T and let $D \subseteq M^n$ be an infinite definable set (say, by the formula $\varphi(\bar{x})$).

(1) D is said to be minimal in $\mathfrak{M}$ if for every definable $Y \subseteq D^n$ either Y or $D \backslash Y$ is finite. Equivalently, if for every $\mathscr{L}(M)$-formula $\psi(\bar{x})$, $(\varphi \wedge \psi)(\mathfrak{M})$ is either finite or co-finite. We also say that the formuly $\varphi(\bar{x})$ is minimal in $\mathfrak{M}$.
(2) The set D and the formula $\varphi(\bar{x})$ are said to be strongly minimal, if for every elementary extension $\mathfrak{N}$ of $\mathfrak{M}$, the formula $\varphi(\bar{x})$ is minimal in $\mathfrak{N}$.
(3) A theory T is said to be strongly minimal if the universe of every model $\mathfrak{M}$ of T is a strongly minimal set. That is, every definable set in $\mathfrak{M}$ is either finite, or co-finite. This is equivalent to saying that the formula $x \doteq x$ is strongly minimal.

LEMMA 2.2. (Exchange Principle) Suppose $D \subseteq M$ is strongly minimal, $A \subseteq D$ and $a, b \in D$. If $a \in \text{acl}(A \cup \{b\}) \backslash \text{acl}(A)$, then $b \in \text{acl}(A \cup \{a\})$.

PROOF. Let $a \in \text{acl}(A \cup \{b\}) \backslash \text{acl}(A)$ and let $\varphi(x, b)$ be a $\mathscr{L}(A \cup \{b\})$-algebraic formula where the parameter b is explicitly shown such that $\mathfrak{M} \models \varphi(a, b)$. Then in particular for some $n \in \omega$ we must have

$$|\{x \in D : \mathfrak{M} \models \varphi(x, b)\}| = n.$$

Let $\delta(x)$ be the formula defining D.

Now consider w a parameter, and let $\psi(w)$ be the formula stating that there are exactly n elements in D realising $\varphi(x,w)$. Thus in particular $\mathfrak{M} \models \psi(b)$. That is $\psi(w)$ is the formula: $\exists x_1 \cdots \exists x_n \Big((\wedge_{i<j}^{n} x_i \neq x_j) \wedge (\wedge_{i=1}^{n} \delta(x_i)) \wedge (\wedge_{i=1}^{n} \varphi(x_i,w)) \Big) \wedge \neg \exists x_1 \cdots \exists x_{n+1} \Big((\wedge_{i \neq j}^{n+1} x_i \neq x_j) \wedge (\wedge_{i=1}^{n+1} \delta(x_i)) \wedge (\wedge_{i=1}^{n+1} \varphi(x_i,w)) \Big)$.

How many elements w are in D such that $\mathfrak{M} \models \psi(w)$? That is, for how many elements $w \in D$, do we have

$$|\{x \in D : \mathfrak{M} \models \varphi(x,w)\}| = n?$$

Consider the formula $(\psi \wedge \delta)(w)$. If $|(\psi \wedge \delta)(\mathfrak{M})| < \aleph_0$, then b is algebraic over A, which implies that a is algebraic over A, contradiction. Therefore $|(\psi \wedge \delta)(\mathfrak{M})| \geq \aleph_0$. But D is strongly minimal, which implies that $(\neg \psi \wedge \delta)(\mathfrak{M})$ is finite and so $\psi(w)$ defines a cofinite set in D.

If $\{y \in D : \mathfrak{M} \models \varphi(a,y) \wedge \psi(y)\}$ is finite, then

$$\varphi(a,y) \wedge \psi(y) \wedge \delta(y)$$

is algebraic over $A \cup \{a\}$ and b satisfies it. Thus $b \in \operatorname{acl}(A \cup \{a\})$ and we have achieved our goal!

Suppose $\{y \in D : \mathfrak{M} \models \varphi(a,y) \wedge \psi(y)\}$ is not finite. Then $|D \backslash \{y \in D : \mathfrak{M} \models \varphi(a,y) \wedge \psi(y)\}| = l$ for some $l \in \omega$. Let $\chi(x)$ be a formula expressing this. Then, in particular $\mathfrak{M} \models \chi(a)$. If $(\chi \wedge \delta)(\mathfrak{M})$ is finite, then a is algebraic over A, which is a contradiction. By minimality of D, $(\chi \wedge \delta)(\mathfrak{M})$ must be co-finite in D. Then in particular $(\chi \wedge \delta)(\mathfrak{M})$ is infinite and so there are $\{a_i\}_{i=1}^{n+1}$ distinct such that $(\chi \wedge \delta)(a_i)$. For each i consider the co-finite set

$$B_i = \{w \in D : \mathfrak{M} \models \varphi(a_i,w) \wedge \psi(w)\}.$$

However the intersection of finitely many co-finite sets is non-empty and so there is

$$\hat{b} \in \bigcap_{i=1}^{n+1} B_i.$$

But then $\mathfrak{M} \models \varphi(a_i,\hat{b})$ holds for each i. Thus,

$$|\{x \in D : \mathfrak{M} \models \varphi(x,\hat{b})\}| \geq n+1,$$

which is a contradiction to $\psi(\hat{b})$. □

CHAPTER 16

Independent Sets

1. Independent Sets

DEFINITION 1.1. We say that a set $A \subseteq D$ is independent, if $a \notin \mathrm{acl}(A \backslash \{a\})$ for all $a \in A$. If $C \subseteq D$, we say that A is independent over C if $a \notin \mathrm{acl}(C \cup A \backslash \{a\})$ for all $a \in A$.

LEMMA 1.2. Suppose $\mathfrak{M}, \mathfrak{N}$ are models of T, $\varphi(x)$ is strongly minimal with parameters from A, where either $A = \emptyset$ or $A \subseteq M_0$, $\mathfrak{M}_0 \prec \mathfrak{M}$ and $\mathfrak{M}_0 \prec \mathfrak{N}$. If $a_1, \cdots, a_n$ are in $\varphi(\mathfrak{M})$ are independent over A and $b_1, \cdots, b_n$ are in $\varphi(\mathfrak{N})$ are independent over A, then

$$\mathrm{tp}^{\mathfrak{M}}(\bar{a}/A) = \mathrm{tp}^{\mathfrak{N}}(\bar{b}/A).$$

PROOF. By induction on n.

Assume $n = 1$. suppose $\psi \in \mathrm{tp}^{\mathfrak{M}}(a/A)$, i.e. $\mathfrak{M} \models \psi(a)$. Since φ is strongly minimal and $a \notin \mathrm{acl}(A)$, we must have $(\varphi \wedge \neg\psi)(\mathfrak{M})$ is finite. But then, $\varphi \wedge \neg\psi$ is algebraic over A and so if $\mathfrak{N} \models (\varphi \wedge \neg\psi)(b)$, then $b \in \mathrm{acl}(A)$, which is a contradiction. Thus $\mathfrak{N} \models (\varphi \wedge \psi)(b)$ and so $\mathfrak{N} \models \psi(b)$, i.e. $\psi \in \mathrm{tp}^{\mathfrak{N}}(b/A)$. To see that $\mathrm{tp}^{\mathfrak{N}}(b/A) \subseteq \mathrm{tp}^{\mathfrak{M}}(a/A)$ switch the roles of $\mathfrak{M}$ and $\mathfrak{N}$.

Suppose the statement is true for n and fix

$$\{a_1, \cdots, a_{n+1}\} \subseteq \varphi(\mathfrak{M}), \{b_1, \cdots, b_{n+1}\} \subseteq \varphi(\mathfrak{N})$$

independent over A. Take $\bar{a} = (a_1, \cdots, a_n)$, $\bar{b} = (b_1, \cdots, b_n)$. By induction hypothesis $\mathrm{tp}^{\mathfrak{M}}(\bar{a}/A) = \mathrm{tp}^{\mathfrak{N}}(\bar{b}/A)$. Take $\psi \in \mathrm{tp}^{\mathfrak{M}}(\bar{a}^\frown a_{n+1}/A)$. That is $\psi(\bar{y}, y) \in \mathscr{L}(A)$ and $\mathfrak{M} \models \psi(\bar{a}, a_{n+1})$. Again, since $a_{n+1} \notin \mathrm{acl}(A \cup \{a_1, \cdots, a_n\})$, $\varphi(\mathfrak{M}) \cap \psi(\bar{a}, \mathfrak{M})$ is infinite, which implies that $\varphi(\mathfrak{M}) \backslash \psi(\bar{a}, \mathfrak{M}) = \varphi(x) \wedge \neg\psi(\bar{a}, x)(\mathfrak{M})$ is finite. Thus there is n such that $\mathfrak{M} \models |\{x : \varphi(x) \wedge \neg\psi(\bar{a}, x)\}| = n$. Because

$$\mathfrak{M}_0 \prec \mathfrak{M}, \mathfrak{M}_0 \prec \mathfrak{N}$$

and $\mathrm{tp}^{\mathfrak{M}}(\bar{a}/A) = \mathrm{tp}^{\mathfrak{N}}(\bar{b}/A)$, we get

$$\mathfrak{N} \models |\{x : \varphi(x) \wedge \neg\psi(\bar{b}, \bar{x})\}| = n.$$

Because $b_{n+1} \notin \mathrm{acl}(A, \bar{b})$ and $b_{n+1} \in \varphi(\mathfrak{N})$ we must have $\mathfrak{N} \models \psi(\bar{b}, b_{n+1})$. Therefore, $\psi \in \mathrm{tp}^{\mathfrak{N}}(\bar{b}^\frown b_{n+1}/A)$ and so

$$\mathrm{tp}^{\mathfrak{M}}(\bar{a}^\frown a_{n+1}/A) \subseteq \mathrm{tp}^{\mathfrak{N}}(\bar{b}^\frown b_{n+1}/A).$$

To get equality, switch the roles of $\mathfrak{M}$ and $\mathfrak{N}$. □

COROLLARY 1.3. Let $\mathfrak{M}, \mathfrak{N}$ be models of T, $A \subseteq M_0$ where M_0 is the universe of a model $\mathfrak{M}_0$ such that $\mathfrak{M}_0 \prec \mathfrak{M}$ and $\mathfrak{M}_0 \prec \mathfrak{N}$. Let $\varphi(x)$ be a strongly minimal formulas with parameters from A, where either $A = \emptyset$ or $A \subseteq M_0$. Let $B \subseteq \varphi(\mathfrak{M})$ and $C \subseteq \varphi(\mathfrak{N})$ be both infinite and independent over A.

Then B and C are infinite sets of indiscernibles of the same type over A, where for a sequence $\langle m_i : i \in I \rangle$ of order indiscernibles in $\mathfrak{M}$,

$$\mathrm{tp}(I) = \{\varphi(x_1, \cdots, x_n) : \mathfrak{M} \models \varphi(m_{i_1}, \cdots, m_{i_n}), i_1 < \cdots < i_n \text{ in } I, n \in \omega\}.$$

DEFINITION 1.4. A subset A of Y, where $Y \subseteq D$ and D is a strogly minimal, is a basis for Y, if A is independent and $\mathrm{acl}(A) = \mathrm{acl}(Y)$.

LEMMA 1.5. Let $D \subseteq \mathfrak{M}$ be strongly minimal and let $A, B \subseteq D$ be independent with $A \subseteq \mathrm{acl}(B)$.

(1) Let $A_0 \subseteq A$, $B_0 \subseteq B$ and let $A_0 \cup B_0$ be a basis for $\mathrm{acl}(B)$, $a \in A \backslash A_0$. Then there is $b \in B_0$ such that $A_0 \cup \{a\} \cup (B_0 \backslash \{b\})$ is a basis for $\mathrm{acl}(B)$.
(2) $|A| \leq |B|$ and so if A, B are basis for $Y \subseteq D$, then $|A| = |B|$.

PROOF. (1) Note that $a \in \mathrm{acl}(B)$. Since $A_0 \cup B_0$ is a basis for $\mathrm{acl}(B)$, we can find $C \subseteq B_0$ such that $|C|$ is least with $a \in \mathrm{acl}(A_0 \cup C)$. Since A is independent and $a \notin A_0$, $C \neq \emptyset$. Thus, take any $b \in C$ and apply the Exchange Principle. Since $a \in \mathrm{acl}(A_0 \cup C) \backslash \mathrm{acl}(A_0)$, $b \in \mathrm{acl}(A_0 \cup \{a\} \cup C \backslash \{b\})$ and thus $b \in \mathrm{acl}(A_0 \cup \{a\} \cup (B_0 \backslash \{b\}))$, which implies

$$\mathrm{acl}(A_0 \cup \{a\} \cup (B_0 \backslash \{b\})) = \mathrm{acl}(B).$$

It remains to show that $A_0 \cup \{a\} \cup (B_0 \backslash \{b\})$ is independent, for which it is sufficient to show that $a \notin \mathrm{acl}(A_0 \cup (B_0 \backslash \{b\}))$. Suppose to the contrary, that $a \in \mathrm{acl}(A_0 \cup (B_0 \backslash \{b\}))$. But then $b \in \mathrm{acl}(A_0 \cup (B_0 \backslash \{b\}))$, which is a contradiction to $A_0 \cup B_0$ being a basis and so, being independent.

(2) We consider two cases:

Case 1: B is finite. Thus $B = \{b_i\}_{i=1}^n$ for some n. Suppose $|A| \not\leq |B|$, i.e. A has at least $n+1$ distinct elements $\{a_i\}_{i=1}^{n+1}$. Start with $A_0 = \emptyset$, $B_0 = B$. Since B is independent, B is a basis for $\mathrm{acl}(B)$. Thus, we can apply part (1) to a_1. That is, there is $b_{i_1} \in B$ such that $\{a_1\} \cup (B \backslash \{b_{i_1}\})$ is a basis for $\mathrm{acl}(B)$. Proceed inductively, to find $\{b_{i_j}\}_{j=1}^n$ distinct such that

$$\{a_i\}_{i=1}^n \cup (B \backslash \{b_{i_j}\}_{j=1}^n) = \{a_i\}_{i=1}^n$$

is a basis for $\mathrm{acl}(B)$. But, by hypothesis $A \subseteq \mathrm{acl}(B)$ and so $a_{n+1} \in \mathrm{acl}(\{a_i\}_{i=1}^n)$, which is a contradiction to A being independent.

Case 2: Suppose B is infinite. Then for any finite $B_0 \subseteq B$, $A \cap \mathrm{acl}(B_0)$ is finite and so

$$A = A \cap \mathrm{acl}(B) = A \cap \bigcup_{B_0 \subseteq B, B_0 \text{ finite}} \mathrm{acl}(B_0) = \bigcup_{B_0 \subseteq B, B_0 \text{ finite}} A \cap \mathrm{acl}(B_0).$$

Then, clearly $|A| \leq |B|$. The remaind of (2) holds, just because the roles of A and B can be switched. □

2. Dimension

DEFINITION 2.1. Let $D \subseteq \mathfrak{M}$ be strongly minimal. Then define $\dim(Y) = |A|$ where A is a basis for Y.

By the previous theorem, the above definition is correct.

THEOREM 2.2. *Let T be a strongly minimal theory. Let $\mathfrak{M}, \mathfrak{N}$ be models of T. Then*

$$\mathfrak{M} \cong \mathfrak{N} \text{ if and only if } \dim(\mathfrak{M}) = \dim(\mathfrak{N}).$$

PROOF. Let B be a basis for $\varphi(\mathfrak{M})$, C a basis for $\varphi(\mathfrak{N})$. By hypothsis, $|B| = |C|$ and so there is a bijection $f : B \to C$. Since B, C realise the same types, the mapping f must be elementary. Now, consider the set of all possible partial elementary extensions of f, i.e. the set

$$I = \{g : B' \to C' : B \subseteq B' \subseteq \varphi(\mathfrak{M}), C \subseteq C' \subseteq \varphi(\mathfrak{N}), f \subseteq g \text{ elementary}\}.$$

By Zorn's Lemma, there is a maximal element of I, call it $g : B' \to C'$, Suppose $b \in \varphi(\mathfrak{M})\backslash B'$. The element b is algebraic over B' and so the type $\mathrm{tp}^{\mathfrak{M}}(b/B')$ is isolated by some formula $\psi(x, \bar{d})$. The mapping g is partial elementary and so there is $c \in \varphi(\mathfrak{N})$ such that

$$\mathfrak{N} \models \psi(c, g(\bar{d})).$$

Then

$$\mathrm{tp}^{\mathfrak{M}}(b/B') = \mathrm{tp}^{\mathfrak{N}}(c/C')$$

and $g \cup \{(b,c)\}$ is also elementary, contradiction to the hypothesis of g being maximal. Thus $\varphi(\mathfrak{M}) = B'$. If we switch the roles of $\mathfrak{M}$ and $\mathfrak{N}$, we get $\varphi(\mathfrak{N}) = C'$. □

COROLLARY 2.3. If T is countable and strongly minimal, then it is categorical in all uncountable cardinalities.

PROOF. Suppose $\mathfrak{M}_1, \mathfrak{M}_2$ are models of T, $|\mathfrak{M}_1| = |\mathfrak{M}_2| = \kappa \geq \aleph_1$ and let B_1, B_2 be basis for $\mathfrak{M}_1$, $\mathfrak{M}_2$ respectively.

Suppose that $\max(|T|, |B_i|) < \mathrm{acl}(B_i)$. Then by the Löwenheim-Skolem theorem there is a proper elementary submodel $\mathfrak{N}$ of $\mathfrak{M}_i$ such that

$$|\mathfrak{N}| = \max(|T|, |B_i|) \text{ and } B_i \subseteq N.$$

However, elementary substructures are algebraically closed and so $\mathrm{acl}(B_i) \subseteq N$, which is a contradiction. Thus $|\mathrm{acl}(B_i)| \leq \max(|T|, |B_i|)$. Then

$$\kappa = |\mathrm{acl}(B_i)| \leq \max(|T|, |B_i|) \leq |B_i| \leq \kappa$$

and so $|B_i| = \kappa$ for $i = 1, 2$.

Let $f : B_1 \to B_2$ be a bijection. However $\mathrm{type}(B_1) = \mathrm{type}(B_2)$ and so f is elementary. But then f extends to an isomorphism $\mathfrak{M}_1 \cong \mathfrak{M}_2$. □

CHAPTER 17

The categoricity theorem

1. Existence of Strongly Minimal Sets

LEMMA 1.1. Let T be an ω-stable theory.

(1) Suppose $\mathfrak{M}$ is a model of T. Then there is a minimal formula in $\mathfrak{M}$.
(2) Moreover, if in addition to $\mathfrak{M} \models T$, we have that $\mathfrak{M}$ is $\aleph_0$-saturated and $\varphi(\bar{x},\bar{a})$ is a minimal formula in $\mathfrak{M}$, then $\varphi(\bar{x},\bar{a})$ is strongly minimal.

PROOF. (1) Suppose not. Inductively, we will construct a binary tree of consistent formulas as follows. Let $\varphi_\emptyset$ be $x \doteq x$. Now, suppose we have defined φ_σ for some $\sigma \in 2^{<\omega}(=\bigcup_{n\in\omega} {}^n2)$ such that $\varphi_\sigma(\mathfrak{M})$ is infinite. Since φ_σ is not minimal, there is $\psi(x)$ such that both

$$(\varphi_\sigma \wedge \psi)(\mathfrak{M}) \text{ and } (\varphi_\sigma \wedge \neg\psi)(\mathfrak{M}) \text{ are infinite.}$$

Then define

$$\varphi_{\sigma^\frown 0} = \varphi_\sigma \wedge \psi \text{ and } \varphi_{\sigma^\frown 1} = \varphi_\sigma \wedge \neg\psi.$$

Proceeding inductively, we can construct a binary tree of consistent formulas. However, an ω-stable theory is totally transcedental and so by definition, T has no model $\mathfrak{M}$ with a binary tree of consistent $\mathscr{L}(\mathfrak{M})$-formulas. Thus, we reached a contradiction!

(2) Suppose $\varphi(\bar{x},\bar{a})$ is minimal, but not strongly minimal. Thus, there is a proper elementary extension $\mathfrak{N}$ of $\mathfrak{M}$ and a formula $\psi(\bar{x},\bar{b})$ for some tuple $\bar{b}$ of elements in N such that

$$\Big(\psi(\bar{x},\bar{b}) \wedge \varphi(\bar{x},\bar{a})\Big)(\mathfrak{N}) \text{ and } \Big(\neg\psi(\bar{x},\bar{b}) \wedge \varphi(\bar{x},\bar{a})\Big)(\mathfrak{N})$$

are infinite subsets of $\varphi(\bar{x},\bar{a})(\mathfrak{N})$. However $\mathfrak{M}$ is $\aleph_0$-saturated and so there is a tuple $\bar{b}'$ in $\mathfrak{M}$ such that

$$\mathrm{tp}^{\mathfrak{M}}(\bar{a},\bar{b}') = \mathrm{tp}^{\mathfrak{N}}(\bar{a},\bar{b}).$$

Then $\psi(\bar{x},\bar{b}')$ defines an infinite, co-infinite subset of $\varphi(\bar{x},\bar{a})(\mathfrak{M})$, i.e. both

$$\Big(\psi(\bar{x},\bar{b}') \wedge \varphi(\bar{x},\bar{a})\Big)(\mathfrak{M}) \text{ and } \Big(\neg\psi(\bar{x},\bar{b}') \wedge \varphi(\bar{x},\bar{a})\Big)(\mathfrak{M}),$$

which is a contradiction. □

THEOREM. If T is a theory with no Vaughtian pairs, then any minimal formula is strongly minimal.

For the proof of the above theorem, we will need the following Lemma.

LEMMA 1.2. Let T be a $\mathscr{L}$-theory with no Vaughtian pairs. Let $\mathfrak{M} \models T$ and let $\varphi(\bar{x},\bar{y})$ be a $\mathscr{L}(M)$-formula, where $\bar{x} = (x_1,\cdots,x_k)$ and $\bar{y} = (y_1,\cdots,y_m)$.

Then there is $n \in \mathbb{N}$ such that for all m-tuples $\bar{a}$ in M,

$$\text{if } |\varphi(\bar{x},\bar{a})(\mathfrak{M})| > n \text{ then } |\varphi(\bar{x},\bar{a})(\mathfrak{M})| \geq \aleph_0.$$

PROOF. Note that the implication from the statement of the theorem can be rewritten as: $\exists n \in \mathbb{N}$ such that for every m-tuple $\bar{a}$ in $\mathfrak{M}$

$$\text{either } |\varphi(\bar{x},\bar{a})(\mathfrak{M})| \leq n \text{ or } \varphi(\bar{x},\bar{a})(\mathfrak{M}) \text{ is infinite.}$$

Assume the claim of the theorem is not true. Thus, assume that for every $n \in \mathbb{N}$ there is an m-tuple $\bar{a}_n$ such that

$$n < |\varphi(\bar{x},\bar{a}_n)(\mathfrak{M})| < \aleph_0.$$

Consider the expanded language $\mathscr{L}^* = \mathscr{L}(\mathscr{U})$ where $\mathscr{U}$ is a unary predicate symbol intended to denote an elementary submodel of a given structure. Let $\bar{y} = (y_1, \cdots, y_m)$ be an m-tuple of variables and let $\Gamma'(\bar{y})$ be the following set of $\mathscr{L}(\mathscr{U})$-formulas:

(1) Formulas implying that $\mathscr{U}^{\mathfrak{A}}$ is a proper $\mathscr{L}$-elementary submodel of $\mathfrak{A}$[1];
(2) $\wedge_{i=1}^{m} \mathscr{U}(y_i)$;
(3) For each $j \in \mathbb{N}$ there are more than j many k-tuples $\bar{x}$ such that $\varphi(\bar{x},\bar{y})$.
Here, in fact we have countably many formulas. One for each j. Note that any model realizing these countably many formulas will have infinitely many k-tuples $\bar{x}$ realizing $\varphi(\bar{x},\bar{y})$.
(4) $\varphi(\bar{x},\bar{y}) \to \wedge_{i=1}^{k} \mathscr{U}(x_i)$. Thus, in particular, any k-tuple realizing $\varphi(\bar{x},\bar{y})$ is already in the interpretation of $\mathscr{U}$.

CLAIM 1.3. The set $\Gamma'(\bar{y})$ is finitely satisfiable.

PROOF. Let $\mathfrak{N}$ be a proper elementary extension of $\mathfrak{M}$ such that $\mathscr{U}^{\mathfrak{N}} = \mathfrak{M}$. Since

$$|\varphi(\mathfrak{M},\bar{a}_n)| < \aleph_0,$$

i.e. $\varphi(\bar{x},\bar{a}_n)$ is algebraic over $\mathfrak{M}$ and $\mathfrak{M} \prec \mathfrak{N}$, we must have

$$\varphi(\mathfrak{M},\bar{a}_n) = \varphi(\mathfrak{N},\bar{a}_n).$$

If $\Delta \subseteq \Gamma'(\bar{y})$ is finite, then we can find $\bar{a}_n$ realising Δ in $(\mathfrak{N},\mathfrak{M})$. □

Now, extend $\Gamma'(\bar{y})$ to a maximal consistent set of formulas, i.e. to a type $\Gamma(\bar{y}) \in S_m(T)$. Let $\mathfrak{N}'$ be a model of T realizing $\Gamma(\bar{y})$. Thus in particular,

(1) $\mathfrak{M}' = \mathscr{U}^{\mathfrak{N}}$ is a proper elementary submodel of $\mathfrak{N}'$ and
(2) there is an m-tuple $\bar{a}$ in $\mathfrak{M}'$ which realises $\Gamma(\bar{y})$ in $\mathfrak{N}'$.

Then $\varphi(\bar{x},\bar{a})(\mathfrak{M}')$ is infinite and $\varphi(\bar{x},\bar{a})(\mathfrak{M}') = \varphi(\bar{x},\bar{a})(\mathfrak{N}')$. Thus, $(\mathfrak{N}',\mathfrak{M}')$ is a Vaughtian pair for T, which is a contradiction. □

QUESTION 1.4. Why in the above proof:

(1) $|\varphi(\bar{x},\bar{a})(\mathfrak{M}')| \geq \aleph_0$?
(2) $\varphi(\bar{x},\bar{a})(\mathfrak{M}') = \varphi(\bar{x},\bar{a})(\mathfrak{N}')$?

REMARK 1.5. Let $\mathfrak{M}$, n be as in the previous Lemma. Then for every elementary extension $\mathfrak{N}$ of $\mathfrak{M}$ and every m-tuple $\bar{b}$ in $\mathfrak{N}$ if

$$|\varphi(\mathfrak{N},\bar{b})| > n$$

then

$$|\varphi(\mathfrak{N},\bar{b})| \geq \aleph_0.$$

Otherwise, we can proceed as in the previous Lemma and obtain a Vaugthian pair for T.

THEOREM 1.6. *If T has no Vaughtian pairs, then any minimal formula is strongly minimal.*

[1]Here $\mathfrak{A}$ is an arbitrary $\mathscr{L}(\mathscr{U})$-structure.

PROOF. Proceed by contradiction. Suppose $\varphi(\bar{x})$ is minimal for $\mathfrak{M}$, $\mathfrak{M} \models T$ and $\varphi(\bar{x})$ is not strongly minimal. Thus, there is a proper elementary extension $\mathfrak{N}$ of $\mathfrak{M}$ and a formula $\psi(\bar{x},\bar{y})$ such that for some tuple $\bar{b}$ in $\mathfrak{N}$ both

$$\psi(x,\bar{b})(\mathfrak{N}) \cap \varphi(\mathfrak{N}) \text{ and } \neg\psi(x,\bar{b})(\mathfrak{N}) \cap \varphi(\mathfrak{N})$$

are infinite.

By Lemma 1.2 there is $n \in \mathbb{N}$ such that for every proper elementary extension $\mathfrak{N}'$ of $\mathfrak{M}$ and every tuple $\bar{a}$ in N' the following holds:

$$|\psi(\mathfrak{N}',\bar{a}) \cap \varphi(\mathfrak{N}')| \geq \aleph_0 \text{ and } |\neg\psi(\mathfrak{N}',\bar{a}) \cap \varphi(\mathfrak{N}')| \geq \aleph_0$$

if and only if

$$|\psi(\mathfrak{N}',\bar{a}) \cap \varphi(\mathfrak{N}')| > n \text{ and } |\neg\psi(\mathfrak{N}',\bar{a}) \cap \psi(\mathfrak{N}')| > n.$$

By minimality of φ in $\mathfrak{M}$, we must have that $\psi(\mathfrak{M},\bar{y}) \cap \varphi(\mathfrak{M})$ or $\neg\psi(\mathfrak{M},\bar{y}) \cap \varphi(\mathfrak{M})$ is finite (and this is true for each parameter $\bar{y}$). That is, by Lemma 1.2

$$\mathfrak{M} \models \forall\bar{y}\Big(|\psi(\mathfrak{M},\bar{y}) \cap \varphi(\mathfrak{M})| \leq n \vee |\neg\psi(\mathfrak{M},\bar{y}) \cap \varphi(\mathfrak{M})| \leq n\Big).$$

But then $\mathfrak{N}$ satisfies the same sentence and so

$$\mathfrak{N} \models \forall\bar{y}\Big(|\psi(\mathfrak{N},\bar{y}) \cap \varphi(\mathfrak{N})| \leq n \vee |\neg\psi(\mathfrak{N},\bar{y}) \cap \varphi(\mathfrak{N})| \leq n\Big),$$

which is a contradiction to the choice of ψ, $\bar{b}$ and $\mathfrak{N}$. □

COROLLARY 1.7. If T is ω-stable and has no Vaughtian pairs, then for any $\mathfrak{M} \models T$, there is a strongly minimal formula over $\mathfrak{M}$.

PROOF. By Lemma 1.1 and Theorem 1.6. □

2. The Categoricity Theorem

LEMMA 2.1. Suppose T has no Vaughtian pairs, $\mathfrak{M} \models T$ and $X \subseteq M^n$ is infinite and definable. Then no proper elementary submodel of $\mathfrak{M}$ contains X. Moreover, if in addition T is ω-stable, then $\mathfrak{M}$ is prime over X.

PROOF. Let $\varphi(\bar{x})$ define X. If $\mathfrak{N}$ is a proper elementary submodel of $\mathfrak{M}$ and $X \subseteq \mathfrak{N}$ then $\varphi(\mathfrak{N}) = \varphi(\mathfrak{M}) = X$. Therefore $(\mathfrak{M},\mathfrak{N})$ is a Vaugthian pair for T.

Suppose in addition that T is ω-stable. Then there is a prime extension $\mathfrak{N}$ of X such that $\mathfrak{N} \prec \mathfrak{M}$. By the same argument as above $\varphi(\mathfrak{M}) = \varphi(\mathfrak{N})$ and since there are no Vaughtian pairs for T, we obtain that $\mathfrak{M} = \mathfrak{N}$ and so $\mathfrak{M}$ is prime over X. □

THEOREM 2.2. *(Baldwin-Lachlan) Let T be a complete theory in a countable language with infinite models and let $\kappa \geq \aleph_1$ be a cardinal. Then T is κ-categorical if and only if T is ω-stable and has no Vaughtian pairs.*

PROOF. $(\Rightarrow)$ If T is κ-categorical, then T is ω-stable and has no Vaughtian pairs.

$(\Leftarrow)$ Now, suppose T is ω-stable and has no Vaughtian pairs. As we just showed, T has a prime model $\mathfrak{M}_0$. By Corollary 1.7 there is a strongly minimal formula $\varphi(x)$ possibly with parameters in $\mathfrak{M}_0$.

Consider any two models $\mathfrak{M},\mathfrak{N}$ of T such that $|\mathfrak{M}| = |\mathfrak{N}| = \kappa$. Since $\mathfrak{M}_0$ is prime, we can assume that $\mathfrak{M}_0 \prec \mathfrak{M}$, $\mathfrak{M}_0 \prec \mathfrak{N}$ and so

$$\dim(\varphi(\mathfrak{M})) = \dim(\varphi(\mathfrak{N})) = \kappa,$$

which means that there is an elementary bijection

$$f : \varphi(\mathfrak{M}) \to \varphi(\mathfrak{N}).$$

By Lemma 2.1, the model $\mathfrak{M}$ is prime over $\varphi(\mathfrak{M})$. Thus the mapping f can be extended to an elementary map f' from $\mathfrak{M}$ to $\mathfrak{N}$. However by Lemma 2.1, $\mathfrak{N}$ has no proper elementary submodels containing $\varphi(\mathfrak{N})$. Thus, f' is surjective and f' is an isomorphism. □

COROLLARY 2.3. (Theorem of Morley) Let κ be an uncountable cardinal. Then

$$T \text{ is } \aleph_1\text{-categorical if and only if } T \text{ is } \kappa\text{-categorical.}$$

PROOF. Note that the Baldwin-Lachlan characterisation of κ-categoricity for $\kappa \geq \aleph_1$ does not depend on κ. □

Part 6

Additional Topics in Model Theory

CHAPTER 18

Morley Rank

1. $\mathfrak{M}$-Morley Rank

In the following T is a complete theory with infinite models.

DEFINITION 1.1. ($\mathfrak{M}$-Morley Rank) Suppose $\mathfrak{M}$ is an $\mathscr{L}$-structure, $\varphi(\bar{x})$ is an $\mathscr{L}(M)$-formula. First we define $\mathrm{MR}^{\mathfrak{M}} \geq \alpha$ for $\alpha \in \mathbb{ON}$, recursively on the ordinals:

(1) $\mathrm{MR}^{\mathfrak{M}}(\varphi) \geq 0$ if and only if $\varphi(\mathfrak{M})$ is non-empty.
(2) If α is a limit ordinal, then $\mathrm{MR}^{\mathfrak{M}}(\varphi) \geq \alpha$ if and only if $\mathrm{RM}^{\mathfrak{M}}(\varphi) \geq \beta$ for all $\beta < \alpha$.
(3) $\mathrm{MR}^{\mathfrak{M}}(\varphi) \geq \alpha+1$ if and only if there are $\mathscr{L}(M)$-formulas $\{\varphi_i(\bar{x})\}_{i\in\omega}$ such that $\{\varphi_i(\mathfrak{M})\}_{i\in\omega}$ is a family of pairwise disjoint non-empty subsets of $\varphi(\mathfrak{M})$ such that $\mathrm{MR}^{\mathfrak{M}}(\varphi_i) \geq \alpha$ for each i.

Finally, define:

(1) $\mathrm{MR}^{\mathfrak{M}}(\varphi) = -1$ if $\varphi(\mathfrak{M}) = \emptyset$,
(2) $\mathrm{MR}^{\mathfrak{M}}(\varphi) = \alpha$ if $\mathrm{MR}^{\mathfrak{M}}(\varphi) \not\geq \alpha+1$ and $\mathrm{MR}^{\mathfrak{M}}(\varphi) \geq \alpha$.
(3) $\mathrm{MR}^{\mathfrak{M}}(\varphi) = \infty$ if for all ordinals α, $\mathrm{MR}^{\mathfrak{M}}(\varphi) \geq \alpha$.

LEMMA 1.2. Suppose that $\theta(\bar{x},\bar{y})$ is a $\mathscr{L}$-formula, $\mathfrak{M}$ an $\aleph_0$-saturated model, $\bar{a}$ and $\bar{b}$ are tuples in M and $\mathrm{tp}^{\mathfrak{M}}(\bar{a}) = \mathrm{tp}^{\mathfrak{M}}(\bar{b})$. Then

$$\mathrm{MR}^{\mathfrak{M}}(\theta(\bar{x},\bar{a})) = \mathrm{MR}^{\mathfrak{M}}(\theta(\bar{x},\bar{b})).$$

PROOF. By transfinite induction on α we show that if $\theta(\bar{x},\bar{y})$ is any $\mathscr{L}$-formula and $\mathrm{tp}^{\mathfrak{M}}(\bar{a}) = \mathrm{tp}^{\mathfrak{M}}(\bar{b})$, then

$$\mathrm{MR}^{\mathfrak{M}}(\theta(\bar{x},\bar{a})) \geq \alpha \text{ iff } \mathrm{MR}^{\mathfrak{M}}(\theta(\bar{x},\bar{b})) \geq \alpha.$$

Base case: Because $\mathrm{tp}^{\mathfrak{M}}(\bar{a}) = \mathrm{tp}^{\mathfrak{M}}(\bar{b})$, $\theta(\mathfrak{M},\bar{a}) = \emptyset$ if and only if $\theta(\mathfrak{M},\bar{b})) = \emptyset$. Thus $\mathrm{MR}^{\mathfrak{M}}(\theta(\bar{x},\bar{a})) \geq 0$ if and only if $\mathrm{MR}^{\mathfrak{M}}(\theta(\bar{x},\bar{b})) \geq 0$.

Successor case: Suppose the claim is true for α and suppose $\mathrm{MR}^{\mathfrak{M}}(\theta(\bar{x},\bar{y})) \geq \alpha+1$. Thus, by definition there are $\mathscr{L}(M)$-formulas $\{\psi_i\}_{i\in\omega}$ such that $\{\psi_i(\mathfrak{M})\}_{i\in\omega}$ is an infinite sequence of pairwise disjoint subsets of $\theta(\mathfrak{M},\bar{a})$ and $\mathrm{RM}^{\mathfrak{M}}(\psi_i) \geq \alpha$ for all i. Now, we write all parameters appearing in the formula $\psi_i(\bar{x})$ explicitly. That is for all i,

$$\psi_i(\bar{x}) = \chi_i(\bar{x},\bar{c}_i)$$

where $\bar{c}_i$ is an m_i-tuple in M. Using the fact that $\mathfrak{M}$ is $\aleph_0$-saturated and a back-and-forth argument, find a sequence $\{\bar{d}_i\}_{i\in\omega}$ such that

$$\mathrm{tp}^{\mathfrak{M}}(\bar{a},\bar{c}_1,\cdots,\bar{c}_m) = \mathrm{tp}^{\mathfrak{M}}(\bar{b},\bar{d}_1,\cdots,\bar{d}_m)$$

for all $m < \omega$. Then $\{\chi_i(\mathfrak{M},\bar{d}_i)\}_{i\in\omega}$ is an infinite sequence of pairwise disjoint subsets of $\theta(\mathfrak{M},\bar{b})$. Furthermore, by inductive hypothesis, since $\mathrm{MR}^{\mathfrak{M}}(\chi_i(\mathfrak{M},\bar{c}_i)) \geq \alpha$ for each $i \in \omega$, we have $\mathrm{MR}(\chi_i(\mathfrak{M},\bar{d}_i)) \geq \alpha$. But then by definition $\mathrm{MR}^{\mathfrak{M}}(\theta(\bar{x},\bar{b})) \geq \alpha+1$.

Switching the roles of $\bar{a}$, $\bar{b}$ we get that if $\mathrm{RM}^{\mathfrak{M}}(\theta(\bar{x},\bar{b})) \geq \alpha+1$, then $\mathrm{MR}^{\mathfrak{M}}(\theta(\bar{x},\bar{b})) \geq \alpha+1$. Therefore

$$\mathrm{MR}^{\mathfrak{M}}(\theta(\bar{x},\bar{a})) \geq \alpha \text{ if and only if } \mathrm{MR}^{\mathfrak{M}}(\theta(\bar{x},\bar{b})) \geq \alpha$$

for all α. Thus, we have:

$$\mathrm{MR}^{\mathfrak{M}}(\theta(\bar{x},\bar{a})) = \mathrm{MR}^{\mathfrak{M}}(\theta(\bar{x},\bar{b})).$$

Limit case: Suppose α is a limit and the claim is true for all $\beta < \alpha$. Then

$$\mathrm{MR}^{\mathfrak{M}}(\theta(\bar{x},\bar{a})) \geq \alpha \text{ if and only if } \mathrm{MR}^{\mathfrak{M}}(\theta(\bar{x},\bar{a})) \geq \beta$$

for all $\beta < \alpha$. However, by the inductive hypothesis the latter is equivalent to

$$\mathrm{MR}^{\mathfrak{M}}(\theta(\bar{x},\bar{b})) \geq \beta$$

for all $\beta < \alpha$. By definition this is equivalent to $\mathrm{MR}^{\mathfrak{M}}(\theta(\bar{x},\bar{b})) \geq \alpha$. □

LEMMA 1.3. Let $\mathfrak{M}$ and $\mathfrak{N}$ be $\aleph_0$-saturated models of T such that $\mathfrak{M} \prec \mathfrak{N}$. Let φ be a $\mathscr{L}(M)$-formula. Then

$$\mathrm{MR}^{\mathfrak{M}}(\varphi) = \mathrm{MR}^{\mathfrak{N}}(\varphi).$$

PROOF. By induction on α show that

$$\mathrm{MR}^{\mathfrak{M}}(\varphi) \geq \alpha \text{ if and only if } \mathrm{MR}^{\mathfrak{N}}(\varphi) \geq \alpha.$$

Since $\mathfrak{M}$ is elementary in $\mathfrak{N}$, we have that $\varphi(\mathfrak{N}) = \emptyset$ if and only if $\varphi(\mathfrak{N}) = \emptyset$. Thus

$$\mathrm{MR}^{\mathfrak{M}}(\varphi) \geq 0 \text{ if and only if } \mathrm{MR}^{\mathfrak{N}}(\varphi) \geq 0.$$

Now, suppose α is a limit ordinal. Then $\mathrm{MR}^{\mathfrak{M}}(\varphi) \geq \alpha$ if and only if $\mathrm{MR}^{\mathfrak{M}}(\varphi) \geq \beta$ for all $\beta < \alpha$. However, by inductive hypothesis, the latter holds if and only if $\mathrm{MR}^{\mathfrak{N}}(\varphi) \geq \beta$ for all $\beta < \alpha$, which by definition is equivalent to $\mathrm{MR}^{\mathfrak{N}}(\varphi) \geq \alpha$.

Next, we consider the successor case. That is, assume $\mathrm{MR}^{\mathfrak{M}}(\varphi) \geq \alpha+1$. Thus, there is a family $\{\psi_i\}_{i\in\omega}$ of $\mathscr{L}(M)$-formulas such that $\{\psi_i(\mathfrak{M})\}_{i\in\omega}$ are pairwise disjoint subsets of $\varphi(\mathfrak{M})$ and $\mathrm{MR}^{\mathfrak{M}}(\psi_i) \geq \alpha$. Since $\mathfrak{M} \prec \mathfrak{N}$ we obtain that $\{\psi_i(\mathfrak{N})\}_{i\in\omega}$ is a family of pairwise disjoint subsets of $\varphi(\mathfrak{N})$. Moreover for each $i \in \omega$ by inductive hypothesis, $\mathrm{MR}^{\mathfrak{N}}(\psi_i) \geq \alpha$. That is $\mathrm{MR}^{\mathfrak{N}}(\varphi) \geq \alpha+1$.

Now, suppose $\mathrm{MR}^{\mathfrak{N}}(\varphi) \geq \alpha+1$. Thus, there is a family $\{\psi_i\}_{i\in\omega}$ of $\mathscr{L}(N)$-formulas such that for each i, $\mathrm{MR}^{\mathfrak{N}}(\psi_i) \geq \alpha$ and $\{\psi_i(\mathfrak{N})\}_{i\in\omega}$ is a family of pairwise disjoint subsets of $\varphi(\mathfrak{M})$. Now, we have to account for all parameters! Let $\bar{a}$ be the set of parameters in M occurring in φ and for each $i \in \omega$ let $\bar{b}_i$ be the parameters of ψ_i from N. Thus, $\psi_i = \theta_i(\bar{x},\bar{b}_i)$, where θ_i is a $\mathscr{L}$-formula. Since $\mathfrak{M}$ is $\aleph_0$-saturated, we can find tuples $\{\bar{c}_i\}_{i\in\omega}$ in M such that

$$\mathrm{tp}^{\mathfrak{N}}(\bar{a},\bar{b}_1,\cdots,\bar{b}_m) = \mathrm{tp}^{\mathfrak{M}}(\bar{a},\bar{c}_1,\cdots,\bar{c}_m) = \mathrm{tp}^{\mathfrak{N}}(\bar{a},\bar{c}_1,\cdots,\bar{c}_m)$$

for each $m \in \omega$. By the previous Lemma,

$$\mathrm{MR}^{\mathfrak{N}}(\theta_i(\bar{x},\bar{c}_i)) \geq \alpha$$

and so by the inductive hypothesis $\mathrm{MR}^{\mathfrak{M}}(\theta_i(\bar{x},\bar{c}_i)) \geq \alpha$. Therefore $\mathrm{MR}^{\mathfrak{M}}(\varphi) \geq \alpha+1$. □

COROLLARY 1.4. Let $\mathfrak{M}$ be a $\mathscr{L}$-structure, $\mathfrak{N}_0$, $\mathfrak{N}_1$ are $\aleph_0$-saturated elementary extensions of $\mathfrak{M}$, φ a $\mathscr{L}(M)$-formula. Then

$$\mathrm{MR}^{\mathfrak{N}_0}(\varphi) = \mathrm{MR}^{\mathfrak{N}_1}(\varphi).$$

PROOF. Find $\mathfrak{N}_2$ such that $\mathfrak{N}_1 \prec \mathfrak{N}_2$ and $\mathfrak{N}_0 \prec \mathfrak{N}_2$ (amalgamation). Let $\mathfrak{N}_3$ be $\aleph_0$-saturated such that $\mathfrak{N}_2 \prec \mathfrak{N}_3$. By the previous Lemma

$$\mathrm{MR}^{\mathfrak{N}_0}(\varphi) = \mathrm{MR}^{\mathfrak{N}_3}(\varphi) = \mathrm{MR}^{\mathfrak{N}_1}(\varphi).$$

□

2. Morley Rank

Throughout T is a complete theory with infinite models.

DEFINITION 2.1. (Morley rank of a formula) Let $\mathfrak{M}$ be a $\mathscr{L}$-structure, φ - a $\mathscr{L}(M)$-formula. The *Morley rank* of φ, denoted $\mathrm{MR}(\varphi)$, is defined as $\mathrm{MR}^{\mathfrak{N}}(\varphi)$ where $\mathfrak{N}$ is any $\aleph_0$-saturated elementary extension of $\mathfrak{M}$.

DEFINITION 2.2. (Morley rank of a definable set) Let $\mathfrak{M}$ be a $\mathscr{L}$-structure and let $X \subseteq M^n$ be defined by the $\mathscr{L}(M)$-formula $\varphi(\bar{x})$. The *Morley rank* of X, denoted $\mathrm{MR}(X)$, is defined as the Morley rank of φ.

REMARK 2.3. In particular, if $\mathfrak{M}$ is $\aleph_0$-saturated and $X \subseteq M^n$ is definable, then

$$\mathrm{MR}(X) \geq \alpha + 1$$

if and only if there is a family $\{Y_i\}_{i\in\omega}$ of pairwise disjoint definable subsets of X each of Morley rank at least α.

LEMMA 2.4. (Properties of Morley Rank) Let $\mathfrak{M}$ be an $\mathscr{L}$-structure, X, Y definable subsets of M^n.

(1) If $X \subseteq Y$, then $\mathrm{MR}(X) \leq \mathrm{MR}(Y)$.
(2) $\mathrm{MR}(X \cup Y) = \max\{\mathrm{MR}(X), \mathrm{MR}(Y)\}$.
(3) If $X \neq \emptyset$, then $\mathrm{MR}(X) = 0$ if and only if X is finite.

PROOF. (1) Note that if $\{\psi_i(\mathfrak{N})\}_{i\in\omega}$ are pairwise disjoint subsets of X and serve as a witness to $\mathrm{MR}(X) \geq \alpha + 1$, then they also witness $\mathrm{MR}(Y) \geq \alpha + 1$.

(2) By induction on α. Notice that if $\{\psi_i(\mathfrak{N})\}_{i\in\omega}$ are pairwise disjoint, witnessing that

$$\mathrm{MR}(X \cup Y) \geq \alpha + 1,$$

then they will give rise to a countable family of pairwise disjoint sets witnessing

$$\mathrm{MR}(X) \geq \alpha + 1 \text{ or } \mathrm{MR}(Y) \geq \alpha + 1.$$

(3) ($\Leftarrow$) Suppose X is finite. Then clearly X does not contain infinitely many pairwise disjoint non-empty sets and so $\mathrm{MR}(X) \ngeq 1$. Thus $\mathrm{MR}(X) = 0$.

($\Rightarrow$) Let $\mathrm{MR}(X) = 0$. Suppose by way of contradiction that X is infinite. Thus there is a sequence $\{a_n\}_{n\in\omega}$ of pairwise distinct elements of X. However each singleton is definable and so we get a family of ω-many pairwise disjoint, definable subsets of X, namely $\{\{a_n\}\}_{n\in\omega}$. Thus $\mathrm{MR}(X) \geq 1$. Contradiction! □

COROLLARY 2.5. A theory T is totally transcendental if and only if for all $\mathfrak{M} \models T$ and all $\mathscr{L}(M)$-formulas φ, $\mathrm{MR}(\varphi) < \infty$.

Hint: Show that a model $\mathfrak{M}$ of T has no binary tree of consistent $\mathscr{L}(M)$-formulas if and only if for every $\mathscr{L}(M)$-formula φ, $\mathrm{MR}(\varphi) < \infty$.

CHAPTER 19

Large saturated structures

Throughout T is a complete theory with infinite models.

1. Large Saturated Models

DEFINITION 1.1.

(1) $\mathfrak{M} \models T$ is said to be κ-universal for T if for all $\mathfrak{N} \models T$ with $|N| < \kappa$, there is an elementary embedding of $\mathfrak{N}$ into $\mathfrak{M}$.
(2) A model $\mathfrak{M}$ is said to be universal for T if it is $|M|^+$-universal for T.

LEMMA 1.2. Let $\kappa \geq \aleph_0$. If $\mathfrak{M}$ is κ-saturated, then $\mathfrak{M}$ is κ^+-universal.

PROOF. Let $\mathfrak{N} \models T$ with $|\mathfrak{N}| \leq \kappa$. Let $\{n_\alpha\}_{\alpha<\kappa} = N$ and let $A_\alpha = \{n_\beta\}_{\beta<\alpha}$. Build a sequence $\{f_\alpha\}_{\alpha\in\kappa}$ of partial elementary maps such that $f_\alpha : A_\alpha \to \mathfrak{M}$, $f_\alpha \subset f_{\alpha+1}$.
Base case: $f_0 = \emptyset$.
Limit case: $f_\alpha = \bigcup_{\beta<\alpha} f_\beta$.
Successor case: Let $f_\alpha : A_\alpha \to \mathfrak{M}$ be a partial embedding and let

$$\Gamma(x) = \{\varphi(x, f_\alpha(\bar{a})) : \mathfrak{M} \models \varphi(n_\alpha, \bar{a})\}.$$

Since f_α is elementary, $|A_\alpha| < \kappa$ and $\mathfrak{M}$ is κ-saturated, there is $b \in M$ realising $\Gamma(x)$. Then define $f_{\alpha+1} = f_\alpha \cup \{(n_\alpha, b)\}$. Then $f_{\alpha+1}$ is partial elementary and extends f_α. Now, define $f = \bigcup_{\alpha\in\kappa} f_\alpha$. Then f is an elementary embedding from $\mathfrak{N}$ into $\mathfrak{M}$. □

THEOREM 1.3. *Let $\mathfrak{M}$ be a model of T. Then T has a κ^+-saturated model $\mathfrak{N}$ such that*

$$\mathfrak{M} \prec \mathfrak{N} \text{ and } |N| \leq |M|^\kappa.$$

PROOF. We start with proving the following claim.

CLAIM. For any $\mathfrak{M}$ there is $\mathfrak{M}'$ such that:

(1) $\mathfrak{M} \prec \mathfrak{M}'$,
(2) $|M'| \leq |M|^\kappa$ and
(3) for every $A \subseteq M$ with $|A| \leq \kappa$, every $p \in S_1^{\mathfrak{M}}(A)$ is realised in $\mathfrak{M}'$.

PROOF. Note that if $A \subseteq M$, $|A| \leq \kappa$, then $|S_1^{\mathfrak{M}}(A)| \leq 2^\kappa$. Thus, we can list

$$S_1^{\mathfrak{M}}(A) = \{p_\alpha\}_{\alpha<|M|^\kappa}.$$

Build an elementary chain $\{\mathfrak{M}_\alpha : \alpha < |M|^\kappa\}$ such that:

(1) $\mathfrak{M}_0 = \mathfrak{M}$,
(2) $\mathfrak{M}_\alpha = \bigcup_{\beta<\alpha} \mathfrak{M}_\beta$ for α limit,
(3) $\mathfrak{M}_\alpha \prec \mathfrak{M}_{\alpha+1}$ with $|\mathfrak{M}_{\alpha+1}| = |\mathfrak{M}_\alpha|$ and $\mathfrak{M}_{\alpha+1}$ realises p_α.

Inductively, one can provide that $|\mathfrak{M}_\alpha| \leq |\mathfrak{M}|^\kappa$ for all α. Let

$$\mathfrak{M}' = \bigcup_{\alpha < |M|^\kappa} \mathfrak{M}_\alpha.$$

Then $|\mathfrak{M}'| \leq |\mathfrak{M}|^\kappa$ and $\mathfrak{M}'$ is as desired. □

Now, build an elementary chain $\{\mathfrak{N}_\alpha\}_{\alpha \in \kappa^+}$ such that $|\mathfrak{N}_\alpha| \leq |\mathfrak{M}|^\kappa$ and

(1) $\mathfrak{N}_0 = \mathfrak{M}$,
(2) $\mathfrak{N}_\alpha = \bigcup_{\beta<\alpha} \mathfrak{N}_\beta$, whenever α is a limit,
(3) At successor steps $\alpha+1$ find a $\mathfrak{N}_{\alpha+1}$ such that $\mathfrak{N}_\alpha \prec \mathfrak{N}_{\alpha+1}$, $|\mathfrak{N}_{\alpha+1}| \leq |\mathfrak{M}|^\kappa$ and if $A \subseteq \mathfrak{N}_\alpha$ with $|A| \leq \kappa$, $p \in S_n^{\mathfrak{N}_\alpha}(A)$, then p is realised in $\mathfrak{N}_{\alpha+1}$.

Finally, take $\mathfrak{N} = \bigcup_{\alpha<\kappa^+} \mathfrak{N}_\alpha$. Since $\kappa^+ \leq |M|^\kappa$, we have $|N| \leq |M|^\kappa$. Consider any set of parameters $A \subseteq N$ such that $|A| \leq \kappa$ and let $p \in S_n^{\mathfrak{N}}(A)$. Using the regularity of κ^+ find $\alpha < \kappa^+$ with $A \subseteq \mathfrak{N}_\alpha$ and note that the type p is realised in $\mathfrak{N}_{\alpha+1} \prec \mathfrak{N}$.

Thus, $\mathfrak{N}$ is κ^+-saturated. □

COROLLARY 1.4. Suppose $2^\kappa = \kappa^+$. Then T has a saturated model of cardinality $2^\kappa = \kappa^+$. Moreover, if GCH holds, then T has a saturated model of size κ^+ for all κ.

REMARK 1.5. Suppose $|S_n(T)| = 2^{\aleph_0}$. Then if $\mathfrak{M} \models T$ and $\mathfrak{M}$ is $\aleph_0$-saturated, then

$$|\mathfrak{M}| \geq 2^{\aleph_0}.$$

Therefore, if $\aleph_1 < 2^{\aleph_0}$, then T has no saturated model of cardinality $\aleph_1$.

COROLLARY 1.6. Suppose $\kappa \geq \aleph_1$ is regular and $2^\lambda \leq \kappa$ for $\lambda < \kappa$. Then T has a saturated model of size κ. In particular, if $\kappa \geq \aleph_1$ is strongly inaccessible, then T has a saturated model of cardinality κ.

PROOF. Let $\mathfrak{M} \models T$, $|M| = \kappa$.

If $\kappa = \lambda^+$ for $\lambda < \kappa$, then the result follows from the previous Corollary.

Thus, assume κ is a limit cardinal. Recursively build an elementary chain

$$\{\mathfrak{M}_\lambda\}_{\lambda<\kappa, \lambda \text{ cardinal}}$$

of models of cardinality κ as follows. Start with $\mathfrak{M}_0 = \mathfrak{M}$ and whenever λ is a limit cardinal take

$$\mathfrak{M}_\lambda = \bigcup_{\mu<\lambda, \mu \text{ cardinal}} \mathfrak{M}_\mu.$$

At successor steps, given $\mathfrak{M}_\lambda$, find an elementary extension $\mathfrak{M}_{\lambda^+}$ of $\mathfrak{M}_\lambda$ such that

$$\mathfrak{M}_{\lambda^+} \text{ is } \lambda^+\text{-saturated and } |\mathfrak{M}_{\lambda^+}| \leq \kappa^\lambda = \kappa.$$

Finally let

$$\mathfrak{N} = \bigcup_{\lambda<\kappa, \text{cardinal}} \mathfrak{M}_\lambda.$$

Now, if $A \subseteq N$ and $|N| < \kappa$, then there is $\lambda < \kappa$ such that $A \subseteq \mathfrak{M}_\lambda$. Thus, if $p \in S_n^{\mathfrak{N}}(A)$ then p is realized in $\mathfrak{M}_{\lambda^+} \prec \mathfrak{N}$. □

The above construction is captured by the following definition:

DEFINITION 1.7. A structure $\mathfrak{M}$ of cardinality $\kappa \geq \omega$ is said to be *special* if

$$\mathfrak{M} = \bigcup_{\lambda<\kappa, \lambda \text{cardinal}} \mathfrak{M}_\lambda,$$

where $\{\mathfrak{M}_\lambda\}_{\lambda<\kappa,\lambda \text{cardinal}}$ is an elementary chain of models such that for each λ, $\mathfrak{M}_\lambda$ is λ^+-saturated. We refer to this elementary chain as a *specialising chain*.

REMARK 1.8. The structure $\mathfrak{M}$ constructed in Corollary 1.6 is special. More generally, if $\mathfrak{M}$ is a special and its cardinality is regular, then $\mathfrak{M}$ is saturated.

We have already seen that two elementarily equivalent saturated structures of the same cardinality are isomorphic. Below, we extend the result to elementarily equivalent special structures of the same cardinality.

THEOREM 1.9. *Let $\mathfrak{M}$ and $\mathfrak{N}$ be elementarily equivalent, special structures of cardinality κ. Then*

$$\mathfrak{M} \cong \mathfrak{N}.$$

PROOF. (Outline) Let $\{\mathfrak{M}_\lambda\}_{\lambda<\kappa,\lambda \text{ cardinal}}$ and $\{\mathfrak{N}_\lambda\}_{\lambda<\kappa,\lambda \text{ cardinal}}$ be specializing chains of $\mathfrak{M}$ and $\mathfrak{N}$ respectively. The universes of $\mathfrak{M}$ and $\mathfrak{N}$ can be enumerated as $\{m_\alpha\}_{\alpha<\kappa}$ and $\{n_\alpha\}_{\alpha<\kappa}$ respectively, so that for each α, the element m_α belongs to the universe of $\mathfrak{M}_{|\alpha|}$ and the element n_α belongs to the universe of $\mathfrak{N}_{|\alpha|}$ (see remark 1.10). Construct an increasing family $\{f^\alpha\}_{\alpha\in\kappa}$ of elementary maps such that

$$f^\alpha : M^\alpha \to N^\alpha$$

where for all limit ordinals $\alpha < \kappa$, for $\alpha = 0$ and for all natural numbers $i \in \mathbb{N}$,

$$m_{\alpha+i} \in M^{\alpha+2i}, n_{\alpha+i} \in N^{\alpha+2i+1},$$

$|M^\alpha| \leq |\alpha|$, $M^\alpha \subseteq M_{|\alpha|}$, $|N^\alpha| \leq |\alpha|$, $N^\alpha \subseteq N_{|\alpha|}$. Then

$$f = \bigcup_{\alpha<\kappa} f^\alpha : \mathfrak{M} \cong \mathfrak{N}.$$

□

REMARK 1.10. For the existence of the enumeration see Lemma A.3.7 of Tent and Ziegler's "A course in Model Theory".

Recall the following definitions:

DEFINITION 1.11. Let $\mathfrak{M}$ be a structure.

(1) $\mathfrak{M}$ is said to be κ-homogenous, if for every subset A of M such that $|A| < \kappa$ and every $a \in M$, every elementary map $A \to M$ extends to an elementary map $A \cup \{a\} \to M$.
(2) $\mathfrak{M}$ is said to be strongly κ-homogenous if for every subset A of M such that $|A| < \kappa$, every elementary map $A \to M$ can be extended to an automorphism of $\mathfrak{M}$.

THEOREM 1.12. *Let $\mathfrak{M}$ be a special structure of cardinality κ. Then:*

(1) $\mathfrak{M}$ is κ^+-universal.
(2) $\mathfrak{M}$ is $cf(\kappa)$-homogeneous.

PROOF. (1) If κ is regular, then $\mathfrak{M}$ is κ-saturated and we have already obtained the result in Lemma 1.2. In the general case, proceed as in the proof of Theorem 1.9.

(2) Fix $A \subseteq M$, there $|A| < \mathrm{cf}(\kappa)$. Let $\{\mathfrak{M}_\lambda\}$ be a specializing chain for $\mathfrak{M}$. Then we can find λ^* such that the universe of $\mathfrak{M}_{\lambda^*}$ contains the set A. Now, the sequence $\{\mathfrak{M}^*_\lambda\}$, where $\mathfrak{M}^*_\lambda = (\mathfrak{M}_\lambda, a)_{a\in A}$ for $\lambda \geq \lambda^*$ and $\mathfrak{M}^*_\lambda = (\mathfrak{M}_{\lambda^*}, a)_{a\in A}$ for $\lambda < \lambda^*$ is a specializing sequence for $(\mathfrak{M}, a)_{a\in A}$.

Now, using the same idea, we can show that $(\mathfrak{M}, f(a))_{a\in A}$ is special. More precisely, find $\lambda^{**} < \kappa$ such that $\{f(a)\}_{a\in A}$ is contained in the universe of $M_{\lambda^{**}}$ and define $\mathfrak{M}^{**}_\lambda$ to be $(\mathfrak{M}_\lambda, f(a))_{a\in A}$ whenever $\lambda \geq \lambda^{**}$ and $(\mathfrak{M}_{\lambda^{**}}, f(a))_{a\in A}$ whenever $\lambda < \lambda^{**}$. Thus $\{\mathfrak{M}^{**}_\lambda\}$ is a specializing sequence for $(\mathfrak{M}, f(a))_{a\in A}$.

Thus $(\mathfrak{M}, a)_{a\in A}$ and $(\mathfrak{M}, f(a))_{a\in A}$ are special, elementarily equivalent structures of the same cardinality. The proof of Theorem 1.9 can be modified to produce an automorphism of $\mathfrak{M}$ extending f. Indeed, reproduce the proof of Theorem 1.9 working with the specializing chains $\{(\mathfrak{M}^*_\lambda, a)_{a\in A}\}$ and $\{(\mathfrak{M}^{**}_\lambda, f(a))_{a\in A}\}$ and starting the construction of the (intended) isomorphism with the given elementary mapping f. Note that without loss of generality $\lambda^* = \lambda^{**}$. □

QUESTION 1.13. Why can we assume $\lambda^* = \lambda^{**}$ in the above proof? Are specializing chains unique?

2. Good large structures

THEOREM 2.1. *Let κ be a regular cardinal. If T is κ-stable, then there is a saturated $\mathfrak{M} \models T$ with $|\mathfrak{M}| = \kappa$. Indeed, if $\mathfrak{M}_0 \models T$ with $|M_0| = \kappa$, then there is a saturated elementary extension $\mathfrak{M}$ of $\mathfrak{M}_0$ with $|\mathfrak{M}| = \kappa$.*

PROOF. Find an elementary chain $\{\mathfrak{M}_\alpha\}_{\alpha\in\kappa}$, where $|\mathfrak{M}_\alpha| = \kappa$ for each α, such that $\mathfrak{M}_{\alpha+1}$ realises every type in $S_1^{\mathfrak{M}_\alpha}(M_\alpha)$. Then $\mathfrak{M} = \bigcup_{\alpha<\kappa} \mathfrak{M}_\alpha$ is saturated of cardinality κ. □

In particular, if T is ω-stable, then there are saturated models of size κ for all regular cardinals κ. In general, to claim the existence of arbitrarily large saturated models of a theory T (not necessarily ω-stable) it is sufficient (by Corollary 1.6) to assume that for every cardinal λ there is an inaccessible cardinal $\kappa > \lambda$. Alternatively, we work in Bernays-Gödel with Global Choice (abbreviated GBC) to claim the existence of an appropriate (in the sense that it has all desirable properties) and sufficiently large model of T (in fact this model is a proper class; see Corollary 2.4 and Discussion 2.5).

THEOREM 2.2. *(GBC) There is a class-size model $\mathbb{M}$ of T such that all types over all subsets of the universe of $\mathbb{M}$ are realised in $\mathbb{M}$. The model $\mathbb{M}$ is unique up to isomorphism.*

PROOF. Construct a continuous elementary chain $\{\mathfrak{M}\}_{\alpha\in\mathbb{ON}}$ (here $\mathbb{ON}$ denotes the class of all ordinals) of models of T with the property that every type over $\mathfrak{M}_\alpha$ is realized in $\mathfrak{M}_{\alpha+1}$. Then

$$\mathbb{M} = \bigcup_{\alpha\in\mathbb{ON}} \mathfrak{M}_\alpha.$$

□

DEFINITION 2.3. (GBC) The model $\mathbb{M}$ from Theorem 2.2 is called *the monster model* of T.

COROLLARY 2.4. (GBC) Let $\mathbb{M}$ be the monster model of T.

(1) The universe of $\mathbb{M}$ can be well-ordered.
(2) The model $\mathbb{M}$ is κ-saturated for all cardinals κ.
(3) If $\mathfrak{M} \models T$, then $\mathfrak{M}$ can be embedded in $\mathbb{M}$.
(4) Let A and B be subsets of $\mathbb{M}$ and let $f : A \to B$ be a bijection. Then f extends to an automorphism of $\mathbb{M}$.

DISCUSSION 2.5. We can think of the monster model $\mathbb{M}$ as a sufficiently large (set) model of T which has all desirable properties. In particular:

(1) $\mathbb{M}$ is saturated.

(2) Any model of T that we are interested in or we are ever to consider is of cardinality strictly smaller than the cardinality of $\mathbb{M}$ and moreover any such model is an elementary submodel of $\mathbb{M}$.

(3) Every set of parameters which we are ever to consider is contained in the universe of $\mathbb{M}$ and so for any model $\mathfrak{M}$ of T,

$$\mathfrak{M} \models \varphi(\bar{a}) \text{ if and only if } \mathbb{M} \models \varphi(\bar{a}).$$

Thus, it is justified to say that $\varphi(\bar{a})$ holds if and only if $\mathbb{M} \models \varphi(\bar{a})$.

(4) A set of formulas is consistent with T if and only if it is realised in $\mathbb{M}$.

(5) We can speak about $\mathrm{tp}(\bar{a}/A)$, by which it is meant $\mathrm{tp}^{\mathbb{M}}(A)$. Similarly we speak about $S_n(A)$, by which we mean $S_n^{\mathbb{M}}(A)$.

(6) If A is a subset of the universe of $\mathbb{M}$ of cardinality strictly smaller than the cardinality of $\mathbb{M}$ and $f : A \to \mathbb{M}$ is a partial elementary mapping, then f extends to an automorphism of $\mathbb{M}$. Note that without loss of generality $\lambda^* = \lambda^{**}$.

3. Morley Rank Once Again

DEFINITION 3.1. Let $\mathbb{M}$ be the Monster model and φ a formula with parameters in $\mathbb{M}$.

(1) $\mathrm{MR}(\varphi) \geq 0$ if φ is consistent,

(2) $\mathrm{MR}(\varphi) \geq \alpha + 1$ if there is an infinite sequence $\{\psi_i\}_{i\in\omega}$ of pairwise inconsistent formulas, each of which implying φ and such that $\mathrm{MR}(\psi_i) \geq \alpha$ for all i.

(3) If α is a limit ordinal, $\mathrm{MR}(\varphi) \geq \alpha$ if $\mathrm{MR}(\varphi) \geq \beta$ for each $\beta < \alpha$.

DISCUSSION 3.2. If we think of $\mathbb{M}$ as a (very large) set, the above is equivalent to:

(1) $\mathrm{MR}(\varphi) \geq 0$ if $\varphi(\mathbb{M}) \neq \emptyset$,

(2) $\mathrm{MR}(\varphi) \geq \alpha + 1$ if there is a sequence $\{\psi_i\}_{i\in\omega}$ of $\mathscr{L}(\mathbb{M})$-formulas, such that $\{\psi_i(\mathbb{M})\}_{i\in\omega}$ are pairwise disjoint subsets of $\varphi(\mathbb{M})$ and $\mathrm{MR}(\psi_i) \geq \alpha$ for each i.

(3) If α is a limit ordinal, $\mathrm{MR}(\varphi) \geq \alpha$ if $\mathrm{MR}(\varphi) \geq \beta$ for each $\beta < \alpha$.

With this we can define the Morley rank of a formula $\varphi(x)$.

DEFINITION 3.3.

(1) If there is no α with $\mathrm{MR}(\varphi) \geq \alpha$, we say that $\mathrm{MR}(\varphi) = -\infty$.

(2) If $\mathrm{MR}(\varphi) \geq \alpha$ for all α, then we say $\mathrm{MR}(\varphi) = \infty$.

(3) Otherwise $\{\alpha : \mathrm{MR}(\varphi) \geq \alpha\}$ has a largest element and we define

$$\mathrm{MR}(\varphi) = \max\{\alpha : \mathrm{MR}(\varphi) \geq \alpha\}.$$

4. Morley Degree

If X is definable and $\mathrm{MR}(X) = \alpha$, then X can not be partitioned into infinitely many pairwise disjoint definable subsets of Morley rank α. In fact, there is a finite upper bound on the number of pairwise disjoint definable subsets of X into which X can be partitioned.

THEOREM 4.1. *Let φ be a $\mathscr{L}(\mathbb{M})$-formula, $MR(\varphi) = \alpha$, $\alpha \in \mathbb{ON}$. Then there is a natural number $d \in \mathbb{N}$ such that:*

if $\{\psi_i\}_{i=1}^n$ are $\mathscr{L}(\mathbb{M})$-formulas, $\{\psi_i(\mathbb{M})\}_{i=1}^n$ are pairwise disjoint non-empty subsets of $\varphi(\mathbb{M})$ and $MR(\psi_i) = \alpha$ for all i, then $n \leq d$.

The least such number d is called the Morley degree *of φ and is denoted $deg_{\mathbb{M}}(\varphi)$.*

PROOF. Consider the full binary three $2^{<\omega}$. We will construct a subtree S of $2^{<\omega}$ (i.e. a subset $S \subseteq 2^{<\omega}$ with the property that for all $\sigma \in S$ if $\tau \subseteq \sigma$ then $\tau \in S$) and for each $\sigma \in S$, choose a formula φ_σ as follows:

Take $\varphi_\emptyset = \varphi$. Now suppose $\sigma \in S$ and φ_σ is defined. Exactly one of the following two options holds: Either there is a $\mathscr{L}(\mathbb{M})$-formula ψ such that

$$\mathrm{MR}(\varphi_\sigma \wedge \psi) = \mathrm{MR}(\varphi_\sigma \wedge \neg\psi) = \alpha,$$

or for every $\mathscr{L}(\mathbb{M})$-formula ψ we have

$$\mathrm{MR}(\varphi_\sigma \wedge \psi) < \alpha, \text{ or } \mathrm{MR}(\varphi_\sigma \wedge \neg\psi) < \alpha.$$

In the former case, define

$$\varphi_{\sigma^\frown 0} = \varphi_\sigma \wedge \psi, \varphi_{\sigma^\frown 1} = \varphi_\sigma \wedge \neg\psi.$$

In the latter case, declare $\sigma \in S$ to be a terminal node of S (that is there is no $\tau \in S$ such that $\sigma \subset \tau$).

Now, suppose S is infinite. Since S is finitely branching by König's Lemma there is a branch through the tree, i.e. there is a function $f : \omega \to 2$ such that $f \restriction n \in S$ for all $n \in \omega$. Let

$$\psi_n := \varphi_{f\restriction n} \wedge \neg\varphi_{f\restriction n+1}.$$

Then $\mathrm{MR}(\psi_n) = \alpha$ and $\psi_n(\mathbb{M}) \cap \psi_{n+1}(\mathbb{M}) = \emptyset$, since

$$\psi_n(\mathbb{M}) \subseteq \neg\varphi_{n+1}(\mathbb{M}) \text{ and } \psi_{n+1}(\mathbb{M}) \subseteq \varphi_{f\restriction n+1}(\mathbb{M}).$$

Thus $\{\psi_n(\mathbb{M})\}_{n\in\omega}$ is an infinite family of pairwise disjoint subsets of $\varphi(\mathbb{M})$ each of rank α. Therefore $\mathrm{MR}(\varphi) \geq \alpha + 1$, which is a contradiction.

Therefore the tree is finite and we can consider the set of terminal nodes of T, i.e. the set

$$\mathrm{TN} = \{\sigma \in S : \forall \tau \in 2^{<\omega} \text{ if } \sigma \subsetneq \tau \text{ then } \tau \notin S\}.$$

We know that $|\mathrm{TN}| < \omega$. Take $d = |\mathrm{TN}|$ and let $\{\psi_i\}_{i=1}^d = \{\varphi_\sigma : \sigma \in \mathrm{TN}\}$. Then

$$\bigcup\{\psi_i(\mathbb{M})\}_{i=1}^d = \varphi(\mathbb{M}).$$

Note that every level of the tree induces in a natural way a partition of $\varphi(\mathbb{M})$ and

$$\mathrm{MR}(\psi_i) \geq \alpha$$

for all i. Furthermore, since these are terminal nodes in the tree for all $i \in \{1, \cdots, d\}$ and every χ

$$\text{either } \mathrm{MR}(\psi_i \wedge \chi) < \alpha \text{ or } \mathrm{MR}(\psi_i \wedge \neg\chi) < \alpha.$$

Now, consider a family of pairwise disjoint definable subsets $\{\theta_j(\mathbb{M})\}_{j=1}^n$ of $\varphi(\mathbb{M})$ where $\mathrm{MR}(\theta_j) = \alpha$ for each j.

CLAIM 4.2. For all $i \in \{1, \cdots, d\}$ there is at most one $j \in \{1, \cdots, n\}$ such that

$$\mathrm{MR}(\psi_i \wedge \theta_j) = \alpha.$$

PROOF. Suppose not. Thus there is $i \in \{1, \cdots, d\}$ and there are $j_1 \neq j_2$ in $\{1, \cdots, n\}$ such that

$$\mathrm{MR}(\psi_i \wedge \theta_{j_1}) = \alpha \text{ and } \mathrm{MR}(\psi_i \wedge \theta_{j_2}) = \alpha.$$

Since $\theta_{j_1}(\mathbb{M}) \cap \theta_{j_2}(\mathbb{M}) = \emptyset$ and $(\psi_i \wedge \theta_{j_1})(\mathbb{M})$, $(\psi_i \wedge \theta_{j_2})(\mathbb{M})$ are contained in $\psi_i(\mathbb{M})$ we get:

$$(\psi_i \wedge \theta_{j_1})(\mathbb{M}) \cup (\psi_i \wedge \neg\theta_{j_1})(\mathbb{M}) = \psi_i(\mathbb{M})$$

and so

$$(\psi_i \wedge \theta_{j_2})(\mathbb{M}) \subseteq (\psi_i \wedge \neg\theta_{j_1})(\mathbb{M}) \subseteq \psi_i(\mathbb{M}).$$

But then, by monotonicity of the Morley rank, we obtain:

$$\mathrm{MR}(\psi_i \wedge \neg\theta_{j_1}) = \alpha$$

and so $\mathrm{MR}(\psi_i \wedge \theta_{j_1}) = \mathrm{MR}(\psi_i \wedge \neg\theta_{j_1}) = \alpha$ which is a contradiction to the choice of the terminal nodes of S. $\square$

Suppose $n > d$ and for each $i \in \{1, \cdots, d\}$ choose $j_i \in \{1, \cdots, n\}$ such that

$$\mathrm{MR}(\psi_i \wedge \theta_{j_i}) = \alpha$$

(if there is such). Then clearly

$$|\{j_i\}_{i=1}^d| \leq d$$

and so there is $j^* \in n \backslash \{j_i\}_{i=1}^d$. But then θ_{j_*} has the property that $\mathrm{MR}(\theta_{j^*} \wedge \psi_i) < \alpha$ for each $i \in \{1, \cdots, n\}$. However $\theta_{j^*}(\mathbb{M}) \subseteq \varphi(\mathbb{M}) = \bigcup_{i=1}^n \psi_i(\mathbb{M})$ and so

$$\theta_{j^*}(\mathbb{M}) = \bigcup_{i=1}^n (\theta_{j^*} \wedge \psi_i)(\mathbb{M}).$$

However $\mathrm{MR}(X \cup Y) = \max\{\mathrm{MR}(X), \mathrm{MR}(Y)\}$ and so

$$\mathrm{MR}(\theta_{j^*}) < \alpha,$$

which is a contradiction to the choice of θ_{j^*}. $\square$

COROLLARY 4.3. A formula φ is strongly minimal if and only if

$$\mathrm{MR}(\varphi) = \deg_{\mathbb{M}}(\varphi) = 1.$$

PROOF. ($\Rightarrow$) Suppose φ is strongly minimal. Then if $(\varphi \wedge \psi)(\mathbb{M})$ is infinite (and clearly definable), we must have $(\varphi \wedge \neg\psi)(\mathbb{M})$ is finite and so $\deg_{\mathbb{M}}(\varphi) = 1$.

By definition of strong minimality $\varphi(\mathbb{M})$ is infinite. Thus $\mathrm{MR}(\varphi) \geq 1$. We claim that $\mathrm{MR}(\varphi) \not\geq 2$. Otherwise, there is a family $\{\psi_i\}_{i\in\omega}$ such that $\{\psi_i(\mathbb{M})\}_{i\in\omega} \subseteq \varphi(\mathbb{M})$ and $\{\psi_i(\mathbb{M})\}_{i\in\omega}$ are infinite, pairwise disjoint. But then

$$(\bigcup_{i\geq 2} \psi_i)(\mathbb{M}) \subseteq \neg\psi_1(\mathbb{M})$$

and so in particular, both $\psi_1(\mathbb{M}) = (\psi_1 \wedge \varphi)(\mathbb{M})$ and $(\neg\psi_1 \wedge \varphi)(\mathbb{M})$ are infinite, which is a contradiction to strong minimality of φ. Thus $\mathrm{MR}(\varphi) = 1$.

($\Leftarrow$) Now, suppose $\mathrm{MR}(\varphi) = \deg_{\mathbb{M}}(\varphi) = 1$. Since $\mathrm{MR}(\varphi) = 1$, we know that $\varphi(\mathbb{M})$ is infinite. Also $\deg_{\mathbb{M}}(\varphi) = 1$ and so if $\{\psi_i(\mathbb{M})\}_{i=1}^n$ are infinite pairwise disjoint subset of $\varphi(\mathbb{M})$ then $n \leq 1$. That is, every infinite definable subset of $\varphi(\mathbb{M})$ is co-finite in $\varphi(\mathbb{M})$. Thus, φ is strongly minimal. $\square$

CHAPTER 20

The Paris-Harrington Principle

1. The Finite Ramsey Theorem

The last topic in the course is the Paris-Harrington Principle. We will make us of the finite Ramsey theorem.

Notation: For cardinals κ, η, μ and λ we write $\kappa \to (\eta)^{\mu}_{\lambda}$ if whenever $|X| \geq \kappa$ and $f : [X]^{\mu} \to \lambda$, then there is is $Y \subseteq X$ such that $|Y| \geq \eta$ and Y is homogenous for f.

THEOREM 1.1. *(Finite Ramsey Theorem) For all $k,n,m < \omega$ there is $l < \omega$ such that $l \to (m)^{n}_{k}$-*

A restatement of the theorem:

THEOREM. Given $k,n,m < \omega$ there is $l < \omega$ such that whenever $X \subseteq \omega$, $|X| \geq l$ and $f : [X]^n \to k$ then there is $Y \subseteq X$ such that $|Y| \geq m$ and Y is homogenous for f (that is $f \restriction [Y]^n$ is a constant).

PROOF. Suppose there is no such l. Then for each $l \in \omega$ the set T_l consisting of all

$$f : [\{0, \cdots, l-1\}]^n \to k$$

such that there is no $X \subseteq \{0, \cdots, l-1\}, |X| \geq m$ which is homogenous for f is non-empty. Each T_l is finite and if $f \in T_{l+1}$ then there is a unique $g \in T_l$ such that $g \subset f$. Thus, we can order $T = \bigcup_{l \in \omega} T_l$ by inclusion and consider it as a finitely branching tree. Then, by König's Lemma there is an infinite branch through T, i.e. there is a sequence $\{f_n\}_{n \in \omega} \subseteq T$ such that for each n, $f_n \subset f_{n+1}$. Then $f : [\mathbb{N}]^n \to k$ and by Ramsey's Theorem there is $X \in [\mathbb{N}]^n$ such that X is homogenous for f. Let $x_1, \cdots, x_m$ be the first m elements of X and let $s > x_m$. Then $X_0 = \{x_1, \cdots, x_m\} \subseteq \{0, \cdots, s\}$ and $f \restriction [X_0]^n = f_s \restriction [X_0]^n$ is a constant. Thus, X_0 is a homogenous set for f_s, which is a contradiction. $\square$

DISCUSSION 1.2. Note that the finite Ramsey theorem can be proven directly, without use of the infinite Ramsey theorem. In fact, the finite Ramsey Theorem is a theorem of Peano Arithmetic (PA). What we are going to see, is a slight modification of the theorem, which is true in the standard model of PA, but can not be derived from PA. That is, we will construct a non-standard model of PA in which the theorem will fail. To construct the model, we will use the notion of indiscernibles.

THEOREM 1.3. *(Paris-Harrington Principle) Fo each n,k,m in $\mathbb{N}$ there is $l \in \mathbb{N}$ such that if $f : [l]^n \to k$ then there is $Y \subseteq l$ such that:*

(1) $f \restriction [Y]^n$ is constant,
(2) $|Y| \geq m$,
(3) $|Y| \geq \min Y$.

Note that in comparison with the Finite Ramsey Theorem only the last requirement is new (addition). The proof is next time!

2. The Paris-Harrington Principle

THEOREM 2.1. *(Paris-Harrington Principle) For each n,k,m in $\mathbb{N}$ there is $l \in \mathbb{N}$ such that if $f : [l]^n \to k$ then there is $Y \subseteq l$ such that:*

(1) $f \restriction [Y]^n$ is constant,
(2) $|Y| \geq m$,
(3) $|Y| \geq \min Y$.

PROOF. Suppose there is no such l. Then for each $l < \omega$, let T_l be the set of all functions

$$f : [\{0, \cdots, l-1\}]^n \to k$$

for which there is no $Y \subseteq l$ with the property that $f \restriction [Y]^n$ is a constnat and $|Y| \geq \max\{m, \min Y\}$. Again each T_l is finite and if $f \in T_{l+1}$ then there is a unique $g \in T_l$ such that $g \subset f$. Thus $T = \bigcup T_l$ ordered by inclusion is a finite branching tree. Since $\mathrm{ht}(T) = \omega$, there must be an infinite branch through T, i.e. there is a sequence $\{t_i\}_{i\in\omega}$ such that $t_i \subsetneq t_{i+1}$ and $t_i \in T_i$ for each i. Let $t = \bigcup_{i\in\omega} t_i$. Thus, $t : [\mathbb{N}]^n \to k$ and by the Theorem of Ramsey there is $X \in [\mathbb{N}]^\omega$ such that $t \restriction [X]^n$ is a constant. Let $x_1 = \min X$. Pick $l \geq x_1, m$ and let $x_1, \cdots, x_l$ be the first l elements of X: Let $s > x_l$. Then $Y = \{x_1, \cdots, x_l\}$ is homogenous for f_s and $|Y| \geq \max\{m, \min Y\}$, which is a contradiction to the choice of f_s. □

DEFINITION 2.2. Let $X \subseteq \omega$.

(1) We say that $f : [X]^n \to \omega$ is regressive if $f(A) < \min A$ for all $A \in [X]^n$.
(2) We say that $Y \subseteq X$ is min-homogenous for f, if for all $A, B \in [Y]^n$ with

$$\min A = \min B$$

we have $f(A) = f(B)$.

Now, consider the following combinatorial principle, which we denote (CP($\star$)):

(CP($\star$)) For all natural numbers c, m, n, k there is a natural number d such that if $f_1, \cdots, f_k$ are regressive functions from $[d]^n \to d$, then there is $Y \subseteq [c, d]$ such that $|Y| \geq m$ and Y is min-homogenous for each f_i.

REMARK 2.3. Note that:

(1) The Paris-Harrington Principle (abbreviated PHP) implies (CP($\star$)) and has a proof using finite combinatorics which can be formulated in PA.
(2) We will construct a model of PA in which CP($\star$) fails. Thus PHP is not provable in PA.

DEFINITION 2.4. Let $\mathscr{L}_\mathbb{N}$ be the language of arithmetic. Let Γ be a finite set of formulas in $\mathscr{L}_\mathbb{N}$ and let $\mathfrak{M} \vDash \mathrm{PA}$. A set $I \subseteq M$ is said to be a sequence of *diagonal indiscernibles for* Γ if whenever $\varphi(x_1, \cdots, x_m, y_1, \cdots, y_n) \in \Gamma$, $\{u_0, u_1, \cdots, u_n, v_1, \cdots, v_n\} \subseteq I$ are such that

$$u_0 < u_1 < \cdots < u_n, v_0 < v_1 < \cdots < v_n$$

and $a_1, \cdots, a_m$ are all strictly smaller than u_0, then for $\bar{a} = (a_1, \cdots, a_m)$ we have:

$$\mathfrak{M} \vDash \varphi(\bar{a}, u_1, \cdots, u_n) \leftrightarrow \varphi(\bar{a}, v_1, \cdots, v_n).$$

REMARK 2.5. CP($\star$) will allow us to find a set of diagonal indiscernibles in the standard model $\mathbb{N}$ of the set of Δ_0-formulas.

LEMMA 2.6. For any l,m,n such that $l > m > 2n$ and formulas

$$\{\varphi_i(x_1,\cdots,x_k,y_1,\cdots,y_n)\}_{i=1}^l$$

in $\mathscr{L}_{\mathbb{N}}$ there is a set I of diagonal indiscernibles (in the standard model) for $\{\varphi_i\}_{i=1}^l$ with

$$|I| \geq m.$$

PROOF. By the finite Ramsey theorem, there is w such that $w \to (m+n)_{l+1}^{2n+1}$. By CP($\star$) there is $s \in \mathbb{N}$ such that whenever $f_1,\cdots,f_k : [s]^n \to s$ are regressive, there is $Y \subseteq s$ with $|Y| \geq w$ such that Y is min-homogenous for each f_j. Without loss of generality $s > l$, $s > w$.

Define regressive functions $f_j : [s]^{2n+1} \to l$ for $j = 1,\cdots,k$ and a partition

$$g : [s]^{2n+1} \to l+1$$

as follows. Let $X = \{u_0,\cdots,u_{2n}\}$ where $u_0 < u_1 < \cdots < u_{2n} < s$.

<u>If for all</u> $i \leq l$ <u>and all</u> $a_1,\cdots,a_k < \min\{l,u_0\}$ we have

$$\varphi_i(\bar{a},u_1,\cdots,u_n) \leftrightarrow \varphi_i(\bar{a},u_{n+1},\cdots,u_{2n})$$

then define $f_j(X) = 0$ for each $j = 1,\cdots,k$ and $g(X) = 0$.

<u>Otherwise</u> there are $i \leq l$ and $\bar{a} < \min\{l,u_0\}$ such that

$$\varphi_i(\bar{a},u_1,\cdots u_n) \not\leftrightarrow \varphi_i(\bar{a},u_{n+1},\cdots,u_{2n}).$$

Then define $(f_1(X),\cdots,f_k(X)) = \bar{a}$ and $g(X) = i$.

With this the functions $\{f_j\}_{j=1}^k$ are defined.

Since $\{f_j\}_{j=1}^k$ are regressive, there is $Y \subseteq s$ which is min-homogenous for all of them with $|Y| \geq w$. By choice of w, there are $X \subseteq Y$ and $i \leq l$ such that

$$|X| \geq m+n \text{ and } g(A) = i \text{ for all } A \in [X]^{2n+1}.$$

Now, suppose $i > 0$. Since $m > 2n$, $|X| \geq m+n$ we get $|X| > 3n$. Thus, there are $\{u_i\}_{i=0}^{3n} \subseteq X$ enumerated in strictly increasing order. Since X is min-homogenous for each f_j, we can find $a_j < u_0$ such that

$$a_j = f_j(u_0,\cdots,u_{2n}) = f_j(u_0,u_1,\cdots,u_n,u_{2n+1},\cdots,u_{3n}) = f_j(u_0,u_{n+1},\cdots,u_{3n}).$$

Note that, the latter two equalities in the above formula hold by definition of min-homogeneity.

Let $\bar{a} = (a_1,\cdots,a_k)$. But then:

- Since $g(\{u_l\}_{l=0}^{2n}) = i > 0$ we have:

$$\varphi_i(\bar{a},u_1,\cdots,u_n) \not\leftrightarrow \varphi_i(\bar{a},u_{n+1},\cdots,u_{2n}),$$

- Since $g(\{u_l\}_{l=0}^{n} \cup \{u_l\}_{l=2n+1}^{3n}) = i > 0$, we have

$$\varphi_i(\bar{a},u_1,\cdots,u_n) \not\leftrightarrow \varphi_i(\bar{a},u_{2n+1},\cdots,u_{3n}),$$

- and since $g(\{u_l\}_{l=0}^{n} \cup \{u_l\}_{l=2n+1}^{3n}) = i > 0$ we have

$$\varphi_i(\bar{a},u_{n+1},\cdots,u_{2n}) \not\leftrightarrow \varphi_i(\bar{a},u_{2n+1},\cdots,u_{3n}).$$

However, this is impossible, because at least two of the formulas must have the same value. Thus $i = 0$. Let $w_1 < \cdots < w_n$ be the largest n many elements of X. Let $I = X \setminus \{w_i\}_{i=1}^n$. Then $|I| \geq m$.

CLAIM 2.7. *I is the desired set of diagonal indiscernibles.*

PROOF. Suppose $u_0 < u_1 < \cdots < u_n$ and $v_1 < \cdots < v_n$ are sequences from I with $u_0 < v_1$ and $\bar{a} < u_0$. Then for any $i \leq k$

$$\varphi_i(\bar{a}, u_1, \cdots, u_n) \leftrightarrow \varphi_i(\bar{a}, w_1, \cdots, w_n)$$

and

$$\varphi_i(\bar{a}, u_1, \cdots, u_n) \leftrightarrow \varphi_i(\bar{a}, v_1, \cdots, v_n)$$

and so I is a set of diagonal indiscernibles. □

□

3. PA does not prove PHP

REMARK 3.1. (Δ_0-formulas) Recall that the class of Δ_0-formulas is the smallest set of formulas in $\mathscr{L}_{\mathbb{N}}$ which contains the quantifier free formulas and is closed under $\wedge, \vee, \neg$ and bounded quantification.

We will make use of the following lemma:

LEMMA 3.2. Suppose $\mathfrak{M} \models \mathrm{PA}$ and $\{u_i\}_{i \in \omega, <}$ is a sequence of diagonal indiscernibles for all Δ_0-formulas. Let

$$N = \{v \in M : \exists i \in \omega (v < u_i)\}.$$

Then N is closed under $+, \cdot, S$ and if $\mathfrak{N}$ is the substructure of $\mathfrak{M}$ with underlying set N, then

$$\mathfrak{N} \models \mathrm{PA}.$$

THEOREM 3.3. *CP($\star$) and PHP are not provable in PA.*

PROOF. Let $\mathfrak{M}$ be a non-standard model of PA, $c \in M$ non-standard. Suppose $\mathfrak{M} \models \mathrm{CP}(\star)$.

The finite Ramsey Theorem is provable in PA and so there is $w \in M$ such that

$$\mathfrak{M} \models w \to (3c+1)^{2c+1}_c.$$

Let $d \in M$ be least given by CP($\star$) such that whenever

$$f_1, \cdots, f_c : [d]^{2c+1} \to d$$

are regressive functions, then there is $Y \subseteq (c,d)$ such that $|Y| \geq w$ and Y is min-homogenous for each f_i.

Let TR_{Δ_0} be a unary predicate such that

$$\mathfrak{M} \models \mathrm{TR}_{\Delta_0}(\ulcorner \varphi \urcorner)$$

if and only if

$$\mathfrak{M} \models \varphi \text{ and } \varphi \text{ is a } \Delta_0\text{-formula,}$$

where $\ulcorner \varphi \urcorner$ is the Gödel code of φ. Then, we can find $I \subseteq (c,d)$ such that $|I| \geq c$ and

$$\mathfrak{M} \models I \text{ is a set of diagonal indiscernibles for all } \Delta_0\text{-formulas.}$$

Let $u_0 < u_1 < \cdots$ be an initial segment of I, let

$$N = \{v \in M : \exists i (v < u_i)\}$$

and let $\mathfrak{N}$ be a substructure of $\mathfrak{M}$ with universe N. Then $\mathfrak{N} \models \mathrm{PA}$. Since $I \subseteq (c,d)$ we get $c \in I$ and $d \notin I$.

CLAIM 3.4. $w \in N$.

PROOF. The finite version of Ramsey Theorem is provable in PA and so there is $w' \in N$ such that $\mathfrak{N} \models w' \to (3c+1)^{2c+1}_c$. Since all functions from $[w']^{2c+1} \to C$ and all subsets of w' that are coded in $\mathfrak{M}$ are also coded in $\mathfrak{N}$, $\mathfrak{M} \models w' \to (3c+1)^{2c+1}_c$. However w was chosen to be minimal in M and so $w \leq w'$. Thus $w \in N$. □

Similarly, one can show that if $d' \in N$ and in $\mathfrak{N}$ for all regressive $f_1, \cdots, f_c : [d']^{2c+1} \to d'$ there is $Y \subseteq (c, d')$ which is min-homogenous for each f_i and $|Y| \geq w$, then the same holds in $\mathfrak{M}$. Thus, by choice of d, $d \leq d'$. But $d \notin N$ and so we have a contradiction. Therefore CP($\star$) fails in $\mathfrak{N}$ and so CP($\star$) is not provable from PA. $\square$