MASTERARBEIT / MASTER'S THESIS

Titel der Masterarbeit / Title of the Master's Thesis

Towers and Pseudointersections

verfasst von / submitted by

Jonathan Schilhan, BSc

angestrebter akademischer Grad / in partial fulfilment of the requirements for the degree of

Master of Science (MSc)

Wien, 2018 / Vienna 2018

Studienkennzahl lt. Studienblatt / degree programme code as it appears on the student record sheet:	A 066 821
Studienrichtung lt. Studienblatt / degree programme as it appears on the student record sheet:	Masterstudium Mathematik
Betreut von / Supervisor:	Privatdoz. Dr. Vera Fischer

Abstract

The topic of this thesis is towers, families with the strong finite intersection property and pseudointersections, which are fundamental objects in combinatorial set theory. Particularly it deals with the two cardinal characteristics $\mathfrak{p}$ and $\mathfrak{t}$ which are related to these notions. It provides an overview on some of the classical results concerning these cardinals and gives an exposition of the recent proof of Malliaris and Shelah ([17]) stating that $\mathfrak{p} = \mathfrak{t}$. Furthermore some aspects of the generalized notion of pseudointersection for uncountable regular cardinals κ and the induced characteristics, among which we find $\mathfrak{p}(\kappa)$ and $\mathfrak{t}(\kappa)$, are studied.

Zusammenfassung

Die vorliegende Arbeit beschäftigt sich mit Türmen, Filterbasen und Pseudo-Durchschnitten, welche fundamentale Untersuchungsobjekte der kombinatorischen Mengenlehre sind. Insbesondere werden die damit verbundenen Kardinalzahl-Charakteristiken $\mathfrak{p}$ und $\mathfrak{t}$ behandelt. Die Arbeit bietet einen Überblick über klassische Resultate bezüglich dieser Begrifflichkeiten und gibt den Beweis von Malliaris und Shelah ([17]), der besagt, dass $\mathfrak{p} = \mathfrak{t}$, wieder. Des weiteren werden einige Aspekte der Verallgemeinerung von Türmen und Pseudo-Durchschnitten auf überabzählbare reguläre Kardinalzahlen κ und deren Charakteristiken $\mathfrak{p}(\kappa)$ und $\mathfrak{t}(\kappa)$ studiert.

Aknowledgements

I would like to express my gratitude towards Vera Fisher for the constant support she has given me ever since I met her. She is the one who introduced me to set theory in the first place. Moreover I need to thank the KGRC and all its members for the wonderful environment I was provided for writing my thesis.

Some parts of this thesis were presented at the annual Winter School in Hejnice, 2018, for which I got funding. Special thanks go to the organizers of this amazing gathering.

Further I would like to thank both of my parents for having given me the opportunity of completing my studies in the best possible conditions – a valuable privilege.

Last but not least, I am deeply grateful to Kathi, who has accompanied me during all of my studies, for her love, patience and understanding. And to Noah and Klara, for all the joy they give me.

Contents

Chapter 1

Introduction

An ongoing phenomenon that is observed in the study of combinatorial set theory is that given a structure consisting of objects with "finite" and "countably infinite information", taking the quotient by the finite objects yields an extremely rich structure. Examples of such quotients include the Calkin algebra, relevant in the theory of C^* algebras or the partial order $(\omega^\omega, <^*)$, that describes growth rates of functions. The example that we will be interested in mostly is the Boolean algebra $\mathcal{P}(\omega)/\operatorname{fin}$. More specifically, we will be preoccupied with the relation of almost inclusion ($\subseteq^*$, Definition 1.2.5) between subsets of the naturals, that give rise to the notion of towers and pseudointersections.

Cardinal characteristics are typically defined as the least size of some sort of "maximal" family in a quotient such as $\mathcal{P}(\omega)/\operatorname{fin}$. For example, the pseudointersection number $\mathfrak{p}$ is the least size of a (downwards) directed subset of $[\omega]^\omega/\operatorname{fin}$ with no lower bound (a so called pseudointersection). The tower number $\mathfrak{t}$ can be characterized as the least size of a maximal decreasing chain in $[\omega]^\omega/\operatorname{fin}$. The main objective in the study of cardinal characteristics is to understand the relationship between different notions of "maximality" in different quotient structures. The respective cardinal serves as a characteristic for the specific notion of maximality. Typically, given two cardinal characteristics[1] $\mathfrak{x}, \mathfrak{y}$ there are three possibilities:

(1) $\mathfrak{x} = \mathfrak{y}$

[1]Note that when we say "cardinal characteristic" we usually don't refer to its particular value (i.e. a cardinal in the proper sense) but its definition.

(2) one of them is greater or equal to the other but we know a forcing extension in which they are separated

(3) we have a forcing extension in which $\mathfrak{x} < \mathfrak{y}$ and one in which $\mathfrak{y} < \mathfrak{x}$

In the last case $\mathfrak{x}, \mathfrak{y}$ are often, informally, called independent. Independence usually implies that the two notions of maximality are more or less unrelated, although there might be a relation like $\mathrm{cf}(\mathfrak{x}) = \mathrm{cf}(\mathfrak{y})$. Thus we often look for a stronger notion of independence where given any uncountable regular cardinals κ, λ, there is a forcing extension in which $\mathfrak{x} = \kappa$ and $\mathfrak{y} = \lambda$.

In the case of $\mathfrak{p}$ and $\mathfrak{t}$ we have that $\mathfrak{p} \leq \mathfrak{t}$ but it was a long standing open problem of whether $\mathfrak{p} = \mathfrak{t}$. The general feeling, conveyed by difficult results such as the consistency of $\mathfrak{d} < \mathfrak{a}$ ([25]), was that it is possible to separate $\mathfrak{p}$ and $\mathfrak{t}$, i.e. get a model where $\mathfrak{p} < \mathfrak{t}$. On these grounds, the recent proof of Malliaris and Shelah ([17]), stating that $\mathfrak{p} = \mathfrak{t}$, came very surprising. A big part of the thesis is devoted to retracing their argument.

We outline the rough structure of the thesis:

Part of the introduction is a preliminaries section that contains all important definitions and facts that will be used throughout the thesis. In the next section we give the formal definition of $\mathfrak{p}$, $\mathfrak{t}$, $\mathfrak{b}$ and $\mathfrak{d}$ and mention (with or without proof) how they relate.

In Chapter 2, we review in detail (with proof) some of the well known aspects surrounding towers and pseudointersections that preceded the proof of $\mathfrak{p} = \mathfrak{t}$. For instance we will show how pseudointersections relate to a forcing axiom, that will serve as a useful black box in some of the later combinatorial arguments.

In the next chapter, we provide the proof of $\mathfrak{p} = \mathfrak{t}$. We give an outline of the argument at the beginning of this chapter.

The last chapter deals with the generalization of pseudointersections to larger cardinals. We will show how some of the results of Chapter 2 lift to this case. Apart from Section 4.4, all major results in this chapter are new. For instance, we give a characterization of the generalized bounding number in terms of pseudointersections. This will shed light on some of the apparent distinctions that exist between the combinatorics on ω and on uncountable κ and lead to results such as $\mathfrak{s}(\kappa) \leq \mathfrak{b}(\kappa)$. Moreover we will provide a proof of $\mathfrak{t}(\kappa) \leq \mathrm{add}(\mathcal{M}_\kappa)$, where $\mathrm{add}(\mathcal{M}_\kappa)$ is the additivity of the generalized meager ideal.

1.1 Preliminaries

Set Theory

Our set theory is ZFC (Zermelo-Fraenkel with the Axiom of Choice). This is a first order theory in the language $\{\in\}$ containing one binary relation symbol, intended to have its usual meaning. This theory does a great job in capturing our use of sets in mathematics and is widely accepted as a foundation for the abstract mathematical universe. For a lot of natural questions, ZFC has given a definite answer, i.e. it either proved or disproved a certain statement, but one of the most important topics in modern set theoretic research is: which problems can ZFC decide and which can it not decide?

Our notation is standard, as used in the relevant literature. For a good introduction to set theory in general, we refer to [16] or [14].

ω denotes the set of natural numbers, or equivalently the first limit ordinal, or the first infinite cardinal and we will write $|X| = \omega$ to say that X is countably infinite and $|X| < \omega$ when X is finite. In general we will rather write ω_n instead of $\aleph_n$, even when talking about cardinality. $|X|$ is the cardinality of X. The size of the continuum, $2^{\aleph_0}$, is denoted $\mathfrak{c}$.

For any set X and a cardinal number κ, $[X]^\kappa$ denotes the set of subsets of X of size κ. $[X]^{<\kappa}$ and $[X]^{\leq\kappa}$ are defined analogously. For example $[\omega]^{<\omega}$ is the set of all finite and $[\omega]^\omega$ the set of all infinite subsets of the naturals. Moreover, as usual, $\mathcal{P}(X)$ is just the powerset of X.

HF is the set of hereditarily finite sets. Intuitively these are all sets that can be written down making use of only finitely many brackets "{" and "}". Also it is V_ω in the von Neumann hierarchy. It is defined as follows:

- $V_0 = \emptyset$
- $V_{n+1} = \mathcal{P}(V_n)$
- $V_\omega = \bigcup_{n<\omega} V_n$

$(HF, \in)$ can be used as a model of "finite set theory" and indeed it satisfies all the axioms of ZFC except for the Axiom of Infinity. Thus $(HF, \in)$ is powerful enough to talk about objects that encode finite information. With this we mean e.g. functions $f : n \to m$ for some natural numbers n and m, finite partial orders, finite graphs etc...

Forcing is an indispensable technique in set theory and will appear in various places throughout the thesis. It is a method used to extend a given model of set theory by adjoining new elements to it. Explaining the exact details of forcing is out of the scope for this thesis. The standard references for an introduction to forcing are [16] or [14]. For a particularly gentle introduction, accessible to non-set-theorists, we refer to [32].

We will use the following notation. A stronger condition will be smaller, i.e. we write $p \leq q$ whenever p extends q. Furthermore $p \perp q$ means that p and q are incompatible and $p \parallel q$ means they are compatible. Names usually wear dots, as in "$\dot{x}$", "$\dot{f}$", etc... We will avoid using checks (e.g. "$\check{n}$") to name fixed ground model elements as much as possible except when there is risk of confusion.

We will very often deal with σ-centered forcing notions. These are partial orders $\mathbb{P}$ that can be written as $\bigcup_{n\in\omega} C_n$, where each C_n is centered, i.e. $\forall P \in [C_n]^{<\omega} \exists p \in \mathbb{P} \forall q \in P (p \leq q)$.

Model Theory and Ultrapowers

We review now some of the few convenient notions of Model Theory that we will use. For a more extensive exposition to Model Theory see [11].

Definition 1.1.1. Let $\mathcal{M}$ be a model in a language $\mathcal{L}$. Then

– a type over $\mathcal{M}$ is a set $p(x)$ of $\mathcal{L}$ formulas with parameters in $\mathcal{M}$ and x as a free variable, so that for each $q(x) \in [p(x)]^{<\omega}$ there is $a \in \mathcal{M}$ with $\mathcal{M} \models \bigwedge_{\varphi\in q(x)} \varphi(a)$ (i.e. $p(x)$ is finitely realized)

– a type $p(x)$ is realized iff there is $a \in \mathcal{M}$ so that $\forall \varphi \in p(x)[\mathcal{M} \models \varphi(a)]$

– $\mathcal{M}$ is κ-saturated iff any type $p(x)$ with $|p(x)| < \kappa$ is realized

A filter on a set X is a non-empty family $\mathcal{F} \subseteq \mathcal{P}(X) \setminus \{\emptyset\}$, closed under intersections and supersets. It is an ultrafilter if for any $A \subseteq X$, either $A \in \mathcal{U}$ or $X \setminus A \in \mathcal{U}$. We will mostly consider ultrafilters on ω which extend $\{\omega \setminus n : n \in \omega\}$. These ultrafilters are referred to as non-principal.

Ultrafilters or filters in general are usually thought of as giving a notion of "largeness". So when a property of $x \in X$ holds for all elements of a filter set we think of it as holding "almost everywhere".

Given a filter $\mathcal{F}$ on X and a n-ary relation R on a set Y (i.e. $R \subseteq Y^n$) we define a relation $R_\mathcal{F}$ on Y^X, by

$$R_\mathcal{F}(f_0, \dots, f_n) \text{ iff } \{x \in X : R(f_0(x), \dots, f_n(x))\} \in \mathcal{F}$$

For example $f =_{\mathcal{F}} g$ if f and g agree almost everywhere, i.e. on a filter set. It is clear that this yields an equivalence relation. We write $[f]_{\mathcal{F}}$ to denote the equivalence class of f under $=_{\mathcal{F}}$.

This is the basis of the ultrapower construction.

Definition 1.1.2. Suppose $\mathcal{U}$ is an ultrafilter on X, $\mathcal{M}$ a model in a language $\mathcal{L}$. Then the ultrapower $\mathcal{M}^X/\mathcal{U}$ of $\mathcal{M}$ over $\mathcal{U}$ is defined by:

- $\{[f]_{\mathcal{U}} : f \in M^X\}$ is the underlying universe,
- for a relation symbol $r \in \mathcal{L}$, $r([f_0]_{\mathcal{U}}, \ldots, [f_n]_{\mathcal{U}})$ is interpreted as $R_{\mathcal{U}}(f_0, \ldots, f_n)$ where r is interpreted as R in $\mathcal{M}$,
- for a function symbol $h \in \mathcal{L}$, $h([f_0]_{\mathcal{U}}, \ldots, [f_n]_{\mathcal{U}})$ is interpreted as $[g]_{\mathcal{U}}$ where $g(x) = H(f_0(x), \ldots, f_n(x))$ where h is interpreted as H in $\mathcal{M}$.

It is easy to check that the definitions above are independent of the choice of representatives for $[f_0]_{\mathcal{U}}$, ..., $[f_n]_{\mathcal{U}}$.

The reason why ultrapowers are so powerful and widely used comes from the following famous theorem of Łoś.

Theorem 1.1.3 (Łoś' Theorem)**.** *Suppose $\mathcal{U}$ is an ultrafilter on X, $\mathcal{M}$ a model in a language $\mathcal{L}$. Then for any $\mathcal{L}$ formula $\varphi(v_0, \ldots, v_n)$ and any $[f_0]_{\mathcal{U}}, \ldots [f_n]_{\mathcal{U}} \in \mathcal{M}^X/\mathcal{U}$,*

$$\mathcal{M}^X/\mathcal{U} \models \varphi([f_0]_{\mathcal{U}}, \ldots [f_n]_{\mathcal{U}}) \textit{ iff } \{x \in X : \mathcal{M} \models \varphi(f_0(x), \ldots f_n(x))\} \in \mathcal{U}$$

Łoś' Theorem implies that any ultrapower of $\mathcal{M}$ satisfies the same theory as $\mathcal{M}$ and that $\mathcal{M}$ can be elementarily embedded by the map $a \mapsto c_a$ where $c_a \colon X \to M$ constantly maps to a.

1.2 Countable modulo finite

Definition 1.2.1. Let $f, g \in \omega^\omega$. We write

- $f =^* g$ iff $|\{n \in \omega : f(n) \neq g(n)\}| < \omega$,
- $f <^* g$ iff $|\{n \in \omega : g(n) \leq f(n)\}| < \omega$,
- $f \leq^* g$ iff $|\{n \in \omega : g(n) < f(n)\}| < \omega$.

Notice that $<^*$ and $=^*$ are just $<_{\mathrm{cofin}}$ and $=_{\mathrm{cofin}}$ where cofin is the filter of cofinite sets.

Definition 1.2.2. A family $\mathcal{F} \subseteq \omega^\omega$ is called unbounded if there is no $f \in \omega^\omega$ so that $g <^* f$ for all $g \in \mathcal{F}$. It is called dominating if for every $f \in \omega^\omega$ there is $g \in \mathcal{F}$ so that $f <^* g$.

Definition 1.2.3. The bounding number $\mathfrak{b}$ is the least size of an unbounded family. The dominating number $\mathfrak{d}$ is the least size of a dominating family.

Clearly we have that $\mathfrak{b} \leq \mathfrak{d}$. Also it is easy to prove that $\omega < \mathfrak{b}$. All provable relationships between $\mathfrak{b}$ and $\mathfrak{d}$ are summarized in the following theorem:

Theorem 1.2.4 (see [4, Theorem 2.5])**.** $\omega_1 \leq \mathfrak{b} = \mathrm{cf}(\mathfrak{b}) \leq \mathrm{cf}(\mathfrak{d}) \leq \mathfrak{d} \leq \mathfrak{c}$ *and there are no other restrictions on what* $\mathfrak{b}$ *and* $\mathfrak{d}$ *can be in a forcing extension.*

Definition 1.2.5. Let $A, B \in [\omega]^\omega$. We write

- $A =^* B$ iff $|A \triangle B| < \omega$, i.e. $A \setminus B \cup B \setminus A$ is finite.
- $A \subseteq^* B$ iff $A \setminus B$ is finite.
- A sequence $\langle A_\alpha : \alpha < \delta \rangle$ is called a tower if $\alpha < \beta$ implies $A_\beta \subseteq^* A_\alpha$.

Definition 1.2.6. Let $\mathcal{B} \subseteq [\omega]^\omega$. We say that $\mathcal{B}$ has the strong finite intersection property (SFIP for short) iff $\forall \mathcal{F} \in [\mathcal{B}]^{<\omega}(|\bigcap \mathcal{F}| = \omega)$. Whenever $X \subseteq^* B$ for all $B \in \mathcal{B}$ we call X a pseudointersection of $\mathcal{B}$.

Families with the SFIP are strongly related to filters: they are exactly the families that generate a filter extending the filter of cofinite sets.

Definition 1.2.7. The pseudointersection number $\mathfrak{p}$ is the least size of a family $\mathcal{F}$ with the SFIP but no pseudointersection.

The tower number $\mathfrak{t}$ is the least length of a tower with no pseudointersection. Such a tower is called maximal.

The existence of a maximal tower is clear by a recursive construction, or just using Zorn's Lemma. For a family with the SFIP but no pseudointersection you can consider an ultrafilter.

The tower number is obviously a regular cardinal, as any cofinal subsequence of a maximal tower is still maximal. Clearly a maximal tower also is a family with the SFIP but no pseudointersection. Thus we can infer that $\mathfrak{p} \leq \mathfrak{t}$. Also it follows by an easy diagonalization argument that $\mathfrak{p} > \omega$. We include a proof for completeness:

Theorem 1.2.8. $\mathfrak{p}$ *is uncountable.*

Proof. Assume that $\mathcal{B}$ has the SFIP and $|B| \leq \omega$. Write $\mathcal{B} = \{B_n : n \in \omega\}$. For every $n \in \omega$, choose $x_n \in \bigcap_{i \leq n} B_i$ different from all previously chosen x_i, $i < n$. This is possible by the SFIP and clearly $\{x_n : n \in \omega\}$ is a pseudointersection. □

Theorem 1.2.9. $\omega_1 \leq \mathfrak{p} \leq \mathfrak{t} \leq \mathfrak{b} \leq \mathfrak{d} \leq \mathfrak{c}$.

Proof. It suffices to show that $\mathfrak{t} \leq \mathfrak{b}$. Assume $\{f_\alpha : \alpha < \kappa\} \subseteq \omega^\omega$ and $\kappa < \mathfrak{t}$. We want to show that there is $f \in \omega^\omega$ so that $f_\alpha <^* f$ for every $\alpha < \kappa$. First construct a tower $\langle A_\alpha : \alpha < \kappa \rangle$ so that whenever $g_\alpha \in \omega^\omega$ is the increasing enumeration of A_α, then $g_\alpha > f_\alpha$. The construction of such a tower is straightforward using that $\kappa < \mathfrak{t}$. Now given $\langle A_\alpha : \alpha < \kappa \rangle$ we find a pseudointersection A. Note that whenever $X \subseteq Y$ then the increasing enumeration of X dominates (in the sense of $\geq$) the increasing enumeration of Y. Thus if we define for every $n \in \omega$, h_n to be the enumeration of $A \setminus n$, then for any $\alpha < \kappa$ there is $n \in \omega$ so that $h_n \geq g_\alpha > f_\alpha$. The collection $\{h_n : n \in \omega\}$ is countable, so we find $f \in \omega^\omega$ with $h_n <^* f$ for every $n \in \omega$. In particular we get that $f_\alpha <^* f$ for every $\alpha < \kappa$. □

Chapter 2

Classical results

The aim of this chapter is to present some of the well known aspects of $\mathfrak{p}$ and $\mathfrak{t}$.
Most proofs in this chapter are taken either from [4] or [31].

2.1 Towers and cardinal arithmetic

$\mathfrak{t}$ has consequences for cardinal arithmetic below $\mathfrak{c}$ which are rather unusual for cardinal characteristics. Usually, for a characteristic $\mathfrak{x}$, it is consistent that $\mathfrak{x}$ attains any allowed value (maybe it has to be regular as in the case of $\mathfrak{b}$) between ω_1 and $\mathfrak{c}$ and cardinal arithmetic has no restrictions other than the obvious ones.

Theorem 2.1.1. *Assume $\kappa < \mathfrak{t}$ is an infinite cardinal. Then $2^\kappa = \mathfrak{c}$.*

Proof. It is clear that $2^\kappa \geq \mathfrak{c}$, so we are finished if we find 2^κ many distinct reals (i.e. we have an injection from 2^κ to $\mathfrak{c}$). To achieve this we are constructing a map $c : 2^{<\kappa} \to [\omega]^\omega$ with the property that

(1) $\forall s, t \in 2^{<\kappa}(s \subseteq t \to c(t) \subseteq^* c(s))$

(2) $\forall s \in 2^{<\kappa}(c(s^\frown 0) \cap c(s^\frown 1) = \emptyset)$

Whenever we have such a map, each $f \in 2^\kappa$ corresponds to a tower $\langle c(f \restriction \alpha) : \alpha < \kappa\rangle$. As $\kappa < \mathfrak{t}$ we can find a pseudointersection c_f of this tower. But by (2) above, the family $\{c_f : f \in 2^\kappa\}$ is almost disjoint. In particular $c_f \neq c_g$ for $f \neq g$.

The map is constructed by recursion on the height of $s \in 2^{<\kappa}$. Start with $c(\emptyset) = \omega$. At successor steps nodes have the form $s = t^\frown i$ for $i \in 2$. For each such t partition the already defined $c(t)$ in two infinite sets X_0,X_1 and let $c(t^\frown i) = X_i$. At limit steps, given s, let $c(s)$ be a pseudointersection of $\langle c(s \restriction \alpha) : \alpha < \operatorname{lth} s\rangle$. This exists because $\operatorname{lth} s < \kappa < \mathfrak{t}$. □

Corollary 2.1.2. $\mathfrak{t} \leq \operatorname{cf}(\mathfrak{c})$.

Proof. If $\operatorname{cf}(\mathfrak{c}) < \mathfrak{t}$ then $2^{\operatorname{cf}(\mathfrak{c})} = \mathfrak{c}$ which contradicts König's Lemma (see [16, Theorem I.13.12]). □

2.2 A forcing axiom

Forcing axioms generally express that the universe is somehow locally closed under forcing. In this way they are examples for natural maximality principles. Forcing axioms have shown a wide range of applications. For example they are often used for special constructions in general topology (see e.g. [10]). In this way they provide a good testing ground for consistency results (especially under ¬CH). The first one of these forcing axioms was Martin's axiom (MA, introduced in [18]). MA asserts that given a ccc poset $\mathbb{P}$ and a collection $\langle D_i : i < \kappa\rangle$ of less than $\mathfrak{c}$ many dense subsets of $\mathbb{P}$ there is a filter $G \subseteq \mathbb{P}$ that intersects every D_i (i.e. $G \cap D_i \neq \emptyset$). It is trivially true under CH but it is also consistent under ¬CH (although it gives some restriction on what $\mathfrak{c}$ might be, see [18] for more).

Later the same sort of axiom was considered for various other classes of posets. For example it is well known that MA(countable) (MA for countable posets) is equivalent to $\operatorname{cov}(\mathcal{M}) = \mathfrak{c}$, where $\operatorname{cov}(\mathcal{M})$ is the least number of meager sets needed to cover the real line.

More generally, for a class $\mathcal{K}$ of posets, we can consider the axiom MA($\mathcal{K}$). Furthermore we can associate with it a cardinal $\mathfrak{m}(\mathcal{K})$ which is the least size κ of a family of dense subsets of some poset in $\mathcal{K}$ which has no generic filter. It is actually not too difficult to see that $\mathfrak{m}(\text{countable}) = \mathfrak{m}(\{\mathbb{C}\}) = \operatorname{cov}(\mathcal{M})$, where $\mathbb{C}$ is Cohen forcing (this appears e.g. in [4, 7.13]).

We will consider the case of σ-centered posets. Thus we write MA(σ-centered) to say that for any σ-centered poset $\mathbb{P}$ and for a collection $\langle D_i : i < \kappa\rangle$ of less than $\mathfrak{c}$ many dense subsets of $\mathbb{P}$ there is a filter intersecting all these sets. Interestingly, the cardinal $\mathfrak{p}$ is closely related to MA(σ-centered).

The next Theorem was proven by M. Bell ([3]).

Theorem 2.2.1 (Bell's Theorem)**.** *$\mathfrak{m}(\sigma$-centered$) = \mathfrak{p}$, i.e. whenever $\mathbb{P}$ is σ-centered and $\langle D_i : i < \kappa \rangle$ are dense subsets, where $\kappa < \mathfrak{p}$, then there is a filter on $\mathbb{P}$ intersecting these dense sets. In particular, MA(σ-centered) is equivalent to $\mathfrak{p} = \mathfrak{c}$*

Proof. Assume $\mathbb{P}$ is σ-centered and $\langle D_i : i < \kappa \rangle$ are open dense[1] subsets, $\kappa < \mathfrak{p}$. We want to show that there is a filter $G \subseteq \mathbb{P}$ so that $G \cap D_i \neq \emptyset$ for every $i < \kappa$.

First we note that we can wlog assume that $|\mathbb{P}| \leq \kappa$. Namely, applying the Löwenheim-Skolem Theorem (see [11]), we find $\mathbb{Q} \subseteq \mathbb{P}$ of size $\leq \kappa$, so that $D_i \cap \mathbb{Q}$ is dense in $\mathbb{Q}$ for every $i < \kappa$, and if $p, q \in \mathbb{Q}$ then $p \parallel q$ iff $\exists r \in \mathbb{Q}(r \leq p, q)$. If G is a filter on $\mathbb{Q}$ that intersects each $D_i \cap \mathbb{Q}$, then $\{p \in \mathbb{P} : \exists q \in G(q \leq p)\}$ is a filter on $\mathbb{P}$ that intersects every D_i.

Furthermore it suffices to show that there is $G \subseteq \mathbb{P}$ that is linked and intersects every D_i. The reason is that for every $p \in \mathbb{P}$ we can define the dense open set $D_p = \{q \in \mathbb{P} : q \leq p \vee q \perp p\}$. Now if $p, r \in G$ where G is linked and $q \in D_p \cap D_r \cap G$, then $q \leq p, r$.

So assume that $|\mathbb{P}| \leq \kappa$ and $\mathbb{P} = \bigcup_{n \in \omega} C_n$, where $C_n \neq \emptyset$ is centered for $n \in \omega$. Also wlog $\mathbb{P}$ is atomless. For any $p \in \mathbb{P}$ and $i < \kappa$ we define the set $A(p, i) = \{n \in \omega : \exists q \in C_n(q \in D_i \wedge q \leq p)\}$.

We claim that for each $n \in \omega$, the collection

$$\mathcal{F}_n = \{A(p, i) : p \in C_n, i < \kappa\}$$

has the SFIP (and in particular, each $A(p, i)$ is infinite). To see this, let $F \in [C_n]^{<\omega}$ and $G \in [\kappa]^{<\omega}$. Then F has a lower bound p. Also notice that $D = \bigcap_{i \in G} D_i$ is also open dense. So let A be an infinite antichain in D below p (which exists because $\mathbb{P}$ is atomless). Then the set $\{n \in \omega : \exists q \in A(q \in C_n)\} \subseteq \bigcap_{r \in F} A(r, i)$ and it is infinite (because A was an infinite antichain).

Thus we find pseudointersections A_n of $\mathcal{F}_n$. Define a labeling $T \colon \omega^{<\omega} \to \omega$, so that $T(\emptyset) = 0$ and $T(s^\frown n) =$ the n'th element of $A_{T(s)}$. Further, for each $i < \kappa$, we define a labeling $T_i \colon \omega^{<\omega} \to \mathbb{P}$ such that

(1) $T_i(\emptyset) \in C_0$,

(2) $T_i(s^\frown n) \in C_{T(s^\frown n)} \cap D_i$ and $T_i(s^\frown n) \leq T_i(s)$, if this is possible

(3) $T_i(s^\frown n) \in C_{T(s^\frown n)}$ arbitrary else

[1] Clearly the distinction between dense and open dense is inessential here, as filters are upwards closed.

for every $s \in \omega^{<\omega}, n \in \omega$.

Notice that for each $i < \kappa$ and for each $s \in \omega^{<\omega}$, the set $\{n \in \omega : \exists q \leq T_i(s)(q \in C_{T(s^\frown n)} \cap D_i)\}$ is cofinite (as $A_{T(s)} \subseteq^* A(T_i(s), i)$). Thus we can define for every $i < \kappa$ a function $f_i\colon \omega^{<\omega} \to \omega$ so that $\omega \setminus f_i(s) \subseteq \{n \in \omega : \exists q \leq T_i(s)(q \in C_{T(s^\frown n)} \cap D_i)\}$ for every s.

As $\kappa < \mathfrak{p} \leq \mathfrak{b}$, there is $f\colon \omega^{<\omega} \to \omega$ so that for every $i < \kappa$, $f(s) \geq f_i(s)$ for all but finitely many $s \in \omega^{<\omega}$.

Define $x\colon \omega \to \omega$ by $x(0) = f(\emptyset)$ and $x(n+1) = f(x \restriction n+1)$. Now for each $i < \kappa$, there is some $n \in \omega$ so that $\forall m \geq n(T_i(x \restriction m) \in D_i)$ and $\langle T_i(x \restriction m) : m \geq n\rangle$ is decreasing. Pick such n and let $p_i = T_i(x \restriction n)$.

It suffices to show that $\{p_i : i < \kappa\}$ is linked. Let $i, j < \kappa$ and $n, m < \omega$, so that $p_i = T_i(x \restriction n)$ and $p_j = T_j(x \restriction m)$. Wlog assume that $n \leq m$. Then $T_i(x \restriction m) \leq p_i$ and $T_i(x \restriction m) \in C_{T(x\restriction m)}$. But also $p_j \in C_{T(x\restriction m)}$ and $C_{T(x\restriction m)}$ is centered. Thus p_j and p_i are compatible. □

In the following we give two applications of Bell's Theorem. One is about the cofinality of $\mathfrak{p}$, the other is about topology.

Theorem 2.2.2. $\mathfrak{p}$ *is regular.*

Proof. Assume $\mathrm{cf}(\mathfrak{p}) = \lambda < \mathfrak{p}$ and $\mathcal{B}$ has the SFIP and no pseudointersection and is of size $\mathfrak{p}$. Write $\mathcal{B} = \bigcup_{\alpha<\lambda} \mathcal{B}_\alpha$, where $|\mathcal{B}_\alpha| < \mathfrak{p}$ for each $\alpha < \lambda$. We will construct a tower $\langle C_\alpha : \alpha < \lambda\rangle$ so that each C_α is a pseudointersection of $\mathcal{B}_\alpha$. This tower has to be maximal and puts a contradiction to $\mathfrak{p} \leq \mathfrak{t}$.

The construction is done recursively. Along the recursion we assume inductively that for each $\alpha < \lambda$, $\{C_\alpha\} \cup \mathcal{B}$ has the SFIP. Suppose $\mathcal{C}_\gamma = \{C_\alpha : \alpha < \gamma\}$ was already constructed. By assumption $\mathcal{C}_\gamma \cup \mathcal{B}_\gamma \cup \mathcal{B}_\alpha$ has the SFIP for each $\alpha < \lambda$. Furthermore $|\mathcal{C}_\gamma \cup \mathcal{B}_\gamma \cup \mathcal{B}_\alpha| < \mathfrak{p}$, thus there is a pseudointersection Z_α.

Let $\mathcal{F}$ be the filter generated by $\mathcal{C}_\gamma \cup \mathcal{B}_\gamma$. We define a σ-centered poset $\mathbb{P}$ as follows: $\mathbb{P}$ will consist of pairs (s, X) where $s \in [\omega]^{<\omega}$ and $X \in \mathcal{F}$. $(s, X) \leq (t, Y)$ if $t \subseteq s$, $s \setminus t \subseteq Y$ and $X \subseteq Y$. This forcing is usually called Mathias forcing relative to the filter $\mathcal{F}$. This is σ-centered because if $(s, X_0), \dots, (s, X_{n-1})$ are conditions with same first coordinate, then $(s, X_0 \cap \dots \cap X_{n-1}) \leq (s, X_i)$ for every $i \in n$.

Define the sets $D_{\alpha,n}$ for each $n \in \omega$ and $\alpha < \lambda$ as follows

$$D_{\alpha,n} = \{(s, X) \in \mathbb{P} : \exists m \geq n(m \in s \cap Z_\alpha)\}.$$

We show that each $D_{\alpha,n}$ is dense. For this let (s, X) be arbitrary. As Z_α was a pseudointersection of $\mathcal{C}_\gamma \cup \mathcal{B}_\gamma$ every element of $\mathcal{F}$ has infinite intersection with Z_α. In particular, there is $m \geq n$ so that $m \in X \setminus s$ and we have that $(s \cup \{m\}, X) \leq (s, X)$ and $(s \cup \{m\}, X) \in D_{\alpha,n}$.

Further the sets $E_B = \{(s, X) : X \subseteq B\}$ for $B \in \mathcal{C}_\gamma \cup \mathcal{B}_\gamma$ are dense. Thus let G be a filter intersecting all the sets in $\{D_{\alpha,n} : n \in \omega, \alpha < \lambda\} \cup \{E_B : B \in \mathcal{C}_\gamma \cup \mathcal{B}_\gamma\}$. Then $C = \bigcup\{s \in [\omega]^{<\omega} : \exists X((s, X) \in G)\}$ is easily seen to be a pseudointersection of $\mathcal{C}_\gamma \cup \mathcal{B}_\gamma$ and for any $\alpha < \lambda$, $Z_\alpha \cap C$ is infinite. As Z_α was a pseudointersection of $\mathcal{C}_\alpha$, for every $B \in \mathcal{B}_\alpha$, $C \cap B$ is infinite. Thus $\{C\} \cup \bigcup_{\alpha<\lambda} \mathcal{B}_\alpha = \{C\} \cup \mathcal{B}$ has the SFIP. Let $C_\gamma = C$. □

Remark. Theorem 2.2.2, and actually most applications of Theorem 2.2.1, can also be proven using some clever applications of $\mathfrak{p} \leq \mathfrak{b}$ but Bell's Theorem is a really useful black box that makes these sorts of arguments straightforward for someone used to forcing.

Definition 2.2.3. A subset $A \subseteq X$ of a topological space X is called comeager if it is the intersection of countably many open dense sets in X.

Theorem 2.2.4. *Assume $\mathcal{D}$ is a collection of less then $\mathfrak{p}$ many comeager subsets in a second countable space X. Then the intersection of $\mathcal{D}$ is still comeager.*

Proof. Let $\mathcal{O}$ be a countable base for X. Note that it suffices to the prove the claim for $\mathcal{D}$ consisting only of open dense sets. So assume $\mathcal{D}$ is a collection of less than $\mathfrak{p}$ many open dense subsets of X. We define a σ-centered poset $\mathbb{P}$ as follows. $\mathbb{P}$ consists of pairs $(a, \mathcal{Y})$ where a is a finite subset of $\mathcal{O}$ and $\mathcal{Y}$ is a finite subset of $\mathcal{D}$. We say that $(a, \mathcal{Y}) \leq (b, \mathcal{X})$ if $b \subseteq a$, $\mathcal{X} \subseteq \mathcal{Y}$ and whenever $O \in a \setminus b$, then $O \subseteq \bigcap \mathcal{X}$. Then $\mathbb{P}$ is a σ-centered partial order. Consider the dense sets $E_D = \{(a, \mathcal{Y}) \in \mathbb{P} : D \in \mathcal{Y}\}$, $F_O = \{(a, \mathcal{Y}) \in \mathbb{P} : \exists O' \in a(O' \subseteq O)\}$ for $D \in \mathcal{D}$ and $O \in \mathcal{O}$. It is trivial to check that the sets E_D are dense in $\mathbb{P}$. For F_O note that for finite $\mathcal{X} \subseteq \mathcal{D}$, $\bigcap \mathcal{X}$ is still open and dense.

Finally let G be a filter having non-empty intersection with all sets of the form E_D or F_O. Let $A = \bigcup\{a \subseteq \mathcal{O} : \exists \mathcal{X}((a, \mathcal{X}) \in G)\}$. Consider $\mathcal{B} = \{\bigcup(A \setminus a) : a \in [\mathcal{O}]^{<\omega}, \bigcup(A \setminus a) \text{ is dense}\}$. Then $\mathcal{B}$ is a countable collection of open dense sets. We claim that for any $D \in \mathcal{D}$, there is some finite $a \subseteq \mathcal{O}$ so that $\bigcup(A \setminus a)$ is a dense open subset of D. To see this, assume $(b, \mathcal{X}) \in \mathbb{P}$ is such that $D \in \mathcal{X}$. Let $a = \{O \in b : O \not\subseteq D\}$.

Then $\bigcup(A \setminus a)$ is dense: Let $O \in \mathcal{O}$ be arbitrary. Then there is $U \in \mathcal{O}$ so that $U \subseteq D \cap O$ because D is open dense. Thus if $(c, \mathcal{Y}) \in F_U \cap G$, there is $U' \subseteq U$ so that $U' \in c$ and therefore $U' \in A \setminus a$.

And $\bigcup(A \setminus a) \subseteq D$: If $O \in A \setminus a$, then there is $(c, \mathcal{Y}) \in G$ extending $(b, \mathcal{X})$ so that $O \in c$. But then we have that either $O \in c \setminus b$ and it follows that $O \subseteq D \in \mathcal{X}$ (definition of $\leq$), or we have that $O \in b \setminus a$ which also means that $O \subseteq D$ by definition of a.

Thus we have shown that the comeager set $\bigcap \mathcal{B}$ is a subset of every $D \in \mathcal{D}$. This proves the claim. □

In the special case where $X = 2^\omega$ is the Cantor space we have shown that $\mathfrak{p} \leq \mathrm{add}(\mathcal{M})$, where $\mathrm{add}(\mathcal{M})$ is the least size of a collection of comeager sets (meager sets) with non-comeager (non-meager) intersection (union) in 2^ω (or ω^ω or equivalently any uncountable Polish space).

Corollary 2.2.5. $\mathfrak{p} \leq \mathrm{add}(\mathcal{M})$.

We note that using a more sophisticated proof we can also show that $\mathfrak{t} \leq \mathrm{add}(\mathcal{M})$ (see [4]). This will also follow from Theorem 3.4.2.

2.3 Rothberger's Theorem

This section will give a first approximation to the much more general theorem we will prove in the next chapter.

Theorem 2.3.1. *Assume* $\mathfrak{p} = \omega_1$ *then also* $\mathfrak{t} = \omega_1$.

This theorem was first implicitly proven by Rothberger in 1948 ([22]). It is striking how short the proof of $\mathfrak{p} = \mathfrak{t}$ is given the assumption that $\mathfrak{p} = \omega_1$. The general case will require a much longer proof and has been found only sixty years later.

Before we head to the proof of the theorem we show a lemma that is interesting in its own right.

Lemma 2.3.2. *Assume* $\mathcal{B} \subseteq [\omega]^\omega$ *has the SFIP,* $|\mathcal{B}| < \mathfrak{d}$ *and* $\mathcal{C} \subseteq \mathcal{B}$ *is countable. Then there is* X*, a pseudointersection of* $\mathcal{C}$ *so that* $\forall Y \in \mathcal{B}(|X \cap Y| = \omega)$.

Proof. Let $\mathcal{C} = \{C_n : n \in \omega\}$ and define for every $n \in \omega$, $D_n := (\bigcap_{i \leq n} C_i) \setminus n$. For each $B \in \mathcal{B}$ we can define a function $f_B : \omega \to \omega$ so that $f_B(n) \cap D_n \cap B \neq \emptyset$ for every $n \in \omega$. By assumption $|\mathcal{B}| < \mathfrak{d}$, thus there is $f \in \omega^\omega$ so that $f \not<^* f_B$ for every B. Let $X = \bigcup_{n \in \omega} D_n \cap f(n)$.

First note that $X \subseteq^* C_n$ for every n, because whenever $m \geq n$, then $D_m \cap f(m) \subseteq D_m \subseteq C_n$.

Furthermore we have that for any $B \in \mathcal{B}$, there are infinitely many n such that $f_B(n) \leq f(n)$ and thereby $D_n \cap f_B(n) \subseteq D_n \cap f(n)$. By definition of f_B, this means that for infinitely n there is $m \geq n$ so that $m \in D_n \cap B \cap f(n)$ and thus $m \in X$. We follow that $|X \cap B| = \omega$. □

The lemma has the following interesting corollary:

Corollary 2.3.3. *Assume $\mathcal{U}$ is an ultrafilter generated by less then $\mathfrak{d}$ many sets*[2]*. Then $\mathcal{U}$ is a P-point.*

Proof. Suppose $\mathcal{C} \subseteq \mathcal{U}$ is countable and $\mathcal{B}$ is a base of size $< \mathfrak{d}$. By the lemma there is X a pseudointersection of $\mathcal{C}$ so that $X \cap Y$ is infinite for every $Y \in \mathcal{B}$. Then $X \in \mathcal{U}$ as $\mathcal{U}$ is an ultrafilter. □

Proof of Theorem 2.3.1. Assume $\mathcal{B} = \{B_\alpha : \alpha < \omega_1\}$ witnesses $\mathfrak{p} = \omega_1$, i.e. has the SFIP but no pseudointersection. We can assume that $\mathcal{B}$ is closed under finite intersections. Our goal will be to find a tower $\langle A_\alpha : \alpha < \omega_1 \rangle$ which somehow refines the family $\mathcal{B}$ and thus must also be maximal. More formally we will construct a tower $\langle A_\alpha : \alpha < \omega_1 \rangle$ so that for each $\alpha < \omega_1$, $A_\alpha \subseteq B_\alpha$. Aiming for a contradiction we may also assume $\mathfrak{p} < \mathfrak{t}$. Note that this means in particular that $\mathfrak{p} < \mathfrak{d}$. In order to construct our sequence we require the following inductive assumption on α:

$$\forall B \in \mathcal{B}(|A_\alpha \cap B| = \omega)$$

This will make sure that we can continue in successor steps. Our construction goes as follows:

- $A_0 := B_0$, the inductive hypothesis is fulfilled.
- $A_{\alpha+1} = A_\alpha \cap B_{\alpha+1}$, then $|A_{\alpha+1} \cap B| = |A_\alpha \cap B_{\alpha+1} \cap B| = \omega$ for any $B \in \mathcal{B}$.
- For α a limit ordinal, apply Lemma 2.3.2 to get X a pseudointersection hitting all $B \in \mathcal{B}$. Note that α is countable. Finally let $A_\alpha = X \cap B_\alpha$.

□

[2]There are models in which $\mathfrak{u}$, the least size of an ultrafilter base, is strictly below $\mathfrak{d}$. In fact $\mathfrak{u} = \kappa < \mathfrak{d} = \lambda$ is consistent for arbitrary regular uncountable $\kappa < \lambda$, see [6].

Note that the proof above is actually a proof by contradiction. Thus, although it tells us that there is some tower of size $\mathfrak{p}$, in general it doesn't tell us how to find it. Still we can observe that the proof does tell us something important.

Definition 2.3.4. Assume $\mathcal{B}$ has the SFIP. We say that a tower $\langle A_\alpha : \alpha < \kappa \rangle$ refines $\mathcal{B}$ if for any $B \in \mathcal{B}$ there is $\alpha < \kappa$ so that $A_\alpha \subseteq^* B$.

Note that if X is a pseudointersection of $\mathcal{B}$, then $\langle X \rangle$ is a tower refining $\mathcal{B}$. The essence of Theorem 2.3.1 is actually the following.

Proposition 2.3.5. *Assume $\mathcal{B}$ has the SFIP and is of size ω_1, where $\omega_1 < \mathfrak{d}$. Then there is a tower refining $\mathcal{B}$.*

In general, without assuming $\mathfrak{p} < \mathfrak{d}$, there does not need to be a refining tower for a given witness of $\mathfrak{p}$.

Observation 2.3.6. *There is a family $\mathcal{B} \subseteq [\omega]^\omega$ with the SFIP of size $\mathfrak{d}$, with no refining tower.*

Proof. Let $\mathcal{D} \subseteq \omega^\omega$ be a dominating family. For any $f \in \mathcal{D}$ we define the set $B_f = \{(n,m) \in \omega \times \omega : f(n) \leq m\}$ and for $i \in \omega$ we let $C_i = \{(n,m) \in \omega \times \omega : i \leq n\}$. Then $\{C_i : i \in \omega\} \cup \{B_f : f \in \mathcal{D}\}$ has the SFIP. But whenever X is a pseudointersection of $\{C_i : i \in \omega\}$, then there is $f \in \mathcal{D}$ so that $f(n) > \max\{m : (n,m) \in X\}$ for almost all n and in particular $|X \cap B_f| < \omega$. $\square$

We will later show (at the end of Section 3.4) that this example is somehow optimal. More precisely, we will show that is consistent assuming $\mathfrak{p} = \mathfrak{b} = \kappa < \mathfrak{d} = \lambda$ that every witness for $\mathfrak{p}$ has a refining tower, where κ and λ are arbitrary regular uncountable cardinals.

Chapter 3

The proof of $\mathfrak{p} = \mathfrak{t}$

The following proof of $\mathfrak{p} = \mathfrak{t}$ is entirely based on the proof given by Malliaris and Shelah in [17]. In [17] a theory of "cofinality spectrum problems" is developped, whose initial goal was to solve problems in Model Theory related to Keisler's order on ultrafilters. But it turned out that this framework, together with preceding work by Shelah in [26], could be used to settle the question of whether $\mathfrak{p} = \mathfrak{t}$.

Here our goal is not to present the whole framework developped by Malliaris and Shelah, but to focus only on what is needed to understand the proof of $\mathfrak{p} = \mathfrak{t}$. This has the advantage that a lot of the proofs in [17] get simplified because we only need to consider a very special case. On the other hand this has the side effect that some of our theorems will become valueless once we have proven $\mathfrak{p} = \mathfrak{t}$, because they start with the assumption that $\mathfrak{p} < \mathfrak{t}$ (especially the main Theorem 3.3.6).

Let us give an overview of the argument. In the first section, we show how to attempt a general proof of $\mathfrak{p} = \mathfrak{t}$ based on the idea for the special case where $\mathfrak{p} = \omega_1$ (Theorem 2.3.1). Analyzing where this attempt might break, will lead to another question that we reduce to a problem related to special gaps in $(\omega^\omega, <^*)$. In order to solve this problem we work in an ultrapower of HF by a generically added ultrafilter. The reason for this is that the gap problem we want to solve can be translated to a problem about the linear order of the ultrapower's version of the naturals. It can be shown that the ultrapower has nice saturation properties which allow for special constructions that will be used throughout. Finally Theorem 3.3.6 will settle the question about the gaps in $(\omega^\omega, <^*)$. We can conlude that indeed $\mathfrak{p} = \mathfrak{t}$.

Before we get to the actual proof let us remark that the use of ultrapowers and forcing as outlined above is in no way essential. It is definitely possible to translate the argument into a purely combinatorial one that can be squeezed in a few pages. But we felt that this would make the proof more difficult to follow, unless it was built on a completely new idea. Ultrapowers and forcing here realy just provide us with a language that most of us are already fluent in.

3.1 Reducing the problem

This section is based on [26].

How would one go about proving $\mathfrak{p} = \mathfrak{t}$? Remember the proof of $\mathfrak{p} = \omega_1 \rightarrow \mathfrak{t} = \omega_1$. There we took a witness $\mathcal{B} = \{B_\alpha : \alpha < \omega_1\}$ for $\mathfrak{p} = \omega_1$ and we used it to construct a tower which somehow refines the family $\mathcal{B}$. Additionaly we assumed $\mathfrak{p} < \mathfrak{t}$ to get a contradiction. So what goes wrong if $\mathfrak{p} = \lambda > \omega_1$? It was crucial that each limit step of the construction was of countable cofinality in order to apply $\mathfrak{p} < \mathfrak{d}$.

Let's just informally try again to construct the sequence $\langle A_\alpha : \alpha < \lambda \rangle$ by induction on α. Again we will make the additional (necessary) inductive hypothesis that $\forall \beta < \lambda$, $|A_\alpha \cap B_\beta| = \omega$.

- For $\alpha = 0$ just take $A_\alpha = \omega$, the inductive hypothesis is fulfilled.
- For $\alpha + 1$, take $A_{\alpha+1} = A_\alpha \cap B_\alpha$ which is infinite by the inductive hypothesis.
- What to do at a limit step γ?

On one hand we want to get a "small" set, i.e. $A_\gamma \subseteq^* A_\alpha$ for every $\alpha < \gamma$, on the other hand we want A_γ to be big enough to have unbounded intersection with all B_β. Whenever γ has countable cofinality and we assume $\mathfrak{p} < \mathfrak{t}$ in order to get a contradiction, we can apply Lemma 2.3.2. So the interesting and difficult cases are when γ has uncountable cofinality. By passing to a subsequence we could also assume wlog that $\gamma = \kappa$ is a regular cardinal. So our actual question is the following:

Question. Assume $\kappa < \lambda = \mathfrak{p} < \mathfrak{t}$, where κ is a regular uncountable cardinal. Assume $\{B_\alpha : \alpha < \lambda\}$ has the SFIP and $\langle A_\alpha : \alpha < \kappa \rangle$ is a tower so that $A_\alpha \cap B_\beta$ is infinite for every $\alpha < \kappa$, $\beta < \lambda$. Is there an $A_\kappa \in [\omega]^\omega$ so that

- $\forall \alpha < \kappa (A_\kappa \subseteq^* A_\alpha)$

– $\forall\beta < \lambda(|A_\kappa \cap B_\beta| = \omega)$?

We are going to reduce this question to another one. For the rest of this section assume $\mathfrak{p} = \lambda < \mathfrak{t}$, $\{B_\beta : \beta < \lambda\}$ has the SFIP and $\langle A_\alpha : \alpha < \kappa\rangle$ is a tower with $A_\alpha \cap B_\beta$ infinite for all α, β, where κ is regular.

Definition 3.1.1. Let $\mathbf{S} := \{s : s\colon X \to 2^{<\omega} \text{ where } X \in [\omega]^\omega\}$. For $s \in \mathbf{S}$ let $\operatorname{set}(s) := \bigcup_{n\in\operatorname{dom} s}\{n + i : s(n)(i) = 1\}$. We write $s \leq^* s'$ iff $\operatorname{dom} s' \subseteq^* \operatorname{dom} s$ and $\forall^\infty n \in \operatorname{dom} s(s(n) \subseteq s'(n))$.

Lemma 3.1.2. *There is a sequence $\langle s_\beta : \beta < \lambda\rangle$ in $\mathbf{S}$ so that $\forall\beta < \lambda$:*

(1) $\forall\beta < \beta'(s_\beta \leq^* s_{\beta'})$*,*

(2) $\forall\alpha < \kappa(\operatorname{set}(s_\beta) \subseteq^* A_\alpha)$*, i.e.* $\operatorname{set}(s_\beta)$ *is a pseudointersection of* $\langle A_\alpha : \alpha < \kappa\rangle$*,*

(3) $\forall^\infty n \in \operatorname{dom} s_\beta \exists i \in \operatorname{dom} s_\beta(n)[s_\beta(n)(i) = 1 \wedge n + i \in B_\beta]$.

Proof. The sequence is constructed recursively. Assume $\langle s_\beta : \beta < \delta\rangle$ is given (possibly $\delta = 0$). Then define a poset $\mathbb{P}_\delta$ consisting of pairs $(s, \mathcal{Y})$ where

– $s\colon \operatorname{dom} s \to 2^{<\omega}, \operatorname{dom} s \in [\omega]^{<\omega}$

– $\forall n \in \operatorname{dom} s \exists i[s(n)(i) = 1 \wedge n + i \in B_\delta]$

– $\mathcal{Y} \in [\delta \cup \kappa]^{<\omega}$

$(s, \mathcal{Y}) \leq (s', \mathcal{Y}')$ iff $s \supseteq s'$, $\mathcal{Y} \supseteq \mathcal{Y}'$ and for $n \in \operatorname{dom} s \setminus \operatorname{dom} s'$ and for $\alpha \in \mathcal{Y}'$ the following holds:

– if $\alpha < \kappa$ then $\{n + i : s(n)(i) = 1\} \subseteq A_\alpha$

– if $\alpha < \delta$ then $n \in \operatorname{dom} s_\alpha$ and $s_\alpha(n) \subseteq s(n)$.

$\mathbb{P}_\delta$ is clearly σ-centered and the sets $D_\alpha := \{(s, \mathcal{Y}) : \alpha \in \mathcal{Y}\}$ are dense. Also the sets $E_k := \{(s, \mathcal{Y}) : \exists m \geq k(m \in \operatorname{dom} s)\}$ for $k \in \omega$ are dense. Because consider $(s, \mathcal{Y}) \in \mathbb{P}_\delta$ with $\operatorname{dom} s \subseteq k$. Let $m \geq k$ be so that $\forall n \geq m \forall\alpha, \beta \in \mathcal{Y}$:

– if $\alpha < \kappa$, $\beta < \delta$ and $n \in \operatorname{dom} s_\beta$ then $\{n + i : s_\beta(n)(i) = 1\} \subseteq A_\alpha$

– if $\alpha < \beta < \delta$ and $n \in \operatorname{dom} s_\beta$ then $n \in \operatorname{dom} s_\alpha$ and $s_\alpha(n) \subseteq s_\beta(n)$

– if $\alpha < \beta < \kappa$ then $A_\beta \setminus n \subseteq A_\alpha$

Let $\beta := \max \mathcal{Y} \cap \delta$ and $n \in \operatorname{dom} s_\beta$, $n \geq m$. Also let $\alpha := \max \mathcal{Y} \cap \kappa$. Extend $s_\beta(n) \in 2^{<\omega}$ to $\sigma \in 2^{<\omega}$, so that $\{n + i : \sigma(i) = 1\} \subseteq A_\alpha$ and $\{n + i : \sigma(i) = 1\} \cap B_\delta \neq \emptyset$. This is possible because by assumption $A_\alpha \cap B_\delta$ is infinite.

Put $s' = s \cup \{(n, \sigma)\}$. Then $(s', \mathcal{Y}) \leq (s, \mathcal{Y})$.

We have that $|\{D_\alpha : \alpha < \delta \cup \kappa\} \cup \{E_k : k \in \omega\}| < \lambda = \mathfrak{p}$. Thus by Theorem 2.2.1 there is a filter G on $\mathbb{P}_\delta$ that intersects all these dense sets. Let $s_\delta := \bigcup_{p \in G} \operatorname{dom} p$. It is clear that (1), (2) and (3) hold true. □

Given a sequence $\langle s_\beta : \beta < \lambda \rangle$ as in Lemma 3.1.2, let $X \subseteq^* \operatorname{dom} s_\beta$ for all $\beta < \lambda$. This is possible because we assumed $\lambda = \mathfrak{p} < \mathfrak{t}$ (and this is the first time we use this assumption).

Then define for each $\beta < \lambda$ a function $d_\beta : X \to \omega$, $d_\beta(n) = \operatorname{lth} s_\beta(n)$ if this is defined (which is the case for almost every n) and $d_\beta(n) = 0$ else.

As $\lambda < \mathfrak{t} \leq \mathfrak{b}$, there is a function $d : X \to \omega$ so that for all $\beta < \lambda$, $d_\beta <^* d$.

Now consider for each $\beta < \lambda$ the sets $S_\beta^n := \{\langle n, \sigma \rangle : \sigma \in 2^{<d(n)}, \sigma \supseteq s_\beta(n)\}$ and $S_\beta := \bigcup_{n \in X} S_\beta^n$. We have

$$\forall \beta < \alpha < \lambda (S_\alpha \subseteq^* S_\beta).$$

Because let m be so that $\forall n \in X \setminus m$, $s_\beta(n) \subseteq s_\alpha(n)$, then $\forall n \in X \setminus m$, $S_\alpha^n \subseteq S_\beta^n$.

Let $S \subseteq^* S_\beta$ for all $\beta < \lambda$ and find $s \in \mathbf{S}$ with $\forall n \in \operatorname{dom} s(\langle n, s \rangle \in S)$.

Define for every $\alpha < \kappa$, $f_\alpha : \operatorname{dom} s \to \omega$ by

$$f_\alpha(n) = \max\{k < d(n) : \{n + i : s(n)(i) = 1 \wedge i < k\} \subseteq A_\beta\}$$

Define for every $\beta < \lambda$, $g_\beta : \operatorname{dom} s \to \omega$ by

$$g_\beta(n) = \min\{i < d(n) : s(n)(i) = 1 \wedge n + i \in B_\beta\}$$

Then $\forall \alpha < \kappa, \forall \beta < \lambda$,

$$g_\beta <^* f_\alpha$$

Because let m be so that $\forall n \in \operatorname{dom} s \setminus m$

- $\mathrm{lth}(s_\beta(n)) < d(n)$
- $\{n+i : s_\beta(n)(i) = 1\} \subseteq A_\alpha$
- $\{n+i : s_\beta(n)(i) = 1\} \cap B_\beta \neq \emptyset$
- $s_\beta(n) \subseteq s$

then clearly $\forall n \in \operatorname{dom} s \setminus m$, $g_\beta(n) < \operatorname{lth} s_\beta(n) \leq f_\alpha(n)$.

Lemma 3.1.3. *Assume there is $X \in [\operatorname{dom} s]^\omega$ and $h \colon X \to \omega$ so that $\forall \alpha < \kappa, \forall \beta < \lambda$, $g_\beta \upharpoonright X <^* h <^* f_\alpha \upharpoonright X$. Then there is $A_\kappa \in [\omega]^\omega$ so that*

- $\forall \alpha < \kappa (A_\kappa \subseteq^* A_\alpha)$
- $\forall \beta < \lambda (|A_\kappa \cap B_\beta| = \omega)$

Proof. Let $A_\kappa := \bigcup_{n \in X} \{n+i : i \leq h(n) \wedge s(n)(i) = 1\}$. If $\alpha < \kappa$ then as $h <^* f_\alpha \upharpoonright X$, we have that $\forall^\infty n \in X$,

$$\{n+i : i \leq h(n) \wedge s(n)(i) = 1\} \subseteq \{n+i : i < f_\alpha(n) \wedge s(n)(i) = 1\} \subseteq A_\alpha.$$

And if $\beta < \lambda$ then, as $g_\beta \upharpoonright X <^* h$, $\forall^\infty n \in X$,

$$\{n+i : i \leq h(n) \wedge s(n)(i) = 1\} \cap B_\beta \supseteq \{n+i : i \leq g_\beta(n) \wedge s(n)(i) = 1\} \cap B_\beta \neq \emptyset$$

□

This mirrors the intuition we already gave once before: We want to find a set which is small enough to be a pseudointersection of the A_α ($h <^* f_\alpha$) but big enough to hit all B_β ($g_\beta <^* h$).

We now get the following proposition:

Proposition 3.1.4. *Assume $\lambda = \mathfrak{p} < \mathfrak{t}$. Then there is some $\kappa < \lambda$ regular, $\mathcal{F} \subseteq \omega^\omega$ and $\mathcal{G} \subseteq \omega^\omega$ with*

- $|\mathcal{F}| = \kappa$
- $|\mathcal{G}| = \lambda$
- $\forall f \in \mathcal{F}, \forall g \in \mathcal{G} (g <^* f)$

so that there is no $X \in [\omega]^\omega$ *and* $h : X \to \omega$ *with* $\forall f \in \mathcal{F}, \forall g \in \mathcal{G}(g \restriction X <^* h <^* f \restriction X)$.

Proof. Assume $\lambda = \mathfrak{p} < \mathfrak{t}$ and the conclusion of the proposition fails. Then, given a family $\{B_\beta : \beta < \lambda\}$ with the SFIP but no pseudointersection, we could construct a maximal tower $\langle A_\alpha \rangle_{\alpha<\lambda}$ of length λ using the construction described above. But this clearly contradicts $\lambda < \mathfrak{t}$. □

In [26], Shelah actually showed something stronger. Namely we can get $\mathcal{F} = \langle f_i : i < \kappa \rangle$ to be a decreasing, $\mathcal{G} = \langle g_i : i < \lambda \rangle$ an increasing sequence with respect to $<^*$ so that $\forall h \in \omega^\omega [\forall i < \kappa(h <^* f) \to \exists j < \lambda(h <^* g_j)]$ and $\forall h \in \omega^\omega [\forall i < \lambda(g_i <^* h) \to \exists j < \kappa(f_j <^* h)]$. This was called a (λ, κ)-peculiar cut. Furthermore it can be shown that the existence of a $(\mathfrak{p}, \kappa)$-peculiar cut for $\kappa < \mathfrak{p}$ is independent of ZFC (see [28]). This stronger form of Proposition 3.1.4 will not be needed.

3.2 The generic ultrapower of $\mathbf{HF}$

Definition 3.2.1. HF is the set of hereditarily finite sets. We denote with $\mathbf{HF}$ the structure $(HF, \in)$.

The following facts are easy and well known.

Lemma 3.2.2. *Assume* G *is* $([\omega]^\omega, \subseteq^*)$*-generic over* $\mathbf{V}$*. Then the following hold true:*

- $([\omega]^\omega, \subseteq^*)$ *is* $\mathfrak{t}$*-closed,*
- *thus* $\mathbf{V}$ *and* $\mathbf{V}[G]$ *have the same reals,*
- *and* $\mathbf{V}$ *and* $\mathbf{V}[G]$ *have the same* $< \mathfrak{t}$*-length sequences of reals,*
- $\mathfrak{p}^{\mathbf{V}[G]} = \mathfrak{p}^{\mathbf{V}}$, $\mathfrak{t}^{\mathbf{V}[G]} = \mathfrak{t}^{\mathbf{V}}$,
- G *generates an ultrafilter* $\mathcal{U}$ *on* ω *in* $\mathbf{V}[G]$.

Theorem 3.2.3. *Assume* G *is* $([\omega]^\omega, \subseteq^*)$*-generic over* $\mathbf{V}$ *and* $\mathcal{U} \in \mathbf{V}[G]$ *is the ultrafilter generated by* G*. In* $\mathbf{V}[G]$ *let* $\mathbf{HF}^* := \mathbf{HF}^\omega/\mathcal{U}$ *be the ultrapower of* $\mathbf{HF}$ *using* $\mathcal{U}$*. Then* $\mathbf{HF}^*$ *is* $\mathfrak{p}$ *saturated.*

For the definition of saturation we refer to the preliminaries section of the introduction (1.1).

Proof. Assume $p(x)$ is a type of size $< \mathfrak{p}$ and assume wlog that it is closed under conjunctions. Work in $\mathbf{V}$. Then $p(x)$ is an object in $\mathbf{V}$ by $\mathfrak{t}$ closedness of $([\omega]^\omega, \subseteq^*)$.

Let $A \in G$ force that $p(x)$ is finitely realized in $\mathbf{HF}^*$. This means that $\forall\varphi(x,\bar{a}) \in p(x)$ and $\forall^\infty n \in A$, $\mathbf{HF} \models \exists x\varphi(x,\bar{a}(n))$ ($\bar{a}$ are the parameters occurring in φ). Why? Because if there was an infinite set $B \subseteq A$ and $\varphi(x,\bar{a}) \in p(x)$ so that $\forall n \in B$, $\mathbf{HF} \not\models \exists x\varphi(x,\bar{a}(n))$, then B would force that $p(x)$ is not finitely realized and hence A couldn't force it is.

In order to make a genericity argument, let $B \subseteq^* A$ be arbitrary. Define a σ-centered poset consisting of pairs $(s,\mathcal{Y})$ where s is a finite partial function from B to HF and $\mathcal{Y} \in [p(x)]^{<\omega}$. The extension relation is defined by $(s,\mathcal{Y}) \leq (t,\mathcal{X})$ iff $s \supseteq t$, $\mathcal{Y} \supseteq \mathcal{X}$ and $\forall n \in \operatorname{dom} s \setminus \operatorname{dom} t$, $\forall\varphi(x,\bar{a}) \in \mathcal{X}$, $\mathbf{HF} \models \varphi(s(n),\bar{a}(n))$. Applying Theorem 2.2.1 to the dense sets $D_n := \{(s,\mathcal{Y}) : \max\operatorname{dom} s > n\}$ and $E_\varphi := \{(s,\mathcal{Y}) : \varphi \in \mathcal{Y}\}$ we get a function $f : C \to HF$ where $C \subseteq B$, $|C| = \omega$ and $\forall\varphi(x,\bar{a}) \in p(x)$, $\forall^\infty n \in C$, $\mathbf{HF} \models \varphi(f(n),\bar{a}(n))$. We can extend f arbitrarily to a function $\omega \to HF$ and it is obvious that $C \Vdash [f] \in \mathbf{HF}^*$ realizes $p(x)$. $\square$

Remark. The above theorem applies to **HF** replaced by any countable model.

Theorem 3.2.4. *Let* HF^* *be as in Theorem 3.2.3 and* $\delta < \mathfrak{t}$.
Let $\mathcal{T} \in \mathbf{HF}^*$ *be a partial order, i.e.* $\mathbf{HF}^* \models \varphi(\mathcal{T})$ *where* $\varphi(x)$ *is a formula in the language* $\{\in\}$ *expressing that* x *is a partial order.*
Let $\langle s_\alpha : \alpha < \delta\rangle$ *be an increasing sequence in* $\mathcal{T}$*, i.e.* $\forall\alpha < \delta(\mathbf{HF}^* \models s_\alpha \in \mathcal{T})$ *and* $\forall\alpha < \beta < \delta(\mathbf{HF}^* \models s_\alpha \leq_{\mathcal{T}} s_\beta)$.
Then $\langle s_\alpha : \alpha < \delta\rangle$ *has an upper bound* s*, i.e.* $\forall\alpha < \delta(\mathbf{HF}^* \models s_\alpha \leq_{\mathcal{T}} s)$.

Proof. We have that $\langle s_\alpha : \alpha < \delta\rangle \in \mathbf{V}$ and $\mathcal{T} \in \mathbf{V}$. Each s_α is a function from ω to HF so we will write $s_\alpha(n)$ for the n'th component [1]. The same applies to $\mathcal{T}$ and $\mathcal{T}_n = \mathcal{T}(n)$.

Let $A \in \mathcal{U}$ force that $\mathcal{T}$ is a partial order and that $\langle s_\alpha : \alpha < \delta\rangle$ is increasing in $\mathcal{T}$. This means that $\forall\alpha < \beta$, $\forall^\infty n \in A$, $\mathcal{T}_n$ is a (finite) partial order, $s_\alpha(n) \in \mathcal{T}_n$ and $s_\alpha(n) \leq_{\mathcal{T}_n} s_\beta(n)$.

[1] Formally the elements of $\mathbf{HF}^*$ are equivalence classes but we will always treat them as specific representatives for their class.

Now consider for each $\alpha < \delta$ the set

$$S_\alpha := \{(n, x) : n \in A \wedge x \in \mathcal{T}_n \wedge s_\alpha(n) \leq_{\mathcal{T}_n} x\}$$

It is clear that $S_\beta \subseteq^* S_\alpha$ when $\alpha < \beta$. So let $S \subseteq^* S_\alpha$ for every $\alpha < \delta$ be an infinite set. As $\mathcal{T}_n \in HF$ is finite, it must be that $\operatorname{dom} S$ is infinite. Pick for each $n \in \operatorname{dom} S$ a x_n so that $(n, x_n) \in S$. Let $C := \operatorname{dom} S$, $s : \omega \to HF$ a function so that $s(n) = x_n$ when $n \in C$. Then $C \Vdash [s]$ is an upper bound of $\langle s_\alpha : \alpha < \delta \rangle$ in $\mathcal{T}$. □

The last two theorems express that both, $\mathfrak{p}$ and $\mathfrak{t}$, are related to a sort of saturation property in the structure $\mathbf{HF}^*$.

3.3 Cuts in the generic ultrapower of HF

We have discussed in the last section some properties of the ultrapower of **HF** by a generic ultrafilter. Throughout this section we fix a $([\omega]^\omega, \subseteq^*)$-generic G over **V** and we let $\mathbf{HF}^*$ be as before.

Definition 3.3.1. Let $(X, <)$ be a linear order. Let $\{a_i : i < \delta\}, \{b_\alpha : \alpha < \gamma\} \subseteq X$. Then $(\langle a_i : i < \delta \rangle, \langle b_\alpha : \alpha < \gamma \rangle)$ is a $((\delta, \gamma)$-$)$ precut if

- $\langle a_i : i < \delta \rangle$ is strictly increasing in X
- $\langle b_\alpha : \alpha < \gamma \rangle$ is stricly decreasing in X
- $\forall i < \delta, \forall \alpha < \gamma (a_i < b_\alpha)$

It is filled if $\exists x \in X, \forall i < \delta, \forall \alpha < \gamma (a_i < x < b_\alpha)$. If it is not filled we call it a $((\delta, \gamma)$-$)$ cut.

Notice that whenever $\{a_i : i \leq \delta\}$ is a strictly increasing sequence in a linear order X (so it has an upper bound a_δ), we can always find a strictly decreasing sequence $\{b_\alpha : \alpha < \gamma\}$ so that $(\langle a_i : i < \delta \rangle, \langle b_\alpha : \alpha < \gamma \rangle)$ is a cut. Moreover when γ is infinite, if we pass to a cofinal subsequence $\langle \alpha_i : i < \operatorname{cf}(\gamma) \rangle$ of γ, then $(\langle a_i : i < \delta \rangle, \langle b_{\alpha_i} : i < \operatorname{cf}(\gamma) \rangle)$ is a cut.

We are going to be interested in cuts in the linear order $\mathbb{N}^* := (\mathbb{N})^{\mathbf{HF}^*}$, that is, in the natural numbers of $\mathbf{HF}^*$.

Definition 3.3.2. We let $\mathcal{C} := \{(\kappa, \lambda) : \kappa, \lambda$ are regular and there is a (κ, λ) cut in $\mathbb{N}^*\}$

Basic facts

Notice the following very easy observation about $\mathcal{C}$:

Lemma 3.3.3. *For all regular* κ, λ, $(\kappa, \lambda) \in \mathcal{C}$ *iff* $(\lambda, \kappa) \in \mathcal{C}$.

Proof. When $(\langle a_i : i < \kappa\rangle, \langle b_\alpha : \alpha < \lambda\rangle)$ is a cut in $\mathbb{N}^*$ then $(\{b_0 - b_\alpha : \alpha < \lambda\}, \{b_0 - a_i : i < \kappa\})$ is also a cut. For this just notice that in $\mathbb{N}$, if $x < y \leq z$ then $z - x > z - y \geq 0$. □

Lemma 3.3.4. *There are no* $\kappa, \lambda < \mathfrak{p}$ *so that* $(\kappa, \lambda) \in \mathcal{C}$.

Proof. Apply the $\mathfrak{p}$-saturation of $\mathbf{HF}^*$ (Theorem 3.2.3). More precisely, given a precut $(\langle a_i : i < \kappa\rangle, \langle b_\alpha : \alpha < \lambda\rangle)$ in $\mathbb{N}^*$ define the type $p(x)$ containing formulas expressing:

- x is a natural number

and for all $i < \kappa$, $\alpha < \lambda$:

- $a_i < x$
- $x < b_\alpha$

This type is finitely realized and of size $< \mathfrak{p}$. Thus the precut $(\langle a_i : i < \kappa\rangle, \langle b_\alpha : \alpha < \lambda\rangle)$ is filled. □

Uniqueness

Theorem 3.3.5. *Let* $\kappa \leq \mathfrak{p}$ *and* $\kappa < \mathfrak{t}$. *Then there is a unique* λ *so that* $(\kappa, \lambda) \in \mathcal{C}$.

Proof. For the existence just notice that we can find a strictly increasing sequence $\langle a_i : i < \kappa\rangle \in \mathbb{N}^*$ with an upper bound using $\kappa < \mathfrak{t} \leq \mathfrak{b}$.

For the uniqueness assume $(\kappa, \lambda) \in \mathcal{C}$ and $(\kappa, \lambda') \in \mathcal{C}$ and assume $\lambda \neq \lambda'$. This means that there is

- $(\langle a_i : i < \kappa\rangle, \langle b_\alpha : \alpha < \lambda\rangle)$ a cut in $\mathbb{N}^*$
- $(\langle a'_i : i < \kappa\rangle, \langle b'_\alpha : \alpha < \lambda'\rangle)$ a cut in $\mathbb{N}^*$

Let $b := b_0$ and $b' := b'_0$. Also let $\mathbf{n}_\infty$ be a distinguished element of $\mathbb{N}^*$ so that $\mathbf{HF}^* \models k < \mathbf{n}_\infty$ for every standard natural number $k \in \omega$. Let $\mathcal{T}$ be the tree consisting of $s \in (b \times b')^n$ for $n < \mathbf{n}_\infty$ which are increasing in both coordinates, i.e. $\forall m < m' < n[s(m)(0) \leq s(m')(0) \wedge s(m)(1) \leq s(m')(1)]$.

We are going to construct an increasing sequence $\langle s_i : i < \kappa \rangle$ in $\mathcal{T}$ so that, putting $n_i := \max \operatorname{dom} s_i$: $s_i(n_i)(0) = a_i$, $s_i(n_i)(1) = a'_i$ and for each $k \in \omega$, $n_i < \mathbf{n}_\infty - k$. This last property will be used to be able to continue our recursion in the successor step.

- For $i = 0$, let $n_i = 0$, $s_i(0)(0) = a_0$ and $s_i(0)(1) = a'_0$.
- For $i = j + 1$ let $s_i = s_j {}^\frown (a_i, a'_i)$. This works because then $n_i = n_j + 1 < \mathbf{n}_\infty - k$ for each k.
- For i limit, first apply Theorem 3.2.4 to get an upper bound s of $\langle s_j : j < i \rangle$. Let $n := \max \operatorname{dom} s$. Consider the precut $(\langle n_j : j < i \rangle, \langle n - k : k \in \omega \rangle)$. By Lemma 3.3.4 it is filled, say by m. Now let $s' := s \restriction m$ and let $s_i := s' {}^\frown (a_i, a'_i)$. By construction $n_i = m < n - k \leq \mathbf{n}_\infty - k$ for every $k \in \omega$.

Finally let s be an upper bound of $\langle s_i : i < \kappa \rangle$ in $\mathcal{T}$ and $n := \max \operatorname{dom} s$. This again exists by Theorem 3.2.4 because we assumed $\kappa < \mathfrak{t}$. Consider

- $l_\alpha := \max\{m \leq n : s(m)(0) \leq b_\alpha\}$ for $\alpha < \lambda$
- $l'_\alpha := \max\{m \leq n : s(m)(1) \leq b'_\alpha\}$ for $\alpha < \lambda'$

Notice that these sequences are decreasing and they can't be eventually constant (else we could fill our initial cut). Thus we can pass to strictly decreasing subsequences $\langle m_\alpha : \alpha < \lambda \rangle$, $\langle m'_\alpha : \alpha < \lambda' \rangle$.

Then we have that both $(\langle n_i : i < \kappa \rangle, \langle m_\alpha : \alpha < \lambda \rangle)$ and $(\langle n_i : i < \kappa \rangle, \langle m'_\alpha : \alpha < \lambda' \rangle)$ are cuts. But then $\langle m_\alpha : \alpha < \lambda \rangle$ is coinitial in $\langle m'_\alpha : \alpha < \lambda' \rangle$ and vice-versa. So $\lambda = \lambda'$ by regularity. $\square$

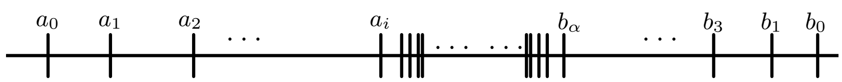
No cuts below $\mathfrak{p}$

The following is the main theorem of the proof of $\mathfrak{p} = \mathfrak{t}$ and corresponds to Theorem 8.5 in [17].

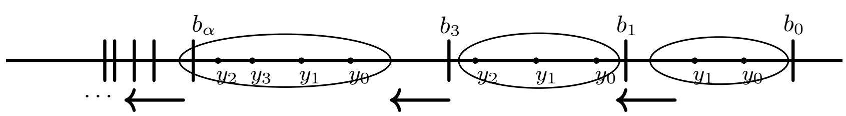
Theorem 3.3.6. *Suppose $\kappa < \mathfrak{p} < \mathfrak{t}$. Then $(\kappa, \mathfrak{p}) \notin \mathcal{C}$.*

Since the proof is much more technical than the ones before, we will first give an informal proof in which we describe the setting that will later be constructed formally.

Informal proof. The proof is by contradiction. Let $\mathfrak{p} = \lambda$ and assume that $(\kappa, \lambda) \in \mathcal{C}$. This means that there is a cut $(\langle a_i \rangle_{i<\kappa}, \langle b_\alpha \rangle_{\alpha<\lambda})$ in $\mathbb{N}^*$. This cut is represented in the following picture:



We imagine a set of points on this line living below b_0. We are going to push these points more and more to the left, but while doing so we make sure that if two points were initialy some distance appart, they will still stay at least this far from each other. While moving the points we will also constantly add new ones. And also these new points need to preserve the initial distance they had to other points when they were first added.



Thus we carefully move points more and more to the left while adding new ones and impose distances between the points to be preserved. This process is carried out for λ many steps and at each step $\alpha < \lambda$ the points will live only below b_α. Then Theorem 3.2.4 will tell us that we can find a limit of this process. Furthermore we will have pushed the points so far to the left whereupon every point has to be completely below all the b_α. As $(\langle a_i \rangle_{i<\kappa}, \langle b_\alpha \rangle_{\alpha<\lambda})$ was a cut this means each point lives now below some a_i. By a pigeonhole argument a lot of them can be found below one fixed a_i. But they also still need to be far appart from each other and the way we introduced them tells us that there is simply no place for that many points to live below a_i and still obey the distance imposition. This leads to a contradiction. □

We now give the real proof and hope that the informal one is helpful to the reader in order to get a clearer picture of the definitions and notation that is to come.

Proof. Fix for the whole proof $\kappa < \mathfrak{p} = \lambda$ and assume that $(\kappa, \lambda) \in \mathcal{C}$.

Claim 1. There is a cut $(\langle a_i \rangle_{i<\kappa}, \langle b_\alpha \rangle_{\alpha<\lambda})$ in $\mathbb{N}^*$ with the additional property that:

– $\forall j < i < \kappa (2 \cdot a_i > a_j)$.

For technical reasons we will use a cut of this special form.

Proof of Claim 1. Using $\kappa < \mathfrak{b}$ construct a sequence $\langle f_i : i < \kappa \rangle$ in ω^ω so that $\forall i < j < \kappa, \forall^\infty n \in \omega (2 \cdot f_i(n) < f_j(n))$. This induces a sequence $\langle a_i \rangle_{i<\kappa}$ in $\mathbb{N}^*$ with the required property. We can then find $\langle b_\alpha \rangle_{\alpha<\lambda'}$ strictly decreasing so that $(\langle a_i \rangle_{i<\kappa}, \langle b_\alpha \rangle_{\alpha<\lambda'})$ is a cut and λ' regular. By Theorem 3.3.5, $\lambda' = \lambda$. □

Let $(\langle a_i \rangle_{i<\kappa}, \langle b_\alpha \rangle_{\alpha<\lambda})$ be according to Claim 1. As before, we fix a distinguished element $\mathbf{n}_\infty$ so that $k < \mathbf{n}_\infty$ for every standard natural number $k \in \omega$, i.e. $\mathbf{n}_\infty$ is "big enough" for our purposes.

In order to implement the picture given in the informal proof above, we make the following definitions:

Let $\mathcal{F} \in \mathbf{HF}^*$ be the tree (or rather, what $\mathbf{HF}^*$ thinks is the tree) consisting of $x : n \to (b_0 + 1)^{\subseteq \mathbf{n}_\infty}$ for $n < \mathbf{n}_\infty$ ($(b_0 + 1)^{\subseteq \mathbf{n}_\infty}$ is the set of partial functions from $\mathbf{n}_\infty$ to $b_0 + 1 = \{a \in \mathbb{N}^* : a \leq b_0\}$). Thus an element of $\mathcal{F}$ is a sequence of functions from a subset of $\mathbf{n}_\infty$ to $b_0 + 1$. x is below y in this tree if $\forall n \in \operatorname{dom} x (x(n) = y(n))$. As usual, we write $\operatorname{lth} x$ for the (possibly nonstandard) natural number n which equals the domain of x.

We imagine the position of points on the line $\mathbb{N}^*$ to be coded by a function that maps "names" or "labels" for the points to their position. In our case these labels are simply numbers below $\mathbf{n}_\infty$. The process of moving points and adding new ones (maybe even removing old ones) is then captured by a sequence of such functions, i.e. an element of $\mathcal{F}$. In the informal proof we also indicated that we want certain distances to be preserved along this process. This leads us to the following definitions:

For $x \in \mathcal{F}$ define $D(x) := \bigcup_{n<\operatorname{lth} x} \operatorname{dom} x(n)$. Furthermore define the partial function $d_x : [D(x)]^2 \to \mathbf{n}_\infty$ as $d_x(a, b) := |x(n)(a) - x(n)(b)|$ where $n := \min\{m < \operatorname{lth} x : \{a, b\} \subseteq \operatorname{dom} x(m)\}$ whenever this is defined. This is the distance of the images of a, b when they first appeared (if this ever happened) in the domain of a function in the sequence of functions x.

Coming back to the analogy of our informal proof, $D(x)$ is the set of all possible (labels of) points that appeared somewhere in x. $d_x(a,b)$ is the distance of the points a,b when they first appeared together (if they ever did).

Remember that all the defitions above are made internally in $\mathbf{HF}^*$. We hope this will always be clear enough from context.

Now let $\mathcal{T} \in \mathbf{HF}^*$ be the definable subtree of $\mathcal{F}$ which consists of all $x \in \mathcal{F}$ with the additional property that:

(a) $\forall a,b \in D(x), \forall n < \operatorname{lth} x(\{a,b\} \in \operatorname{dom} d_x \wedge \{a,b\} \subseteq \operatorname{dom} x(n) \rightarrow |x(n)(a) - x(n)(b)| \geq d_x(a,b))$

(b) $\forall n < m < \operatorname{lth} x[\max \operatorname{ran} x(m) \leq \max \operatorname{ran} x(n)]$.

(a) says that the first appearance of a and b sets a lower bound on the later distances between the images of a and b (i.e. the points labeled by a and b).

(b) says that in the sequence x, the range of some function sets an upper bound on the ranges of later functions.

Claim 2. There is a function $g : [\kappa^+]^2 \rightarrow \kappa$ so that for any $W \in [\kappa^+]^{\kappa^+}$, $g''[W]^2$ is unbounded in κ.

Proof. For any $\alpha < \beta$, define $g(\alpha,\beta)$ according to a bijection $\beta \rightarrow \kappa$. Then, given $W \in [\kappa^+]^{\kappa^+}$, there is $\beta \in W$ so that $|W \cap \beta| = \kappa$. But then $\{g(\alpha,\beta) : \alpha \in W \cap \beta\}$ is clearly unbounded in κ. □

Fix such a function g.

Claim 3. There are increasing sequences $\langle s_\alpha : \alpha < \lambda\rangle$ in $\mathcal{T}$, $\langle n_\alpha : \alpha < \lambda\rangle$ and a set $\{y_\alpha : \alpha < \kappa^+\}$, so that for all $\alpha < \lambda$ the following properties hold true of the sequence $\langle s_\beta : \beta \leq \alpha\rangle$:

(1) $n_\alpha = \max \operatorname{dom} s_\alpha$

(2) if $\alpha < \kappa^+$, then $y_\alpha \in \operatorname{dom} s_\alpha(n_\alpha)$

(3) if $\beta < \alpha \cap \kappa^+$, then $\{\gamma \leq \alpha : y_\beta \in \operatorname{dom} s_\alpha(n_\gamma)\} = [\beta, \alpha]$

(4) if $\alpha < \kappa^+$, then for all $\beta < \alpha$, $a_{g(\alpha,\beta)} \leq d_{s_\alpha}(y_\alpha, y_\beta) < a_i$ for some $i < \kappa$

(5) $\operatorname{ran} s_\alpha(n_\alpha) \subseteq b_\alpha + 1 \wedge b_\alpha \in \operatorname{ran} s_\alpha$

(6) $D(s_\alpha) < \mathbf{n}_\infty - k$ for every $k \in \omega$

Proof of Claim 3. We construct $\langle s_\alpha : \alpha < \lambda \rangle$ and $\{y_\alpha : \alpha < \kappa^+\}$ recursively in λ steps. The construction of the y_α will be finished after the first κ^+ many steps (notice that $\kappa^+ = \lambda$ might be the case).

For $\alpha = 0$, take $s_\alpha = s$, where $\operatorname{dom} s = \{0\}$ and $s(0)$ maps 0 to b_0. $y_\alpha = 0$, $n_\alpha = 0$. (1)-(6) are clearly fulfilled.

For $0 < \alpha < \lambda$, first let s be an upper bound for $\langle s_\beta : \beta < \alpha \rangle$ and $n := \max \operatorname{dom} s$. We can assume that $|D(s)| < \mathbf{n}_\infty - k$ for every standard $k \in \omega$ (Consider $(\{n_\beta : \beta < \alpha\}, \{n_k : k < \omega\})$ where $n_k := \max\{m \leq n : |D(s \restriction m)| < \mathbf{n}_\infty - k\}$ and apply Lemma 3.3.4).

Similarly we can assume that $n < \mathbf{n}_\infty - k$ for every $k \in \omega$.

Furthermore we can also assume that $\forall m \leq n$, $\operatorname{ran} s(m) \not\subseteq b_\alpha$ (Let $n' := \max\{m \leq \max \operatorname{dom} s : \operatorname{ran} s(m) \not\subseteq b_\alpha\}$, which exists and is greater than every n_β for $\beta < \alpha$ by induction hypothesis (5) and (b). Then consider $s \restriction (n'+1)$).

Our goal will now be to extend s with one new function f on top so that the inductive assumptions hold true. The construction of f will be split into two cases:

<u>Case 1</u>: $\alpha < \kappa^+$.

Then $|\alpha| \leq \kappa$. We first find a function $H \colon \alpha \to \kappa$ so that:

- $\forall \beta, \beta' < \alpha[\left|a_{H(\beta)} - a_{H(\beta')}\right| > d_s(y_\beta, y_{\beta'})]$
- $\forall \beta < \alpha[a_{H(\beta)} > a_{g(\alpha,\beta)}]$

To construct H fix a bijection $B \colon |\alpha| \to \alpha$. Assume $H(B(\xi))$ has been constructed for all $\xi < \gamma < |\alpha|$. Note that for each $\xi < \gamma$, there is $i < \kappa$ so that $d_s(y_{B(\gamma)}, y_{B(\xi)}) < a_i - a_{H(B(\xi))}$. This is because by (4), $d_s(y_{B(\gamma)}, y_{B(\xi)}) < a_j$ for some j and thus if we let $i' = \max\{j, H(B(\xi))\}$ and $i = i' + 1$, then $a_i - a_{H(B(\xi))} \geq a_i - a_{i'} > a_{i'} \geq a_j > d_s(y_{B(\gamma)}, y_{B(\xi)})$ (remember that $a_i > 2 \cdot a_{i'}$). As $\gamma < |\alpha| \leq \kappa$, we find that there is a large enough $i < \kappa$ so that $d_s(y_{B(\gamma)}, y_{B(\xi)}) < a_i - a_{H(B(\xi))}$ for every $\xi < \gamma$ and $a_{g(\alpha,B(\gamma))} < a_i$. Define $H(B(\gamma)) = i$. Then our requirements are fulfilled.

Now let y_α and y be arbitrary different elements in $\mathbf{n}_\infty \setminus D(s)$. It is possible to chose them because $|D(s)| < \mathbf{n}_\infty - 2$. Consider the following type in the variable f:

(i) $f \in b_0^{\subseteq \mathbf{n}_\infty}$

for every $\beta < \alpha$:

(ii) $y_\beta \in \operatorname{dom} f$ and $f(y_\beta) = a_{H(\beta)}$

(iii) $y_\alpha \in \operatorname{dom} f$ and $f(y_\alpha) = 0$

(iv) $y \in \operatorname{dom} f$ and $f(y) = b_\alpha$

(v) $\operatorname{ran} f \subseteq b_\alpha + 1$

(vi) $\forall a, b \in D(s)(\{a,b\} \in \operatorname{dom} d_s \wedge \{a,b\} \subseteq \operatorname{dom} f \rightarrow |f(a) - f(b)| \geq d_s(a,b))$

for every $k \in \omega$

(vii) $|D(s) \cup \operatorname{dom} f| < \mathbf{n}_\infty - k$

This type has size less than λ and it is finitely satisfiable:

Let $\Gamma \in [\alpha]^{<\omega}$ and consider the function g with $\operatorname{dom} g = \{y_\beta : \beta \in \Gamma\} \cup \{y_\alpha, y\}$, $g(y_\alpha) = 0$, $g(y) = b_\alpha$ and $g(y_\beta) = a_{H(\beta)}$. Then $\operatorname{ran} g \subseteq b_\alpha + 1$ and $|D(s) \cup \operatorname{dom} g| = |D(s)| + 2 < \mathbf{n}_\infty - k$ for every $k \in \omega$. (vi) holds true by the way we defined the function H.

Let f realize this type and let $s_\alpha := s^\frown f$, $n_\alpha := n + 1$.

First $s_\alpha \in \mathcal{T}$: (a) holds true because of (vi). (b) is true because $\max \operatorname{ran} f = b_\alpha \leq \max \operatorname{ran} s(n) \leq \max \operatorname{ran} s(m)$ for any $m \leq n$.

Secondly s_α satisfies (1)-(6). (1)-(3), (5) and (6) are trivial to check. (4) is true because of (ii),(iii) and the way we chose H.

<u>Case 2</u>: $\alpha \geq \kappa^+$.

In particular $\kappa^+ < \lambda = \mathfrak{p}$ and all y_β are already defined. Let $y \in \mathbf{n}_\infty \setminus D(s)$ be arbitrary. Again define a type:

(i) $f \in b_0^{\subseteq \mathbf{n}_\infty}$

for all $\beta < \kappa^+$

(ii) $y_\beta \in \operatorname{dom} f$

(iii) $\operatorname{ran} f \subseteq b_\alpha + 1$ and $f(y) = b_\alpha$

(iv) $\forall a, b \in D(s)(\{a, b\} \in \operatorname{dom} d_s \wedge \{a, b\} \subseteq \operatorname{dom} f \rightarrow |f(a) - f(b)| \geq d_s(a, b))$

for every $k \in \omega$

(v) $|D(s) \cup \operatorname{dom} f| < \mathbf{n}_\infty - k$

This type has size $\kappa^+ < \lambda$ and it is finitely satisfiable:

Let $\Gamma = \{\beta_0 < \cdots < \beta_k\} \in [\kappa^+]^{<\omega}$ and $i < \kappa$ so that $\max\{d_s(y_\beta, y'_\beta) : \beta, \beta' \in \Gamma\} < a_i$ (by (4)). Define the function g with $\operatorname{dom} g := \{y_\beta : \beta \in \Gamma\} \cup \{y\}$, $g(y_{\beta_l}) := l \cdot a_i$ for $l \leq k$ and $g(y) = b_\alpha$. (i)-(iii) are clear. (iv) is because $y \notin D(s)$ and for $l < l'$, $\left|g(y_{\beta_l}) - g(y_{\beta_{l'}})\right| \geq (l' - l) \cdot a_i \geq a_i > d_s(y_{\beta_l}, y_{\beta_{l'}})$. (v) is because $|D(s) \cup \operatorname{dom} g| = |D(s)| + 1 < \mathbf{n}_\infty - k$ for every $k \in \omega$.

Let f satisfy this type and let $s_\alpha := s^\frown f$, $n_\alpha = n + 1$. Again $s_\alpha \in \mathcal{T}$ and (1)-(6) are trivial to check. □

Let $\langle s_\alpha : \alpha < \lambda \rangle$, $\langle n_\alpha : \alpha < \lambda \rangle$ and $\{y_\alpha : \alpha < \kappa^+\}$ be given by Claim 3. As $\lambda < \mathfrak{t}$ there is an upper bound s of $\langle s_\alpha : \alpha < \lambda \rangle$ in $\mathcal{T}$. Let $m = \operatorname{lth} s$. Then by Theorem 3.3.5 there is a decreasing sequence $\langle m_i : i < \kappa \rangle$ below m so that $(\langle n_\alpha : \alpha < \lambda \rangle, \langle m_i : i < \kappa \rangle)$ is a cut in X.

For any $\beta < \kappa^+$ let $l(\beta) := \min\{i < \kappa : y_\beta \in \operatorname{dom} s(m_i)\}$. This is well defined because else if we define in $\mathbf{HF}^*$, $m := \max\{m \geq n_\beta : y_\beta \in \operatorname{dom} s(m)\}$ then m would fill $(\langle n_\alpha : \alpha < \lambda \rangle, \langle m_i : i < \kappa \rangle)$.

By the pigeonhole principle there is an unbounded subset $W \subseteq \kappa^+$ on which l is constant, say with value i. But now notice that $\operatorname{ran} s(m_i) \subseteq a_j$ for some $j < \kappa$ because by the definition of $\mathcal{T}$, $\max \operatorname{ran} s(m_i) \leq \max \operatorname{ran} s(n_\alpha) \leq b_\alpha$ for every $\alpha < \lambda$ and $(\langle a_i \rangle_{i<\kappa}, \langle b_\alpha \rangle_{\alpha<\lambda})$ was a cut. We also have that g is unbounded on $[W]^2$, so there are $\beta < \beta' \in W$ so that $g(\beta, \beta') > j$. Also by definition of $\mathcal{T}$, $|s(m_i)(y_\beta) - s(m_i)(y_{\beta'})| \geq d_s(y_\beta, y_{\beta'}) \geq a_{g(\beta,\beta')} > a_j$, which is contradicting $\operatorname{ran} s(m_i) \subseteq a_j$. □

3.4 Concluding $\mathfrak{p} = \mathfrak{t}$

Theorem 3.4.1. *Assume $\mathfrak{p} = \lambda < \mathfrak{t}$ and assume $\kappa < \lambda$ is regular, $\mathcal{F} \subseteq \omega^\omega$ and $\mathcal{G} \subseteq \omega^\omega$ with*

– $|\mathcal{F}| = \kappa$

– $|\mathcal{G}| = \lambda$

– $\forall f \in \mathcal{F}, \forall g \in \mathcal{G}(g <^* f)$

Then there is $X \in [\omega]^\omega$ *and* $h : X \to \omega$ *so that* $\forall f \in \mathcal{F}, \forall g \in \mathcal{G}(g \upharpoonright X <^* h <^* f \upharpoonright X)$.

Proof. Let G be $([\omega]^\omega, \subseteq^*)$ generic over $\mathbf{V}$. Then κ is still regular, and $|\mathcal{F}| = \kappa$, $|\mathcal{G}| = \lambda$ still hold true. Form the generic ultrapower of $\mathbf{HF}$ using G. Every $f \in \mathcal{F}$ and $g \in \mathcal{G}$ can be interpreted as a natural number in $\mathbf{HF}^*$ and we have that $\mathbf{HF}^* \models g < f$. By applying the results of the last sections we can find $h \in \mathbf{HF}^*$ so that h fills a precut given by $\mathcal{F}$ and $\mathcal{G}$ (define a sequence that is cofinal in $\mathcal{G}$ and one that is coinitial in $\mathcal{F}$). This means that $\mathbf{HF}^* \models g < h < f$ for every $f \in \mathcal{F}$, $g \in \mathcal{G}$. This statement must be forced by some $X \in [\omega]^\omega$ (remember that the forcing didn't add reals, so $h \in \mathbf{V}$). But then $g \upharpoonright X < h \upharpoonright X < f \upharpoonright X$ has to hold true for every $f \in \mathcal{F}$, $g \in \mathcal{G}$ because else we could pass to an extension of X that forces the contrary. □

Theorem 3.4.2. $\mathfrak{p} = \mathfrak{t}$.

Proof. Assume that $\mathfrak{p} < \mathfrak{t}$. Then Theorem 3.4.1 and Proposition 3.1.4 immediately yield a contradiction. □

It should be noted that again, as in the proof of Theorem 2.3.1, when constructing the tower of size $\mathfrak{p}$ we actually assume the whole time that $\mathfrak{p} < \mathfrak{t}$. Thus the proof tells us there is always some tower of size $\mathfrak{p}$ but it doesn't give any explicit example of such a tower. But in contrast to the proof of Theorem 2.3.1 where, in order to construct a refining tower, all we needed was to assume $\mathfrak{p} < \mathfrak{d}$ (which is consistent), here we really used the full assumption of $\mathfrak{p} < \mathfrak{t}$ at several places. This is somewhat unsatisfying because it means that the theorems that we proved become completely unsubstantial. We may thus ask what happens if we only assume $\mathfrak{p} < \mathfrak{d}$ or even $\mathfrak{p} < \mathfrak{b}$. Does this imply that any witness for $\mathfrak{p}$ has a refining tower (especially when $\mathfrak{p} \geq \omega_2$)?

In Observation 2.3.6 we constructed an example of a family with the SFIP of size $\mathfrak{d}$ and no refining tower. Thus obviously we need to assume at least $\mathfrak{p} < \mathfrak{d}$ to get that every witness for $\mathfrak{p}$ can be refined. We will show that consistently this can happen, even when $\mathfrak{b} = \mathfrak{p}$. For this we define a product like forcing that refines any given family with the SFIP (possibly adding a pseudointersection).

Definition 3.4.3. Assume $F\colon \gamma \to \mathcal{B}$ where γ is an ordinal and $\mathcal{B}$ has the SFIP. Then we define the "refining" poset $\mathbb{R}_F$ as follows. $\mathbb{R}_F$ consists of finite partial functions $p\colon \gamma \to 2^{<\omega}$ so that for all $\alpha \in \operatorname{dom} p$, $p(\alpha)(n) = 1$ implies $n \in F(\alpha)$. The extension relation is defined by $q \leq p$ iff $\operatorname{dom} p \subseteq \operatorname{dom} q$, if $\alpha \in \operatorname{dom} p$ then $p(\alpha) \subseteq q(\alpha)$ and for all $\alpha < \beta \in \operatorname{dom} p$ and $n \in \operatorname{dom} q(\beta) \setminus \operatorname{dom} p(\beta)$, $q(\beta)(n) = 1$ implies $q(\alpha)(n) = 1$.

The poset $\mathbb{R}_F$ adds a tower $\langle A_\alpha : \alpha < \gamma \rangle$ so that $A_\alpha \subseteq F(\alpha)$ for every $\alpha < \gamma$. Thus whenever F is surjective it refines $\mathcal{B}$. But note that whenever $F \restriction \alpha$ is already surjective for some $\alpha < \gamma$, then A_α will be a pseudointersection of $\mathcal{B}$. It is a standard delta system argument to show that $\mathbb{R}_F$ is ccc.

Lemma 3.4.4. *$\mathbb{R}_F$ is ccc.*

Proof. Assume $\{p_i : i < \omega_1\} \subseteq \mathbb{R}_F$. Then, by the Delta System Lemma (see [16, III.2.6]), we can assume that there is $r \in [\gamma]^{<\omega}$ so that $\operatorname{dom} p_i \cap \operatorname{dom} p_j = r$ for all $i \neq j < \omega_1$. But then we can further refine $\{p_i : i < \omega_1\}$ to assume that $p_i(\alpha) = p_j(\alpha)$ for every $i, j < \omega_1$ and $\alpha \in r$. But now, if $i \neq j$ are arbitrary, $q := p_i \cup p_j$ is a condition of $\mathbb{R}_F$. And whenever $\beta \in \operatorname{dom} p_i$, then $q(\beta) = p_i(\beta)$ and thus $\operatorname{dom} q(\beta) \setminus \operatorname{dom} p_i(\beta) = \emptyset$. This shows that $q \leq p_i$ and in the same way we have that $q \leq p_j$. Thus $\{p_i : i < \omega_1\}$ cannot be an antichain. $\square$

Lemma 3.4.5. *Assume $F\colon \gamma \to \mathcal{B}$ and $\alpha \leq \gamma$. Then $\mathbb{R}_{F\restriction\alpha} \lessdot \mathbb{R}_F$.*

Proof. We show that the natural inclusion map from $\mathbb{R}_{F\restriction\alpha}$ to $\mathbb{R}_F$ is a complete embedding (see [16, III.3.65] for a definition). It is obvious that this map preserves the extension and incompatibility relation. We only need to show that any maximal antichain A in $\mathbb{R}_{F\restriction\alpha}$ is still maximal in $\mathbb{R}_F$. For this simply note that p, q are compatible iff $p \restriction (\operatorname{dom} p \cap \operatorname{dom} q)$ and $q \restriction (\operatorname{dom} p \cap \operatorname{dom} q)$ are compatible. Thus whenever $p \in \mathbb{R}_F$ is incompatible with every $q \in A$, then $p \restriction \alpha \in \mathbb{R}_{F\restriction\alpha}$ has to be incompatible with every q. $\square$

Before we get to our forcing construction we need to recall some well established preservation results for towers and unbounded families.

Lemma 3.4.6 (Baumgartner, Dordal [2])**.** *Assume $\bar{A} = \langle A_\alpha : \alpha < \kappa \rangle$ is a maximal tower, δ is limit and $\langle \mathbb{P}_i \rangle_{i \leq \delta}$ a ccc finite support iteration so that $\Vdash_{\mathbb{P}_i}$ "$\bar{A}$ is maximal" for every $i < \delta$. Then $\Vdash_{\mathbb{P}_\delta}$ "$\bar{A}$ is maximal".*

Lemma 3.4.7 (e.g. [1])**.** *Assume $\bar{f} = \langle f_\alpha : \alpha < \kappa \rangle$ is an unbounded family well-ordered by $<^*$, δ is limit and $\langle \mathbb{P}_i \rangle_{i \leq \delta}$ a ccc finite support iteration so that $\Vdash_{\mathbb{P}_i}$ "$\bar{f}$ is unbounded" for every $i < \delta$. Then $\Vdash_{\mathbb{P}_\delta}$ "$\bar{f}$ is unbounded".*

Lemma 3.4.8. *Let κ be regular. Assume $\bar{A} = \langle A_\alpha : \alpha < \kappa \rangle$ is a maximal tower and $|\mathbb{P}| < \kappa$. Then $\Vdash_{\mathbb{P}}$ "$\bar{A}$ is maximal". The analogue statement holds for $\langle f_\alpha : \alpha < \kappa \rangle$, an unbounded family well-ordered by $<^*$.*

Proof. Assume $\dot{x}$ is a $\mathbb{P}$ name for a real which is forced to be a pseudointersection of $\bar{A}$. Then, as $|\mathbb{P}| < \kappa$, we find one condition $p \in \mathbb{P}$, one natural number $n \in \omega$ and an unbounded set $X \subseteq \kappa$ so that $p \Vdash \dot{x} \setminus n \subseteq A_\alpha$ for every $\alpha \in X$. But then $p \Vdash \dot{x} \setminus n \subseteq \bigcap_{\alpha \in X} A_\alpha$ which means that $\bigcap_{\alpha \in X} A_\alpha$ has to be infinite. But then $\bigcap_{\alpha \in X} A_\alpha$ is a pseudointersection of $\bar{A}$ because X is unbounded in κ.

The proof for $\langle f_\alpha : \alpha < \kappa \rangle$ is essentially the same. □

Lemma 3.4.9. *Let κ be regular uncountable and suppose that $\bar{A} = \langle A_\alpha : \alpha < \kappa \rangle$ is a maximal tower. Further assume that $F \colon \kappa \to \mathcal{B}$ where $\mathcal{B}$ has the SFIP. Then $\Vdash_{\mathbb{R}_F}$ "$\bar{A}$ is maximal". The analogue statement holds for $\langle f_\alpha : \alpha < \kappa \rangle$, an unbounded family well-ordered by $<^*$.*

Proof. Assume $\dot{x}$ is a name for a real. Then, by the ccc, $\dot{x}$ can be decided using only conditions in $\mathbb{R}_{F \restriction \alpha}$ for $\alpha < \kappa$. We have that $\mathbb{R}_{F \restriction \alpha} \lessdot \mathbb{R}_F$ and thus a $\mathbb{R}_F$ generic induces a $\mathbb{R}_{F \restriction \alpha}$ generic. As $|\mathbb{R}_{F \restriction \alpha}| < \kappa$ we can apply Lemma 3.4.8. Again the proof for $\langle f_\alpha : \alpha < \kappa \rangle$ is the same. □

Theorem 3.4.10. *Assume GCH and let $\kappa < \lambda$ be uncountable regular cardinals. Then there is a ccc forcing extension in which $\mathfrak{p} = \mathfrak{b} = \kappa < \mathfrak{d} = \lambda$ and for every family with the SFIP of size $\mathfrak{p}$ there is a refining tower.*

Proof. Using standard bookkeeping arguments we can construct a ccc finite support iteration $\langle \mathbb{P}_i, \dot{\mathbb{Q}}_i : i < \lambda \rangle$ using posets of the form $\mathbb{R}_F$, so that whenever $\dot{\mathcal{B}}$ is a $\mathbb{P}_\lambda$ name for a family with the SFIP of size less or equal to κ, then there is $i < \lambda$ so that $\dot{\mathbb{Q}}_i$ is of the form $\dot{\mathbb{R}}_{\dot{F}}$ for $\dot{F}$ a $\mathbb{P}_i$ name for a surjection from κ to $\dot{\mathcal{B}}$. Note that whenever $|\mathcal{B}| < \kappa$, then a corresponding poset of the form $\mathbb{R}_F$ adds a pseudointersection of $\mathcal{B}$. This shows that in an extension by $\mathbb{P}_\lambda$, $\mathfrak{p} \geq \kappa$. Further we have that, by a reflection argument, there is $\delta < \lambda$ of cofinality κ so that in $\mathbf{V}^{\mathbb{P}_\delta}$, $\kappa \leq \mathfrak{p} \leq \mathfrak{b} \leq \kappa$, and thus $\mathfrak{p} = \mathfrak{b} = \kappa$ ($\mathfrak{b} \leq \kappa$ because of cofinaly many Cohen reals added). This means that in this extension there is a maximal tower of length κ and an unbounded family of functions well-ordered by $<^*$ of size κ. Using Lemma 3.4.9, Lemma 3.4.6 and Lemma 3.4.7 we find that this tower is still maximal in $\mathbf{V}^{\mathbb{P}_\lambda}$ and that the family of functions stays unbounded. This shows that in $\mathbf{V}^{\mathbb{P}_\lambda}$, $\mathfrak{p} = \mathfrak{b} = \kappa$. Further we have that after forcing with $\mathbb{P}_\lambda$, $\mathfrak{d} = \mathfrak{c} = \lambda$ because we add cofinally many Cohen reals and thus unbounded reals. □

Remark. The same theorem as above but with $\mathfrak{p} = \kappa < \mathfrak{b} = \mathfrak{d} = \lambda$ also holds true. This can be achieved with the same construction, additionally adding Hechler dominating reals. The preservation results from [2] then imply that in this extension $\mathfrak{p} = \kappa$.

Now the really interesting question is whether $\mathfrak{p} < \mathfrak{d}$ or $\mathfrak{p} < \mathfrak{b}$ already imply the conclusion of the above theorem. A positive result would be a remarkable and very surprising new proof of $\mathfrak{p} = \mathfrak{t}$. We believe that the answers to both questions are negative, although we couldn't get any simple construction to work. Also this quite natural question seems, to our best knowledge, to be left untouched by the literature. In [8] the authors investigate which filters contain towers, but our question is not adressed. Also, we ask for filters being refined by towers, which do not need to be contained in the filter.

In [28], the consistency of a ($\mathfrak{p}$,ω_1)-peculiar cut (see the end of Section 3.1) is claimed, where $\omega_1 < \mathfrak{p} < \mathfrak{c}$. Such a cut induces a witness of $\mathfrak{p}$ with no refining tower in the following way. Given a peculiar cut $(\langle f_i : i < \kappa\rangle, \langle g_i : i < \omega_1\rangle)$ with $\kappa > \omega_1$, we can define $F_i = \{(n,m) \in \omega \times \omega : m \geq f(n)\}$ for $i < \kappa$ and $G_i = \{(n,m) \in \omega \times \omega : m \leq g(n)\}$ for $i < \omega_1$. Then $\mathcal{B} = \{F_i, G_j : i < \kappa, j < \omega_1\}$ has the SFIP and if $\langle A_\alpha : \alpha < \kappa\rangle$ refines $\mathcal{B}$ then there is some $\alpha < \kappa$ so that $h <^* g_i$ for all $i < \omega_1$ where $h(n) = \max(\{m \in \omega : (n,m) \in A_\alpha\} \cup \{0\})$. But then $h <^* f_i$ for some $i < \kappa$ which implies that $A_\alpha \cap F_i$ is finite. We do not know the values of $\mathfrak{b}$ and $\mathfrak{d}$ in the model of [28].

One related question is the following:

Question. Does $\mathfrak{p} = \mathfrak{u} < \mathfrak{d}$ (where $\mathfrak{u}$ is the least size of an ultrafilter base) imply that every ultrafilter generated by $\mathfrak{u}$ many sets is generated by a tower?

By Lemma 2.3.2 a positive answer requires $\mathfrak{u} \geq \omega_2$ and in the model of $\omega_2 \leq \mathfrak{u} < \mathfrak{d}$ constructed by Blass and Shelah in [6], the witness for $\mathfrak{u}$ is generated by a tower.

Chapter 4

Generalizations to uncountable κ

The notion of almost inclusion can be generalized to arbitrary regular cardinals κ. For $A, B \in [\kappa]^\kappa$ we write $A \subseteq^* B$ iff $|A \setminus B| < \kappa$. Thus we call $X \in [\kappa]^\kappa$ a pseudointersection of $\mathcal{B} \subseteq [\kappa]^\kappa$ when $X \subseteq^* B$ for every $B \in \mathcal{B}$. Similarly we can generalize the $\leq^*$ relation to κ^κ: $f \leq^* g$ iff $|\{\alpha \in \kappa : g(\alpha) < f(\alpha)\}| < \kappa$, $f <^* g$ is defined analogously.

We may thus generalize the cardinal characteristics that we defined for $\kappa = \omega$ to the more general case of κ regular uncountable. Throughout this chapter we fix κ to be a regular uncountable cardinal.

Definition 4.0.1. The κ-bounding number $\mathfrak{b}(\kappa)$ is defined as

$$\mathfrak{b}(\kappa) := \min\{|\mathcal{B}| : \mathcal{B} \text{ is unbounded wrt } \leq^*\}.$$

The κ-dominating number $\mathfrak{d}(\kappa)$ is defined as

$$\mathfrak{d}(\kappa) := \min\{|\mathcal{D}| : \mathcal{D} \text{ is dominating wrt } \leq^*\}.$$

Note that the definition of $\mathfrak{b}(\kappa)$, $\mathfrak{d}(\kappa)$ also works when κ is singular, but we will only consider the case of regular cardinals which reflect much more of the properties that hold at ω. Some of the cardinal characteristics for singular cardinals have been studied in the author's Bachelor thesis ([23],[24]).

Before we define the analogues of $\mathfrak{p}$ and $\mathfrak{t}$ we like to make the following remark:

Remark. Whenever κ is regular uncountable there is a sequence $\langle A_n : n \in \omega \rangle$, $\forall n(A_n \in [\kappa]^\kappa)$, which is decreasing wrt $\subseteq^*$ and has no pseudointersection. For this just partition κ as $\bigcup_{n\in\omega} X_n$, where $|X_n| = \kappa$, and take $A_n := \bigcup_{m\geq n} X_m$. If

X was a pseudointersection of $\langle A_n : n \in \omega \rangle$ then one finds α so that $X \setminus \alpha \subseteq A_n$ holds for all n, which is impossible as $\bigcap_{n\in\omega} A_n = \emptyset$.

Thus we have to make some restrictions in our general definition.

Definition 4.0.2. A family $\mathcal{B} \subseteq [\kappa]^\kappa$ has the κ-intersection property (κ-IP) if for every $\mathcal{B}' \in [\mathcal{B}]^{<\kappa}$, $\bigcap \mathcal{B}' \in [\kappa]^\kappa$.

A sequence $\langle A_\alpha : \alpha < \delta \rangle$ in $[\kappa]^\kappa$ is called a κ-tower if $\forall \alpha < \beta < \delta (A_\beta \subseteq^* A_\alpha)$ and $\{A_\alpha : \alpha < \delta\}$ has the κ-IP.

The definitions of $\mathfrak{p}(\kappa)$ and $\mathfrak{t}(\kappa)$ now come naturally:

Definition 4.0.3. The κ-pseudointersection number $\mathfrak{p}(\kappa)$ is defined as

$$\mathfrak{p}(\kappa) := \min\{|\mathcal{B}| : \mathcal{B} \text{ has the } \kappa\text{-IP but no pseudointersection}\}.$$

The κ-tower number $\mathfrak{t}(\kappa)$ is defined as

$$\mathfrak{t}(\kappa) := \min\{\delta : \text{there is a } \kappa\text{-tower of length } \delta \text{ that cannot be further extended}\}.$$

The reader may not have noticed this but we actually didn't justify that the sets in the definition above are non-empty. In the case $\kappa = \omega$ this follows by an easy application of Zorn's lemma. But when κ is uncountable this is less clear. Our justification for $\mathfrak{p}(\kappa)$ is that the set of club subsets of κ have the κ-IP and no pseudointersection. For $\mathfrak{t}(\kappa)$ apply Zorn's lemma to construct a κ-tower only consisting of clubs (and notice that the closure of a pseudointersection of clubs is club and still a pseudointersection).

The following follows in a straightforward manner.

Theorem 4.0.4. $\mathfrak{t}(\kappa)$ *is a regular cardinal.* $\kappa^+ \leq \mathfrak{p}(\kappa) \leq \mathfrak{t}(\kappa) \leq 2^\kappa$, $\kappa^+ \leq \mathfrak{b}(\kappa) \leq \mathfrak{d}(\kappa) \leq 2^\kappa$.

It can also be shown that $\mathfrak{p}(\kappa)$ is regular (see [13]) and that $\mathfrak{b}(\kappa)$ is regular (the proof is as for $\kappa = \omega$). Other known results that we won't prove include the generalization of Theorem 2.1.1, which states that if $\kappa^{<\kappa} = \kappa$ and $\kappa \leq \mu < \mathfrak{t}(\kappa)$, then $2^\mu = 2^\kappa$ (see [30]).

We will show in the next section that also $\mathfrak{t}(\kappa) \leq \mathfrak{b}(\kappa)$.

4.1 A characterization of bounding

In this section we want to point out a nice characterization of the bounding and dominating numbers for regular uncountable κ. Our feeling is that this has been used implicitly before but was never mentioned explicitly. This is surprising because we find it very natural and useful. For example the proof of the quite surprising result that $\mathfrak{s}(\kappa) \leq \mathfrak{b}(\kappa)$ for regular uncountable κ will become, a posteriori, almost straightforward. We will also draw conclusions for the generalized version of Mathias forcing.

We begin with the following well known and easy fact. It says that the "closure" points of any $f \in \kappa^\kappa$ make a club set.

Lemma 4.1.1. *For any $f \in \kappa^\kappa$, the set $C_f := \{\alpha \in \kappa : f''\alpha \subseteq \alpha\}$ is a club in κ.*

For now let us define $C_g := \{\alpha \in \kappa : g''\alpha \subseteq \alpha\}$ for any $g \in \kappa^\kappa$.

Lemma 4.1.2. *Whenever $f \leq^* g$, then $C_g \subseteq^* C_f$.*

Proof. Assume $\delta < \kappa$ is such that $f(\alpha) \leq g(\alpha)$ for all $\alpha \geq \delta$ and let additionally $\delta \in C_f$. Now assume $\gamma \in C_g \setminus C_f$, $\gamma \geq \delta$. But this means that γ is not closed under f but closed under g. But then for some i with $\delta \leq i < \gamma$, $f(i) \geq \gamma > g(i)$, which is a contradiction. □

For any club C we can consider the function f_C defined by $f_C(\alpha) := \min C \cap (\alpha, \kappa)$. Then $C_{f_C} = \lim C \cup \{0\}$ ($\lim C$ is the set of limit points of C).

Lemma 4.1.3. *Whenever $C \subseteq^* C'$, then f_C dominates $f_{C'}$.*

Proof. Let δ be such that $C \setminus \delta \subseteq C'$. Let $i > \delta$. Then $f_{C'}(i) = \min C' \cap (i, \kappa) \leq \min C \cap (i, \kappa) = f_C(i)$. □

Lemma 4.1.4. *For any $f \in \kappa^\kappa$, $f_{C_f} > f$.*

Proof. $f_{C_f}(i) = \min C_f \cap (i, \kappa) > f(i)$. □

Now we get:

Proposition 4.1.5. $\mathfrak{b}(\kappa) = \min\{|\mathcal{B}| :$ *$\mathcal{B}$ is a set of clubs with no pseudointersection*$\}$

Notice that when $X \subseteq^* C$ where C is club then $\bar{X} \subseteq^* C$ where $\bar{X}$ is the closure of X.

Proof. For $\mathcal{G}$ an unbounded family, consider $\mathcal{B} := \{C_g : g \in \mathcal{G}\}$. This family has no club pseudointersection, because if C would be one then f_C would dominate all f_{C_g} for $g \in \mathcal{G}$ which in turn dominates g.

On the other hand, if $\mathcal{H}$ is a semanuret of clubs with no pseudointersection, then $\mathcal{G} := \{f_C : C \in \mathcal{H}\}$ is unbounded. Because if g dominates all f_C, then C_g is almost included in all $C_{f_C} = \lim C \cup \{0\}$ for $C \in \mathcal{B}$. $\square$

There is now a dual characterization of $\mathfrak{d}(\kappa)$ in terms of clubs and almost containment, which basically says that $\mathfrak{d}(\kappa)$ is the cofinality of the club filter at κ.

Proposition 4.1.6. $\mathfrak{d}(\kappa) = \min\{|\mathcal{F}| : \mathcal{F} \subseteq [\kappa]^\kappa, \forall C \in \mathcal{C} \exists X \in \mathcal{F}(X \subseteq^* C)\}$, *where $\mathcal{C}$ denotes the club filter.*

Proof. The proof is dual to the one for $\mathfrak{b}(\kappa)$. $\square$

We can also translate the above characterization to the following:

Proposition 4.1.7. $\mathfrak{b}(\kappa)$ *is the smallest size of a collection $\mathcal{F}$ of non stationary sets so that for every $X \in [\kappa]^\kappa$ there is $N \in \mathcal{F}$ with $|N \cap X| = \kappa$.*

Dually:

Proposition 4.1.8. $\mathfrak{d}(\kappa)$ *is the smallest size of a collection $\mathcal{F}$ of clubs so that for every N non-stationary there is $C \in \mathcal{F}$ with $|N \cap C| < \kappa$.*

Theorem 4.1.9. $\kappa^+ \leq \mathfrak{p}(\kappa) \leq \mathfrak{t}(\kappa) \leq \mathfrak{b}(\kappa) \leq \mathfrak{d}(\kappa) \leq 2^\kappa$.

Proof. We only need to show that $\mathfrak{t}(\kappa) \leq \mathfrak{b}(\kappa)$. For this we construct a maximal tower of length $\mathfrak{b}(\kappa)$. Let $\{C_\alpha : \alpha < \mathfrak{b}(\kappa) = \lambda\}$ be a collection of clubs on κ with no pseudointersection as given by Proposition 4.1.5. We construct a tower $\langle A_\alpha : \alpha < \lambda \rangle$ consisting only of clubs as follows:

- $A_0 = C_0$
- $A_{\alpha+1} = A_\alpha \cap C_\alpha$
- If γ is limit then $\langle A_\alpha : \alpha < \gamma \rangle$ has a pseudointersection C' as $\gamma < \lambda = \mathfrak{b}(\kappa)$. Let C be the closure of C'. Then C is still a pseudointersection of $\langle A_\alpha : \alpha < \gamma \rangle$. Let $A_\gamma = C \cap C_\gamma$.

The constructed tower is obviously maximal. $\square$

Definition 4.1.10. Assume $\mathcal{U}$ is a κ-complete ultrafilter on κ (i.e. κ is measurable). Then Mathias forcing for $\mathcal{U}$, $\mathbb{M}(\mathcal{U})$ consists of pairs $(a, U) \in [\kappa]^{<\kappa} \times \mathcal{U}$. The order is defined as $(b, V) \leq (a, U)$ iff $a \subseteq b$, $V \subseteq U$ and $b \setminus a \subseteq U$.

Mathias forcing for a κ-complete ultrafilter is the natural analogue of Mathias forcing for ultrafilters at ω. One of its main features is that it adds unsplit reals and can be used to make the splitting number $\mathfrak{s}$ large. On the other hand it is sometimes possible to construct the ultrafilter in a special way, so that some fixed unbounded family of reals will be preserved unbounded. This can be used to get a model where $\mathfrak{b} < \mathfrak{s}$ (see e.g. [9]).

However at κ the situation seems to be very different. First of all we will show that Mathias forcing for κ-complete ultrafilters always adds dominating reals. Secondly it is a result by Raghavan and Shelah ([20]) that $\mathfrak{s}(\kappa) \leq \mathfrak{b}(\kappa)$ for κ regular uncountable. We will give an intuitive sketch of the proof. The proof of this result was also covered in [23].

Lemma 4.1.11 (Scott, see [15])**.** *Assume $\mathcal{U}$ is a κ-complete ultrafilter on κ. Then there is a function $f \in \kappa^\kappa$, so that the κ-complete ultrafilter $\mathcal{V} = \{X \subseteq \kappa : f^{-1}(X) \in \mathcal{U}\}$ extends the club filter.*

Proposition 4.1.12. *Assume $\mathcal{U}$ is a κ-complete ultrafilter on κ. Then $\mathbb{M}(\mathcal{U})$ adds a κ-dominating real.*

Proof. Let f be as in Lemma 4.1.11. $\mathbb{M}(\mathcal{U})$ adds a pseudointersection X of $\mathcal{U}$. Let C be an arbitrary ground model club on κ. Then $f^{-1}(C) \in \mathcal{U}$ and so $X \subseteq^* f^{-1}(C)$. But then $f''X \subseteq^* C$. Furthermore, as $\mathcal{V}$ contains only unbounded sets, we have that $|f''U| = \kappa$ for all $U \in \mathcal{U}$. Thus by genericity we have that $|f''X| = \kappa$. So we have shown that $f''X$ is a pseudointersection of the ground model club filter. □

Definition 4.1.13. A family $\mathcal{S} \subseteq [\kappa]^\kappa$ is called splitting if $\forall X \in [\kappa]^\kappa \exists Y \in \mathcal{S}(|X \cap Y| = \kappa \wedge |X \setminus Y| = \kappa)$. $\mathfrak{s}(\kappa)$ is the least size of a splitting family.

Theorem 4.1.14. $\mathfrak{s}(\kappa) \leq \mathfrak{b}(\kappa)$.

Proof sketch. Let $\mathcal{B}$ be a family of clubs and M and elementary submodel of some large enough $H(\theta)$ of size $|\mathcal{B}|$ containing all elements of $\mathcal{B}$ and κ.

Suppose $M \cap [\kappa]^\kappa$ is not splitting, i.e. there is $X \in [\kappa]^\kappa$ unsplit over M. Then X generates an ultrafilter $\mathcal{U} = \{Y \in M \cap [\kappa]^\kappa : X \subseteq^* Y\}$ over M, κ-complete over M (this means that any $\mathcal{A} \subseteq \mathcal{U}$ in M of size less then κ has intersection in $\mathcal{U}$).

Using a similar proof as the one of Lemma 4.1.11 we find $f \in \kappa^\kappa \cap M$ so that $\mathcal{V} = \{A \in [\kappa]^\kappa \cap M : f^{-1}(A) \in \mathcal{U}\}$ extends the club filter on M. Then X induces a pseudointersection $(f''X)$ of $\mathcal{V} \supseteq \mathcal{B}$. □

A lot of the combinatorics at ω generalize in a straightforward manner to the uncountable case. But the results in this section indicate a tendency in the other direction. Namely that cardinal characteristics at κ can behave very differently to their analogues at ω. Our feeling is that one of the main reasons for this behavior is the additional existence of clubs at κ. As an additional example we mention that the proof of $\mathfrak{d}(\kappa) = \kappa^+ \to \mathfrak{a}(\kappa) = \kappa^+$ by Blass, Hyttinen and Zhang (see [5]) makes an essential use of club guessing sequences (this result has recently been weakend to $\mathfrak{b}(\kappa) = \kappa^+ \to \mathfrak{a}(\kappa) = \kappa^+$, see [21]). $\mathfrak{a}(\kappa)$ is the almost disjointness number. For $\mathfrak{a} = \mathfrak{a}(\omega)$ it is still unknown whether the analogue implication ($\mathfrak{d} = \omega_1 \to \mathfrak{a} = \omega_1$) holds.

4.2 The generalized meager ideal

The ideal of meager subsets of 2^ω, denoted by $\mathcal{M}$, has been studied a lot over past decades (see [1] for an extensive study of the relationship between the ideals of meager and null sets; [19] is another classic text that treats this topic). In particular, the related cardinal characteristics $\mathrm{add}(\mathcal{M})$, $\mathrm{cov}(\mathcal{M})$, $\mathrm{non}(\mathcal{M})$ and $\mathrm{cof}(\mathcal{M})$ have shown to have interesting characterizations in terms of combinatorial properties of sets of reals. This is used to draw strong connections to the notion of bounding, which appears in the famous Cichoń Diagram whose middle part (the one concerning the meager ideal) is shown below (the arrows indicate inequalities between the cardinals).

The additivity $\mathrm{add}(\mathcal{M})$ of the meager ideal is the least size of of a collection of meager sets whose union is not meager anymore. $\mathrm{cov}(\mathcal{M})$, the covering number, is the least number of meager sets needed to cover the whole real line. $\mathrm{non}(\mathcal{M})$, the uniformity number, is the smallest size of a non-meager set and $\mathrm{cof}(\mathcal{M})$, the cofinality, is the smallest size of a base for the meager ideal. All of these cardinals can easily be seen to lie between ω_1 and $\mathfrak{c}$. Apart from the inequalities shown in Cichoń's Diagram, we have that $\mathrm{add}(\mathcal{M}) = \min\{\mathfrak{b}, \mathrm{cov}(\mathcal{M})\}$ and $\mathrm{cof}(\mathcal{M}) = \max\{\mathfrak{d}, \mathrm{non}(\mathcal{M})\}$.

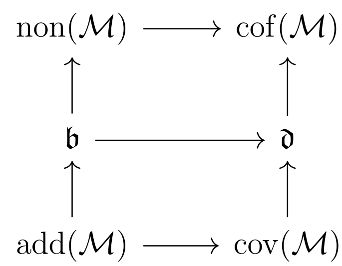


Figure 4.1: The middle part of Cichoń's Diagram

Recently there has been increasing interest in generalizing the classical results (ZFC results as well as independence results) to the space 2^κ for κ regular uncountable ([12],[5],[27],[29],[7]). The topology we put on 2^κ is the κ-box topology, also called bounded topology, i.e. the topology generated by basic open sets of the form $[s] = \{x \in 2^\kappa : s \subseteq x\}$ where $s \in 2^{<\kappa}$. The definition of the meager ideal $\mathcal{M}_\kappa$ on 2^κ and the cardinals $\mathrm{add}(\mathcal{M}_\kappa), \mathrm{cov}(\mathcal{M}_\kappa), \mathrm{non}(\mathcal{M}_\kappa)$ and $\mathrm{cof}(\mathcal{M}_\kappa)$ is then done as in the case of 2^ω (we will later define formally those that we need). One of the main results of [7] is that the diagram pictured in Figure 4.1 holds more generally for κ which is strongly inaccessible. Moreover in this case we still have that $\mathrm{add}(\mathcal{M}_\kappa) = \min\{\mathfrak{b}(\kappa), \mathrm{cov}(\mathcal{M}_\kappa)\}$ and $\mathrm{cof}(\mathcal{M}_\kappa) = \max\{\mathfrak{d}(\kappa), \mathrm{non}(\mathcal{M}_\kappa)\}$.

Another one of the classical results that hold on 2^ω, is the following connection with the tower number:

Theorem 4.2.1. $\mathfrak{t} \leq \mathrm{add}(\mathcal{M})$.

For a direct proof of this inequality we refer to [4]. It also follows from Corollary 2.2.5 and Theorem 3.4.2. We want to generalize this theorem to the case where κ is regular and uncountable and $\kappa^{<\kappa} = \kappa$. The last assumption is needed because by [7, Observation 23], $\mathrm{add}(\mathcal{M}_\kappa)$ is equal to κ^+ whenever $\kappa^{<\kappa} > \kappa$. Let us define more clearly what $\mathrm{add}(\mathcal{M}_\kappa)$ is. We always assume that κ is regular.

Definition 4.2.2. A set $X \subseteq 2^\kappa$ is called nowhere dense, if for any $s \in 2^{<\kappa}$ there is $s' \in 2^{<\kappa}$, $s' \supseteq s$ so that $[s'] \cap X = \emptyset$. A set $X \subseteq 2^\kappa$ is called meager, if it is the union of κ many nowhere dense sets. A set $X \subseteq 2^\kappa$ is called comeager if it is the complement of a meager set. We let $\mathcal{M}_\kappa$ be the collection of meager sets.

Definition 4.2.3. $\mathrm{add}(\mathcal{M}_\kappa) = \min\{|\mathcal{A}| : \mathcal{A} \subseteq \mathcal{M}_\kappa, \bigcup \mathcal{A} \notin \mathcal{M}_\kappa\}$.

Theorem 4.2.4. *Assume $\kappa = \kappa^{<\kappa}$, then $\mathfrak{t}(\kappa) \leq \mathrm{add}(\mathcal{M}_\kappa)$.*

In order to prove the theorem we introduce a notion of "club" subset of $2^{<\kappa}$:

Definition 4.2.5. Let $C \subseteq 2^{<\kappa}$. Then we call C club iff:

– $\forall s \in 2^{<\kappa} \exists s' \supseteq s (s' \in C)$

– for every sequence $\langle s_i : i < \delta \rangle$ where $\delta < \kappa$, $s_i \in C$ for every $i < \delta$ and $s_i \subseteq s_j$ for $i < j$, $\bigcup_{i<\delta} s_i \in C$.

Lemma 4.2.6. *Let $\delta < \kappa$, C_i a club subset of $2^{<\kappa}$ for every $i < \delta$. Then $\bigcap_{i<\delta} C_i$ is also club.*

Proof. We first show that $\bigcap_{i<\delta} C_i$ is dense. Given $s \in 2^{<\kappa}$ construct an increasing sequence (wrt $\subseteq$) $\langle s_\alpha : \alpha < \delta \cdot \omega \rangle$ so that $s_{\delta \cdot n + i} \in C_i$ for every $n \in \omega$ and $i < \delta$ and $s_0 \supseteq s$. Then clearly $\bigcup_{\alpha < \delta \cdot \omega} s_\alpha \in \bigcap_{i<\delta} C_i$.

To show that $\bigcap_{i<\delta} C_i$ is closed is trivial using that each C_i is closed. □

For C, D clubs on $2^{<\kappa}$ let us write $C \subseteq^{**} D$ iff $C \setminus D \subseteq 2^{<\alpha}$ for some $\alpha < \kappa$. Note that $\subseteq^{**}$ agrees with $\subseteq^*$ in case κ is inaccessible. We call a sequence $\langle D_\alpha : \alpha < \gamma \rangle$ where each $D_\alpha \subseteq 2^{<\kappa}$ is club and $D_\alpha \subseteq^{**} D_\beta$ for $\beta < \alpha$, a $\subseteq^{**}$-tower on $2^{<\kappa}$.

Lemma 4.2.7. *Let $\kappa^{<\kappa} = \kappa$. Assume $\langle D_\alpha : \alpha < \gamma \rangle$ is a $\subseteq^{**}$-tower on $2^{<\kappa}$ and $\gamma < \mathfrak{t}(\kappa)$, $\mathrm{cf}(\gamma) \geq \kappa$. Then there is a set $C \in [2^{<\kappa}]^\kappa$ so that $C \subseteq^* D_\alpha$ for every $\alpha < \gamma$ and $C \not\subseteq 2^{<\alpha}$ for any $\alpha < \kappa$.*

Proof. The case where κ is inaccessible is obvious as $\langle D_\alpha : \alpha < \gamma \rangle$ is simply a tower which cannot be maximal.

If κ is not inaccessible we find a sequence $\langle \alpha_i : i < \kappa \rangle$ cofinal in κ so that

$$\left| 2^{<\alpha_i} \setminus \bigcup_{j<i} 2^{<\alpha_j} \right| = \kappa$$

for every $i < \kappa$. Write $X_i := 2^{<\alpha_i} \setminus \bigcup_{j<i} 2^{<\alpha_j}$ and fix bijections $G_i \colon \kappa \to X_i$.

We define for every $\alpha < \gamma$ a function $f_\alpha \in \kappa^\kappa$ as follows:

$$f_\alpha(i) = \begin{cases} \min\{j < \kappa : G_i(j) \in D_\alpha\} & \text{if } X_i \cap D_\alpha \neq \emptyset \\ 0 & \text{else} \end{cases}$$

As $\gamma < \mathfrak{t}(\kappa) \leq \mathfrak{b}(\kappa)$ there is $f \in \kappa^\kappa$ so that $f_\alpha <^* f$ for every $\alpha < \gamma$. Define $C_\alpha = \bigcup_{i<\kappa} D_\alpha \cap G_i'' f(i)$ for every $\alpha < \gamma$. It is easy to see that $\langle C_\alpha : \alpha < \gamma \rangle$ is a tower. Thus we find C a pseudointersection of $\langle C_\alpha : \alpha < \gamma \rangle$. C is as required. □

Lemma 4.2.8. *Let $\kappa^{<\kappa} = \kappa$. Assume $\langle D_\alpha : \alpha < \gamma \rangle$ is a $\subseteq^{**}$-tower on $2^{<\kappa}$ and $\gamma < \mathfrak{t}(\kappa)$. Then there is a club C on $2^{<\kappa}$ so that $C \subseteq^{**} D_\alpha$ for every $\alpha < \gamma$.*

Proof. In case $\mathrm{cf}(\gamma) < \kappa$, pass to a cofinal subsequence of $\langle D_\alpha : \alpha < \gamma \rangle$ of cofinality $\mathrm{cf}(\gamma)$ and take an intersection D which is club by Lemma 4.2.6. D is as required.

Now assume $\mathrm{cf}(\gamma) \geq \kappa$. For each $s \in 2^{<\kappa}$ and $\alpha < \gamma$ let $D^s_\alpha := \{s' \in D_\alpha : s' \supseteq s\}$. Applying Lemma 4.2.7, for each $s \in 2^{<\kappa}$ there is a set C_s which is a pseudointersection of $\langle D^s_\alpha : \alpha < \gamma \rangle$ and $C_s \not\subseteq 2^{<\alpha}$ for any $\alpha < \kappa$.

For each $\alpha < \gamma$ we define a function $f_\alpha \colon 2^{<\kappa} \to \kappa$ so that $C_s \setminus 2^{<f_\alpha(s)} \subseteq D_\alpha$ for every $s \in 2^{<\kappa}$. As $\gamma < \mathfrak{t}(\kappa) \leq \mathfrak{b}(\kappa)$ we can find a single $f \colon 2^{<\kappa} \to \kappa$ so that $|\{s \in 2^{<\kappa} : f_\alpha(s) \geq f(s)\}| < \kappa$ for every $\alpha < \gamma$. For each $s \in 2^{<\kappa}$ choose $\sigma_s \in C_s \setminus 2^{<f(s)}$. The set $C' = \{\sigma_s : s \in 2^{<\kappa}\}$ is a pseudointersection of $\langle D_\alpha : \alpha < \gamma \rangle$. C' is not necessarily club but if we let C be the closure of C', then C is club and for any $\alpha < \gamma$, as $C' \setminus 2^{<\beta} \subseteq D_\alpha$ for some $\beta < \kappa$, we have that $C \subseteq^{**} D_\alpha$.

C is as required. □

Proof of Theorem 4.2.4. Assume $\langle Y_\alpha : \alpha < \lambda \rangle$ are open dense sets in 2^κ and $\lambda < \mathfrak{t}(\kappa)$. The theorem is proven if we show that $\bigcap_{\alpha<\lambda} Y_\alpha$ is comeager.

First note that for each α we can write $Y_\alpha = \bigcup_{s \in S_\alpha} [s]$ where $S_\alpha \subseteq 2^{<\kappa}$ and S_α is upwards closed, i.e. $s' \supseteq s \in S_\alpha$ implies $s' \in S_\alpha$. Each set S_α is clearly club as defined in Definition 4.2.5.

We are going to construct a $\subseteq^{**}$ tower $\langle D_\alpha : \alpha < \lambda \rangle$ of clubs on $2^{<\kappa}$ so that $D_\alpha \subseteq S_\alpha$ for every $\alpha < \lambda$. The construction is as follows:

- $D_0 = S_0$
- $D_{\alpha+1} = D_\alpha \cap S_{\alpha+1}$, this set is still club by Lemma 4.2.6.
- For γ limit we consider two cases:
  - Case 1: $\mathrm{cf}(\gamma) < \kappa$, then pass to a cofinal subsequence of cofinality $\mathrm{cf}(\gamma)$ and take an intersection C which is club by Lemma 4.2.6. Then take $D_\gamma = C \cap S_\gamma$.

– $\mathrm{cf}(\gamma) \geq \kappa$. Then apply Lemma 4.2.8 to get C club with $C \subseteq^{**} D_\alpha$ for all $\alpha < \gamma$. Let $D_\gamma = C \cap S_\gamma$.

Given the sequence $\langle D_\alpha : \alpha < \lambda \rangle$ we can again find a club $C \subseteq^{**} D_\alpha$ for each $\alpha < \lambda$. Finally let

$$Y := \bigcap_{i<\kappa} \bigcup_{\substack{s \in C, \\ \mathrm{lth}\, s \geq i}} [s]$$

For every $i < \kappa$, $\bigcup_{\substack{s \in C, \\ |s| \geq i}} [s]$ is open dense. Thus Y is comeager. Now fix $\alpha < \lambda$. As $C \subseteq^{**} D_\alpha \subseteq S_\alpha$, there is $i < \kappa$ so that $\{s \in C : |s| \geq i\} \subseteq S_\alpha$. In particular $\bigcup_{\substack{s \in C, \\ |s| \geq i}} [s] \subseteq Y_\alpha$ and thus $Y \subseteq Y_\alpha$. □

Note that the notion of a club subset of $2^{<\kappa}$ was essential in the above proof in order to continue the construction at limits of small cofinality. Clubs on $2^{<\kappa}$ are combinatorial objects that seem to be very interesting in their own right. We show that they behave similar to clubs on κ. Namely we will show how to define a diagonal intersection and prove a version of Fodor's Lemma.

Definition 4.2.9. Let $\langle D_\alpha : \alpha < \kappa \rangle$ be clubs on $2^{<\kappa}$. Then we define the diagonal intersection of $\langle D_\alpha : \alpha < \kappa \rangle$ as

$$\triangle_{\alpha<\kappa} D_\alpha = \{s \in 2^{<\kappa} : \forall \alpha < \mathrm{lth}(s)(s \in D_\alpha)\}.$$

Lemma 4.2.10. *If $\langle D_\alpha : \alpha < \kappa \rangle$ are clubs on $2^{<\kappa}$, then $D = \triangle_{\alpha<\kappa} D_\alpha$ is club and $D \subseteq^{**} D_\alpha$ for every $\alpha < \kappa$.*

Proof. Assume $\{s_i : i < \delta\}$ is increasing in D. Let $s = \bigcup_{i<\delta} s_i$ and $\alpha = \mathrm{lth}\, s$. Then we have that for any $i < \alpha$ that for all j with $i < j < \alpha$, $s_j \in D_i$. This shows that for any $i < \alpha$, $s \in D_i$. In particular $s \in D$. Thus D is closed.

Let $s \in 2^{<\kappa}$ be arbitrary. Find an increasing sequence $\langle s_n : n \in \omega \rangle$ with $s_0 = s$ and for every n, $\forall \alpha < \mathrm{lth}\, s_n \exists s'(s_n \subseteq s' \subseteq s_{n+1} \wedge s' \in D_\alpha)$. Then $\bigcup_{n\in\omega} s_n \in D$ and extends s. Thus D is unbounded.

Fix $\alpha < \kappa$. Assume $s \in D \setminus 2^{\leq\alpha}$. Then $s \in D_\alpha$ by definition of D. Thus $D \subseteq^{**} D_\alpha$. □

There is another candidate for a notion of diagonal intersection that we will use in the proof of our version of Fodor's Lemma below.

Definition 4.2.11. Let $\langle D_s : s \in 2^{<\kappa} \rangle$ be clubs. Then we define

$$\triangle_{s \in 2^{<\kappa}} D_s = \{s \in 2^{<\kappa} : \forall s' \subseteq s (s \in D_{s'})\}.$$

The proof that $\triangle_{s \in 2^{<\kappa}} D_s$ is club is analogous to the one of Lemma 4.2.10. Note that $\triangle_{s \in 2^{<\kappa}} D_s$ agrees with $\triangle_{\alpha < \kappa} D_\alpha$ whenever $D_s = D_{\mathrm{lth}\, s}$.

Definition 4.2.12. A set $S \subseteq 2^{<\kappa}$ is called stationary if $S \cap C \neq \emptyset$ for every club C.

A typical example of a stationary set would be one of the form $\{s : \sigma \subseteq s\}$ for some $\sigma \in 2^{<\kappa}$. We can now prove a version of Fodor's Lemma.

Lemma 4.2.13 (Fodor's Lemma for $2^{<\kappa}$)**.** *Let $S \subseteq 2^{<\kappa}$ be stationary and suppose that $f \colon S \setminus \{\emptyset\} \to 2^{<\kappa}$ is such that $f(s) \subseteq s$ for every $s \in 2^{<\kappa}$. Then f is constant on a stationary set.*

Proof. Assume not. This means that for any $s \in 2^{<\kappa}$ there is a club C_s so that $C_s \cap f^{-1}(s) = \emptyset$. Let $C = \triangle_{s \in 2^{<\kappa}} C_s$. Then C is club and in particular $C \cap S$ is non empty. So let $s \in C \cap S$. Then $f(s) \subseteq s$ and $s \in f^{-1}(f(s))$, and so $s \notin C_{f(s)}$. But from the definition of C it follows that $s \in C_{f(s)}$. We have arrived at a contradiction. □

We have proven some structural properties of the collection of clubs on $2^{<\kappa}$. But several other questions can be asked about them. For instance we may introduce a new cardinal $\mathfrak{p}_{2^{<\kappa}}$ as the least size of a family $\mathcal{B}$ of clubs on $2^{<\kappa}$ with no club $\subseteq^{**}$ pseudointersection. We will determine the value of $\mathfrak{p}_{2^{<\kappa}}$ completely in terms of the values in Cichoń's Diagram. This shows that the notion of club on $2^{<\kappa}$ is not simply an artifact of the proof of Theorem 4.2.4 but is strongly related to the combinatorics on the generalized Cantor space and could serve as a useful tool.

We can first show the following:

Proposition 4.2.14. $\mathfrak{p}_{2^{<\kappa}} \leq \mathrm{add}(\mathcal{M}_\kappa)$.

Proof. Assume $\langle Y_\alpha : \alpha < \lambda \rangle$ are open dense subsets of 2^κ. As before, we can write $Y_\alpha = \bigcup_{s \in S_\alpha} [s]$ where $S_\alpha \subseteq 2^{<\kappa}$ and S_α is upwards closed for every $\alpha < \lambda$. Clearly each of the S_α is a club set. If S is club and $S \subseteq^{**} S_\alpha$ for every $\alpha < \lambda$, then $Y := \bigcap_{i<\kappa} \bigcup_{\substack{s \in S, \\ |s| \geq i}} [s]$ is comeager and a subset of each Y_α. □

Moreover, analyzing the proof of Theorem 4.2.4 we find that

Proposition 4.2.15. *If $\kappa^{<\kappa} = \kappa$, then $\mathfrak{t}(\kappa) \leq \mathfrak{p}_{2^{<\kappa}}$.*

The following is also an easy observation:

Proposition 4.2.16. $\mathfrak{p}_{2^{<\kappa}} \leq \mathfrak{b}(\kappa)$.

Proof. Assume $\mathcal{B}$ is a family of clubs on κ. For $B \in \mathcal{B}$, let $C_B = \bigcup_{\alpha \in B} 2^\alpha$ which obviously is a club subset of $2^{<\kappa}$. Assume that there is a club $C \subseteq^{**} C_B$ for every $B \in \mathcal{B}$. Then the set $D = \{\alpha : \exists s \in 2^\alpha \cap C\}$ is a pseudointersection of $\mathcal{B}$. □

Definition 4.2.17. The κ-covering number $\mathrm{cov}(\mathcal{M}_\kappa)$ is the least size of a family of meager sets covering 2^κ. Equivalently it is the least size of a family of comeager sets with empty intersection.

It follows easily from the fact that 2^κ is not meager that $\mathrm{add}(\mathcal{M}_\kappa) \leq \mathrm{cov}(\mathcal{M}_\kappa)$. We can now characterize $\mathfrak{p}_{2^{<\kappa}}$ as follows.

Theorem 4.2.18. *Assume $\kappa^{<\kappa} = \kappa$, then $\mathfrak{p}_{2^{<\kappa}} = \min\{\mathfrak{b}(\kappa), \mathrm{cov}(\mathcal{M}_\kappa)\}$.*

Proof. Let $\{C_\alpha : \alpha < \lambda\}$ be a family of clubs on $2^{<\kappa}$ with $\lambda < \mathrm{cov}(\mathcal{M}_\kappa), \mathfrak{b}(\kappa)$. Consider the sets $Y_\alpha = \bigcap_{i<\kappa} \bigcup_{s \in C_\alpha, |s| \geq i} [s]$. Every Y_α is comeager, thus $Y := \bigcap_{\alpha<\lambda} Y_\alpha$ is dense in 2^κ as $\lambda < \mathrm{cov}(\mathcal{M}_\kappa)$ (It follows from the definition of $\mathrm{cov}(\mathcal{M}_\kappa)$ that Y is non empty. Applying the result to the homeomorphic spaces of the form $[s]$ yields that Y is dense). Thus we can find a dense subset $\{x_i : i < \kappa\} \subseteq Y$ (use that $\kappa^{<\kappa} = \kappa$). Note that for every $i < \kappa$ and every $\alpha < \lambda$, the set $C^i_\alpha = \{j < \kappa : x \restriction j \in C_\alpha\}$ is a club in κ. This follows from the definition of Y and because C_α is club. As $\lambda < \mathfrak{b}(\kappa)$, for each $i < \kappa$, $\mathcal{B}_i = \{C^i_\alpha : \alpha < \lambda\}$ has a pseudointersection $B_i \in [\kappa]^\kappa$. Again applying $\lambda < \mathfrak{b}(\kappa)$ we can find a function $f \in \kappa^\kappa$ so that

$$\forall \alpha < \lambda (|\kappa \setminus \{i \in \kappa : B_i \setminus 2^{<f(i)} \subseteq C^i_\alpha\}| < \kappa).$$

Now enumerate $2^{<\kappa}$ as $\langle s_i : i < \kappa \rangle$ and for every i find $\sigma_i \supseteq s_i$ so that $\sigma_i \in B_j \setminus 2^{<f(j)}$ for some $j > i$. The collection $C' = \{\sigma_i : i \in \kappa\}$ is unbounded in $2^{<\kappa}$. Furthermore we have that $C' \subseteq^* C_\alpha$ for every α. If C is the closure of C', then $C \subseteq^{**} C_\alpha$ for every α. Thus we have shown that $\{C_\alpha : \alpha < \lambda\}$ has a $\subseteq^{**}$ pseudointersection which is club. □

This is very interesting because we have that $\mathrm{add}(\mathcal{M}) = \min\{\mathfrak{b}, \mathrm{cov}(\mathcal{M})\}$, although in the general case $\mathrm{add}(\mathcal{M}_\kappa) = \min\{\mathfrak{b}(\kappa), \mathrm{cov}(\mathcal{M}_\kappa)\}$ was first only shown for κ inaccessible (see [7] and especially Question 29). The author was told that the more general result of $\mathrm{add}(\mathcal{M}_\kappa) = \min\{\mathfrak{b}(\kappa), \mathrm{cov}(\mathcal{M}_\kappa)\}$ for all regular κ is an unpublished result of J. Brendle. Thus this gives the characterization of $\mathrm{add}(\mathcal{M}_\kappa) = \mathfrak{p}_{2^{<\kappa}}$.

In the last section we showed that the bounding number $\mathfrak{b}(\kappa)$ can be characterized using clubs. Now we will do the opposite for $\mathfrak{p}_{2^{<\kappa}}$. We will show that $\mathfrak{p}_{2^{<\kappa}}$ can be characterized as a sort of bounding number.

Definition 4.2.19. Suppose $f\colon 2^{<\kappa} \to 2^{<\kappa}$ is such that $\forall s \in 2^{<\kappa}(f(s) \supseteq s)$. Then we call f normal.

If f, g are normal functions on $2^{<\kappa}$, we write $f <^* g$ whenever

$$|\{\alpha < \kappa : \exists s \in 2^{<\alpha} \forall s' \in 2^{<\kappa}(s \subseteq s' \subseteq g(s) \to f(s') \not\subseteq g(s))\}| < \kappa.$$

We write $\mathfrak{b}_{2^{<\kappa}}$ for the least size of a family of normal functions on $2^{<\kappa}$ unbounded with respect to $<^*$.

Intuitively $f <^* g$ if eventually every "interval" $[s, g(s)]$ contains an "interval" of the form $[s', f(s')]$. This is a very natural notion of bounding and looks similar to notion of an interval partition dominating another interval partition (see [4]).

Proposition 4.2.20. $\mathfrak{b}_{2^{<\kappa}} = \mathfrak{p}_{2^{<\kappa}}$.

The proof is very similar to the one of Proposition 4.1.5.

Proof. We first show $\mathfrak{b}_{2^{<\kappa}} \leq \mathfrak{p}_{2^{<\kappa}}$. For this let $\{C_\alpha : \alpha < \lambda\}$ be a collection of clubs on $2^{<\omega}$. Define for each α an normal function f_α on $2^{<\kappa}$ with the property that $f_\alpha(s) \in C_\alpha$ for every $s \in 2^{<\kappa}$.

Assume that $\{f_\alpha : \alpha < \kappa\}$ can be bounded by f. Consider the set $C = \{s \in 2^{<\kappa} : \forall s' \subsetneq s \exists s''(s' \subseteq s'' \wedge f(s'') \subseteq s)\}$. It is not difficult to see that C is club and is a $\subseteq^{**}$ pseudointersection of $\{C_\alpha : \alpha < \lambda\}$.

We now show that $\mathfrak{p}_{2^{<\kappa}} \leq \mathfrak{b}_{2^{<\kappa}}$. For this let $\{f_\alpha : \alpha < \lambda\}$ be an unbounded collection of normal functions on $2^{<\omega}$. For each $\alpha < \lambda$ define C_α as we did for f above, i.e. $C_\alpha = \{s \in 2^{<\kappa} : \forall s' \subsetneq s \exists s''(s' \subseteq s'' \wedge f_\alpha(s'') \subseteq s)\}$. Again these sets are club. If $C \subseteq^{**} C_\alpha$ for every α and C is club, then any normal function f with $f(s) \in C$ for every $s \in 2^{<\kappa}$ would bound $\{f_\alpha : \alpha < \lambda\}$. $\square$

The next question is a reformulation of Question 84 in [7] where it was asked whether $\mathrm{add}(\mathcal{M}_\kappa)$ and $\mathfrak{b}(\kappa)$ can be separated when κ is (at least) inaccessible.

Question. If κ is inaccessible, is $\mathfrak{b}_{2^{<\kappa}} = \mathfrak{b}(\kappa)$?

Another question, that can be motivated by the proof of Theorem 4.1, is the following:

Question. Is $\mathfrak{s}(\kappa) \leq \mathrm{add}(\mathcal{M}_\kappa)$? Equivalently, knowing $\mathfrak{s}(\kappa) \leq \mathfrak{b}(\kappa)$, is $\mathfrak{s}(\kappa) \leq \mathrm{cov}(\mathcal{M}_\kappa)$?

4.3 Bell's Theorem generalized

In this section we will try to generalize Bell's Theorem (Theorem 2.2.1) to the uncountable case. For this we need to find the right analogue of σ-centered posets.

Definition 4.3.1. A subset $C \subseteq \mathbb{P}$ of a poset $\mathbb{P}$ is called κ-directed if $\forall D \in [C]^{<\kappa} \exists q \forall p \in D (q \leq p)$. A poset $\mathbb{P}$ is κ-centered if $\mathbb{P} = \bigcup_{i<\kappa} C_i$ where each C_i is κ-directed.

Note that it is impossible to get a theorem of the form "for every κ-centered poset and a collection of less than $\mathfrak{p}(\kappa)$ many dense sets, there is a filter intersecting each of them". Simply observe that any poset of size $\leq \kappa$ is κ-centered. In particular if $2^\omega \leq \kappa$, then almost all posets typically considered in set theory of the reals (e.g. tree forcings) are κ-centered.

Before defining the class of posets that we will work with we introduce some notation. Whenever $\bar{p}$ is a decreasing sequence in $\mathbb{P}$ and $q \in \mathbb{P}$ we write $q \leq \bar{p}$ if q is a lower bound for $\bar{p}$. Suppose $\mathbb{P}$ is a poset and $\bar{C} = \langle C_i : i < \kappa \rangle$ is a sequence of subsets of $\mathbb{P}$. For any $s \in \kappa^{<\kappa} \setminus \{\emptyset\}$ we write $\mathbf{S}(s, \bar{C})$ for the set of decreasing sequences $\langle p_\alpha : \alpha < \mathrm{lth}\, s \rangle$ in $\mathbb{P}$ so that $\forall \alpha < \mathrm{lth}\, s (p_\alpha \in C_{s(\alpha)})$.

Definition 4.3.2. A poset $\mathbb{P}$ is κ-specially-centered if $\mathbb{P}$ is κ-centered as witnessed by a collection $\{C_i : i < \kappa\}$ so that additionally for any $s \in \kappa^{<\kappa} \setminus \{\emptyset\}$ and any $P \in [\mathbf{S}(s, \bar{C})]^{<\kappa}$, the sequences in P have a common lower bound, i.e. there is $q \in \mathbb{P}$ so that $q \leq \bar{p}$ for every $\bar{p} \in P$.

Note that in the above definition it is very much possible that some of the sets $\mathbf{S}(s, \bar{C})$ are simply empty, i.e. there are no decreasing sequences $\langle p_i \rangle$ so that $p_i \in C_{s(i)}$. But e.g. when s has length 1, i.e. is of the form $\langle i \rangle$, then $\mathbf{S}(s, \bar{C})$ simply corresponds to C_i. Also note that κ-specially-centered implies κ-closed.

κ-specially-centered expresses a sort of "locality" of the compatibility relation. If $\langle p_i \rangle$ and $\langle q_i \rangle$ are decreasing sequences so that each p_i, q_i always live in a common centered set (in particular are compatible), then they have a common lower bound.

Theorem 4.3.3. *Assume $\mathbb{P}$ is κ-specially-centered and below every $p \in \mathbb{P}$ there is a κ sized antichain. Also assume that $\kappa^{<\kappa} = \kappa$. Then whenever $\{D_\alpha : \alpha < \lambda\}$ is a collection of dense subsets of $\mathbb{P}$, where $\lambda^{<\kappa} = \lambda$ and $\lambda < \mathfrak{p}(\kappa)$, then there is a filter intersecting every D_α.*

The proof follows the same lines as the one of Theorem 2.2.1, but contains some more technicalities. Thus it is highly suggested to first read and understand the (already complicated enough) proof of Theorem 2.2.1.

Proof. We can assume wlog that $|\mathbb{P}| \leq \lambda$. The reason is that given $\mathbb{P}$ and $\{D_\alpha : \alpha < \lambda\}$ dense open we can find $\mathbb{Q} \subseteq \mathbb{P}$ with $|\mathbb{Q}| \leq \lambda$ so that

- for every $\alpha < \lambda$, $D_\alpha \cap \mathbb{Q}$ is dense in $\mathbb{Q}$,
- for every $p, q \in \mathbb{Q}$, $p \parallel q$ in $\mathbb{P}$ iff $\exists r \in \mathbb{Q}(r \leq p, q)$,
- for every $B \in [\mathbb{Q}]^{<\kappa}$, if B has a lower bound in $\mathbb{P}$, then it has one in $\mathbb{Q}$
- for every $p \in \mathbb{Q}$, there is a κ-sized antichain in $\mathbb{Q}$ below p.

Then whenever the Theorem is true for $\mathbb{Q}$, we get a filter on $\mathbb{Q}$, generating one on $\mathbb{P}$ intersecting all D_α.

The construction of $\mathbb{Q}$ is a standard Löwenheim-Skolem argument. We construct recursively a sequence $\langle \mathbb{Q}_i \rangle_{i \leq \kappa}$ so that $|\mathbb{Q}_i| \leq \lambda$ will hold true for every $i \in \kappa + 1$.

- $\mathbb{Q}_0 = \{\mathbb{1}\}$,
- $\mathbb{Q}_\gamma = \bigcup_{i<\gamma} \mathbb{Q}_i$ for γ limit, $|\mathbb{Q}_\gamma| \leq |\gamma| \cdot \lambda \leq \lambda$,
- $\mathbb{Q}_{i+1}$ is obtained by adding to $\mathbb{Q}_i$, κ sized antichains below every $p \in \mathbb{Q}_i$, a condition $q \in D_\alpha$ below p for every $\alpha < \lambda$ and for every $B \in [\mathbb{Q}_i]^{<\kappa}$ a lower bound, if there is one in $\mathbb{P}$. Then $|\mathbb{Q}_{i+1}| \leq \lambda \cdot \kappa + \lambda + \lambda^{<\kappa} \leq \lambda$.

$\mathbb{Q} = \mathbb{Q}_\kappa$ works and is still κ-specially-centered.

Now we also notice that it suffices to get $G \subseteq \mathbb{P}$ which is linked, instead of a filter. The reason is the same as in the proof of Theorem 2.2.1.

Let $\mathbb{P} = \bigcup_{i<\kappa} C_i$ be as witnessed by κ-specially-centeredness. Moreover fix $\mathcal{S}$ to be the set of $s \in \kappa^{<\kappa} \setminus \{\emptyset\}$ so that $\mathbf{S}(s, \bar{C})$ is non-empty.

For each $\alpha < \lambda$ and any $\bar{p}$ a decreasing sequence in $\mathbb{P}$ of length $< \kappa$ we let $A(\alpha, \bar{p}) = \{i \in \kappa : \exists q \leq \bar{p}(q \in D_\alpha \wedge q \in C_i)\}$. For every $s \in \mathcal{S}$ the family

$$\mathcal{F}_s = \{A(\alpha, \bar{p}) : \bar{p} \in \mathbf{S}(s, \bar{C}), \alpha < \lambda\}$$

has the κ-IP. To see this, assume that $P \in [\mathbf{S}(s, \bar{C})]^{<\kappa}, F \in [\lambda]^{<\kappa}$ and $P \neq \emptyset$. Then P has a lower bound $q \in \mathbb{P}$. Notice that by κ-closedness of $\mathbb{P}$, $D = \bigcap_{\alpha \in F} D_\alpha$ is still open dense. So let A be a κ-sized antichain in D below q. Then the set $\{j < \kappa : \exists q \in A(q \in C_j)\} \subseteq \bigcap_{\bar{p} \in P, \alpha \in F} A(\alpha, \bar{p})$ is of size κ.

Thus let A_s be a pseudointersection of $\mathcal{F}_s$ for every $s \in \mathcal{S}$. Also let $A_\emptyset = \kappa$. We define a map $S\colon \kappa^{<\kappa} \to \kappa^{<\kappa}$ recursively with the following requirements:

(1) $S(\emptyset) = \emptyset$,

(2) for any s, $S(s^\frown i) = S(s)^\frown j$ where $j \in A_{S(s)}$ is such that $i < j$ and $S(s)^\frown j \in \mathcal{S}$

(3) and if s has limit length then $S(s) = \bigcup_{s' \subset s} S(s')$.

Note that in order to construct S, we need to make sure that $S(s) \in \mathcal{S}$ inductively, especially in the limit steps. In order to ensure this we can simultaneously define an auxiliary map $T\colon \kappa^{<\kappa} \to \mathbb{P}^{<\kappa}$ with the property that $T(s) \in \mathbf{S}(S(s), \bar{C})$ and $T(s) \subseteq T(s')$ for any s, s' with $s \subseteq s'$. We leave the details to the reader.

Further we define a second labeling $\Phi_\alpha\colon \operatorname{Succ}(\kappa^{<\kappa}) \to \mathbb{P}$, where $\operatorname{Succ}(\kappa^{<\kappa}) = \{s \in \kappa^{<\kappa} : \exists \xi(\operatorname{lth} s = \xi + 1)\}$, for every $\alpha < \lambda$ with the following requirements:

(1) $\Phi_\alpha(s) \in C_{S(s)(\operatorname{lth} s - 1)}$ for any $s \in \operatorname{Succ}(\kappa^{<\kappa})$,

(2) if $s' = s^\frown i$, and there is $\zeta < \operatorname{lth} s$ with the property that $\langle \Phi_\alpha(s \restriction \xi + 1) : \xi \in [\zeta, \operatorname{lth} s) \rangle$ is decreasing, then $\Phi_\alpha(s') \in D_\alpha \cap C_{S(s')(\operatorname{lth} s)}$ and $\Phi_\alpha(s') \leq \langle \Phi_\alpha(s \restriction \xi + 1) : \xi \in [\zeta, \operatorname{lth} s) \rangle$ whenever this is possible,

(3) if $s' = s^\frown i$ and the conditions in (2) are not satisfied, $\Phi_\alpha(s') \in C_{S(s')(\operatorname{lth} s)}$ is arbitrary.

Note that for any $\alpha < \lambda$ and for any $s \in \kappa^{<\kappa}$ so that there is $\zeta < \operatorname{lth} s$ with the property that $\langle \Phi_\alpha(s \upharpoonright \xi + 1) : \xi \in [\zeta, \operatorname{lth} s) \rangle$ is decreasing, the set $\{i \in \kappa : \exists p \in D_\alpha \cap C_{S(s^\frown i)(\operatorname{lth} s)}(p \leq \langle \Phi_\alpha(s \upharpoonright \xi + 1) : \xi \in [\zeta, \operatorname{lth} s)\rangle)\}$ is cobounded in κ. This follows because $A_{S(s)}$ is a pseudointersection of $\mathcal{F}_s$ and thus in particular of $A(\alpha, \langle \Phi_\alpha(s \upharpoonright \xi + 2) : \xi \in [\zeta, \operatorname{lth} s)\rangle)$. This means that $\Phi_\alpha(s^\frown i) \in D_\alpha$ and is below $\langle \Phi_\alpha(s \upharpoonright \xi + 1) : \xi \in [\zeta, \operatorname{lth} s)\rangle$ for almost all $i \in \kappa$.

Thus we can define for every $\alpha < \lambda$ a function $F_\alpha \colon \kappa^{<\kappa} \to \kappa$, so that for any $s \in \kappa^{<\kappa}$ and and any $i \geq F_\alpha(s)$, $\Phi_\alpha(s^\frown i) \in D_\alpha$ and $\Phi_\alpha(s^\frown i) \leq \langle \Phi_\alpha(s \upharpoonright \xi + 1) : \xi \in [\zeta, \operatorname{lth} s)\rangle$ if ζ is such that $\langle \Phi_\alpha(s \upharpoonright \xi + 1) : \xi \in [\zeta, \operatorname{lth} s)\rangle$ is decreasing (e.g. when $\operatorname{lth} s$ is a successor, then $\zeta = \operatorname{lth} s - 1$ works because $|[\zeta, \operatorname{lth} s)| = 1$).

As $|\kappa^{<\kappa}| = \kappa$ and $\lambda < \mathfrak{p}(\kappa) \leq \mathfrak{b}(\kappa)$ we find a single $F \colon \kappa^{<\kappa} \to \kappa$ so that for each $\alpha < \lambda$ and almost all $s \in \kappa^{<\kappa}$, $F_\alpha(s) \leq F(s)$.

Let $x \in \kappa^\kappa$ be defined by letting $x(0) = F(\emptyset)$ and $x(i) = F(x \upharpoonright i)$. Define $Z = \bigcup_{i<\kappa} S(x \upharpoonright i) \in \kappa^\kappa$.

For every $\alpha < \lambda$ there is some i_α large enough so that $\Psi_\alpha(i_\alpha) := \Phi_\alpha(x \upharpoonright i_\alpha + 1) \in D_\alpha$ and for every $j \geq j' \geq i_\alpha$, $\Psi_\alpha(j) := \Phi_\alpha(x \upharpoonright j + 1) \leq \Psi_\alpha(j') := \Phi_\alpha(x \upharpoonright j' + 1)$, i.e. $\langle \Psi_\alpha(j) : j \in [i_\alpha, \kappa)\rangle$ is decreasing.

Define $G = \{\Psi_\alpha(i_\alpha) : \alpha < \lambda\}$. Clearly $G \cap D_\alpha \neq \emptyset$ for every $\alpha < \lambda$. We claim that G is linked. To see this let α, β be such that $i_\alpha \leq i_\beta$. Then $\Psi_\alpha(i_\beta) \leq \Psi_\alpha(i_\alpha)$ and $\{\Psi_\alpha(i_\beta), \Psi_\beta(i_\beta)\} \subseteq C_{Z(i_\beta)}$ and so $\Psi_\alpha(i_\beta), \Psi_\beta(i_\beta)$ are compatible and also $\Psi_\alpha(i_\alpha), \Psi_\beta(i_\beta)$ are compatible.

This proves the theorem. □

We note that all κ-centered, κ-closed posets that are usually considered (e.g. Hechler forcing or Mathias forcing for κ-complete filters) are actually κ-specially-centered and we are not sure whether special centeredness is a real restriction. We are not aware of any example of a κ-centered, κ-closed but not κ-specially-centered poset, although we think that these notions can be separated.

A lot of the usual κ-centered, κ-closed posets fall in the class described by the following Proposition.

Proposition 4.3.4. *Let $\kappa^{<\kappa} = \kappa$. Suppose $\mathcal{F}$ is any set, $S \subseteq \kappa \times \kappa$ is a relation and $R \subseteq \kappa \times \mathcal{F}$ is a relation. Then the poset $\mathbb{P}$ consisting of pairs $(a, \mathcal{X}) \in [\kappa]^{<\kappa} \times [\mathcal{F}]^{<\kappa}$ so that $\forall \alpha, \beta \in a(S(\alpha, \beta))$ together with the order $(b, \mathcal{Y}) \leq (a, \mathcal{X})$ iff $a \subseteq b$, $\mathcal{X} \subseteq \mathcal{Y}$ and $\forall \alpha \in b \setminus a \forall X \in \mathcal{X}(R(\alpha, X))$, is κ-specially-centered.*

Proof. For any $a \in [\kappa]^{<\kappa}$, let $C_a = \{p \in \mathbb{Q} : \operatorname{dom} p = a\}$. We show that $\{C_a : a \in [\kappa]^{<\kappa}\}$ witnesses κ-specially-centeredness. Suppose $s \colon \delta \to$

$[\kappa]^{<\kappa}$ for $\delta < \kappa$ and $\mathbf{S}(s, \bar{C})$ is non empty. If $P \in [\mathbf{S}(s, \bar{C})]^{<\kappa}$ we find that $(\bigcup_{i<\delta} s(i), \bigcup_{\bar{p} \in P} \bigcup_{i<\delta} p_i(1))$ is a lower bound for P. □

For example Mathias forcing for a κ-complete ultrafilter $\mathcal{U}$ corresponds to the poset described above where $\mathcal{F} = \mathcal{U}$, $S = \kappa \times \kappa$ and $R(\alpha, X)$ iff $\alpha \in X$. For Hechler forcing, we identify κ with $\kappa \times \kappa$ and define $S((\alpha, \beta), (\gamma, \delta))$ iff $\alpha \neq \gamma$. Then we let $\mathcal{F} = \kappa^\kappa$ and $R((\alpha, \beta), f)$ iff $\beta > f(\alpha)$.

4.4 Rothberger generalized

In this section we prove the analogue of Theorem 2.3.1. More specifically we will prove:

Theorem 4.4.1. *Assume $\kappa^{<\kappa} = \kappa$. Then $\mathfrak{p}(\kappa) = \kappa^+ \to \mathfrak{t}(\kappa) = \kappa^+$.*

We roughly follow the proof of S. Garti given in [13]. The main difference with the proof of Theorem 2.3.1 is that when constructing a tower on κ we need to additionally ensure that at limit steps of small cofinality ($< \kappa$) we have a pseudointersection (i.e. the tower has the κ-IP). It is in general not clear how to achieve this. For example, even though the proof of Theorem 2.1.1 only takes a few lines, it needs a big effort to get the general result specifically because of these small limits (see [30]).

Proof. Assume $\langle B_\alpha : \alpha < \kappa^+ \rangle \subseteq [\kappa]^\kappa$ has the κ-IP and $\mathfrak{b}(\kappa) > \kappa^+$ (else $\kappa^+ \leq \mathfrak{t}(\kappa) \leq \mathfrak{b}(\kappa) \leq \kappa^+$). We will construct a κ-tower $\langle T_\alpha : \alpha < \kappa^+ \rangle$ so that $T_\alpha \subseteq^* B_\alpha$ for every $\alpha < \kappa^+$. This clearly suffices.

First fix for every $\alpha < \kappa^+$ a pseudointersection A_α of $\langle B_\beta : \beta < \alpha \rangle$. We will construct $\langle T_\alpha : \alpha < \kappa^+ \rangle$ with the additional property that $A_\xi \subseteq^* T_\alpha$, for all $\alpha < \kappa^+$ and $\xi \in [\alpha, \kappa^+)$.

Suppose we have achieved to construct such a tower up to stage $\alpha < \kappa^+$.

If α is of cofinality $< \kappa$, say $\alpha = \sup_{i<\delta} \alpha_i$ for $\delta < \kappa$, then $A_\xi \subseteq^* \bigcap_{i<\delta} T_{\alpha_i}$ for every $\xi \in [\alpha, \kappa^+)$. Because assume $A_\xi \cap \kappa \setminus (\bigcap_{i<\delta} T_{\alpha_i})$ has size κ, then $A_\xi \cap \kappa \setminus T_{\alpha_i}$ has size κ for some $i < \delta$. This contradicts $A_\xi \subseteq^* T_{\alpha_i}$. Thus we may continue by setting $T_\alpha = \bigcap_{i<\delta} T_{\alpha_i}$.

If α is of cofinality κ, say $\alpha = \sup_{i<\kappa} \alpha_i$, then do the following. Let $X_i = \bigcap_{j<i} T_{\alpha_j}$. Then $X_i \subseteq X_j$ whenever $j < i$ and $A_\xi \subseteq^* X_i$ for every $\xi \in [\alpha, \kappa^+)$ (the same argument as above). For any $\xi \in [\alpha, \kappa^+)$ define a function $f_\xi \in \kappa^\kappa$ so that $A_\xi \setminus f_\xi(i) \subseteq X_{i+1}$ for every $i < \kappa$. We assumed $\mathfrak{b}(\kappa) > \kappa^+$. Thus there is $f \in \kappa^\kappa$ so that $f_\xi <^* f$ for every $\xi \in [\alpha, \kappa^+)$. Consider $X = \bigcup_{i<\kappa} X_i \cap f(i)$.

This is clearly a pseudointersection of $\langle X_i : i < \kappa\rangle$ and thus of $\langle T_\beta : \beta < \alpha\rangle$. Fix $\xi \in [\alpha, \kappa^+)$. We show that $A_\xi \subseteq^* X$. To this end let $i < \kappa$ be such that $f(j) > f_\xi(j)$ for every $j > i$. Assume there is $\gamma \in (A_\xi \cap X_i) \setminus X$, then there is a minimal $j > i$ so that $\gamma \notin X_j$ (else $\gamma \in X$). Note that j has to be a successor ordinal, because $X_j = \bigcap_{j'<j} X_{j'}$. So write $j = j' + 1$. But then $\gamma \in X_{j'} \cap f_\xi(j') \subseteq X_{j'} \cap f(j') \subseteq X$, because else $\gamma \geq f_\xi(j')$ but $\gamma \in A_\xi \setminus X_{j'+1}$, contradicting the definition of $f_\xi(j')$. But this means that $\gamma \in X$ contradicting our assumption. We have thus shown that $A_\xi \cap X_i \subseteq X$ and as $A_\xi \subseteq^* X_i$ we follow that $A_\xi \subseteq^* X$. We may take $T_\alpha = X$. □

The main invent of the proof above is to ensure that our tower has some sort of lower bound that guarantees the κ-IP. Apart from that the proof wasn't really different from the case $\kappa = \omega$. Thus it may be conceivable that using a similar trick we could generalize the proof of $\mathfrak{p} = \mathfrak{t}$. But it should be noted that, e.g. when trying to reproduce the reduction in Section 3.1, there are still a lot of places where the problem of having limits of small cofinality is not obvious to overcome. Further one should be aware of the following:

Observation 4.4.2. *There is no sequence $\langle f_n : n \in \omega\rangle \subseteq \kappa^\kappa$ that is decreasing wrt $<^*$.*

Proof. Assume $\langle f_n : n \in \omega\rangle$ was strictly decreasing. Then there is $\alpha < \kappa$ so that $\forall n < m \in \omega(f_m(\alpha) < f_n(\alpha))$. But then $\langle f_n(\alpha) : n \in \omega\rangle$ is a strictly decreasing sequence of ordinals. □

Open questions

From Chapter 3:

Question. Does $\mathfrak{p} < \mathfrak{b}$ or even $\mathfrak{p} < \mathfrak{d}$ imply that every filter base of size $\mathfrak{p}$ has a refining tower?

Question. Does $\mathfrak{p} < \mathfrak{d}$ imply that every ultrafilter base of size $\mathfrak{p}$ is generated by a tower?

From Chapter 4:

Question. Is $\mathrm{add}(\mathcal{M}_\kappa) < \mathfrak{b}(\kappa)$ consistent (assuming $\kappa^{<\kappa} = \kappa$)?

Question. Is $\mathfrak{s}(\kappa) \leq \mathrm{add}(\mathcal{M}_\kappa)$? Equivalently, is $\mathfrak{s}(\kappa) \leq \mathrm{cov}(\mathcal{M}_\kappa)$?

Question. Is $\mathfrak{p}(\kappa) = \mathfrak{t}(\kappa)$?

References

[1] T. Bartoszynski and H. Judah. *Set Theory: On the Structure of the Real Line*. Ak Peters Series. Taylor & Francis, 1995.

[2] James E. Baumgartner and Peter Dordal. Adjoining dominating functions. *The Journal of Symbolic Logic*, 50(1):pp. 94–101, 1985.

[3] Murray Bell. On the combinatorial principle p(c). *Fundamenta Mathematicae*, 114(2):149–157, 1981.

[4] Andreas Blass. *Combinatorial Cardinal Characteristics of the Continuum (Handbook of Set Theory, pp 395-489)*. Springer Science & Business Media, Berlin Heidelberg, 2010.

[5] Andreas Blass, Tapani Hyttinen, and Yi Zhang. Mad families and their neighbors. 2007.

[6] Andreas Blass and Saharon Shelah. Ultrafilters with small generating sets. *Israel J. Math.*, 65(3):259–271, 1989.

[7] Joerg Brendle, Andrew Brooke-Taylor, Sy-David Friedman, and Diana Montoya. Cichon's diagram for uncountable cardinals. 2016.

[8] Jörg Brendle, Barnabás Farkas, and Jonathan Verner. Towers in filters, cardinal invariants, and luzin type families, 2016.

[9] Jörg Brendle and Vera Fischer. Mad families, splitting families and large continuum. *The Journal of Symbolic Logic*, 76:198–208, 3 2011.

[10] Maxim R. Burke. *Forcing Axioms*, pages 1–21. Springer Netherlands, Dordrecht, 1998.

[11] C. C. Chang and H. J. Keisler. *Model theory*, volume 73 of *Studies in Logic and the Foundations of Mathematics*. North-Holland Publishing Co., Amsterdam, third edition, 1990.

[12] James Cummings and Saharon Shelah. Cardinal invariants above the continuum. *Ann. Pure Appl. Logic*, 75(3):251–268, 1995.

[13] Shimon Garti. Pity on lambda. 2011.

[14] Thomas Jech. *Set Theory*. Springer Science & Business Media, Berlin Heidelberg, 2013.

[15] Akihiro Kanamori. *The higher infinite*. Springer Monographs in Mathematics. Springer-Verlag, Berlin, second edition, 2009. Large cardinals in set theory from their beginnings, Paperback reprint of the 2003 edition.

[16] Kenneth Kunen. *Set Theory (Studies in Logic: Mathematical Logic and Foundations)*. College Publications, 2011.

[17] M. Malliaris and S. Shelah. Cofinality spectrum theorems in model theory, set theory, and general topology. *J. Amer. Math. Soc.*, 29(1):237–297, 2016.

[18] D. A. Martin and R. M. Solovay. Internal cohen extensions. *Annals of Mathematical Logic*, 2(2):143–178, 1970.

[19] John C. Oxtoby. Measure and category. A survey of the analogies between topological and measure spaces. 2nd ed., 1980.

[20] Dilip Raghavan and Saharon Shelah. Two inequalities between cardinal invariants. *Fund. Math.*, 237(2):187–200, 2017.

[21] Dilip Raghavan and Saharon Shelah. Two results on cardinal invariants at uncountable cardinals. 2018.

[22] Fritz Rothberger. On some problems of hausdorff and of sierpiński. *Fundamenta Mathematicae*, 35:29–46, 1948.

[23] Jonathan Schilhan. Bounding, splitting and almost disjointness. Bachelor's thesis 1, University of Vienna, 2015.

[24] Jonathan Schilhan. Forcing and applications on boudning, splitting and almost disjointness. Bachelor's thesis 2, University of Vienna, 2016.

[25] Saharon Shelah. Two cardinal invariants of the continuum ($\mathfrak{d} < \mathfrak{a}$) and fs linearly ordered iterated forcing. *Acta Mathematica*, 192(2):187–223, 2004.

[26] Saharon Shelah. A comment on "$\mathfrak{p} < \mathfrak{t}$". *Canad. Math. Bull.*, 52(2):303–314, 2009.

[27] Saharon Shelah. On con($\mathfrak{d}_\lambda > \mathrm{cov}_\lambda(meager)$). 2009.

[28] Saharon Shelah. Large continuum, oracles. *Cent. Eur. J. Math.*, 8(2):213–234, 2010.

[29] Saharon Shelah. A parallel to the null ideal for inaccessible λ. part i. 2012.

[30] Saharon Shelah and Zoran Spasojević. Cardinal invariants $\mathfrak{b}_\kappa$ and $\mathfrak{t}_\kappa$. *Publ. Inst. Math. (Beograd) (N.S.)*, 72(86):1–9, 2002.

[31] Eric Van Douwen. The integers and topology. In Kenneth KUNEN and Jerry E. VAUGHAN, editors, *Handbook of Set-Theoretic Topology*, pages 111 – 167. North-Holland, Amsterdam, 1984.

[32] Nik Weaver. *Forcing for mathematicians*. World Scientific Publishing Co. Pte. Ltd., Hackensack, NJ, 2014.