Vaught's Never Two Theorem

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Abstract

This bachelor thesis provides a short overview of some major results in model theory concerning the spectrum function $I(T, \kappa)$, where T is a first-order theory and κ is a cardinal. In short, $I(T, \kappa)$ tells us how many models of cardinality κ T has up to isomorphism. After summarising some basic definitions and results of elementary predicate logic, we turn our attention to types, both from a model theoretic, a topological and an algebraic perspective. The notions of ω-saturated, atomic and ω-categorical models are discussed in detail and needed in order to prove Robert L. Vaught's Never Two Theorem which states that $I(T, \aleph_0) \neq 2$, when T is a complete theory with infinite models of a countable first-order language.

Contents

1 Basic Facts and Motivation **3**
1.1 Introduction 5

2 Types **7**
2.1 Topological Aspects 10
2.2 Algebraic Aspects 12

3 $\aleph_0$-Categorical Theories, Vaught's Never Two Theorem and beyond **14**
3.1 ω-Saturated Models 14
3.2 Atomic Models 19
3.3 Theories with Several Models 26

References **33**

1 Basic Facts and Motivation

Throughout this thesis we use the letters "ω" and "$\aleph_0$" interchangeably as names for the set of natural numbers. We assume that the reader is familiar with ZFC and basic cardinal arithmetic. Let us first summarize some essential notions and results from elementary model theory which can also be found in [1], [2] and [3]:

- For a given model structure $\mathfrak{M} := \langle M, \dots \rangle$ of the language $\mathcal{L}$ and $A \subseteq M$ the language $\mathcal{L}(A)$ has the same constant, function and relation symbols as $\mathcal{L}$ and for each $a \in A$ an additional new constant symbol c_a. Clearly, for every $\mathcal{L}(A)$-n-formula $\phi(y_1, \dots, y_n)$ there exist a $m \in \mathbb{N}$, a set of variables$\{z_1, \dots, z_m\}$ and a $\mathcal{L}$-$(n+m)$-formula $\psi(y_1, \dots, y_n, z_1, \dots, z_m)$ such that
$$\phi(y_1, \dots, y_n) = \psi(y_1, \dots, y_n, c_{a_1}, \dots, c_{a_m})$$
(the variable z_i is substituted by the new constant symbol c_{a_i}) for some $a_1, \dots, a_m$ in A. This can be checked via induction on the complexity of terms and formulas. The structure $\mathfrak{M}_A$ has the same universe as $\mathfrak{M}$ with the same interpretations of non logical symbols of $\mathcal{L}$. Any additional constant symbol c_a ($a \in A$) is interpreted with a (i.e. $c_a^{\mathfrak{M}_A} = a$).For $A = M$ the theory of $\mathfrak{M}_A$ (notation: $Th(\mathfrak{M}_A)$), that is the set of all sentences true in $\mathfrak{M}_A$, is also denoted as $Diag(\mathfrak{M})$.

- **Replacement Lemma**: Let $\mathfrak{M} := \langle M, \dots \rangle$ be a structure for the language $\mathcal{L}$, β a term assignment on $\mathfrak{M}$, $\phi(y_1, \dots, y_n)$ a formula and $\tau_1, \dots, \tau_n$ terms such that the variable y_i can be substituted by τ_i in ϕ for $1 \le i \le n$. Then
$$\mathfrak{M} \models \phi(y_1/\tau_1, \dots, y_n/\tau_n)(\beta) \text{ iff } \mathfrak{M} \models \phi(\overline{\beta}),$$
where for every variable x
$$\overline{\beta}(x) := \begin{cases} \beta(x), & x \notin \{y_1, \dots, y_n\} \\ \beta(\tau_i), & x = y_i, 1 \le i \le n \end{cases}$$

- **Elementary Submodel**: Let $\mathfrak{N} := \langle N, \dots \rangle$ be a structure, $M \subseteq N$ be closed under interpretations of function symbols and constant symbols (i.e: $c^{\mathfrak{N}} \in M$ for each constant symbol $c \in \mathcal{L}$). The structure $\mathfrak{M} := \langle M, \dots \rangle$, where $R^{\mathfrak{M}} := R^{\mathfrak{N}} \restriction M^n$ for $n \in \mathbb{N}_+$ and a n-ary relation symbol R of $\mathcal{L}$ is called a submodel of $\mathfrak{N}$. It is called elementary submodel of $\mathfrak{N}$ if for all formulas $\phi(x_1, \dots, x_m)$ and all $(a_1, \dots, a_m) \in M^m$ we have:
$$\mathfrak{M} \models \phi(a_1, \dots, a_m) \text{ iff } \mathfrak{N} \models \phi(a_1, \dots, a_m),$$
or equivalently,
$$Diag(\mathfrak{M}) \subseteq Diag(\mathfrak{N})$$

  An important result we will use several times is the
  Tarski-Vaught criterion: Let $\mathfrak{N} := \langle N, \dots \rangle$ be a structure and $M \subseteq N$. Then M is closed under interpretations of function and constant symbols and the structure $\mathfrak{M} := \langle M, \dots \rangle$ is an elementary submodel of $\mathfrak{N}$ if and only if for all formulas $\phi(y, x_1, \dots, x_n)$ and all $(a_1, \dots, a_n) \in M^n$:

$$\text{If } \mathfrak{N} \models \exists y \phi(y, a_1, \ldots, a_n), \text{ then } \mathfrak{N} \models \phi(m, a_1, \ldots, a_n), \text{ for some } m \in M.$$

This can be proven via induction on the complexity of formulas.

- Let λ be an ordinal and $\mathcal{L}$ a language. A sequence $(\mathfrak{M}_\alpha)_{\alpha<\lambda}$ of $\mathcal{L}$-structures is called an elementary chain if $\mathfrak{M}_\alpha \prec \mathfrak{M}_\beta$, whenever $\alpha < \beta < \lambda$. The limit of the chain denoted as $\lim_{\alpha\to\lambda}(\mathfrak{M}_\alpha) =: \mathfrak{N}$ and defined as follows:

  1. It's universe is $N := \bigcup_{\alpha<\lambda} M_\alpha$, where M_α is the universe of $\mathfrak{M}_\alpha$, for $\alpha < \lambda$.
  2. $c^{\mathfrak{N}} := c^{\mathfrak{M}_0}$, for a constant symbol c.
  3. If $n \in \mathbb{N}_+$, $(a_1, \ldots, a_n) \in N^n$ and f is a n-ary function symbol of $\mathcal{L}$, then
  $$f^{\mathfrak{N}}(a_1, \ldots, a_n) := f^{\mathfrak{M}_\alpha}(a_1, \ldots, a_n),$$
  where $\alpha < \lambda$ minimal such that $(a_1, \ldots, a_n) \in M_\alpha^n$.
  4. If $n \in \mathbb{N}_+$, $(a_1, \ldots, a_n) \in N^n$ and R is a n-ary relation symbol, then
  $$R^{\mathfrak{N}}(a_1, \ldots, a_n) :\Leftrightarrow R^{\mathfrak{M}_\alpha}(a_1, \ldots, a_n),$$
  where $\alpha < \lambda$ minimal such that $(a_1, \ldots, a_n) \in M_\alpha^n$.

  It is not difficult to show that $\mathfrak{M}_\alpha \prec \mathfrak{N}$ for all $\alpha < \lambda$.

- The **downward Löwenheim-Skolem theorem** states: If $\mathcal{L}$ is a language of cardinality κ, Σ a set of $\mathcal{L}$-formulas and $\mathfrak{N} := \langle N, \ldots \rangle$ a structure such that $\kappa \le |N|$ and $\mathfrak{N} \models \Sigma(\beta)$ for a certain term assignment β, then for any $A \subseteq N$ there exists an elementary submodel $\mathfrak{M} := \langle M, \ldots \rangle \prec \mathfrak{N}$ with the following properties:

  1. $A \subseteq M$
  2. $|M| \le max\{\kappa, |A|\}$
  3. $\mathfrak{M} \models \Sigma(\beta')$, for a certain term assignment β'.

- For a set of formulas Σ and a formula ϕ we write $\Sigma \models \phi$ if for every structure $\mathfrak{M} := \langle M, \ldots \rangle$ and every term assignment β we have:
  $$\text{If } \mathfrak{M} \models \Sigma(\beta), \text{ then } \mathfrak{M} \models \phi(\beta).$$
  We write $\Sigma \vdash \phi$ if Σ proves ϕ, i.e. there is a sequence of formulas such that each formula is either a logical axiom, an element of Σ or follows from two prior formulas via modus ponens. Furthermore the last formula of the sequence is ϕ. **Gödel's completenes theorem** says that any consistent set of formulas of a predicate logic language has a model. This implies that the notions "$\models$" and "$\vdash$" are equivalent.

- **The Lemma on Constants**: Let $\mathcal{L}$ be a first order language, Γ a $\mathcal{L}$-theory and $\phi(x_1, \ldots, x_n)$ a $\mathcal{L}$-formula. If $\mathcal{C}$ is a set of constant symbols not occurring in $\mathcal{L}$, then for all $c_1, \ldots, c_n$ in $\mathcal{C}$
  $$\Gamma \vdash_{\mathcal{L}} \phi(x_1, \ldots, x_n) \text{ iff } \Gamma \vdash_{\mathcal{L}_2} \phi(c_1, \ldots, c_n),$$

where $\mathcal{L}_2$ is the language generated by the symbols of $\mathcal{L} \cup \mathcal{C}$.

- **Homomorphism**: Let $\mathfrak{M} := \langle M, \dots \rangle$ and $\mathfrak{N} := \langle N, \dots \rangle$ be (model) structures for a language $\mathcal{L}$. A map $h : M \longrightarrow N$ is called homomorphism if the following holds:

  1. $h(c^{\mathfrak{M}}) = c^{\mathfrak{N}}$, for a constant symbol c.

  2. $h(f^{\mathfrak{M}}(a_1, \dots, a_n)) = f^{\mathfrak{N}}(h(a_1), \dots, h(a_n))$, for $n \in \mathbb{N}_{>0}$, $a_1, \dots, a_n$ in M and a n-ary function symbol f.

  3. If $n \in \mathbb{N}_{>0}$, R is a n-ary relation symbol, $a_1, \dots, a_n$ are in M and $R^{\mathfrak{M}}(a_1, \dots, a_n)$, then $R^{\mathfrak{N}}(h(a_1), \dots, h(a_n))$.

  If h is injective and in (3) we have:

  $$R^{\mathfrak{M}}(a_1, \dots, a_n) \Leftrightarrow R^{\mathfrak{N}}(h(a_1), \dots, h(a_n)),$$

  then h is called an embedding. A bijective homomorphism is called **isomorphism**.

  Fact: Let $\mathfrak{M}$, $\mathfrak{N}$ be $\mathcal{L}$-structures and $h : M \longrightarrow N$ surjective. Then h is an isomorphism between $\mathfrak{M}$ and $\mathfrak{N}$ iff for all $n \in \mathbb{N}$, all formulas $\phi(x_1, \dots, x_n)$ and all $(a_1, \dots, a_n) \in M^n$ the following holds:

  $$\mathfrak{M} \models \phi(a_1, \dots, a_n) \text{ iff } \mathfrak{N} \models \phi(h(a_1), \dots, h(a_n))$$

1.1 Introduction

Given a language $\mathcal{L}$ of cardinality λ, a $\mathcal{L}$-theory Γ and a cardinal number $\kappa \in ON$, a natural question for a model theorist is, how many models of power κ does Γ have. That is, if we consider the class of all models of Γ with cardinality κ and define isomorphic models as equivalent, how many equivalence classes are there?

Remark. Under the conditions mentioned above, there are at most $2^{max\{\kappa,\lambda\}}$ many equivalence classes.

Proof. Suppose $(\mathcal{M}_\alpha)_{\alpha<\mu}$ is a sequence of pairwise nonisomorphic models of Γ each of which has cardinality κ. We can assume without loss of generality, that all models have the same universe, otherwise for each $\alpha < \mu$ use a bijection from M_α to M_0 and then interpret constant, function and relation symbols on M_0 accordingly. There are at most κ^λ many possibilities to interpret the constant symbols of $\mathcal{L}$, at most $(\kappa^{\kappa\cdot\omega})^\lambda \leq 2^{\kappa\cdot\lambda}$ many possibilities for the relation and function symbols. Hence there are at most $\kappa^\lambda \cdot 2^{\kappa\cdot\lambda} \cdot 2^{\kappa\cdot\lambda} = 2^{\kappa\cdot\lambda}$ many models of Γ with power κ. □

This observation justifies the definition of the spectrum funtion of a theory.

Definition. Let $\kappa \in ON$ be a cardinal, T a first-order theory and $\mathcal{L}$ the language generated by all symbols occurring in T. On the class of $\mathcal{L}$-models of T with power κ define two models equivalent if and only if they are isomorphic. Then $I(T, \kappa)$ is defined as the cardinality of a set of pairwise nonisomorphic representatives of all equivalence classes. For fixed T the resulting function on the class of cardinal numbers is called the spectrum function of T.

Example 1.1. Let us start with an easy example: Let our language be generated by a single binary relation symbol $\dot{<}$.The $\mathcal{L}$-theory DLO (Dense Linear Order) is given by the following axioms.

1. (A.1) $\forall x_1 \neg(x_1 \dot{<} x_1)$.
2. (A.2) $\forall x_1 \forall x_2 \forall x_3 (x_1 \dot{<} x_2 \dot{<} x_3 \rightarrow x_1 \dot{<} x_3)$.
3. (A.3) $\forall x_1 \forall x_2 [(x_1 \dot{<} x_2) \vee (x_1 \dot{=} x_2) \vee (x_2 \dot{<} x_1)]$.
4. (A.4) $\forall x_1 \exists x_2 (x_1 \dot{<} x_2)$.
5. (A.5) $\forall x_1 \exists x_2 (x_2 \dot{<} x_1)$.
6. (A.6) $\forall x_1 \forall x_2 [(x_1 \dot{<} x_2) \rightarrow \exists x_3 (x_1 \dot{<} x_3 \dot{<} x_2)]$.

Because of axioms 1, 2 and 4, every model of DLO is infinite. An obvious model is $\mathcal{Q} := \langle \mathbb{Q}, < \rangle$. Using a so called "Back and Forth"-argument, we will now show, that every countable model of DLO is isomorphic to $\mathcal{Q}$, i.e $I(DLO, \aleph_0) = 1$.

Let $\mathfrak{M} := \langle M, <' \rangle$ be an arbitrary countable model of DLO, $\mathbb{Q} = \{q_i \mid i \in \mathbb{N}\}$ and $M = \{m_j \mid \ j \in \mathbb{N}\}$. We inductively define a sequence of finite order preserving partial functions h_k ($k \in \mathbb{N}_{>0}$) such that $h_i \subseteq h_j$, for $i < j$. $h : \mathbb{Q} \mapsto M$, defined as $\bigcup_{k=1}^{\infty} h_k$ will be an isomorphism. For $k = 1$ let $i_1 := 1$, $j_1 := 1$ and $h_1(q_{i_1}) := m_{j_1}$.

$\underline{k \rightarrow k+1}$: Suppose k is odd,

$$Q_k := \{q_{i_1}, \ldots, q_{i_k}\} \subseteq \mathbb{Q} \text{ and } M_k := \{m_{j_1}, \ldots, m_{j_k}\} \subseteq M$$

are sets of cardinality k and $h_k : Q_k \mapsto M_k$ is an order preserving bijection. Let $j_{k+1} \in \mathbb{N}$ be minimal such that $m_{j_{k+1}} \notin M_k$, define

$$T_< := \{x \in M_K \mid x <' m_{j_{k+1}}\} \text{ and } T_> := \{x \in M_k \mid m_{j_{k+1}} <' x\}.$$

If $T_<$ and $T_>$ are nonempty, then there exists $max(T_<) =: u$ and $min(T_>) =: o$, since both sets are finite. $\mathcal{Q} \models (A.6)$, hence there is a minimal $i_{k+1} \in \mathbb{N}$ such that $h_k^{-1}(u) < q_{i_{k+1}} < h_k^{-1}(o)$. Then define $h_{k+1} := h_k \cup \{(q_{i_{k+1}}, m_{j_{k+1}})\}$. Clearly h_{k+1} is also order preserving.

If either $T_<$ or $T_>$ is empty, then $q_{i_{k+1}}$ exists, because $\mathcal{Q} \models [(A.4) \wedge (A.5)]$. If k is even, do the same, but start with $q_{i_{k+1}} \in \mathbb{Q} \setminus Q_k$ with minimal index and find $m_{i_{k+1}}$ analogically, using the fact that $\mathfrak{M} \models DLO$.

It follows that the theory is complete: If σ is a sentence and $\mathfrak{M} := \langle M, \ldots \rangle \models DLO \cup \{\sigma\}$, then of course $|M| \geq \aleph_0$ and because of the theorem of Löwnheim-Skolem there is a countable elementary submodel $\mathfrak{M}' \prec \mathfrak{M}$. By the argument above $\mathfrak{M}' \simeq \mathcal{Q}$, hence $\mathcal{Q} \models \sigma$. This means: If a sentence is true in one model of DLO, then it is true in all models of DLO, i.e $DLO \models \sigma$ (or $DLO \vdash \sigma$).

Example 1.2. (Theory of vector spaces over a finite field) Let $p \in \mathbb{N}$ be a prime number, $n \in \mathbb{N}_{>0}$, $q := p^n$ and $\mathbb{F}_q$ the field with cardinality q. The language $\mathcal{L} := \langle \dot{0}, \dot{+}, (\dot{r})_{r \in \mathbb{F}_q} \rangle$ has a constant symbol $\dot{0}$, a binary function symbol $\dot{+}$ and for each $r \in \mathbb{F}_q$ a 1-ary function symbol $\dot{r}$. Consider the theory Γ given by the following axioms:
Axioms for an abelian group with respect to $\dot{+}$, i.e:

1. $\forall x_1 \forall x_2 \forall x_3 ((x_1 \dot{+} x_2) \dot{+} x_3 = x_1 \dot{+} (x_2 \dot{+} x_3))$
2. $\forall x_1 (x_1 \dot{+} \dot{0} = x_1)$
3. $\forall x_1 \exists x_2 (x_1 \dot{+} x_2 = \dot{0})$
4. $\forall x_1 \forall x_2 (x_1 \dot{+} x_2 = x_2 \dot{+} x_1)$

For each $r \in \mathbb{F}_q$: $\forall x_1 \forall x_2 (\dot{r}(x_1 \dot{+} x_2) = \dot{r} x_1 \dot{+} \dot{r} x_2)$
For all $r_1, r_2, r_3 \in \mathbb{F}_q$ such that $r_1 \cdot r_2 = r_3$ (in $\mathbb{F}_q$): $\forall x_1 (\dot{r_1}(\dot{r_2} x_1) = \dot{r_3} x_1)$
For the 1-element $1 \in \mathbb{F}_q$: $\forall x_1 (\dot{1} x_1 = x_1)$
For all $(r_1, r_2, r_3) \in \mathbb{F}_q^3$ such that $r_1 + r_2 = r_3$ (in $\mathbb{F}_q$): $\forall x_1 (\dot{r_1} x_1 \dot{+} \dot{r_2} x_1 = \dot{r_3} x_1)$

An obvious model is the field $\mathbb{F}_q$ itself, i.e $\dot{0}$ is interpreted by the 0-element, $\dot{+}$ by the addition in $\mathbb{F}_q$ and for $r_1, r_2 \in \mathbb{F}_q$ $\dot{r_1} r_2 := r_1 \cdot r_2$ (multiplication in $\mathbb{F}_q$). If κ is an infinite cardinal and $\mathfrak{M} := \langle M, \dots \rangle$, $\mathfrak{N} := \langle N, \dots \rangle$ are models of Γ such that $|M| = |N| = \kappa$, then each basis $\mathcal{B}_M \subseteq M$ of the $\mathbb{F}_q$-vector space $(M, +^{\mathfrak{M}})$ has cardinality κ, since the field is finite. The same holds for $(N, +^{\mathfrak{N}})$ and an arbitrary basis $B_N \subseteq N$. Any bijection between B_M and B_N induces an isomorphism of vector spaces which is also a model isomorphism between $\mathfrak{M}$ and $\mathfrak{N}$. This means $I(\Gamma, \kappa) = 1$ for every infinite cardinal κ. A theory which has exactly one model of cardinality κ up to isomorphism, for a given cardinal κ is called κ-categorical. So Γ is κ-categorical for all $\kappa \geq \omega$.

An obvious infinite model of Γ is $\{f : \mathbb{N} \mapsto \mathbb{F}_q \mid |supp(f)| < \omega\}$ with pointwise addition and scalar multiplication. It follows that Γ is not complete.

If instead of $\mathbb{F}_q$ we take the infinite field $\mathbb{Q}$ and modify Γ accordingly, then this theory is no longer $\aleph_0$-categorical, since $(\mathbb{Q}, +)$ and $(\mathbb{Q}^2, +)$ are countable models of Γ, but not isomorphic. However, Γ is still κ-categorical for all $\kappa \geq \aleph_1$.

2 Types

In this section we stic to [4] if not stated otherwise.

Definition. Let $\mathcal{L}$ be a first order language. A ($\mathcal{L}$-)type is a consistent set of formulas. For a given $n \in \mathbb{N}$ and a set of variables $\{y_1, \dots, y_n\}$ a n-type is a consistent set of formulas all of which have their free variables among $y_1, \dots, y_n$.

- If $\mathfrak{M} = \langle M, \dots \rangle$ is a structure and $a_1, \dots, a_n$ are in M then $type_{\mathfrak{M}}(a_1, \dots, a_n) := \{\phi(x_1, \dots, x_n) \mid \phi$ is a formula with free variables among $x_1, \dots, x_n$ and $\mathfrak{M} \models \phi(a_1, \dots, a_n)\}$.
- A n-type $t(\overline{y})$ for the variable vector $\overline{y} = (y_1, \dots, y_n)$ (i.e. all free variables of its formulas are among $y_1, \dots, y_n$) is called complete if for each n-formula $\phi(\overline{y})$ we have: $\phi \in t$ or $\neg\phi \in t$.

- For a given structure $\mathfrak{M} = \langle M, \dots \rangle$ and a set $A \subseteq M$ a ($\mathfrak{M}$-) type over A is a set of formulas in the language $\mathcal{L}(A)$ consistent with $Th(\mathfrak{M}_A)$.

It is clear that for every structure $\mathfrak{M}$, $n \in \mathbb{N}$ and $a_1, \dots, a_n$ in M $type_{\mathfrak{M}}(a_1, \dots, a_n)$ is a complete n-type.

Consider for example the structure $\mathfrak{N} := \langle \mathbb{N}, 0, <, S, +, \cdot, E \rangle$ and the following set $t(x_1) := \{x_1 > 0, x_1 > S0, x_1 > SS0, \dots\}$. Then $t(x_1)$ is a 1-type over any $A \subseteq \mathbb{N}$ since every finite subset of it is realized in $\mathfrak{N}_A$, but the entire type is omitted in $\mathfrak{N}$. Any complete 1-type over A extending $t(x_1)$ would among others contain the formulas:

$$c_a = c_a \text{ for } a \in A,\ (SS0) \cdot (SSS0) = SSSSSS0,\ (x_1 > 0) \wedge (x_1 > SS0).$$

Gödel's completeness theorem implies that for a structure $\mathfrak{M} = \langle M, \dots \rangle$ and an arbitrary vector of variables $\overline{y} = (y_1, \dots, y_n)$ any n-type $t(\overline{y})$ over $\emptyset$ is realised in some model of $Th(\mathfrak{M}) \cup t(\overline{y})$ under a certain term assignment. We can show even more:

Lemma 2.1. *Let $\mathfrak{M} = \langle M, \dots \rangle$ be a structure, $\overline{y} = (y_1, \dots, y_n)$ a variable vector and $t(\overline{y})$ a n-type over $\emptyset$. Then there is an elementary extension of $\mathfrak{M}$ which realises $t(\overline{y})$.*

Proof. Let $c_1, \dots, c_n$ be new constants not occurring in $\mathcal{L}(M)$ and define $\overline{c} = (c_1, \dots, c_n)$. Consider the theory $\Sigma := Diag(\mathfrak{M}) \cup t(\overline{c})$. We claim it is consistent, for otherwise there are $\mathcal{L}$-formulas $\phi(\overline{y})$ and $\theta(\overline{y})$ such that for some $a_1, \dots, a_n$ in M, $\phi(c_{a_1}, \dots, c_{a_n}) \in Diag(\mathfrak{M})$, $\theta(\overline{c}) \in t(\overline{c})$ and

$$\phi(c_{a_1}, \dots, c_{a_n}) \vdash \neg\theta(\overline{c}).$$

Hence with the lemma on constants we can conclude

$$\phi(c_{a_1}, \dots, c_{a_n}) \vdash \neg(\exists y_1, \dots, \exists y_n \theta(\overline{y})),$$

but $t(\overline{y})$ is a n-type over $\emptyset$ (for the structure $\mathfrak{M}$) and therefore

$$Th(\mathfrak{M}) \vdash \exists y_1, \dots, \exists y_n \theta(\overline{y}).$$

Since $Th(\mathfrak{M}) \subseteq Diag(\mathfrak{M})$ this would mean that $Diag(\mathfrak{M})$ is inconsistent, a contradiction.

$\mathfrak{M}$ can be embedded elementarily into any model of Σ restricted to $\mathcal{L}$ and any such model realises $t(\overline{y})$, hence there is an elementary extension of $\mathfrak{M}$ realising $t(\overline{y})$. □

Sometimes model theorists are interested in finding a model that omits (i.e. does not realise) a given type. This is not always possible: For example, if Γ is a complete theory, $\phi(x_1, \dots, x_n)$ a n-formula consistent with Γ, $t(x_1, \dots, x_n)$ a n-type and $\Gamma \vdash (\phi \to \psi)$ for all $\psi \in t$, then since $\Gamma \vdash \exists x_1 \dots \exists x_n \phi(x_1, \dots, x_n)$ t is realised in every model of Γ.

Definition.
- If ϕ is a formula of the language $\mathcal{L}$, then $Var(\phi)$ is defined as the set of all variables occurring in ϕ and $Varfree(\phi)$ is the set of all free variables of ϕ.

- Let Γ be a theory, $\overline{y} := (y_1, \ldots, y_n)$ a vector of variables and $\Sigma(\overline{y})$ a set of n-formulas in $\overline{y}$. Σ is called isolated over Γ if there is a n-formula $\phi(\overline{y})$ consistent with Γ such that $\Gamma \vdash (\phi \rightarrow \psi)$ for all $\psi \in \Sigma$.

Theorem 2.2. ***(Omitting Types Theorem)****Let $\mathcal{L}$ be a countable language, $n \in \mathbb{N}$, Γ a consistent $\mathcal{L}$-theory and $\Sigma(x_1, \ldots, x_n)$ a set of n-formulas which is not isolated over Γ. Then Γ has model which omits Σ.*

Proof. We will first add countably many new constants $\mathcal{C} := \{c_k \mid k \in \mathbb{N}\}$, thereby getting a new language $\mathcal{L}_2$, then extend our theory Γ to a complete and consistent Henkin-theory $\overline{\Gamma}$ (in $\mathcal{L}_2$), that is: if $\psi(y)$ is a 1-formula such that $\overline{\Gamma} \vdash \exists y \psi(y)$, then $\overline{\Gamma} \vdash \psi(c)$ for some $c \in \mathcal{C}$. Furthermore, using the fact that Σ is not isolated over Γ, for each n-tuple $\overline{c} := (c_1, \ldots, c_n) \in \mathcal{C}^n$ $\overline{\Gamma} \vdash \neg\psi(\overline{c})$ for some $\psi \in \Sigma$. For any model $\mathfrak{M} := \langle M, \ldots \rangle$ of $\overline{\Gamma}$ we will have

$$\mathfrak{M}' := \langle \{c_k^{\mathfrak{M}} \mid k \in \mathbb{N}\}, \ldots \rangle \prec \mathfrak{M}$$

and $\mathfrak{M}'$ omits Σ.

Let $(\overline{c}_m)_{m \in \mathbb{N}}$ be an enumeration of all n-tuples of $\mathcal{C}$, $(\theta_m)_{m \in \mathbb{N}}$ an enumeration of all sentences (closed formulas) of $\mathcal{L}_2$ and $\Gamma_{-1} := \Gamma$. Suppose $k \geq 0$ and we have already consistently constructed $\Gamma_{k-1} \supseteq \Gamma$ using only finitely many new constants of $\mathcal{C}$ and $\Gamma_{k-1} \setminus \Gamma$ is empty or a $\mathcal{L}_2$-sentence σ_{k-1}.

Case $k = 3m$, for some $m \in \mathbb{N}$: If $\Gamma_{k-1} \cup \{\theta_m\}$ is consistent, then $\sigma_k := \sigma_{k-1} \wedge \theta_m$ and $\Gamma_k := \Gamma \cup \{\sigma_k\}$, otherwise $\sigma_k := \sigma_{k-1} \wedge \neg\theta_m$ and $\Gamma_k := \Gamma \cup \{\sigma_k\}$. For $k = 0$ σ_0 is either θ_0 or $\neg\theta_0$.

Case $k = 3m + 1$, for some $m \in \mathbb{N}$: If $\sigma_{k-1} = \sigma_{k-2} \wedge \delta$ (or $\sigma_{k-1} = \delta$ for $k = 1$) and $\delta = \exists y \psi(y)$, then $\sigma_k := \sigma_{k-1} \wedge \psi(c)$, where c is the first constant of $\mathcal{C}$ not occurring in Γ_{k-1} and $\Gamma_k := \Gamma \cup \{\sigma_k\}$. Otherwise $\sigma_k := \sigma_{k-1}$ and $\Gamma_k := \Gamma \cup \{\sigma_k\}$.

Case $k = 3m + 2$, for some $m \in \mathbb{N}$: If $\overline{c}_m = (c_{i_1}, \ldots, c_{i_n})$ and $\Gamma_{k-1} = \Gamma \cup \{\sigma_{k-1}\}$ then define $A := \{c \in \mathcal{C} \mid c \text{ occurs in } \sigma_{k-1} \text{ but not in } \overline{c}_m\}$. From σ_{k-1} we build a $\mathcal{L}$-formula δ' by first replacing each variable $c \in A$ with a new variable $y_c \notin Var(\sigma_{k-1}) \cup \{x_1, \ldots, x_n\}$ and then replacing c_{i_l} in $\overline{c}_m$ $(1 \leq l \leq n)$ with x_j, where $1 \leq j \leq n$ minimal such that $c_{i_j} = c_{i_l}$. We can assume without loss of generality that the latter step is possible, because otherwise we just rename bounded variables which gives us a tautologically equivalent formula.

Now consider the formula

$$\delta(x_1, \ldots, x_n) := \exists y_{c_{s_1}} \ldots \exists y_{c_{s_r}} \delta' \wedge (\bigwedge_{\substack{1 \leq j, l \leq n \\ c_{i_j} = c_{i_l}}} x_j = x_l).$$

Sicne Γ_{k-1} is consistent, it follows that $\Gamma \cup \{\delta\}$ is consistent. Σ is not isolated over Γ, therefore $\Gamma \cup \{\delta \wedge \neg\psi\}$ is consistent for some $\psi \in \Sigma$. Then define $\sigma_k := \sigma_{k-1} \wedge \neg\psi(c_{i_1}, \ldots, c_{i_n})$ and $\Gamma_k := \Gamma \cup \{\sigma_k\}$. It is easy to see that Γ_k is consistent and hence

$$\overline{\Gamma} := \bigcup_{k \in \mathbb{N}} \Gamma_k$$

is consistent, complete and Henkin.

□

Corollary 2.3. *Let Γ be a consistent theory of the countable language $\mathcal{L}$ and for each $k \in \mathbb{N}$ $\Sigma_k(x_1, \ldots, x_{n_k})$ be a set of n_k-formulas not isolated over Γ. Then there exists a model of Γ omitting Σ_k for all $k \in \mathbb{N}$.*

Proof. Let $\mathcal{C} := \{c_i \mid i \in \mathbb{N}\}$ be a set of new variables, $\mathcal{L}_2$ be the language generated by $\mathcal{L}$ and $\mathcal{C}$ and $(d_m := \langle \theta_m, k_m, \overline{c}_m \rangle)_{m \in \mathbb{N}}$ an enumeration of all triples such that θ_m is a $\mathcal{L}_2$ sentence, $k_m \in \mathbb{N}$ and $\overline{c}_m \in \mathcal{C}^{n_{k_m}}$ where Σ_{k_m} is a set of n_{k_m} formulas.

Similarly to the previous proof we construct a sequence of $\mathcal{L}_2$-sentences $(\sigma_m)_{m \in \mathbb{N}}$ such that the following hold:

1. $\Gamma \cup \{\sigma_m\}$ is consistent.
2. $\sigma_j \vdash \sigma_i$, for $i < j$

In each step $m \in \mathbb{N}$ we do three substeps: Given the sentence σ_{m-1}, we first check whether $\sigma_{m-1} \wedge \theta_m$ is consistent. If so, then $\delta_0 := \sigma_{m-1} \wedge \theta_m$, otherwise $\delta_0 := \sigma_{m-1} \wedge \neg\theta_m$. Then check whether θ_m (or $\neg\theta_m$, if chosen in first substep) is of the form $\exists y \psi(y)$. If so, then $\delta_1 := \delta_0 \wedge \psi(c)$, where $c \in \mathcal{C}$ with minimal index not occurring in δ_0, otherwise $\delta_1 := \delta_0$. Then we define $A_m := \{c \in \mathcal{C} \mid c \text{ occurs in } \delta_1 \text{ but not in } \overline{c}_m\}$, replace every $c \in A_m$ with a new variable y_c which does not occur in δ_1 and every constant c_{i_l} $(1 \leq l \leq n_{k_m})$ of $\overline{c}_k$ with the variable x_j where $1 \leq j \leq n_{k_m}$ minimal such that $c_{i_j} = c_{i_l}$, thereby getting a $\mathcal{L}$-formula δ_1'. The n_{k_m}-formula

$$\phi_m(x_1, \ldots, x_{n_{k_m}}) := \exists y_{d_1} \ldots \exists y_{d_a} \delta_1' \wedge (\bigwedge_{\substack{1 \leq j,l \leq n_{k_m} \\ c_{i_l} = c_{i_j}}} x_j = x_l)$$

is consistent with Γ and since Σ_m is not isolated over Γ it follows that

$$\Gamma \cup \{\phi_m \wedge \neg\psi\}$$

is consistent for some $\psi \in \Sigma_m$. Then it is easy to see that $\sigma_m := \delta_1 \wedge \neg\psi(\overline{c}_m)$ is also consistent with Γ.

The theory $\overline{\Gamma} := \Gamma \cup \{\sigma_m \mid m \in \mathbb{N}\}$ is complete, consistent and Henkin and for every model $\mathfrak{M}$ of $\overline{\Gamma}$ the structure

$$\langle \{c^{\mathfrak{M}} \mid c \text{ is a constant of the language } \mathcal{L}\}, \ldots \rangle \restriction \mathcal{L}$$

is a model of Γ omitting Σ_k for all $k \in \mathbb{N}$. $\square$

2.1 Topological Aspects

In the following section we fix an arbitrary language $\mathcal{L}$, a $\mathcal{L}$-structure $\mathfrak{M} := \langle M, \ldots \rangle$ and a set $A \subseteq M$. For $n \in \mathbb{N}_+$, the variable vector $\overline{x} := (x_1, \ldots, x_n)$ and a n-formula $\phi(\overline{x})$ we often write simply ϕ.

Definition. Let $\mathfrak{M} := \langle M, \ldots \rangle$ be a structure and $A \subseteq M$. For $\overline{x} := (x_1, \ldots, x_n)$ and $n \in \mathbb{N}$ we define $S_n^{\mathfrak{M}}(A) := \{t(\overline{x}) \mid \text{t is a complete n-type over A}\}$.

- If $\phi(\overline{x})$ is a n-formula, then $[\phi] := \{t \in S_n^{\mathfrak{M}}(A) \mid \phi \in t\}$.

One can easily check that for n-formulas $\psi(\overline{x})$ and $\phi(\overline{x})$ we have:

$$[\phi \wedge \psi] = [\phi] \cap [\psi] \quad \text{and} \quad [\phi \vee \psi] = [\phi] \cup [\psi]. \tag{1}$$

This follows immediately from the fact that complete n-types are consistent and deductively closed.

Furthermore $[x_1 = x_1] = S_n^{\mathfrak{M}}(A)$, which together with (1) implies that $\{[\phi(\overline{x})] \mid \phi \text{ is a n- formula in } \overline{x}\}$ is a basis for a topology (the Stone topology) on $S_n^{\mathfrak{M}}(A)$.

If $\phi(\overline{x})$ is a$(\mathcal{L}(A))$-n-formula, $t(\overline{x}) \in S_n^{\mathfrak{M}}(A)$ and $t \notin [\phi]$, then $\neg\phi \in t$ (or $t \in [\neg\phi]$), since t is complete, and $[\neg\phi] \cap [\phi] = \emptyset$. Hence every basic open set is also closed. We now present some basic results for $S_n^{\mathfrak{M}}(A)$.

Lemma 2.4. *For any structure $\mathfrak{M} := \langle M, \dots \rangle$, $n \in \mathbb{N}_{>0}$ and $A \subseteq M$ we have:*

1. *$S_n^{\mathfrak{M}}(A)$ is compact.*
2. *$S_n^{\mathfrak{M}}(A)$ is totally disconnected, that is: For all $t_1, t_2 \in S_n^{\mathfrak{M}}(A)$ with $t_1 \neq t_2$ there exists a clopen $X \subseteq S_n^{\mathfrak{M}}(A)$ such that $t_1 \in X$ and $t_2 \notin X$.*

Proof. 1. We show indirectly that every covering of $S_n^{\mathfrak{M}}(A)$ with basic open sets has a finite subcovering: Suppose there is a cover $\mathcal{C} := \{[\phi_i]\}_{i\in I}$ of $S_n^{\mathfrak{M}}(A)$ with no finite subcover. Then consider the set $F := \{\neg\phi_i\}_{i\in I}$. F is consistent with $Th(\mathfrak{M}_A)$: If $k \in \mathbb{N}_{>0}$ and $\neg\phi_{i_1}, \dots, \neg\phi_{i_k}$ are in F, then there exists a $t \in S_n^{\mathfrak{M}}(A)$, which is by definition consistent with $Th(\mathfrak{M}_(A))$, such that $\neg\phi_{i_j} \in t$ for $1 \leq j \leq k$, since by our assumption $\bigcup_{j=1}^n [\phi_{i_j}] \neq S_n^{\mathfrak{M}}(A)$. So every finite subset of F is consistent with $Th(\mathfrak{M}_A)$, therefore $F \subseteq t$ for some $t \in S_n^{\mathfrak{M}}(A)$. But then $t \notin \bigcup\{[\phi_i] \mid i \in I\}$, contradicting the assumption that $\mathcal{C}$ is a cover of $S_n^{\mathfrak{M}}(A)$.

2. If t_1, t_2 are in $S_n^{\mathfrak{M}}(A)$ and distinct, then since both types are complete there is a n-formula ϕ such that $\phi \in t_1$ and $\neg\phi \in t_2$, which means $t_1 \in [\phi]$ and $t_2 \notin [\phi]$. $[\phi]$ is clopen in $S_n^{\mathfrak{M}}(A)$ as we have pointed out before.

□

Corollary 2.5. *For all $T \subseteq S_n^{\mathfrak{M}}(A)$:*

T is clopen iff $T = [\phi]$ for some n-formula ϕ.

Proof. • ($\Leftarrow$) Already explained.

• ($\Rightarrow$) Since T is closed it is also compact. Furthermore T is open, hence there is a $k \in \mathbb{N}_+$ and n-formulas $\phi_1, \dots, \phi_k$ such that $T = \bigcup_{i=1}^k [\phi_i] = [\bigvee_{i=1}^k \phi_i]$

□

Definition. A complete type $t \in S_n^{\mathfrak{M}}(A)$ is called isolated if $\{t\}$ is open in the Stone topology on $S_n^{\mathfrak{M}}(A)$.

Proposition 2.6. *Let $t \in S_n^{\mathfrak{M}}(A)$. The following are equivalent:*

1. *t is isolated.*
2. *$\{t\} = [\phi(\overline{x})]$ for some $\mathcal{L}(A)$-n-formula $\phi(\overline{x})$.*

3. *There exists a $\mathcal{L}(A)$-n-formula $\phi(\overline{x})$ such that for all $\mathcal{L}(A)$-n-formulas $\psi(\overline{x})$ we have:*

$$\psi \in t \textit{ iff } \phi \vdash_{Th(\mathfrak{M}_A)} \psi.$$

Proof.

- (1) $\Rightarrow$ (2): If $\{t\}$ is open, then there is a $\mathcal{L}(A)$-n-formula $\phi(\overline{x})$ such that $t \in [\phi] \subseteq \{t\} \subseteq [\phi]$.

- (2) $\Rightarrow$ (3): Let $\{t\} = [\phi]$ for some n-formula ϕ and $\psi \in t$. We show that $Th(\mathfrak{M}_A)) \cup \{\phi\} \models \psi$, which is equivalent to $\phi \vdash_{Th(\mathfrak{M}_A)} \psi$: If $\mathfrak{N} := \langle N, \dots \rangle$ and $\mathfrak{N} \models Th(\mathfrak{M}_A) \cup \{\phi\}$ under an assignment $\overline{x} \mapsto \overline{a} \in N^n$, then $type_{\mathfrak{N}}(\overline{a}) \in [\phi] = \{t\}$, which implies $\mathfrak{N} \models \psi(\overline{a})$. Now assume ψ is an arbitrary n-formula and $\phi \vdash_{Th(\mathfrak{M}_A)} \psi$. Since $\{t\} = [\phi]$ (by assumption) and t as a complete n-type is deductively closed and clearly $\phi \in t$, it follows $\psi \in t$.

- (3) $\Rightarrow$ (2): Let ϕ be a n-formula such that (3) holds. Clearly $\phi \vdash \phi$, so $\phi \in t$ and therefore $\{t\} \subseteq [\phi]$. Now let $t' \in [\phi]$. We show that $t \subseteq t'$ which implies $t = t'$, since t and t' are complete n-types: If $\psi \in t$, then by assumption (3) we have $\phi \vdash \psi$ and hence and $\psi \in t'$, because t' is deductively closed. This means $[\phi] \subseteq \{t\}$.

- (2) $\Rightarrow$ (1): This is obvious, since $[\phi]$ is a basic open set in the Stone topology.

$\square$

Remark. Prpposition 2.6 implies that the definition of beeing isolated from a topological perspective is equivalent to that of the previous subsection: A complete n-type over A is isolated if and only if it is isolated over $Th(\mathfrak{M}_A)$.

2.2 Algebraic Aspects

This subsection is based on [5].

A Boolean algebra is a structure $\langle A, 0, 1, +, -, \cdot \rangle$, where $0 \neq 1$ are elements of the set A, $+$ and $-$ are binary operations $A \times A \mapsto A$ and $- : A \mapsto A$ such that for all $x, y, z \in A$ the following holds:

(assiociativity)	$(B1)$ $(x+y)+z = x+(y+z)$,	$(B1')$ $(x \cdot y) \cdot z = x \cdot (y \cdot z)$
(commutativity)	$(B2)$ $x+y = y+x$,	$(B2')$ $x \cdot y = y \cdot x$
(absorption)	$(B3)$ $x+(x \cdot y) = x$,	$(B3')$ $x \cdot (x+y) = x$
(distributivity)	$(B4)$ $x \cdot (y+z) = (x \cdot y)+(x \cdot z)$,	$(B4')$ $x+(y \cdot z) = (x+y) \cdot (x+z)$
(complementation)	$(B5)$ $x+(-x) = 1$,	$(B5')$ $x \cdot (-x) = 0$

On every Boolean algebra $\mathcal{B} := \langle A, \dots \rangle$ there is a partial order defined by

$$a \leq b :\Leftrightarrow a \cdot b = a$$

Definition.

- Let $\mathcal{B} := \langle A, \dots \rangle$ be a Boolean algebra. A filter on $\mathcal{B}$ is a set $F \subseteq A$ with the following properties:

1. $1 \in F$ and $0 \notin F$.
2. If $a, b \in F$, then $a \cdot b \in F$.
3. If $a \in F$, $b \in A$ and $a \leq b$, then $b \in F$.

- A filter F on a Boolean albgebra $\langle A, \dots \rangle$ is called an ultrafilter if for all $a \in A$ either $a \in F$ or $-a \in F$.
- A filter F is called principal if there is an element $a \in A$ such that
$$F = \{b \in A \mid a \leq b\}.$$
- Given a Boolean algebra $\langle A, \dots \rangle$ a set $T \subseteq A$ is called a filter basis if for every non empty finite index set J and $\{a_i \mid i \in J\} \subseteq T$
$$\prod_{i \in J} a_i \neq 0.$$

Remark. 1. If T is a filter basis and T' is defined as the set of all finite products of elements of T, then $\{b \in A \mid c \leq b, \text{for some } c \in T'\}$ is a filter.

2. It is easy to show that a filter is ultra if and only if it is a maximal filter with respect to the inclusion relation $\subseteq$.

Let $\mathcal{L}$ be a language of first order logic , Γ a consistent $\mathcal{L}$-theory, $n \in \mathbb{N}$ and Σ the set of all $\mathcal{L}$-formulas. For $\phi, \psi \in \Sigma$ define
$$\phi \sim \psi :\Leftrightarrow \Gamma \vdash (\phi \leftrightarrow \psi).$$
This gives us an equivalence relation. Now, let $A := \Sigma / \sim$, the quotient space. For every formula ψ, $[\psi]$ denotes the equvivalence class of ψ with respect to $\sim$. Define $\mathcal{B} := \langle A, 0, 1, +, -, \cdot \rangle$, where

1. $0 := [\exists x_1(x_1 \neq x_1)]$ and $1 := [\forall x_1(x_1 = x_1)]$.
2. $- : A \mapsto A$, $-[\psi] := [\neg \psi]$.
3. $+ : A \times A \mapsto A$, $[\phi] + [\psi] := [\phi \vee \psi]$.
4. $\cdot : A \times A$, $[\phi] \cdot [\psi] := [\phi \wedge \psi]$.

We can easily check that $\mathcal{B}$ is a Boolean algebra. If for $n \in \mathbb{N}$ we define Σ_n as the set of all n-formulas with free variables in $x_1, \dots, x_n$ and consider the operations $+, -, \cdot$ on $\Sigma_n / \sim$, then we get a Boolean algebra $\mathcal{B}_{(\Gamma,n)}$ isomorphic to a subalgebra of $\mathcal{B}$. $\mathcal{B}_{(\Gamma,0)}$ is called the Lindenbaum-Tarski algebra of Γ.

Since for any $\mathcal{L}$-formulas ϕ, ψ we have
$$\vdash [(\phi \wedge \psi) \leftrightarrow (\phi)] \leftrightarrow [\phi \rightarrow \psi],$$
$[\phi] \leq [\psi]$ in $\mathcal{B}$ if and only if $\phi \vdash_\Gamma \psi$. If t is an arbitrary type consistent with Γ, then $F_t := \{[\phi] \mid \phi \in t\} \subseteq A$ is a filter basis. If t is deductively closed, then F_t is a filter on $\mathcal{B}$. The map $t \mapsto F_t$ is a bijection from the set of all deductively closed types consistent with Γ to the set of all filters on $\mathcal{B}$, whereby complete types correspond with ultra filters.

3 $\aleph_0$-Categorical Theories, Vaught's Never Two Theorem and beyond

The results of this section can be found in [2].

3.1 ω-Saturated Models

In short, a type for a given theory Γ is a set of formulas describing a property consistent with Γ. As we have seen for the standard model of the natural numbers $\langle \mathbb{N}, 0, S, +, \cdot, < \rangle$ a type is not necessarily realised in all models of Γ. We will now study models which have witnesses for "many" types.

Definition. Let $\mathcal{L}$ be a language and $\mathfrak{M} := \langle M, \dots \rangle$ a $\mathcal{L}$-structure. $\mathfrak{M}$ is called ω-saturated if for every finite $A \subseteq M$ every 1-type over A is realised in $\mathfrak{M}_A$.

Remark. This definition is equivalent to the notion that for every finite $A \subseteq M$ and every $n \in \mathbb{N}_+$ every n-type over A is realised in $\mathfrak{M}_A$.

Proof. Suppose a $\mathcal{L}$-structure is ω-saturated according to our definition and $A \subseteq M$ finite. We show via induction on n, that every n-type over A is realised in $\mathfrak{M}_A$: Case $n = 1$ follows from the definition.

$n \to n+1$: Let $t(x_1, \dots, x_n, x_{n+1})$ be a $n+1$-type over A. Without loss of generality t is deductively closed with respect to $n+1$-formulas, otherwise we do the following argument for the deductive closure of t (for $n+1$-formulas) which is also a $n+1$-type. Consider the set of n-formulas

$$t' := \{\exists x_{n+1} \psi(x_1, \dots, x_{n+1}) \mid \psi(x_1, \dots, x_{n+1}) \in t\}$$

which is consistent with $Th(\mathfrak{M}_A)$, because t is. So t' is a n-type over A. Using the induction hypothesis, there are $a_1, \dots, a_n$ in M such that

$$\mathfrak{M}_A \models \theta(a_1, \dots, a_n)$$

for all $\theta(x_1, \dots, x_n) \in t'$, hence $\mathfrak{M}_B \models \theta(c_{a_1}, \dots, c_{a_n})$, where $B := A \cup \{a_1, \dots, a_n\}$. Using the compactness theorem and the fact that t is deductively closed with respect to $n+1$-formulas, it is easy to see, that

$$\{\psi(c_{a_1}, \dots, c_{a_n}, x_{n+1}) \mid \psi(x_1, \dots, x_{n+1}) \in t\}$$

is a set of 1-formulas in the language $\mathcal{L}(B)$ consistent with $Th(\mathfrak{M}_B)$ and is therefore realised in $\mathfrak{M}_B$ by an element $a_{n+1} \in M$. Then clearly we have

$$\mathfrak{M}_A \models \psi(a_1, \dots, a_{n+1})$$

for all $\psi \in t$. The other direction follows immediately. $\square$

Example 3.1. $\mathcal{Q} := \langle \mathbb{Q}, < \rangle$ is an ω-saturated model of DLO.

Proof. Let $A = \{q_1, \dots, q_n\}$ be a finite subset of $\mathbb{Q}$ and $t(y)$ be a 1-type over A. Then there is a countable model $\mathfrak{M} := \langle M, \dots \rangle$ and a $m \in M$ such that $\mathfrak{M} \models Th(\mathcal{Q}_A) \cup t(m)$. Let $B := \{c_a^{\mathfrak{M}} \mid a \in A\}$. Since $\mathfrak{M} \models Th(\mathcal{Q}_A)$, there is an order preserving bijection $h_0' : A \mapsto B$ and since both $\mathcal{Q}_A$ and $\mathfrak{M}$ are models of DLO, it follows that there is a $q \in \mathbb{Q}$ such that h_0' can be extended to an order preserving bijection $h' : A \cup \{q\} \mapsto B \cup \{m\}$ which in turn can be extended to an isomorphism $h : \mathcal{Q}_A \mapsto \mathfrak{M}$ using the same "Back and Forth" argument we have seen in example 1.1. So $t(y)$ is realised in $\mathcal{Q}_A$. $\square$

Example 3.2. (Theory of infinite vector spaces over a finite field) Let $\mathbb{F}_q$ be a finite field and $\mathcal{L}$ the language described in example 1.2. Consider the theory Γ, which consists of the axioms given in example 1.2 together with the following sentences:

For each $n \in \mathbb{N}_+$ add

$$\forall x_1 \dots \forall x_n \exists x_{n+1} (x_1 \neq x_{n+1} \wedge \dots \wedge x_n \neq x_{n+1})$$

This theory is ω-categorical, as any countable model of it is a vector space with dimension $\aleph_0$. Using the theorem of Löwenheim-Skolem, we can see that it is also complete: Any sentence consistent with Γ is realised in a countable model of which there is only one up to isomorphism.

Let $\mathcal{V}$ be the model of Γ defined by $V := \{f : \mathbb{N} \mapsto \mathbb{F}_q \mid |supp(f)| < \omega\}$ together with pointwise addition and scalar multiplication. If $k \in \mathbb{N}_+$, $A = \{a_1, \dots, a_k\} \subseteq V$ finite, $t(x_1)$ is a type over A and $\mathfrak{M} := \langle M, \dots \rangle$ is a countable model of $Th(\mathcal{V}_A)$ in which $t(x_1)$ is realised, say by an element $m \in M$, then define $M_1' := Span(\{c_a^{\mathfrak{M}} \mid a \in A\})$ (a subspace of M) and choose any subspace M_2' of M such that $M = M_1' \oplus M_2'$. Since M_1' is finite dimensional we have $dim(M_2') = \aleph_0$. Similarily, we can define $V_1' := Span(A)$ and choose a subspace $V_2' \subseteq V$ of dimension $\aleph_0$ such that $V = V_1' \oplus V_2'$. The map $a_i \mapsto c_{a_i}^{\mathfrak{M}}$ induces an isomorphism of vector spaces $h_1 : V_1' \mapsto M_1'$, because $\mathcal{V}_A$ and $\mathfrak{M}$ are models of $Th(\mathcal{V}_A)$. Any bijection from a basis of V_2' to a basis of M_2' yields an isomorphism $h_2 : V_2' \mapsto M_2'$. The map $h_1 \oplus h_2$ then defines an isomorphism $\mathcal{V}_A \mapsto \mathfrak{M}$ and therefore $t(x_1)$ is realised in $\mathcal{V}_A$.

Example 3.3. Let $\mathcal{N} := \langle \mathbb{N}, 0, S, +, \cdot, < \rangle$ be the standard model of the natural numbers, where S denotes the successor operation $x \mapsto x + 1$. Then $\mathcal{N}$ is not ω-saturated, since the set $t(x_1)$ defined as

$$\{x_1 > S^n 0 \mid n \in \mathbb{N}_+\}$$

is a 1-type over $\emptyset$ which is not realised in $\mathcal{N}$. Any model of $Th(\mathcal{N})$ realising $t(x_1)$ is a so called nonstandard model of the theory of the natural numbers.

Proposition 3.4. *Let $\mathcal{L}_1$, $\mathcal{L}_2$ be languages such that $\mathcal{L}_1 \subseteq \mathcal{L}_2$ and $\mathcal{L}_2$ differs form $\mathcal{L}_1$ by a finite set of constant symbols. Let $\mathfrak{M} := \langle M, \dots \rangle$ be a $\mathcal{L}_2$-structure. Then*

$$\mathfrak{M} \text{ is } \omega\text{-saturated iff } \mathfrak{M} \restriction \mathcal{L}_1 \text{ is } \omega\text{-saturated.}$$

Proof.

- ($\Rightarrow$): Let $A \subseteq M$ be finite and $t(x_1)$ a deductutively closed 1-type over A for the structure $\mathfrak{M} \restriction \mathcal{L}_1$. If we define

$$\mathcal{D} := \{c \in \mathcal{L}_2 \setminus \mathcal{L}_1 \mid c \text{ is a constant symbol}\}$$

and $B := A \setminus \{c^{\mathfrak{M}} \mid c \in \mathcal{D}\}$, then B is finite and we can see $t(x_1)$ as a set of formulas in the language $\mathcal{L}_2(B)$. It is consistent with $Th(\mathfrak{M}_B)$, otherwise there is a $n \in \mathbb{N}_{>0}$, $c_1, \dots, c_n$ in $\mathcal{D} \setminus \mathcal{L}_1(A)$ and a $\mathcal{L}_1(A)$-formula $\theta(x_1, \dots, x_n)$ such that

$$\theta(c_1, \dots, c_n) \in Th(\mathfrak{M}_B) \text{ and } \exists x_1 \psi(x_1) \vdash \neg\theta(c_1, \dots, c_n),$$

for some $\psi \in t(x_1)$, then by the lemma on constants

$$\exists x_1 \psi(x_1) \vdash \neg(\exists x_1 \dots \exists x_n \theta(x_1, \dots, x_n))$$

which means that $t(x_1)$ is inconsistent with $Th((\mathfrak{M} \upharpoonright \mathcal{L}_1)_A)$, a contradiction. Any element of M realising $t(x_1)$ in $\mathfrak{M}_B$ also realises it in $(\mathfrak{M} \upharpoonright \mathcal{L}_1)_A$.

- ($\Leftarrow$): If $A \subseteq M$ is finite and $t(x_1)$ is consistent with $Th(\mathfrak{M}_A)$, then we can define $\overline{A} := A \cup \{c^{\mathfrak{M}} \mid c \in \mathcal{D}\}$, which is finite, because $\mathcal{D}$ is. We can see $t(x_1)$ as a set of formulas in the language $\mathcal{L}_1(\overline{A})$ consistent with $Th((\mathfrak{M} \upharpoonright \mathcal{L}_1)_{\overline{A}})$. Any element of M realising $t(x_1)$ in $(\mathfrak{M} \upharpoonright \mathcal{L}_1)_{\overline{A}}$ also realises it in $\mathfrak{M}_A$.

□

Lemma 3.5. *Let Γ be a complete theory of the language $\mathcal{L}$, $\mathfrak{N} := \langle M, \dots \rangle$ an ω-saturated model and $\mathfrak{M} := \langle M, \dots \rangle$ an arbitrary countable model of Γ. Then there exists an elementary embedding $\mathfrak{M} \hookrightarrow \mathfrak{N}$.*

Proof. Let $M = \{m_k \mid k \in \mathbb{N}_{>0}\}$. We inductively construct a countable elementary submodel of $\mathfrak{N}$.

$\underline{k = 1}$: $type_{\mathfrak{M}}(m_1)$ is a 1-type over $\emptyset$ for the structure $\mathfrak{N}$ and is therefore realised by some $n_1 \in N$.

$\underline{k \to k+1}$: Suppose we have $m_1, \dots, m_k$ in M and $n_1, \dots, n_k$ in N such that

$$type_{\mathfrak{M}}(m_1, \dots, m_k) = type_{\mathfrak{N}}(n_1, \dots, n_k).$$

If we add new constant symbols $c_1, \dots, c_k$ to our language $\mathcal{L}$ and define

$$c_i^{\mathfrak{M}} := m_i \text{ and}$$
$$c_i^{\mathfrak{N}} := n_i, \text{ for } 1 \le i \le k,$$

then $Th(\mathfrak{M}_A) = Th(\mathfrak{N}_{A'})$, where $A := \{m_1, \dots, m_k\}$ and $A' := \{n_1, \dots, n_k\}$. Using this fact and the compatness theorem, it follows that

$$t(x_{k+1}) := \{\psi(c_1, \dots, c_k, x_{k+1}) \mid \mathfrak{M}_A \models \psi(c_1, \dots, c_k, m_{k+1})\}$$

is consistent with $Th(\mathfrak{N}_{A'})$, and since $\mathfrak{N}$ is ω-saturated there is a $n_{k+1} \in N$ realising $t(x_{k+1})$ in $\mathfrak{N}_{A'}$. Then clearly we also have

$$type_{\mathfrak{M}}(m_1, \dots, m_{k+1}) = type_{\mathfrak{N}}(n_1, \dots, n_{k+1})$$

The map $h : M \mapsto N$, $m_k \mapsto n_k$ has the following properties.

1. h is injective: If $m_i \neq m_j$, then $\mathfrak{M} \models \neg(x_i = x_j)(m_i, m_j)$, hence

$$\mathfrak{N} \models \neg(x_i = x_j)(n_i, n_j)$$

2. h respects constant symbols: $\mathfrak{M} \models (x_i = c)(m_i) \Rightarrow \mathfrak{N} \models (x_i = c)(n_i)$, for a constant symbol $c \in \mathcal{L}$.

3. h respects relation and function symbols: for example, If $\kappa \in \mathbb{N}_{>0}$ and $R \in \mathcal{L}$ a k-ary relation symbol, then

$$R^{\mathfrak{M}}(m_{j_1}, \dots, m_{j_k}) \Leftrightarrow \mathfrak{M} \models (Rx_{j_1} \dots x_{j_k})(m_{j_1}, \dots, m_{j_k}) \Leftrightarrow \dots$$
$$\dots \Leftrightarrow \mathfrak{N} \models (Rx_{j_1} \dots x_{j_k})(n_{j_1}, \dots, n_{j_k}) \Leftrightarrow R^{\mathfrak{N}}(n_{j_1}, \dots, n_{j_k}).$$

The case for function symbols is similar.

It remains to show that $\langle\{n_k \mid k \in \mathbb{N}_+\}, \dots\rangle$ is an elementary submodel of $\mathfrak{N}$: If $\psi(x_1, \dots, x_k, y)$ is a $\mathcal{L}$-formula and $\mathfrak{N} \models \exists y \psi(n_1, \dots, n_k, y)$, then because of the definition of h, we have $\mathfrak{M} \models \psi(m_1, \dots, m_k, m_i)$, for some $i \in \mathbb{N}_+$. It easily follows $\mathfrak{N} \models \psi(n_1, \dots, n_k, n_i)$, hence $\{n_k \mid k \in \mathbb{N}_+\}$ satisfies the Tarski-Vaught criterion.

$\square$

Corollary 3.6. *Any two countable ω-saturated models of a complete theory are isomorphic.*

Proof. Let $\mathcal{A} := \langle A, \dots\rangle$, $\mathcal{B} := \langle B, \dots\rangle$ be ω-saturated models of Γ, $\{a_k \mid k \in \mathbb{N}_{>0}\}$ an enumeration of A and $\{b_k \mid k \in \mathbb{N}_{>0}\}$ an enumeration of B. We use the construction of the proof of Lemma 3.5 in a "Back and Forth" argument: start with $i_1 := 1$, then there is a $j_1 \in \mathbb{N}_{>0}$ minimal such that

$$type_{\mathcal{A}}(a_{i_1}) = type_{\mathcal{B}}(b_{j_1})$$

If we already have a partial function $a_{i_s} \mapsto b_{j_s}$ for $1 \leq s \leq k$ such that

$$type_{\mathcal{A}}(a_{i_1}, \dots, a_{i_k}) = type_{\mathcal{B}}(b_{j_1}, \dots, b_{j_k}),$$

then we proceed as follows:

<u>k is odd</u>: Let $j_{k+1} := min(\mathbb{N}_{>0} \setminus \{j_1, \dots, j_k\})$. Like before, using the fact that $\mathcal{A}$ is ω-saturated there is a $i_{k+1} \in \mathbb{N}_{>0}$ minimal such that

$$type_{\mathcal{A}}(a_{i_1}, \dots, a_{i_{k+1}}) = type_{\mathcal{B}}(b_{j_1}, \dots, b_{j_{k+1}})$$

<u>k is even</u>: Reverse the roles j_{k+1} and i_{k+1} and use the fact that $\mathcal{B}$ is ω-saturated. The resulting function $h : A \mapsto B$, $a_{i_k} \mapsto b_{j_k}$ is an isomorphism between $\mathcal{A}$ and $\mathcal{B}$ which can be checked similarly to the previous proof. $\square$

Now we are comming to the characterisation of complete theories of a countable language which have a countable ω-saturated model. We will use the following

Fact. Let $\mathfrak{N} := \langle N, \dots\rangle$ be a structure for the language $\mathcal{L}$, $\mathfrak{M} := \langle M, \dots\rangle$ be a submodel of $\mathfrak{N}$ and $A \subseteq M$. Then $\mathfrak{M} \prec \mathfrak{N}$ iff $\mathfrak{M}_A \prec \mathfrak{N}_A$.

Proof.

- $(\Rightarrow)$: Let $\psi(y_1, \dots, y_n)$ be a $\mathcal{L}(A)$-formula, $a_1, \dots, a_m$ in M and $\theta(y_1, \dots, y_n, z_1, \dots, z_m)$ be a $\mathcal{L}$-formula such that

$$\psi(y_1, \dots, y_n) = \theta(y_1, \dots, y_n, c_{a_1}, \dots, c_{a_m})$$

If $b_1, \dots, b_n$ are in M, then by using the replacement lemma we get

$$\mathfrak{M}_A \models \psi(b_1, \dots, b_n) \Leftrightarrow \mathfrak{M} \models \theta(b_1, \dots, b_n, a_1, \dots, a_m) \Leftrightarrow \dots$$
$$\dots \Leftrightarrow \mathfrak{N} \models \theta(b_1, \dots, b_n, a_1, \dots, a_m) \Leftrightarrow \mathfrak{N}_A \models \psi(b_1, \dots, b_n).$$

- $(\Leftarrow)$: This is clear, since for a $\mathcal{L}$-formula $\psi(x_1, \dots, x_n)$ and $b_1, \dots, b_n$ in M we have

$$\mathfrak{M} \models \psi(b_1, \dots, b_n) \Leftrightarrow \mathfrak{M}_A \models \psi(b_1, \dots, b_n)$$

$\square$

Theorem 3.7. *Let Γ be a complete theory of a countable language $\mathcal{L}$. The following are equivalent:*

1. *Γ has a countable ω-saturated model.*
2. *For each $n \in \mathbb{N}$ there are at most $\aleph_0$ many complete n-types extending Γ.*
3. *For every countable model $\mathfrak{M} := \langle M, \dots \rangle$ of Γ, for every finite set $A \subseteq M$ and for each $n \in \mathbb{N}$ there are at most $\aleph_0$ many complete n-types extending $Th(\mathfrak{M}_A)$.*

Proof.

- $(1 \Rightarrow 2)$: Let $\mathcal{A} := \langle A, \dots \rangle$ be a countable ω-saturated model of Γ. $|A| \leq \aleph_0$, hence $|A^n| \leq \aleph_0$ for all $n \in \mathbb{N}$. Each n-tuple $(a_1, \dots, a_n) \in A^n$ realises exactly one complete n-type over $\emptyset$, namely $type_{\mathcal{A}}(a_1, \dots, a_n)$. Since $\mathcal{A}$ is ω-saturated and Γ is complete, each complete n-type extending Γ is realised in $\mathcal{A}$.

- $(2 \Rightarrow 3)$: Let $\mathfrak{M} := \langle M, \dots \rangle$ be a countable model of Γ and $A \subseteq M$ finite. If $A = \emptyset$, then for all $n \in \mathbb{N}$ there are at most $\aleph_0$ many complete n-types extending Γ by assumption, because Γ is complete and therefore a complete n-type contains $Th(\mathfrak{M})$ if and only if it contains Γ. Let $k \in \mathbb{N}_+$ and $A := \{a_1, \dots, a_k\} \subseteq M$ of cardinality k. First note that there is only one complete 0-type over $\mathfrak{M}_A$, namely $Th(\mathfrak{M}_A)$. If $n \in \mathbb{N}_+$ and $t(x_1, \dots, x_n)$ is a complete n-type over A, then define

  $$\bar{t} := \{\theta(x_1, \dots, x_{n+k}) \in \mathcal{L} \mid \theta(x_1, \dots, x_n, c_{a_1}, \dots, c_{a_k}) \in t\}$$

  So $\bar{t}$ is the set of all $\mathcal{L}$-$(n+k)$-formulas which become elements of t, if the variable x_{n+i} is substituted by the new constant c_{a_i} for $1 \leq i \leq k$. Since for every n-formula $\psi(x_1, \dots, x_n)$ of the language $\mathcal{L}(A)$ there is a $(n+k)$-formula $\theta(x_1, \dots, x_{n+k})$ of the language $\mathcal{L}$ such that

  $$\vdash \psi(x_1, \dots, x_n) \leftrightarrow \theta(x_1, \dots, x_n, c_{a_1}, \dots, c_{a_k}),$$

  it follows $t = \{\psi \in \mathcal{L}(A) \mid \vdash \psi \leftrightarrow \theta(x_1, \dots, x_n, c_{a_1}, \dots, c_{a_k}), \text{for some } \theta \in \bar{t}\}$. This argument shows that there is a bijective map from the set of all complete $(n+k)$-types $\bar{t}$ over $\emptyset$ to the set of all complete n-types t over A. By assumption there are at most $\aleph_0$ many complete $(n+k)$-types over $\emptyset$.

- $(3 \Rightarrow 1)$: Start with an arbitrary countable model $\mathfrak{M} := \langle M, \dots \rangle$ of Γ. First fix a finite subset $A \subseteq M$. For $n \in \mathbb{N}_+$ and a complete 1-type $t(x_1)$ over A we know from Lemma 2.1 and the theorem of Löwenheim-Skolem, that there is a countable elementary extension of $\mathfrak{M}_A$ in which $t(x_1)$ is realised. Now let $(t_m)_{m \in \mathbb{N}_+}$ be an enumeration of all complete 1-types over A. Define an elementary chain $(\mathcal{A}_n)_{n \in \mathbb{N}}$ of models of $Th(\mathfrak{M}_A)$ in the following manner: $\mathcal{A}_0 := \mathfrak{M}_A$.

  Given $\mathcal{A}_n$ choose $\mathcal{A}_{n+1}$ as a countable elementary extension of $\mathcal{A}_n$ in which t_{n+1} is realised. $\overline{\mathcal{A}} := lim_{n \to \infty}(\mathcal{A}_n)$ is a countable elementary extension of $\mathfrak{M}_A$ in which t_m is realised for all $m \in \mathbb{N}_+$. Clearly $\mathfrak{M} \prec \overline{\mathcal{A}} \restriction \mathcal{L}$.

  Now let $(A_m)_{m \in \mathbb{N}_+}$ be an enumeration of all finite subsets of M. Define $\mathcal{B}_0 := \mathfrak{M}$.

Given $\mathcal{B}_n$ as a countable elementary extension of $\mathfrak{M}$, using the argument above choose $\mathcal{B}_{n+1}$ as a countable elementary extension of $\mathcal{B}_n$ in which all 1-types over A_{n+1} are realised. $\overline{\mathcal{B}} := lim_{n\to\infty}(\mathcal{B}_n)$ is a countable elementary extension of $\mathfrak{M}$ such that for all finite $A \subseteq M$ every 1-type over A is realised in $\overline{\mathcal{B}}_A$.

Using the second argument, we see that there is an elementary chain of countable models $(\mathcal{C}_n)_{n\in\mathbb{N}}$ such that

1. $\mathcal{C}_0 = \mathfrak{M}$
2. $\mathcal{C}_n \prec \mathcal{C}_{n+1}$ and for each finite $A \subseteq C_n$, where C_n is the universe of $\mathcal{C}_n$, every 1-type over A is realised in $\mathcal{C}_{n+1}$.

$\overline{C} := lim_{n\to\infty}(\mathcal{C}_n)$ is a countable elementary extension of $\mathcal{C}_n$ for all $n \in \mathbb{N}$ and therefore ω-saturated.

□

3.2 Atomic Models

We have seen that an ω-saturated model of a complete theory Γ is "big" in the sense, that any countable model of Γ can be elementarily embedded into it.
We now turn our attention to models which are small in that sense.

Definition. Let Σ be a theory for the language $\mathcal{L}$, $n \in \mathbb{N}$ and $\overline{y} := (y_1, \ldots, y_n)$ a vector of variables. A n-formula $\phi(\overline{y})$ is called n-complete over Σ if for an arbitrary n-formula $\psi(\overline{y})$ exactly one of the following conditions holds:

$$\Sigma \cup \{\phi\} \vdash \psi \text{ or } \Sigma \cup \{\phi\} \vdash \neg\psi.$$

Form now on we will write $\phi \vdash_\Sigma \psi$ instead of $\Sigma \cup \{\phi\} \vdash \psi$.

Example 3.8. If $\tau_1, \ldots, \tau_n$ are closed terms in the language $\mathcal{L}$, $n \in \mathbb{N}$ and Γ is a complete and consistent $\mathcal{L}$-theory, then

$$\phi(x_1, \ldots, x_n) := (x_1 = \tau_1) \wedge \cdots \wedge (x_n = \tau_n)$$

is a n-complete formula over Γ. This is true, because for every n-formula $\psi(x_1, \ldots, x_n)$ we have:

$$\Gamma \vdash \psi(\tau_1, \ldots, \tau_n) \text{ or } \Gamma \vdash \neg\psi(\tau_1, \ldots, \tau_n),$$

but noth both, since Γ is complete and consistent. Furthermore we have:

$$\emptyset \vdash \forall x_1 \ldots \forall x_n[\phi \to (\psi(\tau_1, \ldots, \tau_n) \leftrightarrow \psi(x_1, \ldots, x_n))].$$

Example 3.9. If Γ is defined as the theory of infinite vector spaces over a finite field $\mathbb{F}_q$ (see example 1.2) and $\overline{x} := (x_1, \ldots, x_k)$ for $k \in \mathbb{N}_{>0}$, then the k-formula

$$\phi(\overline{x}) := \bigwedge_{\substack{(r_1,\ldots,r_k)\in\mathbb{F}_q^k \\ (r_1,\ldots,r_k)\neq(0,\ldots,0)}} (\dot{r_1}x_1 + \cdots + \dot{r_n}x_k \neq \dot{0})$$

says that the vectors $x_1, \dots, x_n$ are linearly independent. We show now that it is k-complete over Γ by proving $\Gamma \cup \{\phi\} \models \psi$ for every k-formula $\psi(\overline{x})$ consistent with $\Gamma \cup \{\phi\}$.

Let $\mathfrak{M} := \langle M, \dots \rangle$ and $\mathfrak{N} := \langle N, \dots \rangle$ be models of Γ, $(m_1, \dots, m_k) \in M^k$, $(n_1, \dots, n_k) \in N^k$ and $\psi(x_1, \dots, x_k)$ an arbitrary k-formula such that

$$\mathfrak{M} \models (\phi \wedge \psi)(m_1, \dots, m_k) \text{ and } \mathfrak{N} \models \phi(n_1, \dots, n_k).$$

Because of the theorem of Löwenheim-Skolem there are countable elementary submodels $\mathfrak{M}' := \langle M', \dots \rangle \prec \mathfrak{M}$ and $\mathfrak{N}' := \langle N', \dots \rangle \prec \mathfrak{N}$ such that $(m_1, \dots, m_k) \in M'^k$ and $(n_1, \dots, n_k) \in N'^k$. The vectors $m_1, \dots, m_k$ and $n_1, \dots, n_k$ are linearly independent in $\mathfrak{M}'$ and $\mathfrak{N}'$ respectively. As we have seen before there is an isomorphism between $\mathfrak{M}'$ and $\mathfrak{N}'$ mapping m_i to n_i for $1 \leq i \leq k$. Therefore

$$\mathfrak{N}' \models \psi(n_1, \dots, n_k) \text{ and hence } \mathfrak{N} \models \psi(n_1, \dots, n_k).$$

Example 3.10. Let $\Gamma := DLO$. By a similar argument using countable elementary submodels and the "Back and Forth" technique introduced in example 1.1 we can see that for $k \in \mathbb{N}_{>1}$ the formula

$$x_1 < \dots < x_k$$

is k-complete over Γ.

Definition. Let Γ be a complete theory in the language $\mathcal{L}$. A $\mathcal{L}$-structure $\mathfrak{M} := \langle M, \dots \rangle$ with $\mathfrak{M} \models \Gamma$ is called atomic if for all $n \in \mathbb{N}_+$ and $a_1, \dots, a_n$ in M, $type_{\mathfrak{M}}(a_1, \dots, a_n)$ contains a n-complete formula over Γ.

Clearly, this is equivalent to $type_{\mathfrak{M}}(a_1, \dots, a_n)$ being isolated in $S_n^{\mathfrak{M}}(A)$ for $A := \emptyset$ with respect to the Stone topology.

Example 3.11. Let $\mathfrak{M} := \langle M, \dots \rangle$ be a model for a complete theory Γ. If for all $m \in M$ there exists a closed term τ_m such that $\tau_m^{\mathfrak{M}} = m$, then $\mathfrak{M}$ is atomic. For $m_1, \dots, m_k$ in M and $m_i = \tau_{m_i}^{\mathfrak{M}}$ $(1 \leq i \leq k)$ the k-formula

$$\phi(x_1, \dots, x_k) := \bigwedge_{i=1}^{k} (x_i = \tau_{m_i})$$

is k-complete over Γ as we have seen in example 3.8 and it is an element of $type_{\mathfrak{M}}(m_1, \dots, m_k)$. Therefore the standard model of the natural numbers $\langle \mathbb{N}, \dots \rangle$ is atomic.

Example 3.12. Conisder $\mathfrak{Q} := \langle \mathbb{Q}, < \rangle \models DLO$ and $q_1, \dots, q_n$ in $\mathbb{Q}$. The restriction of $<$ to $\{q_1, \dots, q_n\}$ can be expressed by a n-complete formula similarly to example 3.10. This n-formula is an element of $type_{\mathcal{Q}}(q_1, \dots, q_n)$, so this model is atomic.

Lemma 3.13. *Let Γ be a complete theory of a countable language $\mathcal{L}$, $\mathfrak{M} := \langle M, \dots \rangle$ be a countable atomic model of Γ and $\mathfrak{N} := \langle N, \dots \rangle$ be an arbitrary model of Γ. There exists an elementary embedding $h : \mathfrak{M} \hookrightarrow \mathfrak{N}$.*

Proof. Let $\{m_i \mid i \in \mathbb{N}_+\}$ be an enumeration of M. We will now inductively construct h as an isomorphism between $\mathfrak{M}$ and an elementary submodel $\mathfrak{N}'$ of $\mathfrak{N}$.

$k = 1$: Since $\mathfrak{M}$ is atomic, $type_{\mathfrak{M}}(m_1)$ contains a 1-complete formula $\phi_1(x_1)$. $\mathfrak{N} \models \Gamma$ and Γ is complete, so there exists a $n_1 \in N$ such that $\mathfrak{N} \models \phi_1(n_1)$.

$k \to k+1$: Suppose we have $m_1, \ldots, m_k$ in M and $n_1, \ldots, n_k$ in N and a k-complete formula $\phi_k(x_1, \ldots, x_k)$ such that

$$\mathfrak{M} \models \phi_k(m_1, \ldots, m_k) \text{ and } \mathfrak{N} \models \phi_k(n_1, \ldots, n_k).$$

$type_{\mathfrak{M}}(m_1, \ldots, m_{k+1})$ contains a $(k+1)$-complete formula $\phi_{k+1}(x_1, \ldots, x_{k+1})$. $\phi_k \vdash_\Gamma \exists x_{k+1}\phi_{k+1}$, because $\phi_k \in type_{\mathfrak{M}}(m_1, \ldots, m_{k+1})$ and so there exists a $n_{k+1} \in N$ such that $\mathfrak{N} \models \phi_{k+1}(n_1, \ldots, n_{k+1})$.

We check the Tarski-Vaught criterion for $\{n_k \mid k \in \mathbb{N}_+\} \subseteq N$: Let $k, l \in \mathbb{N}_+$, $k < l$ and $\psi(x_l, x_1, \ldots, x_k)$ a formula such that $\mathfrak{N} \models \exists x_l \psi(x_l, n_1, \ldots, n_k)$. Since for the k-complete formula ϕ_k from our inductive construction we have $\mathfrak{N} \models \phi_k(n_1, \ldots, n_k)$ it follows $\phi_k(x_1, \ldots, x_k) \vdash_\Gamma \exists x_l \psi(x_l, x_1, \ldots, x_k)$ and therefore $\mathfrak{M} \models \exists x_l \psi(x_l, m_1, \ldots, m_k)$. If $\mathfrak{M} \models \psi(m_j, m_1, \ldots, m_k)$ for some $j \in \mathbb{N}_+$, then it follows easily that $\mathfrak{N} \models \psi(n_j, n_1, \ldots, n_k)$.

By a similar argument we can see that the map $m_k \mapsto n_k$ is injective and respects constant , function and relation symbols, which means it is an elementary embedding. □

Corollary 3.14. *If $\mathfrak{A} := \langle A, \ldots \rangle$ and $\mathfrak{B} := \langle B, \ldots \rangle$ are countable atomic models of a complete theory Γ, then $\mathfrak{A} \simeq \mathfrak{B}$.*

Proof. We use the inductive construction in the proof of Lemma 3.13 combined with a "Back and Forth" argument.

Let $\{a_k \mid k \in \mathbb{N}_+\}$ and $\{b_k \mid k \in \mathbb{N}_+\}$ be enumerations of A and B respectively. Let $\phi_1(x_1) \in type_{\mathfrak{A}}(a_1)$ be a 1-complete formula. There exists a $j_1 \in \mathbb{N}$ such that $\mathfrak{B} \models \phi_1(b_{j_1})$, because Γ is complete. This is the first step.

Suppose we already have $a_{i_1}, \ldots, a_{i_k}$ in A, $b_{j_1}, \ldots, b_{j_k}$ in B and a k-complete formula $\phi_k(x_1, \ldots, x_k)$ sucht that $\mathfrak{A} \models \phi_k(a_{i_1}, \ldots, a_{i_k})$ and $\mathfrak{B} \models \phi_k(b_{j_1}, \ldots, b_{j_k})$.If k is odd, then let i_{k+1} be minimal in $\mathbb{N}$ distinct from $i_1, \ldots, i_k$ and

$$\phi_{k+1}(x_1, \ldots, x_{k+1}) \in type_{\mathfrak{A}}(a_{i_1}, \ldots, a_{i_{k+1}})$$

a $(k+1)$-complete formula. Then as before $\phi_k \vdash_\Gamma \exists x_{k+1}\phi_{k+1}$ and so there is a $j_{k+1} \in \mathbb{N}$ such that $\mathfrak{B} \models \phi_{k+1}(b_{j_1}, \ldots, b_{j_{k+1}})$.

If k is even, reverse the roles of $a_{i_1}, \ldots, a_{i_k}$ and $b_{j_1}, \ldots, b_{j_k}$. Like in the previous proof the map $a_{i_k} \mapsto b_{j_k}$ is an elementary embedding. It is also surjective, since in each step we choose the element with the smallest index which has not occurred yet. □

Definition. Let Γ be a theory of the language $\mathcal{L}$ and $n \in \mathbb{N}$. A n-formula $\psi(x_1, \ldots, x_n)$ is called completable over Γ if there exists a formula $\phi(x_1, \ldots, x_n)$ which is n-complete over Γ and $\phi \vdash_\Gamma \psi$. In that case ϕ is also called a completion of ψ over Γ.

We now come to a characterisation of countable complete theories which have an atomic model.

Lemma 3.15. *Let $\mathcal{L}$ be a countable language, Γ a consistent and complete $\mathcal{L}$-theory. Then the following are equivalent:*

1. *Γ has an atomic model.*

2. *For every $n \in \mathbb{N}$: Every n-formula $\psi(x_1, \dots, x_n)$ consistent with Γ has a completion over Γ.*

Proof. • $(1 \Rightarrow 2)$: Let $\mathfrak{M} := \langle M, \dots \rangle$ be an atomic model of Γ and $\psi(x_1, \dots, x_n)$ a n-formula consistent with Γ. Since Γ is complete we have

$$\Gamma \vdash \exists x_1, \dots, \exists x_n \psi(x_1, \dots, x_n)$$

and therefore $\psi \in \mathit{type}_{\mathfrak{M}}(a_1, \dots, a_n)$ for some $a_1, \dots, a_n$ in M. $\mathfrak{M}$ is atomic, therefore $\mathit{type}_{\mathfrak{M}}(a_1, \dots, a_n)$ contains a n-complete n-formula ϕ. This means Γ is consistent with $\psi \wedge \phi$, hence $\phi \vdash_\Gamma \psi$.

- $(2 \Rightarrow 1)$: We shall see two proofs of this direction.

  <u>Proof 1</u>: We use the omitting types theorem: Let $n \in \mathbb{N}$ and define

  $$\Sigma_n(x_1, \dots, x_n) := \{\neg\phi(x_1, \dots, x_n) \mid \phi \text{ is } n\text{-complete over } \Gamma\}.$$

  For all $n \in \mathbb{N}$, Σ_n is not isolated over Γ, because otherwise there was a $n \in \mathbb{N}$ and a formula $\psi(x_1, \dots, x_n)$ consistent with Γ such that

  $$\Gamma \vdash (\psi \to \neg\phi),$$

  for all $\phi(x_1, \dots, x_n)$ n-complete over Γ. But then according to our assumption there is a n-complete ϕ such that $\phi \vdash_\Gamma \psi$, hence $\neg\phi \in \Sigma_n$ and $\phi \vdash_\Gamma \neg\phi$, a contradicion, because n-complete formulas are consistent with Γ.
  Since our language is countable, the "Omitting Types Theorem" guarantees the existence of a model $\mathfrak{M}$ of Γ which omits all Σ_n. This means $\mathfrak{M}$ is atomic.

  <u>Proof 2</u>: First, we show that there is an enumeration $(\psi_n)_{n\in\mathbb{N}_+}$ of all formulas with the following properties:

  1. ψ_n is a n-formula.
  2. If $k \in \mathbb{N}_+$ and θ is a k-formula in which the variable x_k is free, then there is a $n \in \mathbb{N}_{>k}$ such that $\psi_n = \theta(x_k/x_n)$.

  Start with an arbitrary enumeration of all formulas $(\psi_n^{(1)})_{n\in\mathbb{N}_+}$, then define $\psi_n^{(2)} := \psi_k^{(1)}$, where $k \in \mathbb{N}$ minimal such that $Varfree(\psi_k^{(1)}) \subseteq \{x_1, \dots, x_n\}$.
  Let $(p_m)_{m\in\mathbb{N}_+}$ be an enumeration of all odd prime numbers. For $n \in \mathbb{N}_{>0}$ define ψ_n as follows:

  Case 1: If $n = (p_m)^d$ for some $d \in \mathbb{N}_+$, $j \leq m$ maximal such that x_j is free in $\psi_m^{(2)}$ and can be substituted by x_n, then $\psi_n := \psi_m^{(2)}(x_j/x_n)$.

  Case 2: If $n = 2^m$, for some $m > 0$, then $\psi_n := \psi_m^{(2)}$.

  Case 3: If neither case 1 nor case 2 applies to n, then $\psi_n := \psi_n^{(2)}$.

  It is easy to see that $(\psi_n)_{n\in\mathbb{N}_{>0}}$ has the desired properties. Now we can inductively construct a countable atomic model: First, let $\mathfrak{N} := \langle N, \dots \rangle$ be an arbitrary model of Γ and observe that the formula $\theta_1(x_1) := (\exists x_1 \psi_1(x_1)) \to \psi_1(x_1)$ is consistent with Γ. Hence there is a complete

1-formula $\phi_1(x_1)$ such that $\phi_1 \vdash_\Gamma \theta_1$. Since Γ is complete, we have $\mathfrak{N} \models \phi_1(a_1)$, for some $a_1 \in N$.

Suppose we have already found $a_1, \ldots, a_k$ in N and a k-complete formula $\phi_k(x_1, \ldots, x_k)$ such that $\mathfrak{N} \models \phi_k(a_1, \ldots, a_k)$. The $k+1$-formula

$$\theta_{k+1} := \phi_k \wedge [(\exists x_{k+1} \psi_{k+1}(x_1, \ldots, x_{k+1})) \rightarrow \psi_{k+1}(x_1, \ldots, x_{k+1})]$$

is consistent with Γ, otherwise $\phi_k \vdash_\Gamma \exists x_{k+1} \psi_{k+1}(x_1, \ldots, x_{k+1})$ and $\phi_k \vdash_\Gamma \neg\psi_{k+1}(x_1, \ldots, x_{k+1})$. Since the variable x_{k+1} is not free in $\Gamma \cup \{\phi_k\}$, this would imply $\phi_k \vdash_\Gamma \neg\exists x_{k+1} \psi_{k+1}(x_1, \ldots, x_{k+1})$, a contradiction. Let $\phi_{k+1}(x_1, \ldots, x_{k+1})$ be $k+1$-complete over Γ such that $\phi_{k+1} \vdash_\Gamma \theta_{k+1}$. It follows $\phi_k \vdash_\Gamma \exists x_{k+1} \phi_{k+1}(x_1, \ldots, x_{k+1})$. Hence there exists a $a_{k+1} \in N$ such that $\mathfrak{N} \models \phi_{k+1}(a_1, \ldots, a_{k+1})$.

Now define $M := \{a_k \mid k \in \mathbb{N}_+\}$ and check that $\mathfrak{M} := \langle M, \ldots \rangle$ is an elementary submodel of $\mathfrak{N}$: Let $s \in \mathbb{N}_+$, $\{a_1, \ldots, a_s\} \subseteq M$, and

$$\mathfrak{N} \models \exists x_{s+1} \theta(a_1, \ldots, a_s, x_{s+1}),$$

for some $s+1$-formula θ in which x_{s+1} is free. There exists a $m \in \mathbb{N}_{>s}$ such that $\theta = \psi_m^{(2)}$. Thanks to the specific construction of $(\psi_n)_{n \in \mathbb{N}_+}$ (property(2)), we have $\psi_n = \theta(x_1, \ldots, x_s, x_{s+1}/x_n)$ and

$$\phi_n \vdash_\Gamma \phi_{n-1} \wedge ((\exists x_n \psi_n) \rightarrow \psi_n)$$

$\mathfrak{N} \models \phi_n(a_1, \ldots, a_n) \wedge \exists x_n \psi_n(a_1, \ldots, a_n)$, hence $\mathfrak{N} \models \theta(a_1, \ldots, a_s, a_n)$, which verifies the Tarski-Vaught criterion for M. $\mathfrak{M}$ is atomic, since for every tuple $(a_{i_1}, \ldots, a_{i_n})$ and $k > max\{i_1, \ldots, i_n\}$ $\mathfrak{M} \models \phi_k(a_1, \ldots, a_k)$. $\square$

Lemma 3.16. *Let Γ be a complete theory of a countable language $\mathcal{L}$. If Γ has a countable ω-saturated model, then Γ has a countable atomic model.*

Proof. Since Γ has a countable ω-saturated model, we know that for each $n \in \mathbb{N}$ there are at most $\aleph_0$ many complete n-types extending Γ.

Assume indirectly that Γ has no countable atomic model. Then, because of Lemma 3.8, there is a $n \in \mathbb{N}$ and a n-formula $\phi_0(x_1, \ldots, x_n)$ consistent with Γ, which cannot be completed. In particular, ϕ_0 is not n-complete, hence there is a $\psi_0(\overline{x})$, where $\overline{x} = (x_1, \ldots, x_n)$, such that both

$$\phi_{(0,1)} := \phi_0 \wedge \psi_0 \text{ and } \phi_{(0,0)} := \phi_0 \wedge \neg\psi_0$$

is consistent with Γ. We can use the same argument for $\phi_{(0,1)}$ and $\phi_{(0,0)}$: There are four pairwise inconsistent formulas $\phi_{(0,1,1)}, \ldots, \phi_{(0,0,0)}$ such that each of them is consistent with Γ and:

$$\phi_{(0,1,j)} \vdash \phi_{(0,1)}, \text{ for } j \in \{0,1\}, \text{ as well as}$$
$$\phi_{(0,0,j)} \vdash \phi_{(0,0)}, \text{ for } j \in \{0,1\}.$$

This means that there is a binary tree T with root ϕ_0, whose nodes are the formulas $\phi_{(0,a_1,\ldots,a_n)}$, where $n \in \mathbb{N}_+$ and $a_i \in \{0,1\}$ for $1 \leq i \leq n$. Each formula implies its predecessors and each path from ϕ_0 defines a set of formulas consistent with Γ. Because of the compactness theorem, each branch of T is a n-type consistent with Γ and can therefore be extended to a complete n-type which contains Γ. Furthermore, two distinct branches cannot be extended to the same complete n-type, as the union of them is inconsistent. Since T has $2^{\aleph_0}$ many branches, our assumption that Γ has no atomic model has led us to a contradiction. $\square$

Remark. If a complete theory Γ has a countable atomic model, then it does not necessarily have a countable ω-saturated model: $\mathcal{N} := \langle \mathbb{N}, 0, S, +, \cdot, E, < \rangle$ is a countable atomic model of $Th(\mathcal{N})$, but there are continuum many complete types extending this theory, as we will see later, so by theorem 3.7 it cannot have a countable ω-saturated model.

We have already seen two examples of complete $\aleph_0$-categorical theories, namely DLO and the theory of infinite vector spaces over a finite field. Now we have the necessary tools to characterise them.

Recall the bijective correspondence between complete n-types extending a theory Γ and ultra filters on the Boolean algebra $\mathcal{B}_{(\Gamma,n)}$ described in section 2.2.

Theorem 3.17. *Let Γ be a complete theory with infinite models of a countable language $\mathcal{L}$. Then the following are equivalent:*

1. *Γ is $\aleph_0$-categorical.*
2. *Γ has a countable model which is both atomic and ω-saturated.*
3. *For every $n \in \mathbb{N}$ there are only finitely many ultrafilters on $\mathcal{B}_{(\Gamma,n)}$.*
4. *$\mathcal{B}_{(\Gamma,n)}$ is finite for every $n \in \mathbb{N}$.*
5. *Every complete n-type extending Γ is isolated for every $n \in \mathbb{N}$.*

Proof.

- (1 $\Rightarrow$ 2): Let $\mathfrak{M} := \langle M, \dots \rangle$ be a countable model of Γ. Since $\mathcal{L}$ is countable and Γ is $\aleph_0$-categorical and complete, every complete n-type extending Γ is realised in $\mathfrak{M}$ for every $n \in \mathbb{N}$. Furthermore, $|M|^n = \aleph_0$ for all $n \in \mathbb{N}_+$, hence there are at most $\aleph_0$ many such types. With the help of theorem 3.7 it follows that $\mathfrak{M}$ is ω-saturated. Using Lemma 3.16 we see that $\mathfrak{M}$ is also atomic.

- (2 $\Rightarrow$ 3): Let $\mathfrak{M}$ be a countable atomic and ω-saturated model of Γ. Again, because of theorem 3.7 there are at most $\aleph_0$ many complete types in $x_1, \dots, x_n$ extending Γ. We now show indirectly that there are only finitely many such types:

  Let $(t_m)_{m\in\mathbb{N}}$ be a bijective enumeration of all complete n-types in $x_1, \dots, x_n$ extending Γ. Each of these types is realised in the ω-saturated model $\mathfrak{M}$ which is also atomic, hence for all $m \in \mathbb{N}$ there exists a n-formula $\phi_m(x_1, \dots, x_n) \in t_m$ which is n-complete over Γ. If we choose one such ϕ_m for each t_m, we can consider the infinite set

  $$S := \{\neg\phi_m \mid m \in \mathbb{N}\}.$$

  We claim that S is consistent with Γ: Let $k \in \mathbb{N}$, then $\phi_{k+1} \vdash_\Gamma \neg\phi_i$ for all $i \leq k$, otherwise $t_{k+1} = t_j$ for some $j \leq k$, but we assumed our enumeration to be bijective. Since ϕ_{k+1} is consistent with Γ, it follows that $\{\neg\phi_1, \dots, \neg\phi_k\}$ is consistent with Γ. Using the compactness theorem, we conclude S is consistent with Γ and therefore contained in t_m for some $m \in \mathbb{N}$. This would imply $\phi_m \vdash_\Gamma \neg\phi_m$, a contradiction.

- $(3 \Rightarrow 4)$: Let $n \in \mathbb{N}$ and define T_n as the set of all complete n-types in $x_1, \dots, x_n$ extending Γ. Now for a n-formula ψ let $A_\psi := \{t \in T_n \mid \psi \in t\}$. Since T_n is finite, there are only finitely many sets of the form A_ψ. The key observation is that for arbitrary n-formulas ϕ, ψ with free variables in $\{x_1, \dots, x_n\}$ the following holds

  $$A_\phi \subseteq A_\psi \Leftrightarrow \phi \vdash_\Gamma \psi.$$

  Proof of the observation $(\Rightarrow)$: Let $\mathcal{M} := \langle M, \dots \rangle$ be a model of Γ and $a_1, \dots, a_n \in M$ such that $\mathcal{M} \models \phi(a_1, \dots, a_n)$. Then we have

  $$\phi \in type_{\mathcal{M}}(a_1, \dots, a_n) \in A_\phi \subseteq A_\psi,$$

  hence $\mathcal{M} \models \psi(a_1, \dots, a_n)$.

  $(\Leftarrow)$: This follows immediately since complete types are deductively closed.

  Hence $A_\phi = A_\psi$ iff $\Gamma \vdash (\phi \leftrightarrow \psi)$. This means that there is a $k \in \mathbb{N}_+$ and n-formulas $\phi_1, \dots, \phi_k$ such that for every n-formula ψ $A_\psi = A_{\phi_j}$, for some $j \leq k$.

- $(4 \Rightarrow 5)$: Let $n \in \mathbb{N}$, $A_n := \{[\psi_1], \dots, [\psi_m]\}$ be the set of all equivalence classes of $\mathcal{B}_{(\Gamma,n)}$ and $U \subseteq A_n$ an ultrafilter. Since A_n is finite, so is U, hence U is principal, that is for some n-formula ϕ consistent with Γ (i.e. $[\phi] \neq 0$ in $\mathcal{B}_{(\Gamma,n)}$) we have

  $$U = \{[\psi] \in A_n \mid [\phi] \leq [\psi]\}.$$

  Using the correspondence between ultrafilters on $\mathcal{B}_{(\Gamma,n)}$ and complete n-types extending Γ, we see that for every n-formula ψ either $[\phi] \leq [\psi]$, which means $\phi \vdash_\Gamma \psi$ or $[\phi] \leq -[\psi] = [\neg\psi]$, which means $\phi \vdash_\Gamma \neg\psi$. This proves that the corresponding type which contains the n-complete formula ϕ is isolated.

- $(5 \Rightarrow 1)$: If $\mathfrak{M} := \langle M, \dots \rangle$ is an arbitrary countable model of Γ, $n \in \mathbb{N}_+$ and $a_1, \dots, a_n$ are in M, then $type_{\mathfrak{M}}(a_1, \dots, a_n)$ is a complete n-type extending Γ and therefore contains a n-complete formula over Γ. This means that $\mathfrak{M}$ is atomic. Using corollary 3.14, we see that all countable models are isomorphic, hence Γ is $\aleph_0$-categorical.

□

With this we can now prove "Vaught's Never Two Theorem":

Theorem 3.18. *Let Γ be a complete theory with infinite models of a countable language $\mathcal{L}$. Then $I(\Gamma, \aleph_0) \neq 2$.*

Proof. Suppose there is such a theory and let $\mathcal{A} := \langle A, \dots \rangle, \mathcal{B} := \langle B, \dots \rangle$ be two non isomorphic countable models of Γ.

Since Γ is not ω-categorical,theorem 3.17 implies that $\mathcal{B}_{(\Gamma,n)}$ is infinite for some $n \in \mathbb{N}_+$ which means that there is a sequence $(\psi_m)_{m\in\mathbb{N}}$ of n-formulas such that whenever $i, j \in \mathbb{N}$ and $i \neq j$ we have

$$\Gamma \nvdash (\psi_i \leftrightarrow \psi_j).$$

This is equivalent to

$$\Gamma \vdash \neg[\forall x_1 \ldots \forall x_n(\psi_i \leftrightarrow \psi_j)],$$

for $i \neq j$ in $\mathbb{N}$, because Γ is complete.

Every complete n-type extending Γ is realised in $\mathcal{A}$ or $\mathcal{B}$, hence there are at most $\aleph_0$ many such types. It follows from theorem 3.7, that one of the two models, say $\mathcal{B}$, is ω-saturated. Clearly $\mathcal{B}$ cannot be atomic, because then by theorem 3.17 Γ would be ω-categorical. Therefore by Lemma 3.16 $\mathcal{A}$ is atomic. Let $m \in \mathbb{N}_+$ and $E := \{b_1, \ldots, b_m\} \subseteq B$ such that $type_{\mathcal{B}}(b_1, \ldots, b_m)$ does not contain a m-complete formula over Γ. Because of Proposition 3.4 the $\mathcal{L}(E)$-structure $\mathcal{B}_E$ is a ω-saturated model of $Th(\mathcal{B}_E)$. $\mathcal{B}_E$ is not atomic, because otherwise using theorem 3.17 and the completeness of $Th(\mathcal{B}_E)$ we can conclude that $Th(\mathcal{B}_E)$ is ω-categorical and therefore for some $i \neq j$ in $\mathbb{N}$

$$Th(\mathcal{B}_E) \vdash \forall x_1 \ldots \forall x_n(\psi_i \leftrightarrow \psi_j).$$

This is a contradiction, since $\Gamma \subseteq Th(\mathcal{B}_E)$.

Now let $\overline{\mathcal{C}}$ be a countable atomic model of $Th(\mathcal{B}_E)$ and define $\mathcal{C} := \overline{\mathcal{C}} \restriction \mathcal{L}$. Clearly, $\mathcal{C} \models \Gamma$ and $\mathcal{C}$ is not ω-saturated, because otherwise by Proposition 3.4 $\overline{\mathcal{C}}$ is atomic and ω-saturated and $Th(\mathcal{B}_E)$ is $\aleph_0$-categorical.

$\mathcal{C}$ is not atomic either, because since $\overline{\mathcal{C}} \models Th(\mathcal{B}_E)$ we have

$$type_{\mathcal{C}}(c_{b_1}^{\overline{\mathcal{C}}}, \ldots, c_{b_m}^{\overline{\mathcal{C}}}) = type_{\mathcal{B}}(b_1, \ldots, b_m).$$

So we have found a countable model of Γ which is neither atomic nor ω-saturated, therefore it is neither isomorphic to $\mathcal{A}$ nor to $\mathcal{B}$ which is a contradiction to our assumption. □

3.3 Theories with Several Models

In this section we will for each $n \in \mathbb{N}_{>2}$ give an example of a complete theory with exactly n isomorphism classes of countable models. First we prove a useful result:

Lemma 3.19. ***(The Many Automorphisms Lemma)*** *Let $\mathcal{L}$ be an arbitrary first order language, $\mathcal{C} := \langle C, \ldots \rangle$ a $\mathcal{L}$-structure and $A \subseteq C$. Then A is the universe of an elementary submodel of $\mathcal{C}$ if for every finitely generated sublanguage $\mathcal{L}_0$ of $\mathcal{L}$, for every finite set $T \subseteq A$ and for every $b \in C$ there is an automorphism $\pi : \mathcal{C} \restriction \mathcal{L}_0 \mapsto \mathcal{C} \restriction \mathcal{L}_0$ with the following properties:*

1. *$\pi(a) = a$, for all $a \in T$.*
2. *$\pi(b) \in A$.*

Proof. We check the Tarski-Vaught criterion for A: Let $T \subseteq A$ be finite and $\phi(y, \overline{z})$ be a $\mathcal{L}$-formula, where $\overline{z}$ is a possibly empty vector of variables. If

$$\mathcal{C} \models \phi(b, \overline{a}),$$

where $b \in C$ and $\overline{a}$ is a vector of elements of T corresponding to $\overline{z}$, then clearly $\mathcal{C} \restriction \mathcal{L}_0 \models \phi(b, \overline{a})$, where $\mathcal{L}_0$ is the sublanguage of $\mathcal{L}$ generated by the symbols occurring in ϕ. By assumption there is an automorphism $\pi : \mathcal{C} \restriction \mathcal{L}_0 \mapsto \mathcal{C} \restriction \mathcal{L}_0$ satisfying the two conditions and therefore $\mathcal{C} \models \phi(\pi(b), \overline{a})$. □

Remark. Lemma 3.19 provides a sufficient condition for identifying a subset as the universe of an elementary submodel, but this condition is not necessary: Consider $\mathcal{N} := \langle \mathbb{N}, 0, S, +, \cdot, < \rangle$, the standard model of the theory of natural numbers. As we have demonstrated before $\mathcal{N}$ is an atomic model of $Th(\mathcal{N})$. Let $\mathcal{C} := \langle C, \dots \rangle$ be a nonstandard model of $Th(\mathcal{N})$, $S := \{(S^n 0)^{\mathcal{C}} \mid n \in \mathbb{N}\}$ and $c \in C \setminus S$ a nonstandard number. Then S is the universe of an elementary submodel of $\mathcal{C}$, but for every automorphism $\pi : \mathcal{C} \mapsto \mathcal{C}$ we have

$$\pi((S^n 0)^{\mathcal{C}}) = (S^n 0)^{\mathcal{C}},$$

hence $\pi(c) \notin S$.

Example 3.20. A Theory with 3 Models:
The idea of this example was first introduced by Andrzej Ehrenfeucht. Let $\mathcal{L} := \langle (c_j)_{j \in \mathbb{N}_+}, <' \rangle$ be a first order language. The $\mathcal{L}$-theory DLO^+ is defined as $DLO \cup \{c_{j+1} < c_j \mid j \in \mathbb{N}_+\}$, where DLO is the set of sentences described in example 1.1. We can think of $(c_j)_{j \in \mathbb{N}_+}$ as a strictly decreasing sequence. In any model of this theory we can distinguish three cases: 1. The sequence has no lower bound. 2. It has an infimum. 3. It has a lower bound, but no infimum. This observation leads us to the following candidates:

1. $\mathcal{M}_1 := \langle \mathbb{Q}_+, <, (c_j \mapsto \frac{1}{j})_{j \in \mathbb{N}_+} \rangle$
2. $\mathcal{M}_2 := \langle \mathbb{Q}, <, (c_j \mapsto \frac{1}{j})_{j \in \mathbb{N}_+} \rangle$
3. $\mathcal{M}_3 := \langle \mathbb{Q} \setminus \{0\}, <, (c_j \mapsto \frac{1}{j})_{j \in \mathbb{N}_+} \rangle$

Claim 1: These three $\mathcal{L}$-structures are pairwise nonisomorphic models of DLO^+.

Proof. It is easy to check that $\mathcal{M}_i \models DLO^+$, for $1 \le i \le 3$.

- $\mathcal{M}_1 \ncong \mathcal{M}_2$: If $h : \mathcal{M}_2 \mapsto \mathcal{M}_1$ is a homomorphism, then for all $j \in \mathbb{N}_+$ $h(0) < \frac{1}{j}$, since $\mathcal{M}_2 \models (x_1 < c_j)(0)$. This is a contradiction, because $h(0) > 0$.
- $\mathcal{M}_1 \ncong \mathcal{M}_3$: If $h : \mathcal{M}_3 \mapsto \mathcal{M}_1$ is a homomorphism and $q \in \mathbb{Q}_{<0}$, then $h(q) > 0$ and therefore $\mathcal{M}_1 \models (c_j < x_1)(h(q))$ for some $j \in \mathbb{N}_+$ which is incompatible with $\mathcal{M}_3 \models \neg(c_j < x_1)(q)$ for all $j \in \mathbb{N}_+$.
- $\mathcal{M}_2 \ncong \mathcal{M}_3$: If $h : \mathcal{M}_2 \mapsto \mathcal{M}_3$ is an isomorphism, then consider $a := h(0)$: If $a > 0$, we can derive a contradiction like in the two previous arguments. If $a < 0$, then $a < \frac{a}{2} < 0$, hence $0 < h^{-1}(\frac{a}{2})$,but then

  $$\mathcal{M}_2 \models (c_j < x_1)(h^{-1}(\tfrac{a}{2})) \text{ and } \mathcal{M}_3 \models \neg(c_j < x_1)(\tfrac{a}{2})$$

  for some $j \in \mathbb{N}_{>0}$, a contradiction.

$\square$

Claim 2: Every countable model of DLO^+ is isomorphic to one of these three models.

Proof. Let $\mathfrak{M} := \langle M, \dots \rangle$ be a countable model of DLO^{+}, $S := \{\frac{1}{j} \mid j \in \mathbb{N}_{+}\}$, $T := \{c_j^{\mathfrak{M}} \mid j \in \mathbb{N}_{+}\}$ and $h_0 : S \mapsto T$ defined by $h_0(\frac{1}{j}) := c_j^{\mathfrak{M}}$. Clearly h_0 is order preserving and bijective. We will now inductively extend h_0 to an isomorphism between $\mathfrak{M}$ and one of the given models depending on whether S has no lower bound, it has an infimum or it has a lower bound but no infimum:

- (Case 1, T has no lower bound): Let $\{m_k \mid k \in \mathbb{N}_{+}\}$ and $\{q_k \mid k \in \mathbb{N}_{+}\}$ be enumerations of $\mathbb{Q}_{+} \setminus S$ and $M \setminus T$ respectively.

  Suppose $k \in \mathbb{N}_{+}$ and h_{k-1} is a partial order preserving function from $\mathbb{Q}_{+}$ to M extending h_0 such that $dom(h_{k-1}) \setminus S$ and $ran(h_{k-1}) \setminus T$ are finite.

  If k is odd, then let $q \in \mathbb{Q}_{+} \setminus dom(h_{k-1})$ with minimal index. If $1 < q$, then since $\mathfrak{M} \models DLO$ and $\{m \in ran(h_{k-1}) \mid \mathfrak{M} \models (c_1 < x_1)(m)\}$ is finite, we can proceed similarly to the argument of example 1.1 and find a $m \in M \setminus ran(h_{k-1})$ with minimal index such that $h_{k-1} \cup \{(q, m)\} =: h_k$ is an order preserving function. Otherwise we have $\frac{1}{j+1} < q < \frac{1}{j}$ for some $j \in \mathbb{N}_{+}$. Since $\{m \in ran(h_{k-1}) \mid \mathfrak{M} \models (c_{j+1} < x_1 < c_j)(m)\}$ is fintie, using the fact that $\mathfrak{M} \models DLO$, we can extend h_{k-1} to an order preserving function h_k similarily.
  If k is even we take $m \in M \setminus ran(h_{k-1})$ with minimal index and using the fact that $\mathcal{M}_1 \models DLO$, we find a $q \in \mathbb{Q} \setminus dom(h_{k-1})$ with minimal index such that $h_k := h_{k-1} \cup \{(q, m)\}$ is order preserving via a similar case distinction. The function $H := \bigcup_{k=0}^{\infty} h_k$ is an isomorphism between $\mathfrak{M}$ and $\mathcal{M}_1$.

- (Case 2, T has an infimum): First, the infimum is uniquely determined: If u and u' are infima of T, then without loss of generality $u \leq u'$ in $\mathfrak{M}$, but if $u < u'$, then by definition u is not an infimum.
  Now let $u := inf(T)$ and define $h_0' := h_0 \cup \{(0, u)\}$. Similarly to the first case, we start with enumerations $\{m_k \mid k \in \mathbb{N}_{+}\}$ and $\{q_k \mid k \in \mathbb{N}_{+}\}$ of $\mathbb{Q} \setminus (S \cup \{0\})$ and $M \setminus (T \cup \{u\})$. Then, as in case 1, using a "Back and Forth" argument, we inductively define a sequence of partial order preserving functions $(h_k')_{k \in \mathbb{N}_{+}}$ such that

  1. $dom(h_k') \subseteq \mathbb{Q}$ and $ran(h_k') \subseteq M$, for all $k \in \mathbb{N}$.
  2. $h_i' \subseteq h_j'$, for $i < j$.
  3. $dom(h_k') \setminus S$ and $ran(h_k') \setminus T$ are finite for all $k \in \mathbb{N}$.

  In each step $k \in \mathbb{N}$,if k is odd, we take the minimal element $q \in \mathbb{Q}$ we haven't chosen yet, then we compare it to 0. If $0 < q$, then either $\frac{1}{j+1} < q < \frac{1}{j}$ for some$j \in \mathbb{N}_{+}$ or $1 < q$.Since $\mathfrak{M} \models DLO$, we can proceed as in example 1.1 and find a minimal $m \in M$ which has not been chosen so far such that $\mathfrak{M} \models (c_{j+1} < x_1 < c_j)(m)$ or $\mathfrak{M} \models (c_1 < x_1)(m)$ respectively. Similarily if $q < 0$. Then $h_{k+1}' := h_k' \cup \{(q, m)\}$ will be an order preserving extension.

  If k is even we reverse the roles of q and m ("Back and Forth").

  $$H := \bigcup_{k=1}^{\infty} h_k'$$

  is the desired isomorphism.

- (Case 3, T has a lower bound but no infimum): Similarily to the previous cases we construct an isomorphism between $\mathfrak{M}$ and $\mathcal{M}_3$.

$\square$

Claim 3: $\mathcal{M}_1$ is an elementary submodel of both $\mathcal{M}_2$ and $\mathcal{M}_3$.

Proof. We use Lemma 3.19 for the proof:

- $\mathcal{M}_1 \prec \mathcal{M}_2$: Let $A \subseteq \mathbb{Q}_+$ be finite, $b \in \mathbb{Q}_{\leq 0}$ and $\mathcal{L}_0 := \langle <, c_1, \dots, c_n \rangle$, the cases where $\mathcal{L}_0$ is generated from a smaller set are even easier. Define $a := \frac{1}{2} \cdot min\{min(A), \frac{1}{n}\}$, if $A = \emptyset$, then $a := \frac{1}{2n}$ and $h : \mathbb{Q} \mapsto \mathbb{Q}$

$$h(q) := \begin{cases} q + a - b, & q \leq b \\ \frac{a}{2a-b} \cdot q + 2a \cdot \frac{(a-b)}{2a-b}, & b < q \leq 2a \\ q, & \text{else} \end{cases}$$

  Clearly, $h : \mathcal{M}_2 \restriction \mathcal{L}_0 \mapsto \mathcal{M}_2 \restriction \mathcal{L}_0$ is bijective, order preserving and respects the constant symbols $c_1, \dots, c_n$, hence it is an automorphism. Furthermore $h(b) > 0$.

- $\mathcal{M}_1 \prec \mathcal{M}_3$: Let A, $\mathcal{L}_0$ and a be as before and $b \in \mathbb{Q}_{<0}$. Since $\mathbb{Q}_{<b}$ and $\mathbb{Q}_{<a} \setminus \{0\}$ are models of DLO, there is an order preserving bijection h_0 between these two sets (see example 1.1). Using the same argument, there is an order preserving bijection $h_1 : (\mathbb{Q} \setminus \{0\}) \cap (b, 2a) \mapsto (\mathbb{Q} \setminus \{0\}) \cap (a, 2a)$. Then $h : \mathbb{Q} \setminus \{0\} \mapsto \mathbb{Q} \setminus \{0\}$ defined by

$$h(q) := \begin{cases} h_0(q), & q < b \\ a, & q = b \\ h_1(q), & b < q < 2a \\ q, & else \end{cases}$$

  is an automorphism on $\mathcal{M}_3 \restriction \mathcal{L}_0$ such that $h(b) > 0$.

$\square$

It follows directly from claims 2 and 3 and the theorem of Löwenheim-Skolem, that DLO^+ is complete. Since $\mathcal{M}_1, \mathcal{M}_2, \mathcal{M}_3$ represent all countable models of this theory, it follows with theorem 3.7 that one of them is atomic and another one is ω-saturated. We have proven in claim 1, that neither $\mathcal{M}_2$ nor $\mathcal{M}_3$ can be embedded into $\mathcal{M}_1$, therefore using lemma 3.13 we conclude that $\mathcal{M}_1$ is atomic.
Let $A := \{0\}$ and $t(x_1) := \{c_0 < x_1\} \cup \{x_1 < c_j \mid j \in \mathbb{N}_+\}$. $t(x_1)$ is a 1-type over A for the structure $\mathcal{M}_2$, since every finite subset of $t(x_1)$ is realised in $(\mathcal{M}_2)_A$, but clearly the whole type is not realised in $(\mathcal{M}_2)_A$, as $(\frac{1}{j}) \longrightarrow 0$ for $j \to \infty$. Hence $\mathcal{M}_3$ is ω-saturated.

We can use this example to show, that for every $n \in \mathbb{N}_{>2}$ there is a countable language $\mathcal{L}$ and a complete $\mathcal{L}$-theory with exactly n countable models up to isomorphism, but first we prove the following

Fact. Let $D \subseteq \mathbb{Q}$ be dense, that is for all $d_1 < d_2$ in $\mathbb{Q}$ there is a $e \in D$ such that $d_1 < e < d_2$. Then there is a partition of D into countably many dense subsets.

Proof. Let $(t_n)_{n\in\mathbb{N}}$ be an enumeration of $D\times\mathbb{N}\times\mathbb{N}$ and $(d_n)_{n\in\mathbb{N}}$ an enumeration of D, which exist since $|D| = \aleph_0$. We inductively define a function

$$H : D\times\mathbb{N}\times\mathbb{N} \mapsto D.$$

$\underline{n=0}$: If $t_0 = (z, m_0, k_0)$, set $H(t_0) := z$.

$\underline{n\to n+1}$: Suppose we have already defined $H(t_i)$ for $1\le i\le n$. If $t_{n+1} = (z', m, k)$, then $H(t_{n+1}) := d_j$, where $j\in\mathbb{N}$ minimal such that

1. $d_j \neq H(t_i)$, for $1 \le i \le n$.
2. $|z' - d_j| < \frac{1}{2^k}$.

Since D is dense in $\mathbb{Q}$ we can always define $H(t_{n+1})$, given $H(t_1), \dots, H(t_n)$. For $m\in\mathbb{N}$ define $D_m := H(D\times\{m\}\times\mathbb{N})$. Then D_m is dense in $\mathbb{Q}$ for all $m\in\mathbb{N}$, $D_i\cap D_j = \emptyset$, for $i\neq j$ and $D = \bigcup_{m=0}^{\infty} D_m$. ☐

Now let $n\in\mathbb{N}_{>3}$ and the language $\mathcal{L} := \langle (c_j)_{j\in\mathbb{N}_+}; P_1, \dots, P_{n-2}; <\rangle$, where c_j is a constant symbol for $j\in\mathbb{N}_+$, $P_1, \dots, P_{n-2}$ are 1-ary relation symbols and $<$ is a 2-ary relation symbol. The $\mathcal{L}$-theory $DLO^{(n)}$ is defined as

$$DLO^+ \cup \{P_1 c_j \mid j\in\mathbb{N}_+\} \cup \{\sigma_{(n)}\} \cup \{\tau_{(n)}\},$$

where

$$\sigma_{(n)} := \forall x_1[(P_1x_1 \vee \dots \vee P_{n-2}x_1) \wedge (\bigwedge_{j=1}^{n-2}[P_jx_1 \to (\bigwedge_{\substack{i=1\\ i\neq j}}^{n-2} \neg P_ix_1)])]$$

and

$$\tau_{(n)} := \forall x_1\forall x_2[x_1 < x_2 \to (\bigwedge_{j=1}^{n-2} \exists x_3[P_jx_3 \wedge (x_1 < x_3 < x_2)])].$$

In essence $DLO^{(n)}$ describes a set with a dense linear order and a strictly decreasing sequence partitioned into $n-2$ dense subsets. Similar to our study of DLO^+ we can distinguish n cases for any model $\mathfrak{M} := \langle M, \dots\rangle$ of $DLO^{(n)}$.

- Case 1: The sequence $(c_j^{\mathfrak{M}})_{j\in\mathbb{N}_+}$ has no lower bound.
- Case 2: The sequence converges to an element of $P_1^{\mathfrak{M}}$.
- ...
- ...
- Case $(n-1)$: The sequence converges to an element of $P_{n-2}^{\mathfrak{M}}$.
- Case n: The sequence has a lower bound, but does not converge.

We will now show how to construct n representative countable models of that theory. First, notice that $D := \mathbb{Q}\setminus(\{0\}\cup\{\frac{1}{j} \mid j\in\mathbb{N}_+\})$ is dense in $\mathbb{Q}$. Hence there is a partition of D into $n-2$ many dense subsets $D'_1, \dots, D'_{n-2}$. Now define $D_1 := D'_1 \cup \{\frac{1}{j} \mid j\in\mathbb{N}_+\}$ and $D_j := D'_j$, for $2\le j\le n-2$. This gives us a partition of $\mathbb{Q}\setminus\{0\}$ into $n-2$ many dense subsets. Now consider the following models all of which have a subset of $\mathbb{Q}$ as universe, the 2-ary relation symbol "$<$" is interpreted as the canonical linear order and the constant symbol c_k is interpreted as $\frac{1}{k}$ for $k\in\mathbb{N}_+$ in all of them.

- $\mathcal{M}_1 := \langle M_1, \dots \rangle$, where $M_1 := \mathbb{Q}_+$, $P_i^{\mathcal{M}_1} := D_i \cap \mathbb{Q}_+$, for $1 \le i \le n-2$.
- For $2 \le j \le n-1$, $\mathcal{M}_j := \langle M_j, \dots \rangle$, where $M_j := \mathbb{Q}$ and for $1 \le i \le n-2$

$$P_i^{\mathcal{M}_j} := \begin{cases} D_i \cup \{0\}, & i = j-1 \\ D_i, & \textit{else.} \end{cases}$$

- $\mathcal{M}_n := \langle M_n, \dots \rangle$, where $M_n := \mathbb{Q} \setminus \{0\}$ and $P_i^{\mathcal{M}_n} := D_i$, for $1 \le i \le n-2$.

We can use basically the same arguments of our study of DLO^+ (Claim 1) in order to show that these n $\mathcal{L}$-structures are pairwise nonisomorphic models of $DLO^{(n)}$.

Modifying the proof of Claim 2, we see that if $\mathfrak{M}$ is a model of $DLO^{(n)}$ and $1 \le j \le n$ such that case j holds in $\mathfrak{M}$, then $\mathfrak{M} \cong \mathcal{M}_j$. With respect to the back and forth construction of the isomorphism, we have to make sure that the 1-ary relation symbols are respected, which is possible, since they are interpreted as dense subsets of the universe.

Similarily to claim 3 we can prove $\mathcal{M}_1 \prec \mathcal{M}_j$ and that there is no homomorphism of models $\mathcal{M}_j \mapsto \mathcal{M}_1$ for $2 \le j \le n$. It follows that the theory is complete and $\mathcal{M}_1$ is atomic.

For $2 \le j \le n-1$, $\mathcal{M}_j$ is not ω-saturated: Define $A := \{0\}$ and consider the 1-type $t(x_1) := \{(c_0 < x_1) \wedge P_1 x_1\} \cup \{x_1 < c_k \mid k \in \mathbb{N}_+\}$ over A which is not realised in $(\mathcal{M}_j)_A$. Hence $\mathcal{M}_n$ is ω-saturated.

Example 3.21. A Theory with $\aleph_0$ many Models
Let $\mathcal{L} := \langle (c_k)_{k \in \mathbb{N}} \rangle$, $\Gamma_\omega := \{c_i \neq c_j \mid i, j \in \mathbb{N}, i \neq j\}$ and $B := \{2n \mid n \in \mathbb{N}\}$. For $j \in \omega + 1$ define

$$A_j := \begin{cases} B, & j = 0 \\ B \cup \{2m-1 \mid m \in \mathbb{N}, 1 \le m \le j\}, & j \in \mathbb{N}_+ \\ \mathbb{N}, & j = \omega \end{cases}$$

and $\mathcal{M}_j := \langle A_j, \dots \rangle$, where $c_k^{\mathcal{M}_j} := 2k$, for $k \in \mathbb{N}$. Clearly, $\mathcal{M}_j \models \Gamma_\omega$ for all $j \in \omega + 1$. If $\mathfrak{M} := \langle M, \dots \rangle$ is an arbitrary countable model of Γ_ω, then for

$$j := |M \setminus \{c_k^{\mathfrak{M}} \mid k \in \mathbb{N}\}|$$

we have $j \in \omega + 1$ and $\mathfrak{M} \cong \mathcal{M}_j$. The isomorphism is straight forward: First, define $h_0 : B \mapsto \{c_k^{\mathfrak{M}} \mid k \in \mathbb{N}\}$ by

$$h_0(2n) := c_n^{\mathfrak{M}}$$

and let $h_1 : A_j \setminus B \mapsto M \setminus \{c_k^{\mathfrak{M}} \mid k \in \mathbb{N}\}$ be any bijection. Then $H : \mathcal{M}_j \mapsto \mathfrak{M}$ is an isomorphism, where

$$H(x) := \begin{cases} h_0(x), & x \in B \\ h_1(x), & x \in A_j \setminus B \end{cases}$$

By applying Lemma 3.19(Many Automorphisms Lemma), it is easy to prove that $\mathcal{M}_i \prec \mathcal{M}_j$, for all $i, j \in \omega + 1$ and $i \le j$, hence Γ_ω is complete. Clearly,

$\mathcal{M}_j \not\hookrightarrow \mathcal{M}_i$, for $j > i$, as any embedding $g : \mathcal{M}_j \mapsto \mathcal{M}_i$ is the identity when restricted to B and there is no injetion from $A_j \setminus B$ into $A_i \setminus B$.

This also means that the binary relation "$\prec$"(is elementary submodel of) induces a well order on the set of isomorphism classes of countable models of Γ_ω with the order type $\omega + 1$. Notice that this is also the case for $\aleph_0$-categorical theories and DLO^+ with the order types 1 and 3, but not for the theories $DLO^{(n)}$ where $n > 3$. It follows $\mathcal{M}_0$ is atomic and $\mathcal{M}_\omega$ is ω-satruated.

Example 3.22. A Theory with Continuum Many Models
Let $\mathcal{N}$ be the standard model of the theory of natural numbers and $\Gamma := Th(\mathcal{N})$. Let $(p_n)_{n\in\mathbb{N}}$ be an enumeration of all prime numbers. For $n \in \mathbb{N}$ let $\phi_n(x_1) := \exists x_2[(S^m 0) \cdot x_2 = x_1]$, where $m := p_n$. So in essence this 1-formula describes the property of beeing divisible by the n-th prime number. Now for $A \subseteq \mathbb{N}$ consider the following set of 1-formulas

$$T_A := \{\phi_n \mid n \in A\} \cup \{\neg\phi_n \mid n \in \mathbb{N} \setminus A\}.$$

T_A is consistent with Γ for every $A \subseteq \mathbb{N}$, as every finite subset of it is realised in $\mathcal{N}$. Furthermore, if $A \neq A'$, then $T_A \cup T_{A'}$ is inconsistent. Clearly

$$|\{T_A \mid A \subseteq \mathbb{N}\}| = 2^{\aleph_0}$$

and every T_A is realised in a countable model of Γ. If κ is a cardinal and $(\mathfrak{M}_\alpha)_{\alpha<\kappa}$ is a sequence of countable models of Γ, then for all $\alpha < \kappa$

$$|\{A \subseteq \mathbb{N} \mid T_A \text{ is realised in } \mathfrak{M}_\alpha\}| \leq \aleph_0,$$

since every element of the universe of a model realises exactly one of these types. Hence

$$|\{A \subseteq \mathbb{N} \mid \exists\alpha < \kappa \text{ such that } T_A \text{ is realised in } \mathfrak{M}_\alpha\}| \leq \kappa \cdot \aleph_0 = max\{\kappa, \aleph_0\}.$$

It follows, that if each T_A is realised in some $\mathfrak{M}_\alpha$, then $\kappa \geq 2^{\aleph_0}$. We conclude that Γ has continuum many countable models up to isomorphism.

The continuum hypothesis (CH) states that $2^{\aleph_0} = \aleph_1$, where $\aleph_1$ is the least cardinal greater $\aleph_0$. Using the method of forcing, one can see that CH cannot be decided within ZFC, that is by the Zermelo-Fraenkel axioms plus the axiom of choice.

If we assume CH, then clearly there is no countable complete theory T such that $I(T, \aleph_0) = \kappa$, where κ is a cardinal and $\aleph_0 < \kappa < 2^{\aleph_0}$. Robert L. Vaught proposed [6] that this is provable independently of CH.

Using methods of descriptive set theory, M. Morley proved [7], that for a countable complete theory T

$$I(T, \aleph_0) \in (\omega + 1) \cup \{\aleph_1\} \cup \{2^{\aleph_0}\}.$$

Definition. Let T be a complete theory of some first-order language $\mathcal{L}$ and κ an infinite cardinal. T is called κ-stable if for every model $\mathfrak{M} := \langle M, \dots \rangle$ of T and every set $A \subseteq M$ of cardinality κ the set of complete 1-types over A has cardinality κ.

Vaught's conjecture was proved for some special cases such as the theories of trees [8] and varieties [9]. S. Shelah, L. Harrington and M. Makkai gave a proof for $\aleph_0$-stable theories [10], but as of now, in its general form, "Vaught's conjecture" is still an open problem.

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