MAD FAMILIES, SPLITTING FAMILIES, AND LARGE CONTINUUM

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ABSTRACT. Let $\kappa < \lambda$ be regular uncountable cardinals. Using a finite support iteration of ccc posets we obtain the consistency of $\mathfrak{b} = \mathfrak{a} = \kappa < \mathfrak{s} = \lambda$. If μ is a measurable cardinal and $\mu < \kappa < \lambda$, then using similar techniques we obtain the consistency of $\mathfrak{b} = \kappa < \mathfrak{a} = \mathfrak{s} = \lambda$.

1. INTRODUCTION

In the following we will study the bounding, splitting and almost disjointness numbers (for basic definitions and notation see [2]). Following standard notation ${}^{\omega}\omega$ denotes the set of all functions from ω to ω, $[\omega]^{\omega}$ denotes the set of infinite subsets of ω and $\leq^*$ denotes the eventual dominance order on ${}^{\omega}\omega$. That is for f, g in ${}^{\omega}\omega$ $f \leq^* g$ if and only if there is $n \in \omega$ such that for all $i \geq n(f(i) \leq g(i))$. A family $\mathcal{B} \subseteq {}^{\omega}\omega$ is *unbounded* if there is no single real which simultaneously dominates all elements of $\mathcal{B}$. The *bounding number* $\mathfrak{b}$ is the minimal cardinality of an unbounded family. A family $\mathcal{A} \subseteq [\omega]^{\omega}$ is *almost disjoint* (a.d.) if any two distinct elements of $\mathcal{A}$ have finite intersection. An almost disjoint family $\mathcal{A}$ is maximal, called *maximal almost disjoint*, if for every $C \in [\omega]^{\omega}$ there is $A \in \mathcal{A}$ such that $|A \cap C| = \omega$. *The almost disjointness number* $\mathfrak{a}$ is the minimal cardinality of a maximal almost disjoint family. It is well known that $\mathfrak{b} \leq \mathfrak{a}$ (see [2]). A family $S \subseteq [\omega]^{\omega}$ is *splitting* if for every $A \in [\omega]^{\omega}$ there is $B \in S$ such that $|A \cap B| = |A \cap B^c| = \omega$. The *splitting number* $\mathfrak{s}$ is the minimal cardinality of a splitting family.

The bounding and the splitting numbers are independent. The consistency of $\mathfrak{s} < \mathfrak{b}$ was obtained in 1985 by J. Baumgartner and P. Dordal (see [1]). The consistency of $\mathfrak{b} < \mathfrak{s}$ was obtained in 1984 by S. Shelah (see [9]) using a proper forcing notion of size continuum, which is almost ${}^{\omega}\omega$-bounding and adds a real not split by the ground model reals. There is an increased interest in obtaining models in which $\mathfrak{c} \geq \aleph_3$. In 1998 J. Brendle obtained the consistency of $\mathfrak{b} = \kappa < \mathfrak{a} = \kappa^+$ using a finite support iteration of ccc posets (see [5]). The consistency of $\mathfrak{b} = \kappa < \mathfrak{s} = \kappa^+$ was obtained in [7] (see also [6]). In fact the forcing construction of the last two models, can be combined and in an appropriate finite support iteration of ccc posets one obtains the consistency of $\mathfrak{b} = \kappa < \mathfrak{a} = \mathfrak{s} = \kappa^+$.

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In the present paper, we obtain the more general consistency results of $\mathfrak{b} = \mathfrak{a} = \kappa < \mathfrak{s} = \lambda$, for κ, λ arbitrary regular uncountable cardinals (see Theorem 17) and $\mathfrak{b} = \kappa < \mathfrak{s} = \mathfrak{a} = \lambda$ where $\kappa < \lambda$ are arbitrary regular cardinals, above a measurable μ (see Theorem 21). Both of the constructions use the idea of matrix iteration introduced by A. Blass and S. Shelah in [3].

Notation: For an ultrafilter $\mathcal{U}$ on ω, let $\mathbb{M}_{\mathcal{U}}$ denote the associated Mathias forcing (see [2]). That is $\mathbb{M}_{\mathcal{U}}$ is the poset of all $(a, A) \in [\omega]^{<\omega} \times \mathcal{U}$ such that $\max a < \min A$ with extension relation defined as follows: $(a_1, A_1) \leq (a_2, A_2)$ if a_2 is an initial segment of a_1, $a_1 \backslash a_2 \subseteq A_2$ and $A_1 \subseteq A_2$. The Hechler forcing $\mathbb{D}$ (see [2]) consists of all $(s, f) \in {}^{<\omega}\omega \times {}^{\omega}\omega$ with extension relation defined as follows: $(s_1, f_1) \leq (s_2, f_2)$ if s_2 is an initial segment of s_1, for all $i \in \mathrm{dom}(s_1)\backslash\mathrm{dom}(s_2)$, $s_1(i) \geq f_2(i)$ and for all $i \in \omega$ $f_2(i) \leq f_1(i)$. For μ a measurable cardinal and $\mathcal{D}$ a μ-complete ultrafilter on μ, let $\mathbb{P}^\mu/\mathcal{D}$ denote the ultrapower of $\mathbb{P}$ (see [4]), where $\mathbb{P}$ is a given poset. Ultrapowers of posets were introduced by S. Shelah in [10].

2. Adding a mad family

Definition 1 (S. Hechler [8])**.** *For γ an ordinal, $\mathbb{P}_\gamma$ is the poset of all finite partial functions $p : \gamma \times \omega \to 2$ such that dom$(p) = F_p \times n_p$ where $F_p \in [\gamma]^{<\omega}$, $n_p \in \omega$. The order is given by $q \leq p$ if $p \subseteq q$ and $|q^{-1}(1) \cap F^p \times \{i\}| \leq 1$ for all $i \in n_q \backslash n_p$.*

$\mathbb{P}_\gamma$ is ccc. If G is $\mathbb{P}_\gamma$-generic, then the family $\mathcal{A}_\gamma = \{A_\alpha : \alpha < \gamma\}$, where $A_\alpha = \{i : \exists p \in Gp(\alpha, i) = 1\}$ is almost disjoint and for $\gamma \geq \omega_1$ maximal almost disjoint (see [8]). This product like forcing decomposes as a two-step iteration as follows. Let $\gamma < \delta$, G a $\mathbb{P}_\gamma$-generic filter. In $V[G]$, let $\mathbb{P}_{[\gamma,\delta)}$ be the poset of all pairs (p, H) such that $p : (\delta\backslash\gamma) \times \omega \to 2$ is a finite partial function with $\mathrm{dom}(p) = F_p \times n_p$ where $F_p \in [\delta\backslash\gamma]^{<\omega}$, $n_p \in \omega$ and $H \in [\gamma]^{<\omega}$. The order is given by $(q, K) \leq (p, H)$ if $q \leq_{\mathbb{P}_\delta} p$, $H \subseteq K$ and for all $\alpha \in F_p$, $\beta \in H$, $i \in n_q\backslash n_p$ if $i \in A_\beta$, then $q(\alpha, i) = 0$. Observe that $\mathbb{P}_\delta = \mathbb{P}_\gamma * \dot{\mathbb{P}}_{[\gamma,\delta)}$, i.e. $\mathbb{P}_\delta$ is forcing equivalent to the two step iteration of $\mathbb{P}_\gamma$ and $\dot{\mathbb{P}}_{[\gamma,\delta)}$.

Definition 2. *Let $M \subseteq N$ be models of set theory, $\mathcal{B} = \{B_\alpha\}_{\alpha<\gamma} \subseteq M \cap [\omega]^\omega$, $A \in N \cap [\omega]^\omega$. Then $(\star^{M,N}_{\mathcal{B},A})$ holds if for every $h : \omega \times [\gamma]^{<\omega} \to \omega$, $h \in M$ and $m \in \omega$ there are $n \geq m$, $F \in [\gamma]^{<\omega}$ such that $[n, h(n, F))\backslash \bigcup_{\alpha\in F} B_\alpha \subseteq A$.*

Lemma 3. *Let $(\star^{M,N}_{\mathcal{B},A})$ hold, $\mathcal{B} = \{B_\alpha\}_{\alpha<\gamma}$, let $\mathcal{I}(\mathcal{B})$ be the ideal generated by $\mathcal{B}$ and the finite sets and let $B \in M \cap [\omega]^\omega$, $B \notin \mathcal{I}(\mathcal{B})$. Then $|A \cap B| = \aleph_0$.*

Proof. Otherwise $A \cap B \subseteq n$ for some $n \in \omega$. Let $m \geq n$, $F \in [\gamma]^{<\omega}$. Since $B \notin \mathcal{I}(\mathcal{B})$, $B \not\subseteq^* \bigcup_{\alpha\in F} B_\alpha$ and so there is $k_{m,F} \in B\backslash \bigcup_{\alpha\in F} B_\alpha$ greater than m. Define $h(m, F) = k_{m,F} + 1$ for all $m \geq n$, $F \in [\gamma]^{<\omega}$ and $h\restriction n \times [\gamma]^{<\omega} = 0$. Then h is a function in M such that $[m, h(m, F))\backslash \bigcup_{\alpha\in F} B_\alpha \not\subseteq A$ for all $m \geq n$, $F \in [\gamma]^{<\omega}$, contradicting $(\star^{M,N}_{\mathcal{B},A})$. □

The sets A_α added by the forcing $\mathbb{P}_\gamma$ satisfy the above property in the following sense:

Lemma 4. *Let $G_{\gamma+1}$ be $\mathbb{P}_{\gamma+1}$-generic, $G_\gamma = G_{\gamma+1} \cap \mathbb{P}_\gamma$ and $\mathcal{A}_\gamma = \{A_\alpha\}_{\alpha<\gamma}$, where $A_\alpha = \{i : \exists p \in G_{\gamma+1} p(\alpha, i) = 1\}$, $\alpha \leq \gamma$. Then $(\star^{V[G_\gamma],V[G_{\gamma+1}]}_{\mathcal{A}_\gamma,A_\gamma})$ holds.*

Proof. Let $h \in V[G_\gamma]$, $h : \omega \times [\gamma]^{<\omega} \to \omega$, $(p, H) \in \mathbb{P}_{[\gamma,\gamma+1)}$, $m \in \omega$. Then $\mathrm{dom}(p) = \{\gamma\} \times n_p$ where $n_p \in \omega$. Define an extension (q, K) of (p, H) in $\mathbb{P}_{[\gamma,\gamma+1)}$ as follows. Let $n > \max\{m, n_p\}$, $n_q = h(n, H)$. Let $\mathrm{dom}(q) = \{\gamma\} \times n_q$, $K = H$, $q{\restriction}\{\gamma\} \times n_p = p$, $q{\restriction}\{\gamma\} \times [n_p, n) = 0$ and for $i \in [n, n_q)$ let $q(\gamma, i) = 1$ if and only if $i \notin \bigcup_{\alpha\in H} A_\alpha$. Then $(q, K) \le (p, H)$ and $(q, K) \Vdash [n, h(n, H)) \setminus \bigcup_{\alpha\in H} A_\alpha \subseteq A_\gamma$. □

3. Combinatorics and preservation

In addition to $(\star^{M,N}_{\mathcal{B},A})$, we consider one more combinatorial property which will be crucial for the second consistency result to be established, and systemize some preservation theorems for both of these properties.

Definition 5. *If $M \subseteq N$ are models of set theory, $c \in N \cap [\omega]^\omega$ such that for all $f \in M \cap [\omega]^\omega$, $N \models c \not\le^* f$, we will say that $(\star M, N, c)$ holds.*

The following lemma can be found in [3].

Lemma 6 (A. Blass, S. Shelah, [3])**.** *Let $M \subseteq N$ be models of set theory, $\mathcal{U}$ an ultrafilter in M, $c \in \omega^\omega \cap N$ such that $(\star M, N, c)$ holds. Then there is an ultrafilter $\mathcal{V} \supseteq \mathcal{U}$ in N such that:*

(1) *every maximal antichain of $\mathbb{M}_\mathcal{U}$ which belongs to M is a maximal antichain of $\mathbb{M}_\mathcal{V}$ in N,*
(2) *$(\star M[G], N[G], c)$ holds where G is $\mathbb{M}_\mathcal{V}$-generic over N (and thus, by (1), $\mathbb{M}_\mathcal{U}$-generic over M).*

In analogy, we obtain the following.

Crucial Lemma 7. *Let $M \subseteq N$ be models of set theory, $\mathcal{B} = \{B_\alpha\}_{\alpha<\gamma} \subseteq M \cap [\omega]^\omega$, $A \in N \cap [\omega]^\omega$ such that $(\star^{M,N}_{\mathcal{B},A})$ holds. Let $\mathcal{U}$ be an ultrafilter in M. Then there is an ultrafilter $\mathcal{V} \supseteq \mathcal{U}$ in N such that*

(1) *every maximal antichain of $\mathbb{M}_\mathcal{U}$ which belongs to M is a maximal antichain of $\mathbb{M}_\mathcal{V}$ in N,*
(2) *$(\star^{M[G],N[G]}_{\mathcal{B},A})$ holds where G is $\mathbb{M}_\mathcal{V}$-generic over N (and thus, by (1), $\mathbb{M}_\mathcal{U}$-generic over M).*

Proof. Work in N. Let $C \subseteq \mathbb{M}_\mathcal{U}$, $C \in M$, be a maximal antichain, and let $s \in [\omega]^{<\omega}$. We say *X is forbidden by C, s* if (s, X) is incompatible with all conditions from C.

Given an $\mathbb{M}_\mathcal{U}$-name $\dot h : \omega \times [\gamma]^{<\omega} \to \omega$, $\dot h \in M$, there are (in M) maximal antichains $D^{\dot h}_{n,F} \subseteq \mathbb{M}_\mathcal{U}$ and functions $g^{\dot h}_{n,F} : D^{\dot h}_{n,F} \to \omega$ such that p forces that $\dot h(n, F) = g^{\dot h}_{n,F}(p)$ for all $p \in D^{\dot h}_{n,F}$. Say *Y is forbidden by $\dot h, t$* if, for all n and all F, (t, Y) is incompatible with all conditions $p \in D^{\dot h}_{n,F}$ which satisfy $[n, g^{\dot h}_{n,F}(p)) \setminus \bigcup_{\alpha\in F} B_\alpha \subseteq A$. (This means that (t, Y) forces that $[n, \dot h(n, F)) \setminus \bigcup_{\alpha\in F} B_\alpha \not\subseteq A$ for all n, F.)

Let $\mathcal{I}$ be the ideal generated by all forbidden sets.

Main Claim 8. $\mathcal{I} \cap \mathcal{U} = \emptyset$

Once the main claim in proved, we construct $\mathcal{V} \supseteq \mathcal{U}$ such that $\mathcal{V} \cap \mathcal{I} = \emptyset$. Then (1) and (2) easily hold.

Proof. By contradiction. Assume there are forbidden sets

$$X_0, ..., X_{k-1}, Y_0, ..., Y_{k-1}$$

with witnesses

$$C_0, s_0, ..., C_{k-1}, s_{k-1}, \dot{h}_0, t_0, ..., \dot{h}_{k-1}, t_{k-1}$$

such that $Z := \bigcup_{i<k} X_i \cup \bigcup_{i<k} Y_i \in \mathcal{U}$ (in M).

For $t \in [\omega]^{<\omega}$ and $(s, X) \in \mathbb{M}_{\mathcal{U}}$, say (s, X) *permits* t if $s \subseteq t \subseteq s \cup X$. If $C \subseteq \mathbb{M}_{\mathcal{U}}$, say C *permits* t if there is $p \in C$ which permits t. Note that p is compatible with (t, Y) iff there is $u \subseteq Y$ such that p permits $t \cup u$.

Subclaim 9. *There is $h : \omega \times [\gamma]^{<\omega} \to \omega$, $h \in M$, with $h(n, F) \geq n$, such that whenever we partition $Z \cap [n, h(n, F))$ into $2k$ pieces, then at least one piece s has the following property:*

- *for all $i < k$ there is $t \subseteq s$ such that C_i permits $s_i \cup t$,*
- *for all $i < k$ there is $t \subseteq s$ such that some $p \in D^{\dot{h}_i}_{n,F}$ with $g^{\dot{h}_i}_{n,F}(p) < h(n, F)$ permits $t_i \cup t$.*

Proof. The subclaim only mentions objects from M, and is clearly absolute. Therefore we may prove it in M.

Assume the subclaim was false for some n and F. Consider $Z \setminus n$. By a compactness argument (equivalently, by König's Lemma) we could partition $Z \setminus n$ into $2k$ pieces none of which satisfies the conclusion of the subclaim. One of the $2k$ pieces, say W, must belong to $\mathcal{U}$. Since C_i is a maximal antichain, there is $p \in C_i$ such that p and (s_i, W) are compatible. Thus there is $t \subseteq W$ such that p permits $s_i \cup t$. Similarly, there are $p \in D^{\dot{h}_i}_{n,F}$ and $t \subseteq W$ such that p permits $t_i \cup t$. If we choose $h(n, F)$ large enough, $s = W \cap [n, h(n, F))$ has the required properties, contradictory to our assumption about n and F. □

We continue the proof of the main claim. Fix n and F. Consider the partition given by $\{X_i \cap [n, h(n, F)), Y_i \cap [n, h(n, F)) : i < k\}$. Consider a piece $X_i \cap [n, h(n, F))$. Since (s_i, X_i) is incompatible with all conditions from C_i, there is no $t \subseteq X_i \cap [n, h(n, F))$ such that C_i permits $s_i \cup t$. So $X_i \cap [n, h(n, F))$ is not as in the subclaim. Hence one piece $s = Y_i \cap [n, h(n, F))$ satisfies the conclusion of the subclaim. Thus there are $t \subseteq Y_i \cap [n, h(n, F))$ and $p \in D^{\dot{h}_i}_{n,F}$ with $g^{\dot{h}_i}_{n,F}(p) < h(n, F)$ such that p permits $t_i \cup t$. In particular p is compatible with (t_i, Y_i). On the other hand, (t_i, Y_i) is incompatible with all $q \in D^{\dot{h}_i}_{n,F}$ which satisfy $[n, g^{\dot{h}_i}_{n,F}(q)) \setminus \bigcup_{\alpha \in F} B_\alpha \subseteq A$. Thus $[n, g^{\dot{h}_i}_{n,F}(p)) \setminus \bigcup_{\alpha \in F} B_\alpha \not\subseteq A$, and $[n, h(n, F)) \setminus \bigcup_{\alpha \in F} B_\alpha \not\subseteq A$ follows. Unfixing n and F, we see this holds for all n and F. This contradicts $(\star^{M,N}_{\mathcal{B},A})$, and the proof of the main claim and the crucial lemma is complete. □

□

Lemma 10. *Let $\langle \mathbb{P}_{\ell,\eta}, \dot{\mathbb{Q}}_{\ell,\eta} : \eta < \zeta \rangle$, $\ell \in \{0, 1\}$ be finite support iterations such that $\mathbb{P}_{0,\eta}$ is a complete suborder of $\mathbb{P}_{1,\eta}$ for all $\eta < \zeta$. Then $\mathbb{P}_{0,\zeta}$ is a complete suborder of $\mathbb{P}_{1,\zeta}$.*

Proof. It is clear that $\mathbb{P}_{0,\zeta} \subseteq \mathbb{P}_{1,\zeta}$ and that incompatibility is preserved. Let $p \in \mathbb{P}_{1,\zeta}$. Since $\mathbb{P}_{1,\zeta}$ is finite support iteration, there is $\eta < \zeta$ such that $p \in \mathbb{P}_{1,\eta}$. However $\mathbb{P}_{0,\eta} \lessdot \mathbb{P}_{1,\eta}$ and so there is $q \in \mathbb{P}_{0,\eta}$ which is a reduction of p (in $\mathbb{P}_{0,\eta}$). Then q is also a reduction of p in $\mathbb{P}_{0,\zeta}$. Indeed, let $r \leq q$, $r \in \mathbb{P}_{0,\zeta}$. Then $r = r_0 \cup r_1$ where $r_0 \in \mathbb{P}_{0,\eta}$ and $\mathrm{suppt}(r_1) \subseteq [\eta, \zeta)$. Then $r_0 \leq q$ and since q is a reduction of p in $\mathbb{P}_{0,\eta}$, there is $\tilde{r}_0 \in \mathbb{P}_{0,\eta}$ which is a common extension of p and r_0. Then $\tilde{r}_0 \cup r_1$ is a common extension of r and p. Thus q is a reduction of p in $\mathbb{P}_{0,\zeta}$. $\square$

Lemma 11. *Let $M \subseteq N$ be models of set theory, $\mathbb{P} \in M$ a poset such that $\mathbb{P} \subseteq M$, G a $\mathbb{P}$-generic filter over M (and so $\mathbb{P}$-generic over N).*

1. *Let $\mathcal{B} = \{B_\alpha\}_{\alpha\in\gamma} \subseteq M \cap [\omega]^\omega$, $A \in N \cap [\omega]^\omega$ such that $(\star^{M,N}_{\mathcal{B},A})$ holds. Then $(\star^{M[G],N[G]}_{\mathcal{B},A})$ holds.*
2. *Let $c \in N \cap {}^\omega\omega$ such that $(\star M, N, c)$. Then $(\star M[G], N[G], c)$ holds.*

Proof. We give a proof only of (1), since part (2) is proved similarly. If (1) does not hold, then there are $h \in M[G]$, $h : \omega \times [\gamma]^{<\omega} \to \omega$ and $m \in \omega$ such that for all $n \geq m$, $F \in [\gamma]^{<\omega}$, $N[G] \vDash [n, h(n,F)) \setminus \bigcup_{\alpha\in F} B_\alpha \not\subseteq A$. Then there are a $\mathbb{P}$-name $\dot{h}$ for h in M, $p \in G$ and $m \in \omega$ such that

$$p \Vdash_{N,\mathbb{P}} \forall n \geq m \forall F \in [\gamma]^{<\omega}([n, \dot{h}(n,F)) \setminus \bigcup_{\alpha\in F} B_\alpha \not\subseteq A).$$

However for all $n \geq m$, $F \in [\gamma]^{<\omega}$ there are $p_{n,F} \leq p$ (in M) and $k_{n,F} \in \omega$ such that $p_{n,F} \Vdash_{M,\mathbb{P}} \dot{h}(n,F) = k_{n,F}$. Then

$$p_{n,F} \Vdash_{N,\mathbb{P}} [n, k_{n,F}) \setminus \bigcup_{\alpha\in F} B_\alpha \not\subseteq A$$

and so $N \vDash [n, k_{n,F}) \setminus \bigcup_{\alpha\in F} B_\alpha \not\subseteq A$. In M define $h_0 : \omega \times [\gamma]^{<\omega} \to \omega$ as follows. Let $h_0 \restriction m \times [\gamma]^{<\omega} = 0$ and for all $n \geq m$, $F \in [\gamma]^{<\omega}$ let $h_0(n,F) = k_{n,F}$. Then h_0 gives a contradiction to $(\star^{M,N}_{\mathcal{B},A})$. $\square$

Lemma 12. *Let $\langle \mathbb{P}_{\ell,n}, \dot{\mathbb{Q}}_{\ell,n} : n \in \omega\rangle$, $\ell \in \{0,1\}$ be finite support iterations such that $\mathbb{P}_{0,n}$ is a complete suborder of $\mathbb{P}_{1,n}$ for all n. Let $V_{\ell,n} = V^{\mathbb{P}_{\ell,n}}$.*

1. *Let $\mathcal{B} = \{A_\gamma\}_{\gamma<\alpha} \subseteq V_{0,0} \cap [\omega]^\omega$, $A \in V_{1,0} \cap [\omega]^\omega$. If $(\star^{V_{0,n},V_{1,n}}_{\mathcal{B},A})$ holds for all $n \in \omega$, then $(\star^{V_{0,\omega},V_{1,\omega}}_{\mathcal{B},A})$ holds.*
2. *Let $c \in V_{1,0} \cap {}^\omega\omega$. If $(\star V_{0,n}, V_{1,n}, c)$ holds for all $n \in \omega$, then $(\star V_{0,\omega}, V_{1,\omega}, c)$ holds.*

Proof. We will give a proof of (1). The proof of (2) is analogous. Thus suppose the claim of (1) does not hold and let $h : \omega \times [\alpha]^{<\omega} \to \omega$ be a function in $V_{0,\omega}$ such that for some $m \in \omega$, for all $n \geq m$, $F \in [\alpha]^{<\omega}$, $V_{1,\omega} \vDash [n, h(n,F)) \setminus \bigcup_{\gamma\in F} A_\gamma \not\subseteq A$. Then there are a $\mathbb{P}_{0,\omega}$-name $\dot{h}$, $p \in \mathbb{P}_{1,\omega}$ such that $p \Vdash [k, \dot{h}(k,F)) \setminus \bigcup_{\gamma\in F} A_\gamma \not\subseteq A$ for all $k \geq m$, $F \in [\alpha]^{<\omega}$. Since p has finite support, there is $n \in \omega$ such that $p \in \mathbb{P}_{1,n}$. Let $G_{1,n}$ be a $\mathbb{P}_{1,n}$-generic filter containing p and let $h' = \dot{h}/G_{0,n}$ be the quotient name, where $G_{0,n} = G_{1,n} \cap \mathbb{P}_{0,n}$. Let $\mathbb{R}^\ell_{n,\omega}$ be the quotient poset $\mathbb{P}_{\ell,n}/G_{\ell,n}$ in

$V_{\ell,n} = V[G_{\ell,n}]$. Then $h' \in V_{0,n}$ and for all $k \geq m$, $F \in [\alpha]^{<\omega}$

$$V_{n,1} \vDash \Vdash_{\mathbb{R}^1_{n,\omega}} [k, h'(k,F)) \setminus \bigcup_{\gamma \in F} A_\gamma \not\subseteq A.$$

Then for all $k \geq m$, $F \in [\alpha]^{<\omega}$ find $p_{k,F} \in \mathbb{R}^0_{n,\omega}$ and $x_{k,F} \in \omega$ such that $p_{k,F} \Vdash h'(k,F) = x_{k,F}$ and define $h_0(k,F) = x_{k,F}$. Let $h_0 \restriction m \times [\alpha]^{<\omega} = 0$. Then $h_0 \in V_{0,n}$ and $[k, h_0(k,F)) \setminus \bigcup_{\gamma \in F} A_\gamma \not\subseteq A$ for all $k \geq m$, $F \in [\alpha]^{<\omega}$ contradicting $(\star^{V_{0,n},V_{1,n}}_{\mathcal{B},A})$. □

The following Lemma is well-known and often used.

Lemma 13. *Let $\mathbb{P}, \mathbb{Q}$ be partial orders, such that $\mathbb{P}$ is completely embedded into $\mathbb{Q}$. Let $\dot{\mathbb{A}}$ be a $\mathbb{P}$-name for a forcing notion, $\dot{\mathbb{B}}$ a $\mathbb{Q}$-name for a forcing notion such that $\Vdash_{\mathbb{Q}} \dot{\mathbb{A}} \subseteq \dot{\mathbb{B}}$, and every maximal antichain of $\dot{\mathbb{A}}$ in $V^{\mathbb{P}}$ is a maximal antichain of $\dot{\mathbb{B}}$ in $V^{\mathbb{Q}}$. Then $\mathbb{P} * \dot{\mathbb{A}} <\circ \mathbb{Q} * \dot{\mathbb{B}}$.*

Proof. It is sufficient to show that every maximal antichain of $\mathbb{P} * \dot{\mathbb{A}}$ is a maximal antichain of $\mathbb{Q} * \dot{\mathbb{B}}$. Thus let $\{(p_\alpha, \dot{a}_\alpha) : \alpha < \kappa\}$ be a maximal antichain of $\mathbb{P} * \dot{\mathbb{A}}$. Suppose it is not maximal in $\mathbb{Q} * \dot{\mathbb{B}}$ and let $(q, \dot{b})$ be a condition in $\mathbb{Q} * \dot{\mathbb{B}}$ which is incompatible with all $(p_\alpha, \dot{a}_\alpha)$ for $\alpha < \kappa$. Let $\dot{H}$ be the canonical $\mathbb{P}$-name for the $\mathbb{P}$-generic filter and let $\dot{\Omega}$ be a $\mathbb{P}$-name such that $\Vdash \dot{\Omega} = \{\alpha : p_\alpha \in \dot{H}\}$.

Claim. $\Vdash$ "$\{\dot{a}_\alpha : \alpha \in \dot{\Omega}\}$ *is a maximal antichain of* $\dot{\mathbb{A}}$".

Proof. Suppose not. Then, there are $p \in \mathbb{P}$ and a $\mathbb{P}$-name $\dot{a}$ such that $p \Vdash \forall \alpha (\alpha \in \dot{\Omega} \rightarrow \dot{a} \perp \dot{a}_\alpha)$. Then $(p, \dot{a}) \in \mathbb{P} * \dot{\mathbb{A}}$ and so there is $\alpha < \kappa$ such that $(p, \dot{a}) \not\perp (p_\alpha, \dot{a}_\alpha)$. Let $(p', \dot{a}')$ be a common extension. Then $p' \Vdash (\dot{a}' \leq \dot{a}$ and $\dot{a}' \leq \dot{a}_\alpha)$, and since $p' \leq p_\alpha$, $p' \Vdash \alpha \in \dot{\Omega}$. That is, $p' \Vdash (\alpha \in \dot{\Omega}$ and $\dot{a}' \leq \dot{a}, \dot{a}' \leq \dot{a}_\alpha)$ which is a contradiction. □

Let G be $\mathbb{Q}$-generic filter such that $q \in G$. Then since $\mathbb{P} <\circ \mathbb{Q}$, there is a $\mathbb{P}$-generic filter H such that $V[H] \subseteq V[G]$. Now let $b = \dot{b}[G]$, $a_\alpha = \dot{a}_\alpha[G] = \dot{a}_\alpha[H]$ (for α such that $p_\alpha \in H$) and let $\Omega = \dot{\Omega}[G] = \{\alpha < \kappa : p_\alpha \in H\}$. By the above claim $\{a_\alpha : \alpha \in \Omega\}$ is a maximal antichain in $\mathbb{A}$ (in $V[G]$) and so by the hypothesis of the Lemma, it is a maximal antichain of $\mathbb{B}$ (in $V[G]$). So there is $\alpha \in \Omega$ such that $b = \dot{b}[G]$ is compatible with $a_\alpha = \dot{a}_\alpha[H]$. So there is $q' \in G$ such that $q' \leq p_\alpha$, $q' \leq q$ and $q' \Vdash (\alpha \in \dot{\Omega}$ and $\dot{b} \not\perp \dot{a}_\alpha)$. Thus there is a $\mathbb{Q}$-name $\dot{b}'$ such that $q' \Vdash$ "$\dot{b}' \leq \dot{b}$, $\dot{b}' \leq \dot{a}_\alpha$" and so $(q', \dot{b}')$ is a common extension of $(q, \dot{b})$ and $(p_\alpha, \dot{a}_\alpha)$, which is a contradiction. □

4. The consistency of $\mathfrak{b} = \mathfrak{a} = \kappa < \mathfrak{s} = \lambda$

Let $f : \{\eta < \lambda : \eta \equiv 1 \bmod 2\} \rightarrow \kappa$ be an onto mapping, such that for all $\alpha < \kappa$, $f^{-1}(\alpha)$ is cofinal in λ. Recursively define a system of finite support iterations $\langle \langle \mathbb{P}_{\alpha,\zeta} : \alpha \leq \kappa, \zeta \leq \lambda \rangle, \langle \dot{\mathbb{Q}}_{\alpha,\zeta} : \alpha \leq \kappa, \zeta < \lambda \rangle \rangle$ as follows. For all α, ζ let $V_{\alpha,\zeta} = V^{\mathbb{P}_{\alpha,\zeta}}$. We refer to such systems as matrix iterations. Note that this type of iterations appeared for the first time in [3].

(1) If $\zeta = 0$, then for all $\alpha \leq \kappa$, $\mathbb{P}_{\alpha,0}$ is Hechler's poset (see Definition 1) for adding an a.d. family $\mathcal{A}_\alpha = \{A_\beta\}_{\beta < \alpha}$ (note that for $\alpha \geq \omega_1$, $\mathcal{A}_\alpha$ is mad in $V_{\alpha,0}$).

(2) If $\zeta = \eta + 1$, $\zeta \equiv 1 \mod 2$, then $\Vdash_{\mathbb{P}_{\alpha,\eta}} \dot{\mathbb{Q}}_{\alpha,\eta} = \mathbb{M}_{\dot{\mathcal{U}}_{\alpha,\eta}}$ where $\dot{\mathcal{U}}_{\alpha,\eta}$ is a $\mathbb{P}_{\alpha,\eta}$-name for an ultrafilter and for all $\alpha < \beta \leq \kappa$, $\Vdash_{\mathbb{P}_{\beta,\eta}} \dot{\mathcal{U}}_{\alpha,\eta} \subseteq \dot{\mathcal{U}}_{\beta,\eta}$.
(3) If $\zeta = \eta + 1$, $\zeta \equiv 0 \mod 2$, then if $\alpha \leq f(\eta)$, $\dot{\mathbb{Q}}_{\alpha,\eta}$ is a $\mathbb{P}_{\alpha,\eta}$-name for the trivial forcing notion; if $\alpha > f(\eta)$ then $\dot{\mathbb{Q}}_{\alpha,\eta}$ is a $\mathbb{P}_{\alpha,\eta}$-name for $\mathbb{D}^{V_{f(\eta),\eta}}$.
(4) If ζ is a limit, then for all $\alpha \leq \kappa$, $\mathbb{P}_{\alpha,\zeta}$ is the finite support iteration of $\langle \mathbb{P}_{\alpha,\eta}, \dot{\mathbb{Q}}_{\alpha,\eta} : \eta < \zeta \rangle$.

Furthermore the construction will satisfy the following two properties:

(a) $\forall \zeta \leq \lambda \forall \alpha < \beta \leq \kappa$, $\mathbb{P}_{\alpha,\zeta}$ is a complete suborder of $\mathbb{P}_{\beta,\zeta}$.
(b) $\forall \zeta \leq \lambda \forall \alpha < \kappa$ $(\star^{V_{\alpha,\zeta}, V_{\alpha+1,\zeta}}_{\mathcal{A}_\alpha, A_\alpha})$ holds.

Proceed by recursion on ζ. For $\zeta = 0$, $\alpha \leq \kappa$ let $\mathbb{P}_{\alpha,0}$ be the poset from Definition 1. Then by the product property of $\mathbb{P}_{\alpha,0}$ and Lemma 4 respectively, properties (a) and (b) above hold. Let $\zeta = \eta + 1$ be a successor ordinal and suppose $\forall \alpha \leq \kappa$, $\mathbb{P}_{\alpha,\eta}$ has been defined so that properties (a) and (b) above hold.

If $\zeta \equiv 1 \mod 2$ define $\dot{\mathbb{Q}}_{\alpha,\eta}$ by induction on $\alpha \leq \kappa$ as follows. *If* $\alpha = 0$, let $\dot{\mathcal{U}}_{0,\eta}$ be a $\mathbb{P}_{0,\eta}$-name for an ultrafilter, $\dot{\mathbb{Q}}_{0,\eta}$ a $\mathbb{P}_{0,\eta}$-name for $\mathbb{M}_{\dot{\mathcal{U}}_{0,\eta}}$ and let $\mathbb{P}_{0,\zeta} = \mathbb{P}_{0,\eta} * \dot{\mathbb{Q}}_{0,\eta}$. *If* $\alpha = \beta + 1$ and $\dot{\mathcal{U}}_{\beta,\eta}$ has been defined, by the inductive hypothesis and Lemma 7 there is a $\mathbb{P}_{\alpha,\eta}$-name $\dot{\mathcal{U}}_{\alpha,\eta}$ for an ultrafilter such that $\Vdash_{\mathbb{P}_{\alpha,\eta}} \dot{\mathcal{U}}_{\beta,\eta} \subseteq \dot{\mathcal{U}}_{\alpha,\eta}$, every maximal antichain of $\mathbb{M}_{\dot{\mathcal{U}}_{\beta,\eta}}$ in $V_{\beta,\eta}$ is a maximal antichain of $\mathbb{M}_{\dot{\mathcal{U}}_{\alpha,\eta}}$ and $(\star^{V_{\beta,\zeta}, V_{\beta+1,\zeta}}_{\mathcal{A}_\beta, A_\beta})$ holds, where $V_{\beta+1,\zeta} = V^{\mathbb{P}_{\alpha,\zeta}}$, $\mathbb{P}_{\alpha,\zeta} = \mathbb{P}_{\alpha,\eta} * \dot{\mathbb{Q}}_{\alpha,\eta}$ and $\dot{\mathbb{Q}}_{\alpha,\eta}$ is a $\mathbb{P}_{\alpha,\eta}$-name for $\mathbb{M}_{\dot{\mathcal{U}}_{\alpha,\eta}}$. Note that by Lemma 13 $\mathbb{P}_{\beta,\zeta} = \mathbb{P}_{\beta,\eta} * \mathbb{M}_{\dot{\mathcal{U}}_{\beta,\eta}}$ is a complete suborder of $\mathbb{P}_{\alpha,\zeta}$. *If* α *is a limit* ordinal and for all $\beta < \alpha$ $\dot{\mathcal{U}}_{\beta,\eta}$ has been defined (and so $\dot{\mathbb{Q}}_{\beta,\eta}$ is a $\mathbb{P}_{\beta,\eta}$-name for $\mathbb{M}_{\dot{\mathcal{U}}_{\beta,\eta}}$; $\mathbb{P}_{\beta,\zeta} = \mathbb{P}_{\beta,\eta} * \dot{\mathbb{Q}}_{\beta,\eta}$), consider the following two cases. If $\mathrm{cf}(\alpha) = \omega$, find a $\mathbb{P}_{\alpha,\eta}$-name $\dot{\mathcal{U}}_{\alpha,\eta}$ for an ultrafilter such that for all $\beta < \alpha$, $\Vdash_{\mathbb{P}_{\alpha,\eta}} \dot{\mathcal{U}}_{\beta,\eta} \subseteq \dot{\mathcal{U}}_{\alpha,\eta}$ and every maximal antichain of $\mathbb{M}_{\dot{\mathcal{U}}_{\beta,\eta}}$ from $V_{\beta,\eta}$ is a maximal antichain of $\mathbb{M}_{\mathcal{U}_{\alpha,\eta}}$ (in $V_{\alpha,\eta}$) (for the construction of such an ultrafilter in $V^{\mathbb{P}_{\alpha,\eta}}$ see [3], p. 266). If $\mathrm{cf}(\alpha) > \omega$, then let $\dot{\mathcal{U}}_{\alpha,\eta}$ be a $\mathbb{P}_{\alpha,\eta}$-name for $\bigcup_{\beta<\alpha} \mathcal{U}_{\beta,\eta}$. Let $\dot{\mathbb{Q}}_{\alpha,\eta}$ be a $\mathbb{P}_{\alpha,\eta}$-name for $\mathbb{M}_{\dot{\mathcal{U}}_{\alpha,\eta}}$ and let $\mathbb{P}_{\alpha,\zeta} = \mathbb{P}_{\alpha,\eta} * \dot{\mathbb{Q}}_{\alpha,\eta}$. Again by Lemma 13 for all $\beta < \alpha$ $\mathbb{P}_{\beta,\zeta}$ is a complete suborder of $\mathbb{P}_{\alpha,\zeta}$.

If $\zeta \equiv 0 \mod 2$, then for all $\alpha \leq f(\eta)$ let $\dot{\mathbb{Q}}_{\alpha,\eta}$ be a $\mathbb{P}_{\alpha,\eta}$-name for the trivial poset and for $\alpha > f(\eta)$ let $\dot{\mathbb{Q}}_{\alpha,\eta}$ be a $\mathbb{P}_{\alpha,\eta}$-name for $\mathbb{D}^{V_{f(\eta),\eta}}$. Let $\mathbb{P}_{\alpha,\zeta} = \mathbb{P}_{\alpha,\eta} * \dot{\mathbb{Q}}_{\alpha,\eta}$. If $\alpha < \beta \leq f(\eta)$, then $\mathbb{P}_{\alpha,\zeta} = \mathbb{P}_{\alpha,\eta}$, $\mathbb{P}_{\beta,\zeta} = \mathbb{P}_{\alpha,\eta}$ and so by the inductive hypothesis $\mathbb{P}_{\alpha,\zeta}$ is a complete suborder of $\mathbb{P}_{\beta,\zeta}$. If $\alpha \leq f(\eta) < \beta$, then $\mathbb{P}_{\alpha,\zeta} = \mathbb{P}_{\alpha,\eta}$ and $\mathbb{P}_{\alpha,\eta} \lessdot \mathbb{P}_{\beta,\eta} \lessdot \mathbb{P}_{\beta,\eta} * \dot{\mathbb{Q}}_{\beta,\eta}$. Thus $\mathbb{P}_{\alpha,\zeta} \lessdot \mathbb{P}_{\beta,\zeta}$. If $f(\eta) < \alpha < \beta$, then again by Lemma 13 $\mathbb{P}_{\alpha,\zeta} \lessdot \mathbb{P}_{\beta,\zeta}$. Furthermore by Lemma 11.(1) $(\star^{V_{\alpha,\zeta}, V_{\alpha+1,\zeta}}_{\mathcal{A}_\alpha, A_\alpha})$ holds for all $\alpha \leq \kappa$.

If ζ is a limit and for all $\eta < \zeta$, $\mathbb{P}_{\alpha,\eta}$, $\dot{\mathbb{Q}}_{\alpha,\eta}$ have been defined, let $\mathbb{P}_{\alpha,\zeta}$ be the finite support iteration of $\langle \mathbb{P}_{\alpha,\eta}, \dot{\mathbb{Q}}_{\alpha,\eta} : \eta < \zeta \rangle$. By Lemma 10 $\mathbb{P}_{\alpha,\zeta}$ is a complete suborder of $\mathbb{P}_{\beta,\zeta}$ and by Lemma 12.(1) $(\star^{V_{\alpha,\zeta}, V_{\alpha+1,\zeta}}_{\mathcal{A}_\alpha, A_\alpha})$ holds.

Remark 14. *For all* $\alpha < \beta \le \kappa$, $\zeta < \eta \le \lambda$ $\mathbb{P}_{\alpha,\zeta} \lessdot \mathbb{P}_{\beta,\eta}$.

Lemma 15. *For* $\zeta \le \lambda$*:*

(1) *for every* $p \in \mathbb{P}_{\kappa,\zeta}$ *there is* $\alpha < \kappa$ *such that* p *belongs to* $\mathbb{P}_{\alpha,\zeta}$*,*
(2) *for every* $\mathbb{P}_{\kappa,\zeta}$*-name for a real* $\dot{f}$ *there is* $\alpha < \kappa$ *such that* $\dot{f}$ *is a* $\mathbb{P}_{\alpha,\zeta}$*-name.*

Proof. The proof follows just as in the last Lemma of [3], p. 270. For completeness we will give a proof as well. We will prove (1) and (2) simultaneously by induction on ζ. Note that by the ccc property of $\mathbb{P}_{\kappa,\zeta}$ and the fact that κ is regular, uncountable, it is clear that (2) follows immediately from (1). If $\zeta = 0$ the claim follows from the product property of $\mathbb{P}_{\kappa,0}$. If ζ is a limit, $p \in \mathbb{P}_{\kappa,\zeta}$ then since p has a finite support, there is $\eta < \zeta$ such that $p \in \mathbb{P}_{\kappa,\eta}$. By inductive hypothesis there is $\alpha < \eta$ such that $p \in \mathbb{P}_{\alpha,\eta}$ and so in particular $p \in \mathbb{P}_{\alpha,\zeta}$. Let $\zeta = \eta + 1$ be a successor. Then $p \in \mathbb{P}_{\kappa,\zeta}$ is of the form $(p_0, \dot{p}_1)$ where $p_0 \in \mathbb{P}_{\kappa,\eta}$ and $\Vdash_{\mathbb{P}_{\kappa,\eta}} \dot{p}_1 \in \dot{\mathbb{Q}}_{\kappa,\eta}$. If $\zeta \equiv 1 \bmod 2$ then $\dot{p}_1$ is of the form $(a, \dot{A})$ where $\Vdash_{\mathbb{P}_{\kappa,\eta}} \dot{A} \in \dot{\mathcal{U}}_{\kappa,\eta}$, $a \in [\omega]^{<\omega}$. If $\zeta \equiv 0 \bmod 2$, then $\dot{p}_1$ is of the form $(s, \dot{f})$ where $\dot{f}$ is a $\mathbb{P}_{\kappa,\eta}$-name for a real, $s \in {}^{<\omega}\omega$ or $\dot{p}_1$ is trivial. In either of the above cases, the inductive hypothesis for (2) implies that there is $\alpha_1 < \kappa$ such that $\dot{p}_1$ is a $\mathbb{P}_{\alpha_1,\eta}$-name. Again the inductive hypothesis for (1) implies that $p_0 \in \mathbb{P}_{\alpha_0,\eta}$ for some $\alpha_0 < \kappa$. Then $p = (p_0, \dot{p}_1)$ belongs to $\mathbb{P}_{\alpha,\eta}$ where $\alpha = \max\{\alpha_0, \alpha_1\}$. □

Lemma 16. $V_{\kappa,\lambda} \vDash \mathfrak{b} = \mathfrak{a} = \kappa < \mathfrak{s} = \lambda$.

Proof. The family $\{A_\alpha\}_{\alpha\in\kappa}$ remains a maximal almost disjoint family in $V_{\kappa,\lambda}$. Indeed, otherwise there is a set $B \in V_{\kappa,\lambda} \cap [\omega]^\omega$ such that $\forall\alpha < \kappa(|B \cap A_\alpha| < \omega)$. By Lemma 15 there is $\alpha < \kappa$ such that $B \in V_{\alpha,\lambda} \cap [\omega]^\omega$. However $B \notin \mathcal{I}(\mathcal{A}_\alpha)$. On the other hand $(\star^{V_{\alpha,\lambda},V_{\alpha+1,\lambda}}_{\mathcal{A}_\alpha,A_{\alpha+1}})$ holds, and so by Lemma 3 $|B \cap A_{\alpha+1}| = \omega$ which is a contradiction. Therefore $\mathfrak{a} \le \kappa$.

Let $\mathcal{B} \subseteq V_{\kappa,\lambda} \cap {}^\omega\omega$ be of cardinality $< \kappa$. Then by Lemma 15 there are $\alpha < \kappa$, $\zeta < \lambda$ such that $\mathcal{B} \subseteq V_{\alpha,\zeta}$. Since $\{\gamma : f(\gamma) = \alpha\}$ is cofinal in λ, there is $\zeta' > \zeta$ such that $f(\zeta') = \alpha$. Then $\mathbb{P}_{\alpha+1,\zeta'+1}$ adds a real dominating $V_{\alpha,\zeta'} \cap {}^\omega\omega$ (and so $V_{\alpha,\zeta} \cap {}^\omega\omega$ since $V_{\alpha,\zeta} \subseteq V_{\alpha,\zeta'}$). Thus $\mathcal{B}$ is not unbounded. Therefore $V_{\kappa,\lambda} \Vdash \mathfrak{b} \ge \kappa$. However $\mathfrak{b} \le \mathfrak{a}$ (see [2]) and so $V_{\kappa,\lambda} \Vdash \mathfrak{b} = \mathfrak{a} = \kappa$.

To see that $V_{\kappa,\lambda} \vDash \mathfrak{s} = \lambda$, note that if $S \subseteq V_{\kappa,\lambda} \cap [\omega]^\omega$ is a family of cardinality $< \lambda$, then there is $\zeta < \lambda$ such that $\zeta = \eta + 1$, $\zeta \equiv 1 \bmod 2$ and $S \subseteq V_{\kappa,\eta}$. Then $\mathcal{M}_{\mathcal{U}_{\kappa,\eta}}$ adds a real not split by S and so S is not splitting. □

Theorem 17. *Let* $\kappa < \lambda$ *be arbitrary regular uncountable cardinals. Then there is a ccc generic extension in which* $\mathfrak{b} = \mathfrak{a} = \kappa < \mathfrak{s} = \lambda$.

5. The consistency of $\kappa = \mathfrak{b} < \mathfrak{s} = \mathfrak{a} = \lambda$ above a measurable

Let μ be a measurable cardinal, $\mathcal{D}$ a μ-complete ultrafilter on μ. Let $\kappa < \lambda$ be regular such that $\mu < \kappa$. For notation regarding ultrapowers of posets and names, see [4] and [10]. In the Lemma below we show that $(\star M, N, c)$ is preserved under ultrapowers.

Lemma 18. *Let* $\mathbb{P} \lessdot \mathbb{Q}$, $c \in V^{\mathbb{Q}}$ *such that for all* $f \in V^{\mathbb{P}} \cap {}^\omega\omega$, $\Vdash_{\mathbb{P}} c \not\le^* f$. *Let* $\mathbb{Q}' = \mathbb{Q}^\mu/\mathcal{D}$, $\mathbb{P}' = \mathbb{P}^\mu/\mathcal{D}$, $f \in V^{\mathbb{P}'} \cap {}^\omega\omega$. *Then* $\Vdash_{\mathbb{Q}'} c \not\le^* f$.

Proof. Suppose not. Thus there is a $\mathbb{P}'$ name for a real $\dot{f}$ and $[q] \in \mathbb{Q}'$ such that for some $k \in \omega$, for all $i \geq k$, $[q] \Vdash \dot{c}(i) \leq \dot{f}(i)$. Note that $\dot{f}$ is determined by maximal antichains $\{[p_{n,i}]\}_{n,i\in\omega}$ and $\{k_{n,i}\}_{n,i\in\omega} \subseteq \omega$ such that for all n, i $[p_{n,i}] \Vdash_{\mathbb{P}'} \dot{f}(i) = k_{n,i}$. Furthermore we can assume (see [4]) that for all $\alpha \in \mu$ there are maximal antichains $\{p^\alpha_{n,i}\}_{n,i\in\omega}$ in $\mathbb{P}$, such that $[p_{n,i}] = \langle p^\alpha_{n,i} : \alpha < \mu\rangle/\mathcal{D}$, and a $\mathbb{P}$-name for a real $\dot{f}^\alpha$ such that $p^\alpha_{n,i} \Vdash_{\mathbb{P}} \dot{f}^\alpha(i) = k_{n,i}$. By elementarity, $A = \{\alpha : q(\alpha) \Vdash_{\mathbb{Q}} \dot{c}(i) \leq \dot{f}^\alpha(i)\}$ is in $\mathcal{D}$, and so in particular A is non-empty. Here we identify $\dot{c}$ with its ultrapower. Let $\alpha \in A$. Then $\dot{f}^\alpha$ is a $\mathbb{P}$-name and for all $i \geq k$, $q(\alpha) \Vdash_{\mathbb{Q}} \dot{c}(i) \leq f^\alpha(i)$, which is a contradiction. □

Let $f : \{\eta < \lambda : \eta \equiv 1 \mod 3\} \to \kappa$ be an onto mapping such that for all $\alpha < \kappa$, $f^{-1}(\alpha)$ is cofinal in λ. Similarly to the construction from the previous section, recursively define a system of finite support iterations $\langle\langle\mathbb{P}_{\alpha,\zeta} : \alpha \leq \kappa, \zeta \leq \lambda\rangle, \langle\dot{\mathbb{Q}}_{\alpha,\zeta} : \alpha \leq \kappa, \zeta < \lambda\rangle\rangle$ so that properties (1) – (5), as well as (a) – (b) below hold. For all α, ζ let $V_{\alpha,\zeta} = V^{\mathbb{P}_{\alpha,\zeta}}$.

(1) If $\zeta = 0$, then for all $\alpha \leq \kappa$, let $\mathbb{P}_{\alpha,0}$ be the forcing notion for adding α many Cohen reals, $\{c_\gamma\}_{\gamma<\alpha}$.
(2) If $\zeta = \eta + 1$, $\zeta \equiv 1 \mod 3$, then $\Vdash_{\mathbb{P}_{\alpha,\eta}} \dot{\mathbb{Q}}_{\alpha,\eta} = \mathbb{M}_{\dot{\mathcal{U}}_{\alpha,\eta}}$ where $\dot{\mathcal{U}}_{\alpha,\eta}$ is a $\mathbb{P}_{\alpha,\eta}$-name for an ultrafilter and for all $\alpha < \beta \leq \kappa$, $\Vdash_{\mathbb{P}_{\beta,\eta}} \dot{\mathcal{U}}_{\alpha,\eta} \subseteq \dot{\mathcal{U}}_{\beta,\eta}$.
(3) If $\zeta = \eta + 1$, $\zeta \equiv 2 \mod 3$, then if $\alpha \leq f(\eta)$, $\dot{\mathbb{Q}}_{\alpha,\eta}$ is a $\mathbb{P}_{\alpha,\eta}$-name for the trivial forcing notions; if $\alpha > f(\eta)$ then $\dot{\mathbb{Q}}_{\alpha,\eta}$ is a $\mathbb{P}_{\alpha,\eta}$-name for $\mathbb{D}^{V_{f(\eta),\eta}}$.
(4) If $\zeta = \eta + 1$, $\zeta \equiv 0 \mod 3$, then for every $\alpha \leq \kappa$ let $\dot{\mathbb{Q}}_{\alpha,\eta}$ be a $\mathbb{P}_{\alpha,\eta}$-name for the quotient poset of $((\mathbb{P}_{\alpha,\eta})^\mu)/\mathcal{D}$ and $\mathbb{P}_{\alpha,\eta}$.
(5) If ζ is a limit, $\mathbb{P}_{\alpha,\zeta}$ is the finite support iteration of $\langle\mathbb{P}_{\alpha,\eta}, \dot{\mathbb{Q}}_{\alpha,\eta} : \eta < \zeta\rangle$.

Furthermore the construction will satisfy the following two properties:

(a) $\forall\zeta \leq \lambda\forall\alpha < \beta \leq \kappa$, $\mathbb{P}_{\alpha,\zeta}$ is a complete suborder of $\mathbb{P}_{\beta,\zeta}$.
(b) $\forall\zeta \leq \lambda\forall\alpha < \kappa$ $(\star V_{\alpha,\zeta}, V_{\alpha+1,\zeta}, c_{\alpha+1})$ holds.

Proceed by induction on ζ. If $\zeta = 0$, then for all $\alpha \leq \kappa$ let $\mathbb{P}_{\alpha,0}$ be the forcing notion for adding α many Cohen reals, $\{c_\gamma\}_{\gamma<\alpha}$. The properties of Cohen forcing imply that (a) and (b) above hold. Let $\zeta = \eta + 1$ be a successor ordinal and suppose that for all $\alpha \leq \kappa$, $\mathbb{P}_{\alpha,\eta}$ has been defined so that the relevant properties (a) and (b) above hold.

If $\zeta \equiv 1 \mod 3$ define $\dot{\mathbb{Q}}_{\alpha,\eta}$ by induction on $\alpha \leq \kappa$ as follows. *If* $\alpha = 0$ let $\dot{\mathcal{U}}_{0,\eta}$ be a $\mathbb{P}_{0,\eta}$-name for an ultrafilter, $\dot{\mathbb{Q}}_{0,\eta}$ a $\mathbb{P}_{0,\eta}$-name for $\mathbb{M}_{\dot{\mathcal{U}}_{0,\eta}}$ and let $\mathbb{P}_{0,\zeta} = \mathbb{P}_{0,\eta} * \dot{\mathbb{Q}}_{0,\eta}$. *If* $\alpha = \beta + 1$ and $\dot{\mathcal{U}}_{\beta,\eta}$ has been defined, by the inductive hypothesis and Lemma 6 there is a $\mathbb{P}_{\alpha,\eta}$-name $\dot{\mathcal{U}}_{\alpha,\eta}$ for an ultrafilter such that $\Vdash_{\mathbb{P}_{\alpha,\eta}} \dot{\mathcal{U}}_{\beta,\eta} \subseteq \dot{\mathcal{U}}_{\alpha,\eta}$, every maximal antichain of $\mathbb{M}_{\dot{\mathcal{U}}_{\beta,\eta}}$ in $V_{\beta,\eta}$ is a maximal antichain of $\mathbb{M}_{\dot{\mathcal{U}}_{\alpha,\eta}}$ and $(\star V_{\beta,\zeta}, V_{\beta+1,\zeta}, c_{\beta+1})$ holds, where $V_{\beta+1,\zeta} = V^{\mathbb{P}_{\alpha,\zeta}}$, $\mathbb{P}_{\alpha,\zeta} = \mathbb{P}_{\alpha,\eta} * \dot{\mathbb{Q}}_{\alpha,\eta}$ and $\dot{\mathbb{Q}}_{\alpha,\eta}$ is a $\mathbb{P}_{\alpha,\eta}$-name for $\mathbb{M}_{\dot{\mathcal{U}}_{\alpha,\eta}}$. By Lemma 13 $\mathbb{P}_{\beta,\zeta} = \mathbb{P}_{\beta,\eta} * \mathbb{M}_{\dot{\mathcal{U}}_{\beta,\eta}}$ is a complete suborder of $\mathbb{P}_{\alpha,\zeta}$. *If* α *is a limit* ordinal and for all $\beta < \alpha$, $\dot{\mathcal{U}}_{\beta,\eta}$ has been defined, proceed as in the limit case for α, ζ successor, odd ordinal, in the construction of the system of finite support iterations from the previous section.

If $\zeta \equiv 2 \mod 3$ then for all $\alpha \leq f(\eta)$ let $\dot{\mathbb{Q}}_{\alpha,\eta}$ be a $\mathbb{P}_{\alpha,\eta}$-name for the trivial poset and for $\alpha > f(\eta)$ let $\dot{\mathbb{Q}}_{\alpha,\eta}$ be a $\mathbb{P}_{\alpha,\eta}$-name for $\mathbb{D}^{V_{f(\eta),\eta}}$. Let $\mathbb{P}_{\alpha,\zeta} = \mathbb{P}_{\alpha,\eta} * \dot{\mathbb{Q}}_{\alpha,\eta}$. Property (a) can be established for ζ and $\alpha \leq \kappa$ just as in the successor, even case in the construction from the previous section. The inductive hypothesis and Lemma 11.(2) imply that for ζ and $\alpha \leq \kappa$, property (b) holds as well.

If $\zeta \equiv 0 \mod 3$ then for all $\alpha \leq \kappa$ let $\dot{\mathbb{Q}}_{\alpha,\eta}$ be a $\mathbb{P}_{\alpha,\eta}$-name for the quotient poset of $(\mathbb{P}^{\mu}_{\alpha,\eta})/\mathcal{D}$ and $\mathbb{P}_{\alpha,\eta}$. Let $\mathbb{P}_{\alpha,\zeta} = \mathbb{P}_{\alpha,\eta} * \dot{\mathbb{Q}}_{\alpha,\eta}$. Then by the inductive hypothesis and Lemma 5 of [4] for all $\alpha < \beta \leq \kappa$ $\mathbb{P}_{\alpha,\zeta}$ is a complete suborder of $\mathbb{P}_{\beta,\zeta}$. By the inductive hypothesis and Lemma 18 for all $\alpha < \kappa$ $(\star V_{\alpha,\zeta}, V_{\alpha+1,\zeta}, c_{\alpha+1})$ holds.

If ζ is a limit and for all $\eta < \zeta$, $\mathbb{P}_{\alpha,\eta}$ and $\dot{\mathbb{Q}}_{\alpha,\eta}$ have been defined, then let $\mathbb{P}_{\alpha,\zeta}$ be the finite support iteration of $\langle \mathbb{P}_{\alpha,\eta}, \dot{\mathbb{Q}}_{\alpha,\eta} : \eta < \zeta \rangle$. By Lemma 10 for all $\alpha < \beta \leq \kappa$ $\mathbb{P}_{\alpha,\zeta}$ is a complete suborder of $\mathbb{P}_{\beta,\zeta}$ and by Lemma 12.(2) $(\star V_{\alpha,\zeta}, V_{\alpha+1,\zeta}, c_{\alpha+1})$ holds for all $\alpha < \kappa$.

Lemma 19. *For $\zeta \leq \lambda$:*

(1) *for every $p \in \mathbb{P}_{\kappa,\zeta}$ there is $\alpha < \kappa$ such that p belongs to $\mathbb{P}_{\alpha,\zeta}$,*
(2) *for every $\mathbb{P}_{\kappa,\zeta}$-name for a real $\dot{f}$ there is $\alpha < \kappa$ such that $\dot{f}$ is a $\mathbb{P}_{\alpha,\zeta}$-name.*

Proof. If ζ is a limit, proceed as in the limit case of Lemma 15. If $\zeta = \eta + 1$ is a successor and $\zeta \equiv 1 \mod 3$ or $\zeta \equiv 2 \mod 3$ the proof follows as in the successor case of Lemma 15. Let $\zeta \equiv 0 \mod 3$ and let $p \in (\mathbb{P}_{\kappa,\eta})^{\mu}/\mathcal{D}$. Then $p = [f] = \langle f(\gamma) : \gamma < \mu \rangle/\mathcal{D}$ where $f(\gamma) \in \mathbb{P}_{\kappa,\eta}$ for $\gamma < \mu$. By the inductive hypothesis and $\kappa = \mathrm{cf}(\kappa) > \mu$, there is $\alpha < \kappa$ such that $f(\gamma) \in \mathbb{P}_{\alpha,\eta}$ for all $\gamma < \mu$ and so $p = [f]$ belongs to $(\mathbb{P}_{\alpha,\eta})^{\mu}/\mathcal{D} = \mathbb{P}_{\alpha,\zeta}$. □

Lemma 20. $V_{\kappa,\lambda} \vDash \mathfrak{b} = \kappa < \mathfrak{s} = \mathfrak{a} = \lambda$,

Proof. Let $\dot{f} \in V_{\kappa,\lambda} \cap {}^{\omega}\omega$. Then there are $\zeta < \lambda$, $\alpha < \kappa$ such that $f \in V_{\alpha,\zeta} \cap {}^{\omega}\omega$. Since $(\star\ V_{\alpha,\zeta}\ V_{\alpha+1,\zeta}\ c_{\alpha+1})$ holds, $V_{\alpha+1,\zeta} \vDash c_{\alpha+1} \not\leq^* f$ and so $V_{\kappa,\lambda} \vDash c_{\alpha+1} \not\leq^* f$. Therefore $\{c_{\alpha+1}\}_{\alpha<\kappa}$ is unbounded in $V_{\kappa,\lambda}$. If $\mathcal{B} \subseteq V_{\kappa,\lambda} \cap {}^{\omega}\omega$ is a family of reals of cardinality $< \kappa$, then there are $\alpha < \kappa$, $\zeta < \lambda$ such that $\mathcal{B} \subseteq V_{\alpha,\zeta}$. Since $\{\gamma : f(\gamma) = \alpha\}$ is cofinal in λ, there is $\zeta' > \zeta$ such that $f(\zeta') = \alpha$. Therefore $(\mathfrak{b} = \kappa)^{V_{\kappa,\lambda}}$.

Since $\mathfrak{a} \geq \mathfrak{b}$, we have $V_{\kappa,\lambda} \vDash \mathfrak{a} \geq \kappa$. Let $\mathcal{A} \subseteq V_{\kappa,\lambda} \cap [\omega]^{\omega}$ be an almost disjoint family of cardinality ν where $\kappa \leq \nu < \lambda$. Then there is $\zeta < \lambda$ such that $\zeta = \eta + 1$, $\zeta \equiv 0 \mod 3$ and $\mathcal{A} \subseteq V_{\kappa,\eta}$. Then by Lemma 4 of [4], in $V_{\kappa,\zeta}$ there is a real which has a finite intersection with all elements of $\mathcal{A}$ and so $\mathcal{A}$ is not maximal. Therefore $(\mathfrak{a} = \mathfrak{c} = \lambda)^{V_{\kappa,\lambda}}$.

To see that $(\mathfrak{s} = \lambda)^{V_{\kappa,\lambda}}$ note that if $S \subseteq V_{\kappa,\lambda} \cap [\omega]^{\omega}$, $|S| < \lambda$, then there is $\zeta < \lambda$, $\zeta = \eta + 1$, $\zeta \equiv 1 \mod 3$ such that $S \subseteq V_{\kappa,\eta}$. Then in $V_{\kappa,\zeta}$ there is a real which is not split by S (added by $\mathbb{M}_{\mathcal{U}_{\kappa,\eta}}$) and so S is not splitting. □

Thus we obtain the following theorem.

Theorem 21. *Let μ be a measurable cardinal, $\kappa < \lambda$ regular such that $\mu < \kappa$. Then there is a ccc generic extension in which $\mathfrak{b} = \kappa < \mathfrak{s} = \mathfrak{a} = \lambda$.*

6. Comments and questions

The converse consistency $\kappa = \mathfrak{s} < \mathfrak{b} = \mathfrak{a} = \lambda$ is well-known and standard. However of interest remain the following questions:
(1) Is it relatively consistent that $\mathfrak{b} < \mathfrak{a} < \mathfrak{s}$?
(2) Is it relatively consistent that $\mathfrak{b} < \mathfrak{s} < \mathfrak{a}$?
(3) Is it relatively consistent that $\mathfrak{b} < \mathfrak{s} = \mathfrak{a}$ without assuming a measurable?

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