GAMES ON BASE MATRICES

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ABSTRACT. We show that base matrices for $\mathcal{P}(\omega)/\text{fin}$ of regular height larger than $\mathfrak{h}$ necessarily have maximal branches which are not cofinal. The same holds for base matrices of height $\mathfrak{h}$ if $\mathfrak{t}_{\text{Spoiler}} < \mathfrak{h}$, where $\mathfrak{t}_{\text{Spoiler}}$ is a variant of $\mathfrak{t}$ which has been introduced in "Construction with opposition: cardinal invariants and games" by Brendle, Hrušák and Torres-Pérez.

1. INTRODUCTION

A forcing $\mathbb{P}$ is *δ-distributive* if any system of δ many maximal antichains has a common refinement. The *distributivity* of a forcing notion $\mathbb{P}$, denoted by $\mathfrak{h}(\mathbb{P})$, is the least λ such that $\mathbb{P}$ is not λ-distributive. In particular, $\mathfrak{h}(\mathcal{P}(\omega)/\text{fin})$ is the classical cardinal characteristic $\mathfrak{h}$. Note that $\mathfrak{h}(\mathbb{P})$ is actually the least λ such that there is a system of λ many *refining* maximal antichains without common refinement, which gives rise to the following definition:

Definition 1.1. We say that $\mathcal{A} = \{A_\xi \mid \xi < \lambda\}$ is a *distributivity matrix for $\mathbb{P}$ of height λ* if

(1) A_ξ is a maximal antichain in $\mathbb{P}$, for each $\xi < \lambda$,
(2) A_η refines A_ξ whenever $\eta \geq \xi$, i.e., for each $b \in A_\eta$ there exists $a \in A_\xi$ such that $b \leq a$, and
(3) there is no common refinement, i.e., there is no maximal antichain B which refines every A_ξ.

A special sort of distributivity matrices have been considered in the seminal paper [BPS80] of Balcar, Pelant, and Simon, where $\mathfrak{h}$ has been introduced:

Definition 1.2. A distributivity matrix $\{A_\xi \mid \xi < \lambda\}$ for $\mathbb{P}$ is a *base matrix* if $\bigcup_{\xi<\lambda} A_\xi$ is dense in $\mathbb{P}$, i.e., for each $p \in \mathbb{P}$ there is $\xi < \lambda$ and $a \in A_\xi$ such that $a \leq p$.

In [BPS80], the famous base matrix theorem has been shown: there exists a base matrix for $\mathcal{P}(\omega)/\text{fin}$ of height $\mathfrak{h}$. A more general version for a wider class of forcings has been given in [BDH15, Theorem 2.1].

Due to its refining structure, a distributivity matrix $\{A_\xi \mid \xi < \lambda\}$ can be viewed as a tree, with level ξ being A_ξ. Let us say that $\langle a_\xi \mid \xi < \delta\rangle$ is a *branch of the*

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distributivity matrix $\{A_\xi \mid \xi < \lambda\}$ if $a_\xi \in A_\xi$ for each $\xi < \delta$, and $a_\eta \leq a_\xi$ for each $\xi \leq \eta < \delta$. We say that the branch is *maximal* if there is no branch of $\{A_\xi \mid \xi < \lambda\}$ strictly extending it. If $\delta = \lambda$, the branch $\langle a_\xi \mid \xi < \delta\rangle$ is called *cofinal* in $\{A_\xi \mid \xi < \lambda\}$.

A *tower for* $\mathbb{P}$ is a decreasing sequence in $\mathbb{P}$ which does not have a lower bound in $\mathbb{P}$. The minimal length of a tower for $\mathbb{P}$ is denoted by $\mathfrak{t}(\mathbb{P})$. It is well-known that $\mathfrak{t}(\mathbb{P}) \leq \mathfrak{h}(\mathbb{P})$ (see Observation 2.2). Note that each maximal branch of a distributivity matrix for $\mathbb{P}$ which is not cofinal is a tower. So if there are no towers of length strictly less than $\mathfrak{h}(\mathbb{P})$, i.e., if $\mathfrak{t}(\mathbb{P}) = \mathfrak{h}(\mathbb{P})$, all maximal branches of a distributivity matrix of height $\mathfrak{h}(\mathbb{P})$ are cofinal.

The structure of base matrices for $\mathcal{P}(\omega)/\mathrm{fin}$ has been investigated in the literature. Dow showed that in the Mathias model, there exists a base matrix of height $\mathfrak{h}$ without cofinal branches (see [Dow89, Lemma 2.17]). It is actually consistent that *no* base matrix of height $\mathfrak{h}$ has cofinal branches. This was proved by Dordal by constructing a model in which $\mathfrak{h}$ does not belong to the tower spectrum (see [Dor87] or[1] [Dor89, Corollary 2.6]), and has later been shown to hold true also in the Mathias model.

In [FKW], the authors of this paper have shown that consistently there exists a distributivity matrix for $\mathcal{P}(\omega)/\mathrm{fin}$ of regular height larger than $\mathfrak{h}$ in which all maximal branches are cofinal.

In [Bre], Brendle has shown that if $\lambda \leq \mathfrak{c}$ is regular and greater or equal than the splitting number $\mathfrak{s}$ (or, alternatively, there exists no strictly $\subseteq^*$-decreasing sequence of length λ), then there exists a base matrix for $\mathcal{P}(\omega)/\mathrm{fin}$ of height λ. In particular, there always exists a base matrix of height $\mathfrak{c}$ provided that $\mathfrak{c}$ is regular. He mentions that in the Cohen and random models base matrices of height larger than $\mathfrak{h}$ necessarily have maximal branches which are not cofinal (in fact, there are no strictly $\subseteq^*$-decreasing sequences of length larger than ω_1).

We will show below that, in ZFC, any base matrix for $\mathcal{P}(\omega)/\mathrm{fin}$ of regular height larger than $\mathfrak{h}$ has maximal branches which are not cofinal.

2. Main Result

In Theorem 2.3 and Corollary 2.4, we give the connection between a variant of the tower number $\mathfrak{t}$ given in [BHTP19] and branches of base matrices. Let us first introduce the relevant game (where $\mathbb{P}$ is an arbitrary forcing notion):

Definition 2.1. Let $G^\delta_{\mathfrak{t}}(\mathbb{P})$ denote the *tower game of length* δ *for* $\mathbb{P}$:

I	a_0		a_1		$\dots$		a_μ		$a_{\mu+1}$			$\dots$
II		b_0		b_1		$\dots$		b_μ		$b_{\mu+1}$	$\dots$	

The players alternately pick conditions in $\mathbb{P}$ such that the resulting sequence is decreasing, i.e., $b_i \leq a_i$ and $a_j \leq b_i$ for every $i < j < \delta$. Player I starts the game and plays at limits μ. If Player I cannot play at limits (because the sequence played till then has no lower bound), the game ends and Player I wins immediately. If the

[1] Dordal's original model (in which $\mathfrak{c} = \omega_2$) is presented in [Dor87], whereas [Dor89, Corollary 2.6] is a more general result which also gives models satisfying $\mathfrak{h} = \mathfrak{c} > \omega_2$ (but is, interestingly enough, easier to prove).

game continuous for δ many steps, Player II wins if and only if there exists a $b \in \mathbb{P}$ with $b \leq a_i$ for every $i < \delta$.

This game has been considered in [BHTP19], where $\mathfrak{t}_{\text{Spoiler}}$ is defined to be the minimal δ such that Player II does not have a winning strategy in $G^{\delta}_{\mathfrak{t}}(\mathcal{P}(\omega)/\text{fin})$. It is mentioned in [BHTP19] that $\mathfrak{t} \leq \mathfrak{t}_{\text{Spoiler}} \leq \mathfrak{h}$. More generally, let $\mathfrak{t}_{\text{Spoiler}}(\mathbb{P})$ denote the minimal δ such that Player II does not have a winning strategy in $G^{\delta}_{\mathfrak{t}}(\mathbb{P})$. For the convenience of the reader we give the proof of the following well-known fact:

Observation 2.2.

$$\mathfrak{t}(\mathbb{P}) \leq \mathfrak{t}_{\text{Spoiler}}(\mathbb{P}) \leq \mathfrak{h}(\mathbb{P}).$$

Proof. First note that $\mathfrak{t}(\mathbb{P}) \leq \mathfrak{t}_{\text{Spoiler}}(\mathbb{P})$ follows directly from the definition.

Now fix a distributivity matrix $\{A_\alpha \mid \alpha < \mathfrak{h}(\mathbb{P})\}$ of height $\mathfrak{h}(\mathbb{P})$. In particular the conditions intersecting this matrix are not dense in $\mathbb{P}$. Let $a_0 \in \mathbb{P}$ such that no intersecting condition is stronger than a_0. Let us describe a winning strategy σ for Player I. Let $\prec$ be a well-order of $\mathbb{P}$. Let $\sigma(\langle\rangle) := a_0$. Assume Player II played b_α in the αth round of the game (for $\alpha < \mathfrak{h}(\mathbb{P})$). Since A_α is a maximal antichain, there exists $a \in A_\alpha$ which is compatible with b_α. Let $a_{\alpha+1}$ be the $\prec$-minimal witness for the compatibility and let $\sigma(\langle a_0, b_0, \ldots, b_\alpha\rangle) := a_{\alpha+1}$. At limits Player I picks the $\prec$-minimal lower bound of the sequence played so far, if there exists one.

If Player I follows the strategy σ, the game stops after at most $\mathfrak{h}(\mathbb{P})$ many rounds and Player I wins. Indeed, if there exists a run of the game of length $\mathfrak{h}(\mathbb{P})$ where Player I followed σ and has not won the game yet, then there exists a $b \in \mathbb{P}$ such that $b \leq a_{\alpha+1}$ for every $\alpha < \mathfrak{h}(\mathbb{P})$, which implies that b intersects the matrix and $b \leq a_0$, a contradiction.

So Player I has a winning strategy in $G^{\mathfrak{h}(\mathbb{P})}_{\mathfrak{t}}(\mathbb{P})$, therefore Player II does not have one and hence $\mathfrak{t}_{\text{Spoiler}}(\mathbb{P}) \leq \mathfrak{h}(\mathbb{P})$. □

Let us now state the main result and its consequences:

Theorem 2.3. *Let $\mathcal{A} = \{A_\xi \mid \xi < \lambda\}$ be a base matrix for $\mathbb{P}$ such that the length of any of its maximal branches has cofinality at least ν. Then $\nu \leq \mathfrak{t}_{\text{Spoiler}}(\mathbb{P})$.*

Proof. Fix a well-order $\prec$ on $\mathbb{P}$. Let $\delta < \nu$. We will show that Player II has a winning strategy in $G^{\delta}_{\mathfrak{t}}(\mathbb{P})$, which we define as follows. Assume Player I has played $a_i \in \mathbb{P}$. Then Player II picks the $\prec$-minimal $b_i \leq a_i$ with $b_i \in \bigcup_{\xi<\lambda} A_\xi$; this is possible since $\mathcal{A}$ is a base matrix. For each $\mu \leq \delta$, the following holds:

Claim. *The sequence $\langle b_i \mid i < \mu\rangle$ has a lower bound.*

Proof. We can assume that the sequence is not eventually constant. Moreover, we can assume that it is strictly decreasing. It is easy to check that there is a strictly increasing sequence $\langle \xi_i \mid i < \mu\rangle \subseteq \lambda$ with $b_i \in A_{\xi_i}$ for each $i < \mu$. The sequence $\langle b_i \mid i < \mu\rangle$ induces a branch of the matrix of length $\sup(\{\xi_i \mid i < \mu\})$. Since $\mu < \nu$ this branch is not maximal. Consequently, there exists an a (in the matrix) such that $a \leq b_i$ for each $i < \mu$. □

Therefore, for any $i < \delta$, Player I can play some a_i, so the game does not stop before length δ. Furthermore, there exists $b \leq a_i$ for every $i < \delta$, hence Player II wins the game, and the defined strategy is a winning strategy. □

Corollary 2.4. *Let $\mathcal{A} = \{A_\xi \mid \xi < \lambda\}$ be a base matrix for $\mathbb{P}$ of regular height λ all whose maximal branches are cofinal. Then $\lambda = \mathfrak{h}(\mathbb{P}) = \mathfrak{t}_{\text{Spoiler}}(\mathbb{P})$.*

In particular, any base matrix for $\mathbb{P}$ of regular height $\lambda > \mathfrak{h}(\mathbb{P})$ has a maximal branch which is not cofinal.

Proof. It follows from the above theorem that $\lambda \leq \mathfrak{t}_{\text{Spoiler}}(\mathbb{P})$. On the other hand, $\mathfrak{h}(\mathbb{P}) \leq \lambda$, because there exists a base matrix for $\mathbb{P}$ of height λ. Together with the fact that $\mathfrak{t}_{\text{Spoiler}}(\mathbb{P}) \leq \mathfrak{h}(\mathbb{P})$, the equality follows. □

Remark 2.5. Note that it follows from the above corollary that if λ is regular and $\mathbb{P}$ is not $<\lambda$-strategically closed (i.e., using the notation from [BHTP19], $\mathfrak{t}^*_{\text{Spoiler}}(\mathbb{P}) < \lambda$), then any base matrix for $\mathbb{P}$ of height λ has maximal branches which are not cofinal.

For the important case of $\mathcal{P}(\omega)/\text{fin}$, we can now derive the following:

Corollary 2.6. *Let $\mathcal{A} = \{A_\xi \mid \xi < \lambda\}$ be a base matrix for $\mathcal{P}(\omega)/\text{fin}$ of regular height λ, where $\lambda > \mathfrak{t}_{\text{Spoiler}}$ (i.e., $\lambda > \mathfrak{h}$ or $\mathfrak{h} > \mathfrak{t}_{\text{Spoiler}}$). Then for every $a \in \bigcup_{\xi<\lambda} A_\xi$, there is a maximal branch of $\mathcal{A}$ containing a which is not cofinal.*

Proof. Fix a in the matrix (i.e., $a \in \bigcup_{\xi<\lambda} A_\xi$). Let $\mathbb{P} := \{b \mid b \subseteq^* a\}$ be the part of $\mathcal{P}(\omega)/\text{fin}$ below a. Since $\mathcal{P}(\omega)/\text{fin}$ is homogenous, $\mathfrak{h}(\mathbb{P}) = \mathfrak{h}(\mathcal{P}(\omega)/\text{fin}) = \mathfrak{h}$ and $\mathfrak{t}_{\text{Spoiler}}(\mathbb{P}) = \mathfrak{t}_{\text{Spoiler}}$. Note that the part of $\mathcal{A}$ below a is a base matrix for $\mathbb{P}$ of height λ. Since $\lambda > \mathfrak{t}_{\text{Spoiler}}$, it follows by Corollary 2.4 that the part of $\mathcal{A}$ below a has maximal branches which are not cofinal. Any such branch induces a maximal branch of $\mathcal{A}$ containing a which is not cofinal. □

Remark 2.7. In [BHTP19] it has been shown that consistently $\mathfrak{t}_{\text{Spoiler}} < \mathfrak{h}$. In such models *every* base matrix (in particular every of height $\mathfrak{h}$) has (many) maximal branches which are not cofinal. It is an open question of [BHTP19] whether $\mathfrak{t} = \mathfrak{t}_{\text{Spoiler}}$ holds true in ZFC. Note that a positive answer would imply that either all base matrices of height $\mathfrak{h}$ only have cofinal maximal branches, or all base matrices of height $\mathfrak{h}$ have (many) maximal branches which are not cofinal, depending on whether $\mathfrak{t} = \mathfrak{h}$ or not.

Corollary 2.6 actually implies that distributivity matrices for $\mathcal{P}(\omega)/\text{fin}$ of regular height larger than $\mathfrak{h}$ cannot simultaneously have only cofinal maximal branches and be a base matrix. Therefore, Brendle's theorem from [Bre] together with Corollary 2.6 shows that there are distributivity matrices for $\mathcal{P}(\omega)/\text{fin}$ of regular height larger than $\mathfrak{h}$ with maximal branches which are not cofinal provided that $\mathfrak{c} > \mathfrak{h}$ is regular (or $\mathfrak{s} < \mathfrak{c}$).

On the other hand, Corollary 2.6 shows that the generic distributivity matrix of regular height larger than $\mathfrak{h}$ from [FKW] cannot be a base matrix because all its maximal branches are cofinal (this can also be seen by analyzing the forcing construction, see the end of [FKW, Section 7.1]).

Further note that in the model from [FKW], there are both kinds of distributivity matrices for $\mathcal{P}(\omega)/\text{fin}$ of regular height larger than $\mathfrak{h}$: matrices all whose maximal branches are cofinal, and matrices with maximal branches which are not cofinal.

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