Projective Maximal Families of Orthogonal Measures with Large Continuum

Vera Fischer

Kurt Gödel Research Center
University of Vienna

January 2012

The results which we are to consider, concern the definability of certain combinatorial objects on the real line and in particular the question of how low in the projective hierarchy such objects exist.

- (Mathias) There is no Σ^1_1 mad family in $[\omega]^\omega$.
- (Miller) If $V = L$, then there is a Π^1_1 mad family in $[\omega]^\omega$.

Theorem (L. Harrington)

The existence of $\boldsymbol{\Delta}^1_3$-definable wellorder of the reals is consistent with $\mathfrak{c}$ being as large as desired and MA.

Theorem (S. D. Friedman)

The existence of Δ^1_3-definable wellorder of the reals is consistent with $\mathfrak{c} = \omega_2$ and MA.

Note that Δ^1_3 wellorder is optimal for models of $\mathfrak{c} > \aleph_1$, since by a result of Mansfield if there is a Σ^1_2 definable w.o. on $\mathbb{R}$, then all reals are constructible.

The existence of a Δ^1_3-definable w.o. on the reals is consistent with each of the following:

- (F., Friedman) $\mathfrak{d} < \mathfrak{c} = \omega_2$; $\mathfrak{b} < \mathfrak{g} = \mathfrak{c} = \omega_2$; $\mathfrak{b} < \mathfrak{a} = \mathfrak{s} = \mathfrak{c} = \omega_2$;
- (Friedman, Zdomskyy) the existence of a Π^1_2 definable ω-mad family on $[\omega]^\omega$ together with $\mathfrak{b} = \mathfrak{c} = \omega_2$;
- (F., Friedman, Zdomskyy) the existence of a Π^1_2 definable ω-mad family on $[\omega]^\omega$ together with $\mathfrak{b} = \mathfrak{c} = \omega_3$;
- (F., Friedman, Zdomskyy) Martin's Axiom and $\mathfrak{c} = \omega_3$.

Let X be a Polish space, $P(X)$ be the Polish space of Borel probability measures on X, $\mu, \nu \in P(X)$.

- $\mu \perp \nu$ iff $\exists$ Borel set $B \subseteq X$ such that $\mu(B) = 0$ and $\nu(X \setminus B) = 0$.
- A set of measures $\mathcal{A} \subseteq P(X)$ is said to be orthogonal if whenever $\mu, \nu \in \mathcal{A}$ and $\mu \neq \nu$ then $\mu \perp \nu$.
- A maximal orthogonal family, or m.o. family, is an orthogonal family $\mathcal{A} \subseteq P(X)$ which is maximal under inclusion.

Theorem (Preiss, Rataj, 1985)

There are no analytic m.o. families.

Theorem (F., Törnquist, 2009)

If $V = L$ then there is a Π^1_1 m.o. family.

Theorem (F., S.D. Friedman, A. Törnquist)

There are generic extensions of L, in which there is a Δ^1_3-definable w.o. of the reals, a Π^1_2 definable m.o. family, there are no $\mathbf{\Sigma}^1_2$-definable m.o. families and each of the following holds:

- $\mathfrak{d} = \omega_1 < \mathfrak{c} = \omega_2$
- $\mathfrak{b} = \mathfrak{c} = \omega_3$

Let X be a Polish space and let $\mu, \nu \in P(X)$. Then

- μ is absolutely continuous with respect to ν, written $\mu \ll \nu$, if for all Borel $B \subseteq X$, if $\nu(B) = 0$ then $\mu(B) = 0$.
- $\mu, \nu \in P(2^\omega)$ are absolutely equivalent, written $\mu \approx \nu$, if $\mu \ll \nu$ and $\nu \ll \mu$.

- For $s \in 2^{<\omega}$, let $N_s = \{x \in 2^\omega : s \subseteq x\}$ and let $p(2^\omega)$ be the set of all $f : 2^{<\omega} \to [0,1]$ such that $f(\emptyset) = 1 \wedge (\forall s \in 2^{<\omega}) f(s) = f(s^\frown 0) + f(s^\frown 1)$.
- $p(2^\omega)$ and $P(2^\omega)$ are isomorphic via $f \mapsto \mu_f$ where $\mu_f \in P(2^\omega)$ is uniquely determined by $\mu_f(N_s) = f(s)$ for all $s \in 2^{<\omega}$. If $\mu = \mu_f$, then f is called the code for μ.

We describe now a way of coding $z \in 2^\omega$ into a $\mu \in P_c(2^\omega)$.

- For $s \in 2^{<\omega}$, let $t(s, \mu)$ be the lexicographically least $t \in 2^{<\omega}$ such that $s \subseteq t$, $\mu(N_{t^\frown 0}) > 0$ and $\mu(N_{t^\frown 1}) > 0$.
- Recursively define $(t_n^\mu)_{n\in\omega} \subseteq 2^{<\omega}$ by letting $t_0^\mu = \emptyset$ and $t_{n+1}^\mu = t(t_n^{\mu\frown} 0, \mu)$.
- For $f \in p_c(2^\omega)$ and $n \in \omega \cup \{\infty\}$ write t_n^f for $t_n^{\mu_f}$.

Define the relation $R \subseteq p_c(2^\omega) \times 2^\omega$ as follows: $R(f, z)$ holds iff for all $n \in \omega$ we have

$$\big(z(n) = 1 \longleftrightarrow (f(t_n^f{}^\frown 0) = \frac{2}{3} f(t_n^f) \wedge f(t_n^f{}^\frown 1) = \frac{1}{3} f(t_n^f))\big) \wedge$$
$$\big(z(n) = 0 \leftrightarrow f(t_n^f{}^\frown 0) = \frac{1}{3} f(t_n^f) \wedge f(t_n^f{}^\frown 1) = \frac{2}{3} f(t_n^f)\big).$$

Whenever $(f, z) \in R$ we say that f codes z.

Lemma (F.,Törnquist)

There is a recursive function $\bar{r} : p_c(2^\omega) \times 2^\omega \to p_c(2^\omega)$ *such that for all* $f \in p_c(2^\omega)$ *and* $z \in 2^\omega$ *we have:*

$$\mu_{\bar{r}(f,z)} \approx \mu_f \text{ and } R(\bar{r}(f,z), z).$$

Proposition

Let $a \in \mathbb{R}$. If there is either a Cohen or a random real over $L[a]$, then there is no $\Sigma^1_2(a)$ m.o. family.

We proceed with the construction of a generic extension of L in which there is a Δ^1_3 definable well order of the reals, there is a Π^1_2-definable m.o. family, there are no $\mathbf{\Sigma}^1_2$-definable m.o. families and $\mathfrak{d} = \omega_1 < \mathfrak{c} = \omega_2$.

A transitive ZF^- model is suitable if $\omega_2^{\mathcal{M}}$ exists and $\omega_2^{\mathcal{M}} = \omega_2^{L^{\mathcal{M}}}$.

If $S \subseteq \omega_1$ is a stationary, co-stationary set, let $Q(S)$ be the poset of all countable closed subsets of $\omega_1 \backslash S$ with the extension relation end-extension.

Then $Q(S)$ is $\omega_1 \backslash S$-proper, ω-distributive and adds a club disjoint from S.

Let $X \subseteq \omega_1$, $\phi(\omega_1, X)$ a Σ_1-sentence with parameters ω_1, X which is true in all suitable models containing ω_1 and X as elements. Let $\mathcal{L}(\phi)$ be the poset of all $r : |r| \to 2$, where $|r| \in Lim(\omega_1)$ such that

1. if $\gamma < |r|$ then $\gamma \in X$ iff $r(2\gamma) = 1$,
2. if $\gamma \leq |r|$, $\mathcal{M}$ is a countable, suitable model with $r \restriction \gamma \in \mathcal{M}$, $\gamma = \omega_1^{\mathcal{M}}$, then $\mathcal{M} \vDash \phi(\gamma, X \cap \gamma)$.

The extension relation is end-extension.

- $\mathcal{L}(\phi)$ has a countably closed dense subset,
- if G is $\mathcal{L}(\phi)$-generic, $\mathcal{M}$ is countable suitable with $(\bigcup G) \restriction \gamma \in \mathcal{M}$, where $\gamma = \omega_1^{\mathcal{M}}$, then $\mathcal{M} \vDash \phi(\gamma, X \cap \gamma)$.

Let $Y \subseteq \omega_1$ be generic over L, in $L[Y]$ cofinalities have not been changed. Let $\bar{\mu} = \{\mu_i\}_{i\in\omega_1}$ of L-countable ordinals such that μ_i is

$$\min\{\mu : \mu > \sup_{j<i} \mu_j, L_\mu[Y \cap i] \vDash ZF^-, L_\mu \vDash \omega \text{ is the largest cardinal}\}.$$

A real R codes Y below i if $\forall j < i$, $j \in Y$ iff $L_{\mu_j}[Y \cap j, R] \vDash ZF^-$.

- Let $\mathcal{C}(Y)$ be the p.o. of all perfect trees $T \subseteq 2^{<\omega}$ whose branches codes Y below $|T| = \min\{i : T \in L_{\mu_i}[Y \cap i]\}$.
- For T_0, T_1 in $\mathcal{C}(Y)$, let $T_0 \leq T_1$ iff $T_0 \subseteq T_1$.
- $\mathcal{C}(Y)$ is proper and ${}^{\omega}\omega$-bounding.
- $\mathcal{C}(Y)$ adds a real R which codes Y.

Fix a bookkeeping function $F : Lim'(\omega_2) \to L_{\omega_2}$ and a sequence $\vec{S} = (S_\beta : \beta < \omega_2)$ of almost disjoint stationary subsets of ω_1, which are Σ_1-definable over L_{ω_2} with parameter ω_1 and

- $F^{-1}(a)$ is unbounded in $Lim'(\omega_2)$ for every $a \in L_{\omega_2}$
- if $\mathcal{M}, \mathcal{N}$ are suitable such that $\omega_1^{\mathcal{M}} = \omega_1^{\mathcal{N}}$ then $F^{\mathcal{M}}, \vec{S}^{\mathcal{M}}$ agree with $F^{\mathcal{N}}$, $\vec{S}^{\mathcal{N}}$ on $\omega_2^{\mathcal{M}} \cap \omega_2^{\mathcal{N}}$.

Also if $\mathcal{M}$ is suitable and $\omega_1^{\mathcal{M}} = \omega_1$ then $F^{\mathcal{M}}, \bar{S}^{\mathcal{M}}$ equal the restrictions of F, $\vec{S}$ to the ω_2 of $\mathcal{M}$.

- Assume that all names for reals are nice and that for $\alpha < \beta < \omega_2$, all $\mathbb{P}_\alpha$-names for reals precede in the canonical wellorder $<_L$ of L all $\mathbb{P}_\beta$-names for reals, which are not $\mathbb{P}_\alpha$-names.
- If $x \in L[G_\alpha] \cap {}^\omega\omega$ let σ_x^α be the $<_L$-least $\mathbb{P}_\gamma$-name for x, where $\gamma \leq \alpha$ is least so that x has a $\mathbb{P}_\gamma$-name.
- If $x, y \in L[G_\alpha] \cap {}^\omega\omega$ let $x <_\alpha y$ iff $\sigma_x^\alpha <_L \sigma_y^\alpha$.
- Then $<^G = \bigcup_{\alpha<\omega_2} \dot{<}_\alpha^G$ will be the desired wellorder of the reals in $L[G]$.

For every $\alpha \in \omega_2$, let

- W_α be the L-least subset of ω_1 coding α.
- $\dot{F}^0_\alpha$, $\dot{F}^1_\alpha$ be $\mathbb{P}_\alpha$-names for nicely definable bijections

$$F^0_\alpha : 2^\omega \to p_c(2^\omega), F^1_\alpha : (2^\omega)^\omega \to 2^\omega$$

such that for all $i \in \{0, 1\}$ and $\alpha < \beta < \omega_2$ in $L^{\mathbb{P}_\beta}$ we have $F^i_\alpha \subseteq F^i_\beta$ (e.g. take $(F^0_\alpha)^{-1}$, F^1_α to be Cantor diagonalization).

We proceed with the definition of the poset. Let $\mathbb{P}_0$ be the trivial poset. Suppose $\mathbb{P}_\alpha$, $\langle O_\gamma : \gamma < \alpha \rangle$ and $\langle A_\gamma : \gamma < \alpha \rangle$ have been defined. Let G_α be a $\mathbb{P}_\alpha$-generic filter.

Suppose

$$\alpha \in Lim'(\omega_2) = \{\alpha \in Lim(\omega_2) : \alpha = \omega \cdot \omega \cdot \alpha'' \text{ for some } \alpha'' \geq 0\}.$$

We will define $\mathbb{P}_{\alpha+\gamma}$ for $\gamma \in \omega \cdot \omega$ recursively as follows:

Case A.1. If $F(\alpha) = \{\sigma_x^\alpha, \sigma_y^\alpha\}$, then let $x = \sigma_x^\alpha[G_\alpha]$, $y = \sigma_y^\alpha[G_\alpha]$

- if $m \in \Delta(x * y)$ let $\mathbb{Q}_{\alpha+m} = Q(S_{\alpha+m})$ and if $m \notin \Delta(x * y)$ let $\mathbb{Q}_{\alpha+m}$ be the random real forcing.
- In $L^{\mathbb{P}_{\alpha+\omega}}$ let $X_{\alpha+\omega} \subseteq \omega_1$, coding: W_α, (x, y), a level of L in which α has size at most ω_1 and the generic $G_{\alpha+\omega}$, which can be regarded as a subset of an element of L_{ω_2}.
- Let $\mathbb{K}^1_{\alpha+\omega} = \mathcal{L}(\phi_{\alpha+\omega})$, where $\phi_{\alpha+\omega} = \phi_{\alpha+\omega}(\omega_1, X_{\alpha+\omega})$ is the Σ_1-sentence which holds iff $X_{\alpha+\omega}$ codes a $W \subseteq \omega_1$ and a pair (x, y) of reals, such that W is the L-least code for an ordinal $\bar{\alpha} < \omega_2$ and $S_{\bar{\alpha}+m}$ is non-stationary for $m \in \Delta(x * y)$.

- Let $Y_{\alpha+\omega}$ be $\mathbb{K}^1_{\alpha+\omega}$-generic over $L[G_{\alpha+\omega}]$. The even part of $Y_{\alpha+\omega}$ codes $X_{\alpha+\omega}$ and so codes the generic $G_{\alpha+\omega}$. Then in $L[Y_{\alpha+\omega}]$, let $\mathbb{K}^2_{\alpha+\omega} = \mathcal{C}(Y_{\alpha+\omega})$ and let $R_{\alpha+\omega}$ be the real added by $\mathbb{K}^2_{\alpha+\omega}$.
- Then in particular $R_{\alpha+\omega}$ is added by $\mathbb{Q}_{\alpha+\omega} = \mathbb{K}^1_{\alpha+\omega} * \dot{\mathbb{K}}^2_{\alpha+\omega}$ and codes the stationary kill corresponding to $x < y$.
- For every $\gamma \in [\alpha + \omega + 1, \alpha + \omega \cdot \omega)$ let $\mathbb{Q}_{\alpha+\gamma}$ be the random real forcing.

Case A.2. If $F(\alpha) = \{\sigma_x^\alpha\}$, then let $x = \sigma_x^\alpha[G_\alpha]$, $f = F_\alpha^0(x)$. If f is not orthogonal to $\bigcup_{\gamma<\alpha} O_\gamma$, let $\mathbb{Q}_{\alpha+\gamma}$ be the random real forcing, for all $\gamma \in \omega \cdot \omega$. If f is orthogonal to $\bigcup_{\gamma<\alpha} O_\gamma$, define $\mathbb{Q}_{\alpha+\gamma}$ for $\gamma \in \omega \cdot \omega$ recursively as follows:

- Define $\mathbb{Q}_{\alpha+\omega}$ just as in Case A.1, but instead of $\Delta(x * y)$ use $\Delta(x)$ to determine which stationary sets will be destroyed. Let $R_{\alpha+\omega}$ be the generic real added by $\mathbb{Q}_{\alpha+\omega}$.

Suppose $\mathbb{P}_{\alpha+\omega\cdot n+1}$ has been defined and $\mathbb{Q}_{\alpha+\omega\cdot n}$ adds a real $R_{\alpha+\omega\cdot n}$ generic over $L^{\mathbb{P}_{\alpha+\omega\cdot n}}$. Define $\mathbb{Q}_{\alpha+\omega\cdot n+m}$ for $m \geq 1$ as follows:

- If $m-1 \in \Delta(R_{\alpha+\omega\cdot n})$ let $\mathbb{Q}_{\alpha+\omega\cdot n+m} = Q(S_{\alpha+\omega\cdot n+(m-1)})$ and if $m-1 \in \Delta(R_{\alpha+\omega\cdot n})$ let $\mathbb{Q}_{\alpha+\omega\cdot n+m}$ be the random real forcing.
- In $L[G_{\alpha+\omega\cdot n+\omega}]$ let $X_{\alpha+\omega\cdot n+\omega} \subseteq \omega_1$ coding the sets $W_{\alpha+\omega\cdot j}$ where $j \leq n+1$, the real $R_{\alpha+\omega\cdot n}$, a level of L in which $\alpha+\omega\cdot n+\omega$ has size at most ω_1 and the generic $G_{\alpha+\omega\cdot n+\omega}$.

- Let $\mathbb{K}^1_{\alpha+\omega\cdot(n+1)} = \mathcal{L}(\phi^{n+1}_\alpha)$, where $\phi^{n+1}_\alpha(\omega_1, X_{\alpha+\omega\cdot(n+1)})$ is the Σ_1-sentence which holds iff $X_{\alpha+\omega\cdot(n+1)}$ codes a tuple $\langle \bar{W}_j \rangle_{j\leq n+1}$ of subsets of ω_1 and a real z, such that $\bar{W}_{n+1}$ is the L-least code for an ordinal $\bar{\alpha} = \bar{\alpha}_{n+1}$, $\bar{W}_j$ is the L-least code for the largest limit $\bar{\alpha}_j$ strictly smaller than $\bar{\alpha}_{j+1}$ for $j \leq n$, and for every $m \in \Delta(z)$, the set $S_{\bar{\alpha}+m}$ is non-stationary.

- Let $Y_{\alpha+\omega\cdot(n+1)}$ be $\mathbb{K}^1_{\alpha+\omega\cdot(n+1)}$-generic over $L[G_{\alpha+\omega\cdot(n+1)}]$. In $L[Y_{\alpha+\omega\cdot(n+1)}]$ let $\mathbb{K}^2_{\alpha+\omega\cdot(n+1)} = \mathcal{C}(Y_{\alpha+\omega\cdot(n+1)})$ and let $R_{\alpha+\omega\cdot(n+1)}$ be the generic real added by $\mathbb{K}^2_{\alpha+\omega\cdot(n+1)}$.
- Let $\mathbb{Q}_{\alpha+\omega\cdot(n+1)} = \mathbb{K}^1_{\alpha+\omega\cdot(n+1)} * \dot{\mathbb{K}}^2_{\alpha+\omega\cdot(n+1)}$.
- Then in particular $R_{\alpha+\omega\cdot(n+1)}$ is added by $\mathbb{Q}_{\alpha+\omega\cdot(n+1)}$.

In $L^{\mathbb{P}_{\alpha+\omega\cdot\omega}}$ let $u_0^\alpha = x$, $u_n^\alpha = R_{\alpha+\omega\cdot n}$ for $n \geq 1$. Let $\vec{u}_\alpha = (u_n^\alpha)_{n\in\omega}$ and let

$$g_\alpha = \bar{r}(F^0_{\alpha+\omega\cdot\omega}(u_0^\alpha), F^1_{\alpha+\omega\cdot\omega}((u_n^\alpha)_{n\geq 1}))$$

For every $\gamma \in [\alpha, \alpha+\omega\cdot\omega)$ let $O_\gamma = \{g_\alpha\}$. For $n \in \omega$, let $A_{\alpha+\omega\cdot n} = \alpha + \omega\cdot n + \omega \backslash \Delta(u_n^\alpha)$ and for γ successor in $[\alpha, \alpha+\omega\cdot\omega)$, let $A_\gamma = \emptyset$.

Case B. If $\alpha \in Lim(\omega_2) \setminus Lim'(\omega_2)$, or α is a successor which can not be presented in the form $\alpha' + \omega \cdot n + m$ for some $\alpha' \in Lim'(\omega_2)$, $n, m \in \omega$, then let $\mathbb{Q}_\alpha$ the random real forcing. Let $O_\alpha = A_\alpha = \emptyset$.

With this the recursive construction of $\mathbb{P}_{\omega_2}$ is complete. In $L^{\mathbb{P}_{\omega_2}}$, let $O = \bigcup_{\alpha < \omega_2} O_\alpha$, $F^0 = \bigcup_{\alpha \in \omega_2} F^0_\alpha$, $F^1 = \bigcup_{\alpha \in \omega_2} F^1_\alpha$ and for $\vec{z} = (z_n)_{n \in \omega} \in (2^\omega)^\omega$ let $\mathcal{R}(\vec{z}) = \bar{r}(F^0(z_0), F^1((z_n)_{n \geq 1}))$.

Lemma

If G is $\mathbb{P}_{\omega_2}$-generic and $\xi \in \bigcup_{\xi\in\omega_2} \dot{A}^G_\xi$, then S_ξ is stationary in $L[G]$.

Lemma

Let G be $\mathbb{P}_{\omega_2}$-generic and let $g = \mathcal{R}(\vec{z})$, $\vec{z} = (z_n)_{n\in\omega}$. Then $g \in O$ if and only if for every countable suitable model $\mathcal{M}$ such that $g \in \mathcal{M}$, there is $\bar{\alpha} < \omega_2^{\mathcal{M}}$ such that for all $n \in \omega$ the set $S_{\alpha+\omega\cdot n+m}$ is non-stationary in $(L[z_{n+1}])^{\mathcal{M}}$ for $m \in \Delta(z_n)$.

Proof:

If $g \in O$, then $g = g_\alpha = \mathcal{R}(\vec{u}_\alpha)$ for some α. Let $\mathcal{M}$ be a countable suitable such that $g \in \mathcal{M}$. But then $\vec{u}_\alpha \in \mathcal{M}$, and so $Y_{\alpha+\omega\cdot n} \cap \omega_1^{\mathcal{M}} \in \mathcal{M}$ for all n. Thus $X_{\alpha+\omega\cdot n} \cap \omega_1^{\mathcal{M}}$ is also an element of $\mathcal{M}$. By definition of $\mathcal{L}(\phi^n_{\alpha+\omega\cdot n})$, the set $X_{\alpha+\omega\cdot n} \cap \omega_1^{\mathcal{M}}$ codes a tuple $\langle W^n_j \rangle_{j \leq n}$ of subsets of ω_1 such that W^n_n is the L-least code of an ordinal α^n_n in ω_2 and for $j < n$ the set W^n_j is the L-least code for the largest limit ordinal α^n_j below α^n_{j+1}. It remains to observe that $W^n_j = W^m_j$ for $j \leq n < m$ and so α^n_0 does not depend on n (i.e. we have proper decoding). But then $\bar{\alpha} = \alpha^n_0$ is the desired ordinal.

Suppose that for every countable suitable model $\mathcal{M}$ such that $g \in \mathcal{M}$, there is $\bar{\alpha} < \omega_2^{\mathcal{M}}$ with the desired properties. By the Löwenheim-Skolem theorem, the same holds in $\mathbb{H}_\Theta^{\mathbb{P}_{\omega_2}}$ for some large Θ. Therefore there is $\alpha < \omega_2^{\mathcal{M}}$ such that for all $n \in \omega$, the set $S_{\alpha+\omega\cdot n+m}$ is non-stationary iff $m \in \Delta(z_n)$. Since there is no accidental stationary kill, $z_n = u_n^\alpha$ for all n, which implies that $g = \mathcal{R}(\vec{u}_\alpha) = g_\alpha \in O$. $\square$

We will show that O is maximal in $p_c(2^\omega)$. Suppose in $L^{\mathbb{P}_{\omega_2}}$ there is a code f of a measure orthogonal to every measure in the family $\bar{O} = \{\mu_g : g \in O\}$. Choose α minimal in $Lim'(\omega_2)$ such that $f \in L[G_\alpha]$ and let $x = (F^0_\alpha)^{-1}(f)$. Since $F^{-1}(\sigma^\alpha_x)$ is unbounded, there is $\alpha' \geq \alpha$ in $Lim'(\omega_2)$ such that $F(\alpha') = \sigma^\alpha_x (= \sigma^{\alpha'}_x)$. But then $g_{\alpha'}$ is a code of a measure equivalent to μ_f, which is a contradiction. To obtain a Π^1_2-definable m.o. family in $L^{\mathbb{P}_{\omega_2}}$, consider the union of $\bar{O}$ with the set of all point measures.

Similarly one can show that $<$ has a Δ^1_3 definition. More precisely, we have:

Lemma

Let G be $\mathbb{P}_{\omega_2}$-generic and let x, y be reals in $L[G]$. Then $x < y$ iff there is a real R such that for every countable suitable model $\mathcal{M}$, containing R as an element there is an $\bar{\alpha} < \omega_2^{\mathcal{M}}$ such that $S_{\bar{\alpha}+m}$ is non-stationary iff $m \in \Delta(x * y)$.

- Since for every real $a \in L^{\mathbb{P}_{\omega_2}}$ there is a random real over L, in $L^{\mathbb{P}_{\omega_2}}$ there are no $\mathbf{\Sigma}^1_2$ m.o. families.
- The dominating number $\mathfrak{d}$ remains ω_1 in $L^{\mathbb{P}_{\omega_2}}$, since the countable support iteration of S-proper ${}^\omega\omega$-bounding posets is ${}^\omega\omega$-bounding.