Projective Maximal Families of Orthogonal Measures with Large Continuum

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The results which we are to consider, concern the definability of certain combinatorial objects on the real line and in particular the question of how low in the projective hierarchy such objects exist.

- (Mathias) There is no Σ^1_1 mad family in $[\omega]^\omega$.
- (Miller) If $V = L$, then there is a Π^1_1 mad family in $[\omega]^\omega$.

Theorem (L. Harrington)

The existence of $\boldsymbol{\Delta}^1_3$-definable wellorder of the reals is consistent with $\mathfrak{c}$ being as large as desired and MA.

Theorem (S. D. Friedman)

The existence of Δ^1_3-definable wellorder of the reals is consistent with $\mathfrak{c} = \omega_2$ and MA.

Note that Δ^1_3 wellorder is optimal for models of $\mathfrak{c} > \aleph_1$, since by a result of Mansfield if there is a Σ^1_2 definable w.o. on $\mathbb{R}$, then all reals are constructible.

The existence of a Δ^1_3-definable w.o. on the reals is consistent with each of the following:

- (F., Friedman) $\mathfrak{d} < \mathfrak{c} = \omega_2$; $\mathfrak{b} < \mathfrak{g} = \mathfrak{c} = \omega_2$; $\mathfrak{b} < \mathfrak{a} = \mathfrak{s} = \mathfrak{c} = \omega_2$;
- (Friedman, Zdomskyy) the existence of a Π^1_2 definable ω-mad family on $[\omega]^\omega$ together with $\mathfrak{b} = \mathfrak{c} = \omega_2$;
- (F., Friedman, Zdomskyy) the existence of a Π^1_2 definable ω-mad family on $[\omega]^\omega$ together with $\mathfrak{b} = \mathfrak{c} = \omega_3$;
- (F., Friedman, Zdomskyy) Martin's Axiom and $\mathfrak{c} = \omega_3$.

Let X be a Polish space, and let $P(X)$ be the Polish space of Borel probability measures on X.

- If $\mu, \nu \in P(X)$ then μ and ν are said to be orthogonal, written $\mu \perp \nu$, if there is a Borel set $B \subseteq X$ such that $\mu(B) = 0$ and $\nu(X \setminus B) = 0$.
- A set of measures $\mathcal{A} \subseteq P(X)$ is said to be orthogonal if whenever $\mu, \nu \in \mathcal{A}$ and $\mu \neq \nu$ then $\mu \perp \nu$.
- A maximal orthogonal family, or m.o. family, is an orthogonal family $\mathcal{A} \subseteq P(X)$ which is maximal under inclusion.

Theorem (Preiss, Rataj, 1985)

There are no analytic m.o. families.

Theorem (F., Törnquist, 2009)

If $V = L$ then there is a Π^1_1 m.o. family.

We study Π^1_2 m.o. families in the context of $\mathfrak{c} \geq \omega_2$, with the additional requirement that there is a Δ^1_3-definable wellorder of $\mathbb{R}$.

Theorem (F., Freidman, Törnquist, 2011)

It is consistent with $\mathfrak{c} = \mathfrak{b} = \omega_3$ that there is a Δ^1_3-definable well order of the reals, a Π^1_2 definable maximal orthogonal family of measures and there are no $\mathbf{\Sigma}^1_2$-definable maximal sets of orthogonal measures.

There is nothing special about $\mathfrak{c} = \omega_3$. In fact the same result can be obtained for any reasonable value of $\mathfrak{c}$.

Theorem (F., Freidman, Törnquist, 2011)

It is consistent with $\mathfrak{b} = \omega_1$, $\mathfrak{c} = \omega_2$ that there is a Δ^1_3-definable wellorder of the reals, a Π^1_2 definable maximal orthogonal family of measures and there are no $\mathbf{\Sigma}^1_2$-definable maximal sets of orthogonal measures.

Let X be a Polish space. Recall that if $\mu, \nu \in P(X)$ then μ is absolutely continuous with respect to ν, written $\mu \ll \nu$, if for all Borel subsets of X we have that $\nu(B) = 0$ implies that $\mu(B) = 0$. Two measures $\mu, \nu \in P(2^\omega)$ are called absolutely equivalent, written $\mu \approx \nu$, if $\mu \ll \nu$ and $\nu \ll \mu$.

For $s \in 2^{<\omega}$, let $N_s = \{x \in 2^\omega : s \subseteq x\}$. Let $p(2^\omega)$ be the set of all $f : 2^{<\omega} \to [0,1]$ such that

$$f(\emptyset) = 1 \wedge (\forall s \in 2^{<\omega}) f(s) = f(s^\frown 0) + f(s^\frown 1)\}.$$

The spaces $p(2^\omega)$ and $P(2^\omega)$ are isomorphic via the $f \mapsto \mu_f$ where $\mu_f \in P(2^\omega)$ is the measure uniquely determined by $\mu_f(N_s) = f(s)$ for all $s \in 2^{<\omega}$. The unique real $f \in p(2^\omega)$ such that $\mu = \mu_f$ is called the code for μ.

Let $P_c(2^\omega)$ be the set of all non-atomic measures and $p_c(2^\omega)$ the set of all codes for non-atomic measures. We describe now a way of coding a given real $z \in 2^\omega$ into a measure $\mu \in P_c(2^\omega)$.

- Let $\mu \in P_c(2^\omega)$, $s \in 2^{<\omega}$. Then let $t(s,\mu)$ be the lexicographically least $t \in 2^{<\omega}$ such that $s \subseteq t$, $\mu(N_{t^\frown 0}) > 0$ and $\mu(N_{t^\frown 1}) > 0$, if it exists and otherwise we let $t(s,\mu) = \emptyset$.
- Define recursively $t_n^\mu \in 2^{<\omega}$ by letting $t_0^\mu = \emptyset$ and $t_{n+1}^\mu = t(t_n^{\mu\frown}0, \mu)$. Since μ is non-atomic, we have $\mathrm{lh}(t_{n+1}^\mu) > \mathrm{lh}(t_n^\mu)$. Let $t_\infty^\mu = \bigcup_{n=0}^\infty t_n^\mu$.
- For $f \in p_c(2^\omega)$ and $n \in \omega \cup \{\infty\}$ we will write t_n^f for $t_n^{\mu_f}$. Clearly the sequence $(t_n^f : n \in \omega)$ is recursive in f.

Define the relation $R \subseteq p_c(2^\omega) \times 2^\omega$ as follows: $R(f, z)$ holds iff for all $n \in \omega$ we have

$$\big(z(n) = 1 \longleftrightarrow (f(t_n^f{}^\frown 0) = \frac{2}{3} f(t_n^f) \wedge f(t_n^\frown 1) = \frac{1}{3} f(t_n))\big) \wedge$$

$$\big(z(n) = 0 \leftrightarrow f(t_n^f{}^\frown 0) = \frac{1}{3} f(t_n^f) \wedge f(t_n^f{}^\frown 1) = \frac{2}{3} f(t_n^f)\big).$$

Whenever $(f, z) \in R$ we say that f codes z.

Lemma (Coding Lemma)

There is a recursive function $G : p_c(2^\omega) \times 2^\omega \to p_c(2^\omega)$ such that for all $f \in p_c(2^\omega)$ and $z \in 2^\omega$ we have:

- $\mu_{G(f,z)} \approx \mu_f$, *and*
- $R(G(f,z), z)$

Proposition

Let $a \in \mathbb{R}$ and suppose that there either is a Cohen real over $L[a]$ or there is a random real over $L[a]$. Then there is no $\Sigma^1_2(a)$ m.o. family.

We proceed with the construction of a generic extension of L in which there is a Δ^1_3 definable well order of the reals, there is a Π^1_2-definable m.o. family, there are no $\mathbf{\Sigma}^1_2$-definable m.o. families and $\mathfrak{c} = \aleph_3$.

A transitive ZF^- model is suitable if $\omega_3^{\mathcal{M}}$ exists and $\omega_3^{\mathcal{M}} = \omega_3^{L^{\mathcal{M}}}$. If $\mathcal{M}$ is suitable then also $\omega_1^{\mathcal{M}} = \omega_1^{L^{\mathcal{M}}}$ and $\omega_2^{\mathcal{M}} = \omega_2^{L^{\mathcal{M}}}$.

Fix a $\diamondsuit_{\omega_2}(cof(\omega_1))$ sequence $\langle G_\xi : \xi \in \omega_2 \cap cof(\omega_1)\rangle$ which is Σ_1-definable over L_{ω_2}.

- For $\alpha < \omega_3$, let W_α be the L-least subset of ω_2 coding α and
- let $S_\alpha = \{\xi \in \omega_2 \cap cof(\omega_1) : G_\xi = W_\alpha \cap \xi \neq \emptyset\}$.

Then $\vec{S} = \langle S_\alpha : 1 < \alpha < \omega_3\rangle$ is a sequence of stationary subsets of $\omega_2 \cap cof(\omega_1)$, which are mutually almost disjoint.

For every α such that $\omega \leq \alpha < \omega_3$ shoot a club C_α disjoint from S_α via the poset $\mathbb{P}^0_\alpha$, consisting of all closed subsets of ω_2 which are disjoint from S_α with the extension relation being end-extension, and let $\mathbb{P}^0 = \prod_{\alpha<\omega_3} \mathbb{P}^0_\alpha$ be the direct product of the $\mathbb{P}^0_\alpha$'s with supports of size ω_1, where for $\alpha \in \omega$, $\mathbb{P}^0_\alpha$ is the trivial poset. Then $\mathbb{P}^0$ is countably closed, ω_2-distributive and ω_3-c.c.

For every α such that $\omega \leq \alpha < \omega_3$ let $D_\alpha \subseteq \omega_3$ be a set coding the triple $\langle C_\alpha, W_\alpha, W_\gamma \rangle$ where γ is the largest limit ordinal $\leq \alpha$. Let

$$E_\alpha = \{\mathcal{M} \cap \omega_2 : \mathcal{M} \prec L_{\alpha+\omega_2+1}[D_\alpha], \omega_1 \cup \{D_\alpha\} \subseteq \mathcal{M}\}.$$

Then E_α is a club on ω_2. Choose $Z_\alpha \subseteq \omega_2$ such that:

- $Even(Z_\alpha) = D_\alpha$, where $Even(Z_\alpha) = \{\beta : 2 \cdot \beta \in Z_\alpha\}$, and
- if $\beta < \omega_2$ is the $\omega_2^{\mathcal{M}}$ for some suitable model $\mathcal{M}$ such that $Z_\alpha \cap \beta \in \mathcal{M}$, then $\beta \in E_\alpha$.

Then we have

$(*)_\alpha$: If $\beta < \omega_2$, $\mathcal{M}$ is a suitable model such that $\omega_1 \subset \mathcal{M}$, $\omega_2^{\mathcal{M}} = \beta$, and $Z_\alpha \cap \beta \in \mathcal{M}$, then $\mathcal{M} \models \psi(\omega_2, Z_\alpha \cap \beta)$, where $\psi(\omega_2, X)$ is the formula "$Even(X)$ codes a triple $\langle \bar{C}, \bar{W}, \bar{\bar{W}} \rangle$, where $\bar{W}$ and $\bar{\bar{W}}$ are the L-least codes of ordinals $\bar{\alpha}, \bar{\bar{\alpha}} < \omega_3$ such that $\bar{\bar{\alpha}}$ is the largest limit ordinal not exceeding $\bar{\alpha}$, and $\bar{C}$ is a club in ω_2 disjoint from $S_{\bar{\alpha}}$".

Similarly to $\vec{S}$ define a sequence $\vec{A} = \langle A_\xi : \xi < \omega_2 \rangle$ of stationary subsets of ω_1 using the "standard" $\diamondsuit$-sequence. Code Z_α by a subset X_α of ω_1 with the poset $\mathbb{P}^1_\alpha$ consisting of all pairs $\langle s_0, s_1 \rangle \in [\omega_1]^{<\omega_1} \times [Z_\alpha]^{<\omega_1}$ where $\langle t_0, t_1 \rangle \leq \langle s_0, s_1 \rangle$ iff s_0 is an initial segment of t_0, $s_1 \subseteq t_1$ and $t_0 \backslash s_0 \cap A_\xi = \emptyset$ for all $\xi \in s_1$.

Then X_α satisfies the following condition:

$(**)_\alpha$: If $\omega_1 < \beta \leq \omega_2$ and $\mathcal{M}$ is a suitable model such that $\omega_2^{\mathcal{M}} = \beta$ and $\{X_\alpha\} \cup \omega_1 \subset \mathcal{M}$, then $\mathcal{M} \vDash \phi(\omega_1, \omega_2, X_\alpha)$, where $\phi(\omega_1, \omega_2, X)$ is the formula: " Using the sequence $\vec{A}$, X almost disjointly codes a subset $\bar{Z}$ of ω_2, such that $Even(\bar{Z})$ codes a triple $\langle \bar{C}, \bar{W}, \bar{\bar{W}} \rangle$, where $\bar{W}$ and $\bar{\bar{W}}$ are the L-least codes of ordinals $\bar{\alpha}, \bar{\bar{\alpha}} < \omega_3$ such that $\bar{\bar{\alpha}}$ is the largest limit ordinal not exceeding $\bar{\alpha}$, and $\bar{C}$ is a club in ω_2 disjoint from $S_{\bar{\alpha}}$".

Let $\mathbb{P}^1 = \prod_{\alpha<\omega_3} \mathbb{P}^1_\alpha$, where $\mathbb{P}^1_\alpha$ is the trivial poset for all $\alpha \in \omega$, with countable support. Then $\mathbb{P}^1$ is countably closed and has the ω_2-c.c.

Now we shall force a localization of the X_α's. Fix ϕ as in $(**)_\alpha$.

Definition

Let $X, X' \subset \omega_1$ be such that $\phi(\omega_1, \omega_2, X)$, $\phi(\omega_1, \omega_2, X')$ hold in any suitable $\mathcal{M}$ with $\omega_1^{\mathcal{M}} = \omega_1^L$, X, X' in $\mathcal{M}$. Denote by $\mathcal{L}(X, X')$ the p.o. of all $r : |r| \to 2$, where $|r| \in \mathrm{Lim}(\omega_1)$, such that:

1. if $\gamma < |r|$ then $\gamma \in X$ iff $r(3\gamma) = 1$
2. if $\gamma < |r|$ then $\gamma \in X'$ iff $r(3\gamma + 1) = 1$
3. if $\gamma \leq |r|$, $\mathcal{M}$ is countable, suitable, such that $r \restriction \gamma \in \mathcal{M}$ and $\gamma = \omega_1^{\mathcal{M}}$, then $\mathcal{M} \vDash \phi(\omega_1, \omega_2, X \cap \gamma) \wedge \phi(\omega_1, \omega_2, X' \cap \gamma)$.

The extension relation is end-extension.

Set $\mathbb{P}^2_{\alpha+m} = \mathcal{L}(X_{\alpha+m}, X_\alpha)$ for every $\alpha \in Lim(\omega_3)\backslash\{0\}$ and $m \in \omega$. Let $\mathbb{P}^2_{0+m}$ be the trivial poset for every $m \in \omega$ and let

$$\mathbb{P}^2 = \prod_{\alpha \in Lim(\omega_3)} \prod_{m \in \omega} \mathbb{P}^2_{\alpha+m}$$

with countable supports. By the Δ-system Lemma in $L^{\mathbb{P}^0 * \mathbb{P}^1}$ the poset $\mathbb{P}^2$ has the ω_2-c.c.

Observe that the poset $\mathbb{P}^2_{\alpha+m}$, where $\alpha > 0$, produces a generic function from ω_1 (of $L^{\mathbb{P}^0 * \mathbb{P}^1}$) into 2, which is the characteristic function of a subset $Y_{\alpha+m}$ of ω_1 with the following property:

$(***)_\alpha$: For every $\beta < \omega_1$ and any suitable $\mathcal{M}$ such that $\omega_1^{\mathcal{M}} = \beta$ and $Y_{\alpha+m} \cap \beta$ belongs to $\mathcal{M}$, we have

$$\mathcal{M} \vDash \phi(\omega_1, \omega_2, X_{\alpha+m} \cap \beta) \wedge \phi(\omega_1, \omega_2, X_\alpha \cap \beta).$$

Lemma

The poset $\mathbb{P}_0 := \mathbb{P}^0 * \mathbb{P}^1 * \mathbb{P}^2$ *is* ω*-distributive.*

For $\alpha : 1 \leq \alpha < \omega_3$ we will say that there is a stationary kill of S_α, if there is a closed unbounded set C disjoint from S_α. We will say that the stationary kill of S_α is coded by a real, if there is a closed unbounded set constructible from this real.

Let $\vec{B} = \langle B_{\zeta,m} : \zeta < \omega_1, m \in \omega \rangle \subseteq \omega$ be a nicely definable sequence of a. d. sets. We will define a f. s. iteration $\langle \mathbb{P}_\alpha, \dot{\mathbb{Q}}_\beta : \alpha \leq \omega_3, \beta < \omega_3 \rangle$ such that $\mathbb{P}_0 = \mathbb{P}^0 * \mathbb{P}^1 * \mathbb{P}^2$, for every $\alpha < \omega_3$, $\dot{\mathbb{Q}}_\alpha$ is a $\mathbb{P}_\alpha$-name for a σ-centered poset, in $L^{\mathbb{P}_{\omega_3}}$ there is a Δ^1_3-definable wellorder of the reals, a Π^1_2-definable m.o. family and there are no $\mathbf{\Sigma}^1_2$-definable m.o. families.

Along the iteration $\forall \alpha < \omega_3$, in $V^{\mathbb{P}_\alpha}$ we will define a set $O_\alpha \subseteq P_c(2^\omega)$ of orthogonal measures and for $\alpha \in Lim(\omega_3)$, a subset $A_\alpha \subseteq [\alpha, \alpha + \omega)$.

$\mathbb{Q}_\alpha$ will add a generic real u_α. We will have that $L[G_\alpha] \cap {}^\omega\omega = L[\langle \dot{u}_\xi^{G_\alpha} : \xi < \alpha\rangle] \cap {}^\omega\omega$. This gives a canonical w.o. of the reals in $L[G_\alpha]$ which depends only on $\langle \dot{u}_\xi : \xi < \alpha\rangle$, whose $\mathbb{P}_\alpha$-name will be denoted by $\dot{<}_\alpha$. Additionally arrange that for $\alpha < \beta$, $<_\alpha$ is an initial segment of $<_\beta$, where $<_\alpha = \dot{<}_\alpha^{G_\alpha}$ and $<_\beta = \dot{<}_\beta^{G_\beta}$. Then if G is a $\mathbb{P}_{\omega_3}$-generic filter:

- $<^G = \bigcup\{\dot{<}_\alpha^G : \alpha < \omega_3\}$ will be is the desired w.o. of $\mathbb{R}$ and,
- $O = \bigcup_{\alpha<\omega_3} O_\alpha \subseteq P_c(2^\omega)$ will be Π^1_2-definable maximal family of orthogonal measures.

Recursively define $\mathbb{P}_{\omega_3}$ as follows. For $\nu \in [\omega_2, \omega_3)$ let $i_\nu : \nu \cup \{\langle \xi, \eta \rangle : \xi < \eta < \nu\} \to Lim(\omega_3)$ be a fixed bijection. If G_α is a $\mathbb{P}_\alpha$-generic, $<_\alpha = \dot{<}_\alpha^{G_\alpha}$ and $x, y \in L[G_\alpha] \cap {}^\omega\omega$ such that $x <_\alpha y$, let $x * y = \{2n\}_{n \in x} \cup \{2n+1\}_{n \in y}$ and $\Delta(x * y) = \{2n+2 : n \in x * y\} \cup \{2n+1 : n \notin x * y\}$.

Suppose $\mathbb{P}_\alpha$ has been defined and let G_α be a $\mathbb{P}_\alpha$-generic filter. If $\alpha = \omega_2 \cdot \alpha' + \xi$, where $\alpha' > 0$, $\xi \in Lim(\omega_2)$, let $\nu = o.t.(\dot{<}_{\omega_2 \cdot \alpha'}^{G_\alpha})$ and let $i = i_\nu$.

Coding the w.o.. If $i^{-1}(\xi) = \langle \xi_0, \xi_1 \rangle$ for some $\xi_0 < \xi_1 < \nu$, let x_{ξ_0}, x_{ξ_1} be the ξ_0-th, ξ_1-th reals in $L[G_{\omega_2 \cdot \alpha'}]$ according to $\dot{<}^{G_\alpha}_{\omega_2 \cdot \alpha'}$. In $L^{\mathbb{P}_\alpha}$ let

$$\mathbb{Q}_\alpha = \{\langle s_0, s_1 \rangle : s_0 \in [\omega]^{<\omega}, s_1 \in [\bigcup_{m \in \Delta(x_{\xi_0} * x_{\xi_1})} Y_{\alpha+m} \times \{m\}]^{<\omega}\},$$

where $\langle t_0, t_1 \rangle \leq \langle s_0, s_1 \rangle$ iff $s_1 \subseteq t_1$, s_0 is an initial segment of t_0 and $(t_0 \backslash s_0) \cap B_{\zeta,m} = \emptyset$ for all $\langle \zeta, m \rangle \in s_1$. Let u_α be the generic real added by $\mathbb{Q}_\alpha$, $A_\alpha = \alpha + \omega \backslash \Delta(x_{\xi_0} * x_{\xi_1})$ and $O_\alpha = \emptyset$.

Coding the m.o. family. Let $i^{-1}(\xi) = \zeta \in \nu$. If the ζ-th real x_ζ according to $\dot{<}^{G_\alpha}_{\omega_2 \cdot \alpha'}$ is not the code of a measure orthogonal to $O'_\alpha = \bigcup_{\gamma<\alpha} O_\gamma$, let $\mathbb{Q}_\alpha$ be trivial, $A_\alpha = \emptyset$, $O_\alpha = \emptyset$. Otherwise, let

$$\mathbb{Q}_\alpha = \{\langle s_0, s_1\rangle : s_0 \in [\omega]^{<\omega}, s_1 \in [\bigcup_{m\in\Delta(x_\zeta)} Y_{\alpha+m} \times \{m\}]^{<\omega}\},$$

where $\langle t_0, t_1\rangle \leq \langle s_0, s_1\rangle$ iff $s_1 \subseteq t_1$, s_0 is an initial segment of t_0 and $(t_0 \backslash s_0) \cap B_{\zeta,m} = \emptyset$ for all $\langle \zeta, m\rangle \in s_1$. Let u_α be the generic real added by $\mathbb{Q}_\alpha$. In $L^{\mathbb{P}_{\alpha+1}} = L^{\mathbb{P}_\alpha * \mathbb{Q}_\alpha}$ let $g_\alpha = G(x_\zeta, u_\alpha)$ be the code of a measure equivalent to μ_{x_ζ} which codes u_α, let $O_\alpha = \{\mu_{g_\alpha}\}$ and let $A_\alpha = \alpha + \omega \backslash \Delta(u_\alpha)$.

If α is not of the above form, i.e. α is a successor or $\alpha \in \omega_2$, let $\mathbb{Q}_\alpha$ be the following poset for adding a dominating real:

$$\mathbb{Q}_\alpha = \{\langle s_0, s_1\rangle : s_0 \in \omega^{<\omega}, s_1 \in [\text{o.t.}(\dot{<}_\alpha^{G_\alpha})]^{<\omega}\},$$

where $\langle t_0, t_1\rangle \leq \langle s_0, s_1\rangle$ iff s_0 is an initial segment of t_0, $s_1 \subseteq t_1$, and $t_0(n) > x_\xi(n)$ for all $n \in \text{dom}(t_0)\backslash\text{dom}(s_0)$ and $\xi \in s_1$, where x_ξ is the ξ-th real in $L[G_\alpha] \cap \omega^\omega$ according to the wellorder $\dot{<}_\alpha^{G_\alpha}$. Let $A_\alpha = \emptyset$, $O_\alpha = \emptyset$.

With this the definition of $\mathbb{P}_{\omega_3}$ is complete. Let $O = \bigcup_{\alpha<\omega_3} O_\alpha$.

Note in particular, that if $\mu \in O$, then f_μ codes u_α for some $\alpha \in \omega_3$. By definition u_α codes the code f_ν for a measure ν equivalent to μ and the sequence $\langle Y_{\alpha+m} : m \in \Delta(f_\nu)\rangle$. We will write $f_\nu = r(\mu)$.

Lemma A

Let $\gamma \leq \omega_3$ and let G_γ be a $\mathbb{P}_\gamma$-generic filter over L. Then $L[G_\gamma] \cap \omega^\omega = L[\langle \dot{u}_\delta^{G_\gamma} : \delta < \gamma \rangle] \cap \omega^\omega$.

Lemma B

Let G be a $\mathbb{P}$-generic filter over L. Then for $\xi \in \bigcup_{\alpha \in Lim(\omega_3)} \dot{A}^G_\alpha$ there is no real coding a stationary kill of S_ξ.

Corollary A

Let G be $\mathbb{P}$-generic over L and let x, y be reals in $L[G]$. Then

- $x <^G y$ iff there is $\alpha < \omega_3$ such that for all m, the stationary kill of $S_{\alpha+m}$ is coded by a real iff $m \in \Delta(x * y)$.
- $\mu \in O$ iff there is $\alpha < \omega_3$ such that for all m, the stationary kill of $S_{\alpha+m}$ is coded by a real iff $m \in \Delta(r(\mu))$.

Proof:

Let $x <^G y$ and let $\alpha' > 0$ be minimal with $x, y \in L[G_{\omega_2 \cdot \alpha'}]$, $i = i_{o.t.(\dot{<}^G_{\omega_2\cdot\alpha'})}$. Find $\xi \in Lim(\omega_2)$ such that $i(\xi) = (\xi_x, \xi_y)$ where x, y are the ξ_x-th, ξ_y-th real resp. in $L[G_{\omega_2\cdot\alpha'}]$ according to $\dot{<}^G_{\omega_2\cdot\alpha'}$. Let $\alpha = \omega_2 \cdot \alpha' + \xi$. Then $\mathbb{Q}_\alpha$ adds a real coding a stationary kill for $S_{\alpha+m}$ for all $m \in \Delta(x * y)$. On the other hand if $m \notin \Delta(x * y)$, then $\alpha + m \in \dot{A}^G_\alpha = \alpha + (\omega \backslash \Delta(x * y))$ and so by Lemma B, there is no real in $L[G]$ coding the stationary kill of $S_{\alpha+m}$.

Now suppose that there exists α such that the stationary kill of $S_{\alpha+m}$ is coded by a real iff $m \in \Delta(x * y)$. Since the stationary kill of some $\alpha + m$'s is coded by a real in $L[G]$, Lemma B implies that $\dot{\mathbb{Q}}^G_\alpha$ introduced a real coding stationary kill for all $m \in \Delta(a * b)$ for some reals $a \dot{<}^G_\alpha b$, while there are no reals coding a stationary kill of $S_{\alpha+m}$ for $m \notin \Delta(a * b)$. Therefore $\Delta(a * b) = \Delta(x * y)$ and hence $a = x$, $b = y$, consequently $x \dot{<}^G_\alpha y$.

Lemma

Let G be a $\mathbb{P}$-generic real over L, $x, y \in {}^\omega\omega \cap L[G]$ and $\mu \in \mathcal{P}_c(2^\omega) \cap L[G]$. Then

- *$x < y$ iff there is a real r such that for every countable suitable model $\mathcal{M}$ such that $r \in \mathcal{M}$, there is $\bar{\bar{\alpha}} < \omega_3^{\mathcal{M}}$ such that for all $m \in \Delta(x * y)$, $(L[r])^{\mathcal{M}} \vDash S_{\bar{\bar{\alpha}}+m}$ is not stationary.*
- *$\mu \in O$ iff for every countable suitable model $\mathcal{M}$ such that $\mu \in \mathcal{M}$, there is $\bar{\alpha} < \omega_3^{\mathcal{M}}$ such that $S_{\bar{\alpha}+m}$ is nonstationary in $(L[r(\mu)])^{\mathcal{M}}$ for every $m \in \Delta(r(\mu))$*

By Corollary A, there exists $\alpha < \omega_3$ such that $\dot{\mathbb{Q}}^G_\alpha$ adds a real r coding a stationary kill of $S_{\alpha+m}$ for all $m \in \Delta(x * y)$. Let $\mathcal{M}$ be a countable suitable model containing r. It follows that $Y_{\alpha+m} \cap \omega_1^{\mathcal{M}} \in \mathcal{M}$ and hence $X_\alpha \cap \omega_1^{\mathcal{M}}$, $X_{\alpha+m} \cap \omega_1^{\mathcal{M}}$ also belong to $\mathcal{M}$. Observe that these sets are actually in $\mathcal{N} := (L[r])^{\mathcal{M}}$.

Note also that $\mathcal{N}$ is a countable suitable model and consequently by the definition of $\mathcal{L}(X_{\alpha+m}, X_\alpha)$ we have that for every $m \in \Delta(x * y)$, $\mathcal{N} \models$

" Using $\vec{A}$, $X_{\alpha+m} \cap \omega_1$ (resp. $X_\alpha \cap \omega_1$) a. d. codes a subset $\bar{Z}_m$ (resp. $\tilde{Z}_0$) of ω_2, such that $Even(\bar{Z}_m)$ (resp. $Even(\tilde{Z}_0)$) codes $\langle \bar{C}, \bar{W}_m, \bar{\bar{W}}_m \rangle$ (resp. $\langle \tilde{C}, \tilde{W}_0, \tilde{\tilde{W}}_0 \rangle$), where $\bar{W}_m$, $\bar{\bar{W}}_m$ are the L-least codes of ordinals $\bar{\alpha}_m, \bar{\bar{\alpha}}_m < \omega_3$ (resp. $\tilde{W}_0 = \tilde{\tilde{W}}_0$ is the L-least code for a limit ordinal $\tilde{\tilde{\alpha}}_0$) such that $\bar{\bar{\alpha}}_m = \tilde{\tilde{\alpha}}_0$ is the largest limit ordinal not exceeding $\bar{\alpha}_m$ and $\bar{C}$ is a club in ω_2 disjoint from $S_{\bar{\alpha}_m}$."

Note that in particular for every $m \neq m'$ in $\Delta(x * y)$, $\bar{\bar{\alpha}}_m = \bar{\bar{\alpha}}_{m'}$.

Suppose there is such a real r. By Löwenheim-Skolem, r has the property from the formulation with respect to *all* suitable $\mathcal{M}$, and so for $\mathbb{H}^{\mathbb{P}}_{\Theta}$, where Θ is sufficiently large. That is $\exists\alpha < \omega_3$ such that $\forall m \in \Delta(x * y)$ $L_\Theta[r] \vDash S_{\alpha+m}$ is not stationary. Then the stationary kill of at least some $S_{\alpha+m}$ was coded by a real.

By Lemma B, $\dot{\mathbb{Q}}^G_\alpha$ adds a real u_α coding stationary kill for all $m \in \Delta(a * b)$ for some reals $a \dot{<}^G_\alpha b$, while there are no reals coding a stationary kill of $S_{\alpha+m}$ for $m \notin \Delta(a * b)$. Therefore $\Delta(a * b) \supset \Delta(x * y)$, and so $\Delta(a * b) = \Delta(x * y)$. Thus $a = x$, $b = y$ and hence $x \dot{<}^G_\alpha y$. $\square$

Lemma

The family O is maximal in $P_c(2^\omega)$.

Proof:

Suppose in $L^{\mathbb{P}_{\omega_3}}$ there is a code x for a measure orthogonal to every measure in the family O. Choose α minimal such that $\alpha = \omega_2 \cdot \alpha' + \xi$ for some $\alpha' > 0$, $\xi \in Lim(\omega_2)$ and $x \in L[G_{\omega_2 \cdot \alpha'}]$. Let $\nu = o.t.(\dot{<}^{G_\alpha}_{\omega_2 \cdot \alpha'})$ and let $i = i_\nu$. Then $x = x_\zeta$ is the ζ-th real according to the wellorder $\dot{<}^{G_\alpha}_{\omega_2 \cdot \alpha'}$, where $\zeta \in \nu$ and so for some $\xi \in Lim(\omega_2)$, $i^{-1}(\xi) = \zeta$. But then $x_\zeta = x$ is the code of a measure orthogonal to $O'_\alpha = \bigcup_{\gamma < \alpha} O_\gamma$ and so by construction O_α contains a measure equivalent to μ_x, which is a contradiction. $\square$

To obtain a Π^1_2-definable m.o. family in $L^{\mathbb{P}_{\omega_3}}$ consider the union of O with the set of all point measures.

Since $\mathbb{P}_{\omega_3}$ is a finite support iteration, we have added Cohen reals along the iteration cofinally often. Thus for every real a in $L^{\mathbb{P}_{\omega_3}}$ there is a Cohen real over $L[a]$ and so in $L^{\mathbb{P}_{\omega_3}}$ there are no $\boldsymbol{\Sigma}^1_2$ m.o. families. Also note that since cofinally often we have added dominating reals, $L^{\mathbb{P}_{\omega_3}} \models \mathfrak{b} = \omega_3$.

Theorem (F., Friedman, Törnquist)

It is consistent with $\mathfrak{c} = \mathfrak{b} = \omega_3$ that there is a Δ^1_3-definable wellorder of the reals, a Π^1_2 definable maximal orthogonal family of measures and there are no $\mathbf{\Sigma}^1_2$-definable maximal sets of orthogonal measures.

THANK YOU!

Let X be a topological space and $\mu : \mathcal{B}(X) \to [0,1]$ such that $\mu(\emptyset) = 0$, $\mu(X) = 1$ and $\mu(\bigcup_{n\in\omega} A_n) = \sum_{n\in\omega} \mu(A_n)$ for every pairwise disjoint family $\{A_n\}_{n\in\omega} \subseteq \mathcal{B}(X)$.

If s_n enumerates $2^{<\omega}$ and $f_n : 2^\omega \to \mathbb{R}$ is defined as follows:

$$f_n(x) = \begin{cases} 1 & \text{if } s_n \subseteq x \\ 0 & \text{otherwise,} \end{cases}$$

then the metric on $P(2^\omega)$ defined by
$\delta(\mu, \nu) = \sum_{n=0}^{\infty} 2^{-n-1} \frac{|\int f_n d\mu - \int f_n d\nu|}{\|f_n\|_\infty}$ makes the map $f \mapsto \mu_f$ an isometric bijection if we equip $p(2^\omega)$ with the metric $d(f, g) = \sum_{n=0}^{\infty} 2^{-n-1} |f(s_n) - g(s_n)|$.