DEFINABLE TOWERS

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Abstract. We study the definability of maximal towers and of inextendible linearly ordered towers (ilt's), a notion that is more general than that of a maximal tower. We show that there is, in the constructible universe, a Π^1_1 definable maximal tower that is indestructible by any proper Suslin poset. We prove that the existence of a Σ^1_2 ilt implies that $\omega_1 = \omega_1^L$. Moreover we show that analogous results hold for other combinatorial families of reals. We prove that there is no ilt in Solovay's model. And finally we show that the existence of a Σ^1_2 ilt is equivalent to that of a Π^1_1 maximal tower.

1. Sets of reals and their definability

The current work belongs to a line of research studying the definability properties of combinatorial sets of reals, more specifically sets often associated with combinatorial cardinal characteristics of the continuum. An excellent exposition of the combinatorial cardinal characteristics of $\mathbb{R}$ can be found in [3]. Classical examples of such sets are maximal almost disjoint families, maximal cofinitary groups, maximal independent families, ultrafilter bases. Before turning our attention to the main object of study of this paper, definable towers, we give a brief overview of the area and some of its central ideas.

1.1. **Sets of reals and their projective complexity.** Recall that in Gödel's constructible universe L, there is a Σ^1_2 definable well-order of the reals. Moreover, the existence of a Σ^1_2-definable well-order of the reals, implies that every real is constructible (see [22, Theorem 25.39]). Thus, in models of $\neg\mathsf{CH}$, the optimal (lowest possible) projective complexity of a definable well-order on the reals is Δ^1_3. The existence of a nicely definable wellorder of the reals gives rise in a natural way to the existence of nicely definable combinatorial sets of reals as they can be often built inductively along the given well-order. A difficult and highly non-trivial task remains however the construction of nicely definable combinatorial sets of reals, whose projective complexity is strictly smaller then the complexity of the given projective well-order.

A celebrated result of Harrington states that the existence of a $\boldsymbol{\Delta^1_3}$-wellorder of the reals is consistent with MA and $\mathfrak{c} = \aleph_2$ ([19]), while the question if the continuum can be strictly above $\aleph_2$ remained an interesting open problem for more than three decades, until 2013 when it was answered to the positive in [11]. Under MA all combinatorial cardinal characteristics of the real line are equal to $\mathfrak{c}$ and so in a sense *certain* infinitary combinators of the reals trivialize: all maximal almost disjoint families, towers, maximal independent families are of cardinality $\mathfrak{c}$. The work in [11] was shortly preceded by yet another interesting result, see [8]: The existence of a Δ^1_3-definable well-order

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of the reals is consistent with $\mathfrak{c} = \aleph_2$ and each of the following cardinal characteristic constellations: $\mathfrak{d} = \aleph_1 < \mathfrak{c} = \aleph_2$, or $\mathfrak{b} = \aleph_1 < \mathfrak{a} = \mathfrak{s} = \aleph_2$, $\mathfrak{b} < \mathfrak{g}$. Thus, the existence of a projective well-order of the reals, leaves enough room for many interesting and exciting combinatorial possibilities.

The study of nicely definable combinatorial sets of reals is an area of its own interest, which undoubtedly goes beyond the sets associated to the combinatorial cardinal characteristics of $\mathbb{R}$. Good examples are the prominent results of Mathias, that there are no analytic maximal almost disjoint families (see [31]) and of Todorčević, that there are no analytic Hausdorff gaps (more precisely no $(\boldsymbol{\Sigma}^1_1, \cdot)$ Hausdorff gaps; see [39, Corollary 1]). Answering a question of Mauldin, Preiss and Rataj [35] showed that in an uncountable Polish space there are no analytic maximal families of pairwise orthogonal measures, result which was later proved using Hjorth's theory of turbulance by Kechris and Sofronidis, see [27]. Further interestnig examples are given by the fact that there are no analytic maximal independent families [33], no analytic Hamel bases [33], no analytic C^1-small sets (see [20]; recall that an uncountable subset of $\mathbb{R}^2$ is said to be C^1-small if it meets every C^1-curve in countable many points).

Many of the above "types of families" have co-analytic counterparts in L, making Π^1_1 their optimal projective complexity in the constructible universe: there are co-analytic maximal almost disjoint families [33]; co-analytic Hausdorff gaps [29]; co-analytic maximal families of orthogonal measures [16]; co-analytic C^1-small sets [45]; co-analytic maximal independent families (see for example [5]). For many of those results, one can use a technique due to Erdös, Kunen and Mauldin [7], a technique which found broad applications after its appearance in Miller's [33]. A beautiful generalization of the technique giving a general framework for constructing co-analytic sets of reals is developed by Vidnyánszky in [45].

Projective complexity gives an important distinction between various very closely related combinatorial sets of reals. While there are no analytic Hausdorff gaps, there are perfect Luzin gaps [39]. While there are no analytic maximal almost disjoint families, there are Borel maximal cofinitary groups and Borel eventually different families (see [21] and [38]). Note that the latter two results were only gradually obtained and remained open for long time. Major stepping stones towards those results are the construction of a maximal cofinitary group with a co-analytic set of generators [18], followed shortly thereafter by the construction of a co-analytic maximal cofinitary group in [25]. An interesting parallel can be drawn at this point with the study of two-point sets: while in L there are co-analytic two-point sets [33], the existence of a Borel such set remains an open question (the question is due to Erdös).

Another more recent line of investigations, which is of particular interest in the light of [19] and [8], are the definability properties of various combinatorial sets of reals in models of large continuum. Studies of nicely definable maximal almost disjoint families in models of large continuum can be found in [17], [9], [6]. For a Sacks indestructible co-analytic maximal independent families and so a model of $\mathfrak{i} = \mathfrak{d} < \mathfrak{c}$ in which $\mathfrak{i}$ has an optimal projective witness see [5]. A co-analytic maximal eventually different family of cardinality $\aleph_1$ in a model of $\neg\mathsf{CH}$ can be found in [13]. A powerful strengthening of the notion of maximality for eventually different families, the notion of tight eventually different families which proved to be very fruitful in this context, can be found in [15]. A construction of a co-analytic Cohen indestructible maximal cofinitary group and so a model of $\mathfrak{a}_g = \mathfrak{b} < \mathfrak{d} = \mathfrak{c}$ with an optimal projective witness to $\mathfrak{a}_g$ can be found in [14]. A construction of Π^1_2 definable maximal cofinitary groups, as well as Π^1_2-definable maximal almost disjoint families and Π^1_2-definable maximal eventually different families, of cardinality λ where $\aleph_1 < \lambda < \mathfrak{c}$ can be found in [12] (note that for such families Π^1_2 is optimal). Further results regarding projective well-orders and cardinal characteristics constellations can be found in [10], for more recent work, see [2].

1.2. **Maximal Towers and their projective complexity.** In this note we focus solely on the definablity of *maximal towers* and *inextendible linearly ordered towers* (abbreviated as *ilt*). Interesting studies of the correlation between Hausdorff gaps and towers in $\mathcal{P}(\omega)/fin$ which are not necessarily maximal, as well as examples of towers with various properties (e.g. Hausdorff or Suslin) can be found in [4], [23]. For an example of a tower which can not be the half of any gap, see [4, Example 32] and [42] where the proof originates.

A tower will be, as usual, a set $X \subseteq [\omega]^\omega$ which is well ordered with respect to reverse almost inclusion, i.e. the relation $x \leq y$ given by $\exists n \in \omega (y \setminus n \subseteq x)$. A tower is maximal if it has no pseudointersection. In the definition of a linearly ordered tower we drop the requirement that the order is well-founded. An inextendible linearly ordered tower is one that has no top-extension, i.e. has no pseudointersection. Note that neither maximal towers nor ilt's can be analytic. On the other hand in Section 3 we show that Π^1_1 maximal towers do exist in L (Theorem 3.2). In Section 4 we study forcing indestructible, nicely definable towers and show that in L there is a Π^1_1 maximal tower that is indestructible by any proper Suslin partial order (Theorem 4.3). Section 5 deals with the value of ω_1 in models where ilt's can have simple definitions. As a main result we show that the existence of a $\Sigma^1_2(x)$ ilt implies that $\omega_1 = \omega_1^{L[x]}$ (Theorem 5.3). The same has been shown for mad families in [43]. Using similar ideas we show that this holds analogously for maximal independent families, Hamel bases and ultrafilters (Theorem 5.7, 5.9 and 5.11). In [6] Brendle and Khomskii ask whether there is some notion of transcendence over L that is equivalent to the non-existence of a Π^1_1 mad family. The same question can be asked for other families and our observations contribute to this question by giving a sufficient condition of this kind. In Section 6 we show that there is no ilt in Solovay's model (Theorem 6.1). For mad families this was a long standing open question first asked by Mathias in [31] and solved by Törnquist in [43]. For Hausdorff gaps this follows from [41, Theorem 1]. In Section 7 we show that the existence of a Σ^1_2 ilt is equivalent to that of a Π^1_1 ilt which is equivalent to that of a Π^1_1 tower (Theorem 7.3). This theorem fits into a series of results stating that we can canonicaly construct Π^1_1 objects from given Σ^1_2 ones. For mad families this was shown in [44]. For maximal independent families see [5], for maximal eventually different families see [13] and to the best knowledge of the authors still open for maximal cofinitary groups.

We will always stress the difference between lightface $(\Pi^1_1, \Sigma^1_1, \Sigma^1_2)$ and boldface $(\boldsymbol{\Pi}^1_1, \boldsymbol{\Sigma}^1_1, \boldsymbol{\Pi}^1_2)$ definitions as well as definitions relative to a fixed real parameter $(\Pi^1_1(x), \Sigma^1_1(x), \Sigma^1_2(x))$ to stay as general as possible.

2. Towers and Definability

Definition 2.1. A tower is a set $X \subseteq [\omega]^\omega$ which is well ordered with respect to the relation defined by $x \leq y$ iff $y \subseteq^* x$. It is called maximal if it cannot be further extended, i.e. it has no pseudointersection.

Theorem 2.2. *A tower contains no (uncountable) perfect set, i.e. is thin. In particular there is no $\boldsymbol{\Sigma}^1_1$ maximal tower.*

Proof. Assume $X \subseteq [\omega]^\omega$ is a tower and $P \subseteq X$ is a perfect set. The set $R = \{(x, y) : x, y \in P \wedge y \subseteq^* x\}$ is Borel. P is an uncountable Polish space and R is Borel as a subset of $P \times P$. But R is a well order of P, which contradicts R having the Baire property by [26, Theorem 8.48]. A maximal tower must be uncountable and an uncountable analytic set has a perfect subset by the Perfect Set Theorem. Thus there is no analytic maximal tower. $\square$

Theorem 2.3. *Every $\Sigma^1_2(x)$ tower is a subset of $L[x]$ and thus of size at most $\omega_1^{L[x]}$.*

Proof. If X is a $\Sigma^1_2(x)$ tower then it contains no perfect set and is thus a subset of $L[x]$ by the Mansfield-Solovay Theorem [32, Theorem 21.1]. $\square$

Corollary 2.4. The existence of a $\Sigma^1_2(x)$ maximal tower implies that $\omega_1 = \omega_1^{L[x]}$.

All of the proofs above rely mostly on the fact that towers exhibit a well ordered structure and the maximality is inessential. Thus it is natural to ask for a more general version of a tower which is not trivially ruled out by an analytic definition. We call a set $X \subseteq [\omega]^\omega$ an inextendible linearly ordered tower (abbreviated as ilt) if it is linearly ordered with respect to $\subseteq^*$ and has no pseudointersection. We call $Y \subseteq X$ cofinal in X if $\forall x \in X \exists y \in Y (y \subseteq^* x)$.

Theorem 2.5. *There is no $\mathbf{\Sigma}^1_1$ definable inextendible linearly ordered tower.*

Proof. Assume $X = p[T]$ is an ilt where T is a tree on $2 \times \omega$.

Claim 2.6. *There is $T' \subseteq T$ so that for every $(s, t) \in T'$, $p[T'_{(s,t)}]$ is cofinal in X.*

Proof. Let $T' = \{(s, t) : p[T_{(s,t)}]$ is cofinal in $X\}$. For every $(u, v) \in T \setminus T'$, we let $x_{u,v} \in X$ be such that $\forall y \in p[T_{(u,v)}](x_{u,v} \subseteq^* y)$. The collection $\{x_{u,v} : (u, v) \in T \setminus T'\}$ is countable and therefore there is $x \in X$ so that $x \subsetneq^* x_{u,v}$ for every $(u, v) \in T \setminus T'$. Now let $(s, t) \in T'$ be arbitrary and $x' \in X$ such that $x' \subseteq^* x$. As $p[T_{(s,t)}]$ is cofinal in X, there is $y \in p[T_{(s,t)}]$ so that $y \subseteq^* x'$. Say $(y, z) \in [T_{(s,t)}]$. For every $n \in \omega$, $(y \restriction n, z \restriction n) \in T'$ because else we get a contradiction to $y \subseteq^* x$. Thus $y \in p[T'_{(s,t)}]$. $\square$

By the claim we can wlog assume that for every $(s, t) \in T$, $p[T_{(s,t)}]$ is cofinal in X. Now consider T as a forcing notion (which is equivalent to Cohen forcing). The generic real will be a new element of $p[T]$ together with a witness. Let $\dot{c}$ be a name for the generic real. Notice that the statement that $p[T]$ is linearly ordered by $\subseteq^*$ is absolute. Thus for every $y \in X$ there is a condition $(s, t) \in T$ and $n \in \omega$ so that either

$$(s, t) \Vdash \dot{c} \subseteq y \setminus n$$

or

$$(s, t) \Vdash y \subseteq \dot{c} \setminus n.$$

The second option is impossible, because $p[T_{(s,t)}]$ is cofinal in X. We can thus find (s, t), $n \in \omega$ and $Y \subseteq X$ cofinal in X, so that for every $y \in Y$, $(s, t) \Vdash \dot{c} \subseteq y \setminus n$. Let $(x, z) \in [T_{(s,t)}]$ be arbitrary. As Y is cofinal in X, there is $y \in Y$ so that $y \subsetneq^* x$. But this clearly contradicts $(s, t) \Vdash \dot{c} \subseteq y \setminus n$. $\square$

Corollary 2.7. Every $\mathbf{\Sigma}^1_2$ inextendible linearly ordered tower has a cofinal subset of size ω_1.

Proof. Assume X is $\mathbf{\Sigma}^1_2$. Then it is the union of ω_1 many Borel sets (see e.g. [34]). By Theorem 2.5 each of these Borel sets has a lower bound in X. $\square$

Note that the above results can be applied similarly to inextendible linearly ordered subsets of $(\omega^\omega, \leq^*)$.

3. A Π^1_1 definable maximal tower in L

In this section we will show how to construct in L a maximal tower with a Π^1_1 definition. For this we apply the coding technique that has been developed by A. Miller in [33] in order to show the existence of various nicely definable combinatorial objects in L.

Let O be the set of odd and E the set of even natural numbers.

Lemma 3.1. *Suppose $z \in 2^\omega$, $y \in [\omega]^\omega$ and $\langle x_\alpha : \alpha < \gamma\rangle$ is a tower, where $\gamma < \omega_1$, so that $\forall \alpha < \gamma(|x_\alpha \cap O| = \omega \wedge |x_\alpha \cap E| = \omega)$. Then there is $x \in [\omega]^\omega$ so that $\forall \alpha < \gamma(x \subseteq^* x_\alpha)$, $|x \cap O| = \omega$, $|x \cap E| = \omega$, $z \leq_T x$ and $|y \cap \omega \setminus x| = \omega$.*

Proof. It is a standard diagonalization to find x so that $\forall \alpha < \gamma(x \subseteq^* x_\alpha)$, $|x \cap O| = \omega$, $|x \cap E| = \omega$ and $|y \cap \omega \setminus x| = \omega$. We assume that z is not eventually constant, else there is nothing to do. Now given x find $\langle n_i\rangle_{i\in\omega}$ increasing in x so that $n_i \in O$ iff $z(i) = 0$. Let $x' = \{n_i : i < \omega\}$. Then x' works. $\square$

Theorem 3.2. *Assume $V = L$. Then there is a Π^1_1 definable maximal tower.*

In the rest of the paper, $<_L$ will always stand for the canonical global L well-order. Whenever $r \in 2^\omega$, we write $E_r \subseteq \omega^2$ for the relation defined by

$$mE_rn \text{ iff } r(2^m 3^n) = 0.$$

If E_r is a well-founded and extensional relation then we denote with M_r the unique transitive $\in$-model isomorphic to (ω, E_r). Notice that $\{r \in 2^\omega : E_r \text{ is well-founded and extensional}\}$ is Π^1_1.

If E_r is a well-order on ω then $\|r\|$ denotes the unique countable ordinal α so that (ω, E_r) is isomorphic to $(\alpha, \in)$. We also say that r codes α. The set of r so that E_r is a well-order is called WO. WO is obviously Π^1_1.

For any real $x \in 2^\omega$ we define ω_1^x to be the least countable ordinal which has no recursive code in x, i.e. the least ordinal α so that for any recursive function $r\colon 2^\omega \to 2^\omega$, $r(x)$ does not code α.

Proof of Theorem 3.2. Let $\langle y_\xi : \xi < \omega_1\rangle$ enumerate $[\omega]^\omega$ via the canonical well order of L. We will construct a sequence $\langle \delta(\xi), z_\xi, x_\xi : \xi < \omega_1\rangle$, where for every $\xi < \omega_1$:

- $\delta(\xi)$ is a countable ordinal
- $z_\xi \in 2^\omega \cap L_{\delta(\xi)+\omega}$
- $x_\xi \in [\omega]^\omega \cap L_{\delta(\xi)+\omega}$

The sequence is defined by the following requirements for each $\xi < \omega_1$:

(1) $\delta(\xi)$ is the least ordinal δ greater than $\sup_{\nu<\xi} \delta(\nu)$ so that $y_\xi, \langle \delta(\nu), z_\nu, x_\nu : \nu < \xi\rangle \in L_\delta$ and L_δ projects to ω[1].
(2) z_ξ is the $<_L$ least code for the ordinal $\delta(\xi)$.
(3) $\langle x_\nu : \nu < \xi\rangle$ is a tower and $\forall \nu < \xi(|x_\nu \cap O| = \omega \wedge |x_\nu \cap E| = \omega)$.
(4) x_ξ is $<_L$ least so that $\forall \nu < \xi(x_\xi \subseteq^* x_\nu)$, $|x_\xi \cap O| = \omega$, $|x_\xi \cap E| = \omega$, $z_\xi \leq_T x$ and $|y_\xi \cap \omega \setminus x| = \omega$.

[1]This means that over L_δ there is a definable surjection to ω. The set of such δ is unbounded in ω_1.

Notice that z_ξ and x_ξ indeed can be found in $L_{\delta(\xi)+\omega}$ given that $y_\xi, \langle x_\nu : \nu < \xi \rangle \in L_{\delta(\xi)}$, and that $L_{\delta(\xi)}$ projects to ω. It is then straightforward to check that (1)-(4) uniquely determine a sequence $\langle \delta(\xi), z_\xi, x_\xi : \xi < \omega_1 \rangle$ for which $\langle x_\xi : \xi < \omega_1 \rangle$ is a maximal tower.

Claim 3.3. *$\{x_\xi : \xi < \omega_1\}$ is a Π^1_1 subset of 2^ω.*

Proof. Let $\Psi(v)$ be the formula expressing that for some $\xi < \omega_1$, $v = \langle \delta(\nu), z_\nu, x_\nu : \nu \leq \xi \rangle$. More precisely, $\Psi(v)$ says that v is a sequence $\langle \rho_\nu, \zeta_\nu, \tau_\nu : \nu \leq \xi \rangle$ of some length $\xi + 1$, that satisfies the clauses (1)-(4) for every $\nu \leq \xi$.

The formula $\Psi(v)$ is absolute for transitive models of some finite fragment Th of ZFC which holds at limit stages of the L hierarchy. Namely we need absoluteness of the formula $\varphi_1(\xi, y)$ expressing that $y = y_\xi$, $\varphi_2(\delta, M)$ expressing that $M = L_\delta$ projects to ω and $\varphi_3(z, \delta)$ expressing that z is the $<_L$ least code for δ.

Moreover we have that $\langle \delta(\nu), z_\nu, x_\nu : \nu \leq \xi \rangle \in L_{\delta(\xi)+\omega}$ and that

$$L_{\delta(\xi)+\omega} \models \Psi(\langle \delta(\nu), z_\nu, x_\nu : \nu \leq \xi \rangle)$$

for every $\xi < \omega_1$.

Now let $\Phi(r, x)$ be a formula expressing that E_r is a well founded and extensional relation, $M_r \models$ Th and for some $v \in M_r$,

$$M_r \models v \text{ is a sequence } \langle \rho_\nu, \zeta_\nu, \tau_\nu : \nu \leq \xi \rangle \wedge \Psi(v) \wedge \tau_\xi = x.$$

We thus have that $x = x_\xi$ for some $\xi < \omega_1$ iff $\exists r \in 2^\omega \Phi(r, x)$. $\Phi(r, x)$ can clearly be taken as a Π^1_1 formula.

For any $\xi < \omega_1$, the well order $\delta(\xi)$ is coded by z_ξ and $z_\xi \leq_T x_\xi$. Thus $\delta(\xi) + \omega < \omega_1^{x_\xi}$ and there is $r \in L_{\omega_1^{x_\xi}}$ so that $M_r = L_{\delta(\xi)+\omega}$. In particular

$$\exists r \in L_{\omega_1^{x_\xi}} \cap 2^\omega (\Phi(r, x_\xi)).$$

We get that

$$\exists \xi < \omega_1 (x = x_\xi) \leftrightarrow \exists r \in L_{\omega_1^x} \cap 2^\omega (\Phi(r, x)).$$

The right hand side can be expressed by a Π^1_1 formula. □

□

Remark 3.4. By Theorem 2.3 the Π^1_1 tower constructed above is a subset of L. This implies that its definition will evaluate to the same set in any extension of L. As an immediate corollary, we obtain that the existence of a Π^1_1 definable tower is consistent with $\mathfrak{c} > \aleph_1$ (here $\mathfrak{c}$ denotes the continuum), a question which has been of interest for many combinatorial objects of the real line. For some more recent results in this direction regarding maximal independent families and maximal eventually different families of functions, see [5] and [13] respectively.

Corollary 3.5. The existence of a coanalytic tower is consistent with the bounding number $\mathfrak{b}$ being arbitrarily large.

Recall that the bounding number is defined as the least size of an unbounded family in $(\omega^\omega, <^*)$. It is a natural lower bound for many other classical cardinal characteristics.

Proof. It is well known that finite support iterations of Hechler forcing for adding a dominating real preserve all ground model maximal towers to be maximal (see [1] for more details). □

4. Indestructible Towers

Recall that the pseudointersection number $\mathfrak{p}$ is the least cardinal κ so that any set $\mathcal{F} \subseteq [\omega]^\omega$ with the finite intersection property and $|\mathcal{F}| < \kappa$ has a pseudointersection. $\mathcal{F}$ has the finite intersection property if for any $\mathcal{F}' \in [\mathcal{F}]^{<\omega}$, $\bigcap \mathcal{F}'$ is infinite. We obtain the following result.

Theorem 4.1. *Assume $\mathfrak{p} = \mathfrak{c}$. Let $\mathcal{P}$ be a collection of at most $\mathfrak{c}$ many proper posets of size $\mathfrak{c}$. Then there is a maximal tower indestructible by any $\mathbb{P} \in \mathcal{P}$.*

Proof. Let us call a $\mathbb{P}$ name $\dot{x}$ for a real a nice name whenever it has the form $\bigcup_{n\in\omega}\{(p,\check{n}) : p \in A_n\}$ where the A_n's are countable antichains in $\mathbb{P}$. Remember that when $\mathbb{P}$ is proper, then for any $\mathbb{P}$ name $\dot{x}$ for a real and any $p \in \mathbb{P}$, there is a nice name $\dot{y}$ and $q \leq p$ such that $q \Vdash \dot{y} = \dot{x}$. The number of nice $\mathbb{P}$ names is $|\mathbb{P}|^{\aleph_0}$.

Let us enumerate all pairs $\langle(\mathbb{P}_\alpha, p_\alpha, \dot{y}_\alpha) : \alpha < \mathfrak{c}\rangle$ where $p_\alpha \in \mathbb{P}_\alpha$, $\mathbb{P}_\alpha \in \mathcal{P}$ and $\dot{y}_\alpha$ is a nice $\mathbb{P}_\alpha$ name such that $p_\alpha \Vdash \dot{y}_\alpha \in [\omega]^\omega$.

We construct a tower $\langle x_\alpha : \alpha < \mathfrak{c}\rangle$ recursively. At step α we first choose a pseudointersection x of $\langle x_\xi : \xi < \alpha\rangle$ (here we use $\alpha < \mathfrak{p}$). Next we partition x into two disjoint infinite subsets x^0, x^1. Now note that $p_\alpha \Vdash_{\mathbb{P}_\alpha} (\dot{y}_\alpha \subseteq^* x^0 \wedge \dot{y}_\alpha \subseteq^* x^1)$ is impossible. Thus we find $i \in 2$ and $q_\alpha \leq p_\alpha$ such that $q_\alpha \Vdash_{\mathbb{P}_\alpha} \dot{y}_\alpha \not\subseteq^* x^i$. Let $x_\alpha = x^i$.

Now let $\dot{x}$ be an arbitrary $\mathbb{P}$ name for a real for some $\mathbb{P} \in \mathcal{P}$. We see easily that the set $D = \{q \in \mathbb{P} : \exists\alpha < \mathfrak{c}(q \Vdash \dot{x} \not\subseteq^* x_\alpha)\}$ is dense. Namely for any p we find $(\mathbb{P}_\alpha, p_\alpha, \dot{y}_\alpha)$ where $p_\alpha \leq p$ and $p_\alpha \Vdash \dot{x} = \dot{y}_\alpha$. Then we have $q_\alpha \leq p$ with $q_\alpha \in D$. $\square$

Definition 4.2. A forcing notion $(\mathbb{P}, \leq)$ is Suslin if

1. $\mathbb{P} \subseteq 2^\omega$ is analytic,
2. $\leq \subseteq 2^\omega \times 2^\omega$ is analytic,
3. the incompatibility relation $\perp \subseteq 2^\omega \times 2^\omega$ is analytic (and in particular Borel).

The next thing we want to show is that (in L) for $\mathcal{P}$ the collection of all proper Suslin posets, we can get an indestructible maximal tower which is coanalytic.

Theorem 4.3. *(V=L) There is a Π^1_1 maximal tower indestructible by any proper Suslin poset.*

Proof. First let us note that there is a recursive map $f\colon$ Tree $\times[\omega]^\omega \to 2^\omega$, where Tree is the set of trees on $\omega \times \omega$, such that $f(T, y) \in \mathrm{WO}$ iff $\forall x \in p[T](|x \cap (\omega \setminus y)| = \omega)$ (see [34, Theorem 4A.3]). Fix this map f.

For the construction of our tower we now enumerate via the canonical well order of L all trees $\langle T_\alpha : \alpha < \omega_1\rangle$ on $\omega \times \omega$. Now as in the proof of Theorem 3.2 we define a sequence $\langle\delta(\xi), z_\xi, x_\xi : \xi < \omega_1\rangle$ with

- $\delta(\xi)$ is a countable ordinal
- $z_\xi \in 2^\omega \cap L_{\delta(\xi)+\omega}$
- $x_\xi \in [\omega]^\omega \cap L_{\delta(\xi)+\omega}$

and the following properties:

1. $\langle x_\nu : \nu < \xi\rangle$ is a tower and $\forall\nu < \xi(|x_\nu \cap O| = \omega \wedge |x_\nu \cap E| = \omega)$.
2. $\delta(\xi)$ is the least ordinal δ greater than $\sup_{\nu<\xi}\delta(\nu)$ so that
   - $\langle\delta(\nu), z_\nu, x_\nu : \nu < \xi\rangle, T_\xi \in L_\delta$,

– there are disjoint pseudointersections $x^0, x^1 \in L_\delta$ of $\langle x_\nu : \nu < \xi \rangle$ both hitting O and E infinitely,
– either (a) there is $(x, w) \in [T_\xi] \cap L_\delta$ such that $x \subseteq^* x^0$ or (b) $f(T_\xi, x^0) \in \mathrm{WO}$, $\|f(T_\xi, x^0)\| < \delta$ and there is in L_δ an order preserving map $(\omega, E_{f(T_\xi, x^0)}) \to \|f(T_\xi, x^0)\|$,
– and L_δ projects to ω.

(3) z_ξ is the $<_L$ least code for the ordinal $\delta(\xi)$.
(4) x_ξ is $<_L$ least so that $x_\xi \subseteq^* x^1$ or $x_\xi \subseteq^* x^0$ depending on whether (a) or (b) holds true, $|x_\xi \cap O| = \omega$, $|x_\xi \cap E| = \omega$ and $z_\xi \leq_T x_\xi$.

As in the proof of Theorem 3.2 we see that this definition determines a tower $\langle x_\xi : \xi < \omega_1 \rangle$ which is Π^1_1.

Now let us note the following for a proper Suslin poset $\mathbb{P}$. Whenever $\dot{x}$ is a nice $\mathbb{P}$ name for a real and $p \in \mathbb{P}$, then the set

$$\{z \in [\omega]^\omega : \exists q \leq p (n \in z \leftrightarrow q \not\Vdash n \in \omega \setminus \dot{x})\}$$

is analytic ($q \not\Vdash n \in \omega \setminus \dot{x}$ iff $\exists r \in \operatorname{dom} \dot{x} [(r, n) \in \dot{x} \wedge r \not\perp q]$).

Thus for any $\mathbb{P}, p \in \mathbb{P}$ and $\dot{x}$ a nice name there is $\alpha < \omega_1$ so that

$$p[T_\alpha] = \{z \in [\omega]^\omega : \exists q \leq p (n \in z \leftrightarrow q \not\Vdash n \in \omega \setminus \dot{x})\}.$$

Consider x_α and the respective disjoint sets x^0 and x^1 at stage α of the construction. There are two options:

(a) There is $(x, w) \in [T_\alpha]$ such that $x \subseteq^* x^0$. In this case we have chosen $x_\alpha \subseteq^* x^1$ and there is $q \leq p$ so that $|\{n \in \omega : q \not\Vdash n \notin \dot{x}\} \cap x^1| < \omega$. In particular $p \not\Vdash \dot{x} \subseteq^* x_\alpha$.
(b) Or $L_{\delta(\alpha)} \models$ "$(\omega, E_{f(T_\xi, x^0)})$ is isomorphic to an ordinal". This means that $L \models$ "$(\omega, E_{f(T_\xi, x^0)})$ is isomorphic to an ordinal" and this means that for any $x \in p[T_\alpha]$, x has infinite intersection with $\omega \setminus x^0$. In this case we chose $x_\alpha \subseteq^* x^0$. Now if $q \leq p$ and $n \in \omega$ are arbitrary we can find $r \leq q$ and $m \geq n$ such that $r \Vdash m \in \dot{x} \setminus x_\alpha$. This means again that $p \not\Vdash \dot{x} \subseteq^* x_\alpha$.

Thus we have shown that for any proper Suslin poset $\mathbb{P}$, $\dot{x}$ an arbitrary $\mathbb{P}$ name for a real and $p \in \mathbb{P}$, $p \not\Vdash \dot{x}$ is a pseudointersection of $\langle x_\xi : \xi < \omega_1 \rangle$. □

5. ω_1 and $\mathbf{\Sigma}^1_2$ definitions

Definition 5.1. Let $\mathcal{F}$ be a filter on ω containing all cofinite sets. Then Mathias forcing relative to $\mathcal{F}$ is the poset $\mathbb{M}(\mathcal{F})$ consisting of pairs $(s, F) \in [\omega]^{<\omega} \times \mathcal{F}$ such that $\max s < \min F$. The extension relation is defined by $(s, F) \leq (t, E)$ iff $t \subseteq s$, $F \subseteq E$ and $t \setminus s \subseteq E$.

Lemma 5.2. *Assume that X is a $\mathbf{\Sigma}^1_2$ definable subset of $[\omega]^\omega$, linearly ordered with respect to $\subseteq^*$. Then there is a ccc forcing notion $\mathbb{Q}$ consisting of reals so that for any transitive model $V' \supseteq V^{\mathbb{Q}}$ (with the same ordinals), the reinterpretation of X in V' is not an ilt in V'.*

Proof. As X is $\mathbf{\Sigma}^1_2$, X can be written as a union $\bigcup_{\xi < \omega_1} X_\xi$ of analytic sets. Namely whenever $X = p[Y]$ where $Y \subseteq [\omega]^\omega \times 2^\omega$ is coanalytic then Y can be written as $\{(x, w) : f(x, w) \in \mathrm{WO}\}$ for some fixed continuous function f related to the definition of Y (see [34] for more details). Then X_ξ is defined as $\{x \in [\omega]^\omega : \exists w \in 2^\omega (\|f(x, w)\| = \xi)\}$.

Moreover we see that in any model $W \supseteq V$ where $\omega_1^W = \omega_1^V$, the reinterpretation of X is the union of the reinterpretations of the X_ξ.

If X has a pseudointersection x in V, then x will stay a pseudointersection of (the reinterpretation of) X in any extension by absoluteness. The statement $\forall y(y \notin X \vee x \subseteq^* y)$ is $\mathbf{\Pi}^1_2$. In this case let $\mathbb{Q}$ be the trivial poset.

If X is inextendible in V, then for any $\xi < \omega_1$ there is $x_\xi \in X$ so that $\forall y \in X_\xi(x_\xi \subseteq^* y)$. As X is linearly ordered with respect to $\subseteq^*$, $\{x_\alpha : \alpha < \omega_1\}$ generates a non-principal filter $\mathcal{F}$. Let $\mathbb{Q} = \mathbb{M}(\mathcal{F})$. Then in $V^{\mathbb{Q}}$ there is a real x so that $x \subseteq^* x_\alpha$ for every $\alpha < \omega_1$. By absoluteness $\forall y \in X_\xi(x_\xi \subseteq^* y)$ will still hold true in $V^{\mathbb{Q}}$. In particular $\forall y \in X_\xi(x \subseteq^* y)$ will hold true for any $\xi < \omega_1^V$.

As $\mathbb{Q}$ is ccc we have that $\omega_1^{V^{\mathbb{Q}}} = \omega_1^V$. This implies that x is actually a pseudointersection of X in $V^{\mathbb{Q}}$. Again, this will hold true in any extension. $\square$

Theorem 5.3. *If there is a Σ^1_2 ilt, then $\omega_1 = \omega_1^L$. More generally, the existence of a $\Sigma^1_2(x)$ ilt implies $\omega_1 = \omega_1^{L[x]}$.*

Proof. We only prove the first part as the rest follows similarly.

Suppose that X is a Σ^1_2 ilt and $\omega_1^L < \omega_1$. Apply Lemma 5.2 to (the definition of) X in L to get the respective poset $\mathbb{Q}$ in L. As $\omega_1^L < \omega_1$, $V \models |\mathcal{P}(\omega) \cap L| = \omega$. But this means there is a $\mathbb{Q}$ generic $x \in V$ over L. $L[x] \subseteq V$, thus by Lemma 5.2 X has a pseudointersection in V, contradicting our assumption. $\square$

Remark 5.4. We think that the proofs of Lemma 5.2 and Theorem 5.3 showcase something interesting about Schoenfield absoluteness. Recall that Schoenfield's absoluteness theorem says that Σ^1_2 formulas are absolute between any inner models $W \subseteq W'$, but it does not say anything about the relationship between ω_1^W and $\omega_1^{W'}$. In fact in many applications of Σ^1_2 absoluteness W and W' have the same ω_1 (e.g. when W' is a ccc or proper forcing extension of W). But in this case it can be deduced directly from analytic absoluteness and the representation of Σ^1_2 sets as the same ω_1 union of analytic set in any extension with the same ω_1. The reason is that the existential quantifier $\exists \alpha < \omega_1$ stays the same. So the full strength of Schoenfield absoluteness is only needed in the case where ω_1^W is countable in W' and this is the case that we crucially used in the proof of Theorem 5.3.

We also want to remark that the proofs of Lemma 5.2 and Theorem 5.3 are very general and can be applied to many other maximal combinatorial families. For example A. Törnquist has shown the following theorem in [43], using a similar argument.

Theorem 5.5. *If there is a Σ^1_2 mad family, then $\omega_1 = \omega_1^L$. More generally, the existence of a $\Sigma^1_2(x)$ mad family implies $\omega_1 = \omega_1^{L[x]}$.*

The argument for maximal independent families is a bit different. Let us recall the definition of a maximal independent family.

Definition 5.6. A set $X \subseteq [\omega]^\omega$ is called independent if for any $F \in [X]^{<\omega}$ and $G \in [X]^{<\omega}$ where $F \cap G = \emptyset$, $\bigcap_{x \in F} x \cap \bigcap_{y \in G}(\omega \setminus y)$ is infinite. An independent family is called maximal if it is maximal under inclusion.

The set $\bigcap_{x \in F} x \cap \bigcap_{y \in G}(\omega \setminus y)$ is often denoted $\sigma(F, G)$. We will also use this notation below. Note that an independent family X is not maximal iff there is a real x so that $x \cap \sigma(F, G)$ and

$(\omega \setminus x) \cap \sigma(F,G)$ are infinite for all $F, G \in [X]^{<\omega}$ where $F \cap G = \emptyset$. Such a real will be called independent over X.

We obtain the following result.

Theorem 5.7. *If there is a Σ^1_2 maximal independent family, then $\omega_1 = \omega_1^L$. More generally, the existence of a $\Sigma^1_2(x)$ maximal independent family implies $\omega_1 = \omega_1^{L[x]}$.*

In [33] Miller basically proved that a Cohen real is independent over any ground model coded analytic independent family. He did not put his theorem in these words, so before we go on let us repeat his argument.

Lemma 5.8 ([33, Proof of Theorem 10.28])**.** *Let $\varphi(x)$ be a $\boldsymbol{\Sigma}^1_1$ formula defining an independent family and let c be a Cohen real over V. Then in $V[c]$, c is independent over the family defined by $\varphi(x)$.*

Proof. Let X denote the set $\{x \in [\omega]^\omega : \varphi(x)\}$ in any model extending V. Note that in any model X is an independent family by Schoenfield absolutness. Let

$$K = \{x \in [\omega]^\omega : \exists F \in [X]^{<\omega} \exists G \in [X]^{<\omega} (F \cap G = \emptyset \wedge |\sigma(F,G) \cap x| < \omega)\}$$

and

$$H = \{x \in [\omega]^\omega : \exists F \in [X]^{<\omega} \exists G \in [X]^{<\omega} (F \cap G = \emptyset \wedge |\sigma(F,G) \cap (\omega \setminus x)| < \omega)\}.$$

These sets are both analytic. Note that x is independent over X iff $x \notin H \cup K$. To show that any Cohen real c is independent over X, i.e. $c \notin H \cup K$ it suffices to prove that H and K are meager. Why? When $H \cup K$ is meager then there is a meager F_σ set C so that $H \cup K \subseteq C$ and this statement is absolute $(\forall x (x \in H \cup K \to x \in C))$. As c is Cohen, $V[c] \models c \notin C$ and thus $V[c] \models c \notin H \cup K$ which implies that in $V[c]$, c is independent over X.

So let us prove:

Claim. *K and H are meager.*

Proof. Suppose e.g. that H is nonmeager. The argument for K will follow similarly. Because H is analytic it has the Baire property and is thus comeager somewhere. It is well known and easy to see that any comeager set contains a perfect set of almost disjoint reals. So let $P \subseteq H$ be a perfect almost disjoint family. For each $x \in P$ we have F_x and G_x so that $\sigma(F_x, G_x) \subseteq^* x$. By the Delta system lemma, there is a set $S \in [P]^{\omega_1}$ and $R \in [P]^{<\omega}$ so that

$$\forall x \neq y \in S((F_x \cup G_x) \cap (F_y \cup G_y) = R).$$

For any $x \in S$ we define $R^0_x = R \cap F_x$ and $R^1_x = R \cap G_x$. As S is uncountable there is an uncountable $S' \subseteq S$ so that

$$\forall x, y \in S' (R^0_x = R^0_y \wedge R^1_x = R^1_y).$$

But now note that for any $x \neq y \in S'$, $F_x \cap G_y = (R \cap F_x) \cap (R \cap G_y) = R^0_x \cap R^1_y = R^0_x \cap R^1_x = \emptyset$. By symmetry we also have that $F_y \cap G_x = \emptyset$ and this implies that

$$(F_x \cup F_y) \cap (G_x \cup G_y) = \emptyset.$$

In particular we can form $\sigma(F_x \cup F_y, G_x \cup G_y)$. By choice of F_x, G_x, F_y, G_y we have that

$$\sigma(F_x \cup F_y, G_x \cup G_y) \subseteq^* x \cap y =^* \emptyset$$

as P was an almost disjoint family. But this contradicts the independence of X. $\square$

$\square$

Proof of Theorem 5.7. Assume X is a Σ^1_2 maximal independent family. Then in L, X is also independent and it can be written as a union $\bigcup_{\xi<\omega_1^L} X_\xi$ of analytic sets X_ξ. Assuming for a contradiction $\omega_1^L < \omega_1$, there is a Cohen real c over L. We have that $\omega_1^{L[c]} = \omega_1^L$ and in $L[c]$, X still corresponds to the union $\bigcup_{\xi<\omega_1^L} X_\xi$. By the above lemma c is independent over all the X_ξ so in particular c is independent over X. This statement is Π^1_2 and thus absolute between any inner models containing c. In particular in V, X is not maximal. $\square$

Theorem 5.9. *If there is a Σ^1_2 Hamel basis of $\mathbb{R}$, then $\omega_1 = \omega_1^L$. More generally, the existence of a $\Sigma^1_2(x)$ Hamel basis of $\mathbb{R}$ implies $\omega_1 = \omega_1^{L[x]}$.*

A Hamel basis of $\mathbb{R}$ is a maximal set of linearly independent reals over the rationals $\mathbb{Q}$. Again it was Miller who first showed that a Cohen real in $\mathbb{R}$ is independent over any ground model coded analytic linearly independent family.

Lemma 5.10 ([33, Proof of Theorem 9.25])**.** *Assume $A \subseteq \mathbb{R}$ is an analytic set of reals that are linearly independent over the field of rationals. Assume $c \in \mathbb{R}$ is a Cohen real over V. Then in $V[c]$, c is linearly independent over (the reinterpretation of) A.*

Proof. We assume that $A \neq \emptyset$, else the argument is trivial. Let $x \in A \cap V$ be arbitrary. Suppose that $U \Vdash$ "$\dot{c}$ is not independent over A" where $U \subseteq \mathbb{R}$ is some basic open set. Say $x_0, \dots, x_k \in A \cap V$ and $q_0, \dots, q_k \in \mathbb{Q}$ are such that

$$U \Vdash \exists x_{k+1}, \dots, x_n \in A \setminus V \exists q_{k+1}, \dots, q_n \in \mathbb{Q}(\dot{c} = q_0 x_0 + \cdots + q_n x_n)$$

for some $n \in \omega$. Now let $c \in U$ be Cohen over V and $x_{k+1}, \dots, x_n \in A \setminus V, q_{k+1}, \dots q_n \in \mathbb{Q}$ so that

$$c = q_0 x_0 + \cdots + q_n x_n.$$

Let $s \neq 0$ be a small enough rational number so that $c + sx \in U$. Recall that, as $x \in V$, $c + sx$ is also a Cohen real over V. Thus let $y_{k+1}, \dots, y_n \in A \setminus V, r_{k+1}, \dots, r_n \in \mathbb{Q}$ so that

$$c + sx = q_0 x_0 + \cdots + q_k x_k + r_{k+1} y_{k+1} + \cdots + r_n y_n.$$

But now we have that

$$r_{k+1} y_{k+1} + \cdots + r_n y_n - (q_{k+1} x_{k+1} + \cdots + q_n x_n) = sx$$

and so A is not linearly independent in $V[c]$. But this is impossible by absoluteness. $\square$

Proof of Theorem 5.9. Same as the proof of Theorem 5.7. $\square$

For ultrafilters the proof is not much different. It will appear in [37].

Theorem 5.11. *If there is a Σ^1_2 ultrafilter, then $\omega_1 = \omega_1^L$. More generally, the existence of a $\Sigma^1_2(x)$ ultrafilter implies $\omega_1 = \omega_1^{L[x]}$.*

We want to remark the ideas above can also be used to show that under Martin's Axiom none of the families above have $\mathbf{\Sigma}^1_2$ witnesses.

Theorem 5.12. *MA(ω_1) implies that there is no $\mathbf{\Sigma}^1_2$ ilt, mad family, maximal independent family, Hamel basis or ultrafilter.*

Proof. For mad families this was proven in [43]. For ilt's Theorem 2.3 is enough. For ultrafilters it suffices to note that under MA(ω_1) every $\mathbf{\Sigma}^1_2$ set is Lebesgue measurable (see [24]) and an ultrafilter cannot be Lebesgue measurable. The argument for independent families and Hamel bases is the same. Write $X = \bigcup_{\xi<\omega_1} B_\xi$ where the B_ξ's are analytic. Let M be an elementary submodel of size ω_1 containing all the parameters defining the B_ξ's. Then let $c \in V$ be Cohen over M and use Lemma 5.8 or Lemma 5.10 to conclude that c is independent or linearly independent over X. □

6. Solovay's model

In this section we prove the following result.

Theorem 6.1. *There is no ilt in Solovay's model.*

Let us review some basics about Solovay's model. A good presentation of Solovay's model can be found in [22, Chapter 26]. Assuming κ is an inaccessible cardinal in the constructible universe L we first form an extension V of L in which $\omega_1 = \kappa$ using the Lévy collapse (see again [22, Chapter 26]). Then we let $W \subseteq V$ consist of all sets which are hereditarily definable from ordinals and reals as the only parameters. W is then called Solovay's model. The only facts that we use about W are listed below and are well-known.

Suppose $a \in 2^\omega \cap W$ is arbitrary, then

(1) for every poset $\mathbb{P} \in H(\kappa)^{L[a]}$, there is a $\mathbb{P}$ generic filter over $L[a]$ in W,
(2) whenever $x \in 2^\omega \cap W$, there is a poset $\mathbb{P} \in H(\kappa)^{L[a]}$, $\sigma \in H(\kappa)^{L[a]}$ a $\mathbb{P}$ name and $G \in W$ a $\mathbb{P}$ generic over $L[a]$ so that $x = \sigma[G]$.

Suppose $X \in \mathcal{P}(2^\omega) \cap W$. Then there is $a \in 2^\omega \cap W$ and a formula $\varphi(x)$ in the language of set theory using only a and ordinals as parameters so that

(3) for any poset $\mathbb{P} \in H(\kappa)^{L[a]}$, $\sigma \in H(\kappa)^{L[a]}$ a $\mathbb{P}$ name and $G \in W$, $\mathbb{P}$ generic over $L[a]$,
$$\sigma[G] \in X \leftrightarrow \exists p \in G(p \Vdash \varphi(\sigma)).$$

Until the end of the section we are occupied with proving Theorem 6.1. To prove Theorem 6.1, assume that $X \in \mathcal{P}(2^\omega) \cap W$ is linearly ordered with respect to $\subseteq^*$. We will show that X cannot be an ilt. Let $a \in 2^\omega \cap W$ and $\varphi(x)$ be as in (3). To simplify notation we will assume that $a \in L$ and thus $L[a] = L$. From now on let us work in L.

Lemma 6.2. *Let $\mathbb{P} \in H(\kappa)$, $p \in \mathbb{P}$ and σ a $\mathbb{P}$ name so that $p \Vdash \varphi(\sigma)$. Then there is $p_0, p_1 \leq p$ and $n \in \omega$ so that for any $m \geq n$,*
$$\exists r \leq p_0(r \Vdash m \in \sigma) \to p_1 \Vdash m \in \sigma.$$

Proof. Consider $\mathbb{P} \times \mathbb{P} \in H(\kappa)$ and σ_0 and σ_1 the $\mathbb{P} \times \mathbb{P}$ names so that whenever $G_0 \times G_1$ is $\mathbb{P} \times \mathbb{P}$ generic over V then $\sigma_0[G_0 \times G_1] = \sigma[G_0]$, $\sigma_1[G_0 \times G_1] = \sigma[G_1]$.

Note that $(p,p) \Vdash \varphi(\sigma_0) \wedge \varphi(\sigma_1)$, because whenever $G_0 \times G_1$ is $\mathbb{P} \times \mathbb{P}$ generic over V with $(p,p) \in G_0 \times G_1$ then G_0 and G_1 are $\mathbb{P}$ generic over V with $p \in G_0, G_1$. But then there must be $(p_0, p_1) \leq (p,p)$ and $n \in \omega$ so that either,
$$(p_0, p_1) \Vdash \sigma_0 \setminus n \subseteq \sigma_1$$

or

$$(p_0, p_1) \Vdash \sigma_1 \setminus n \subseteq \sigma_0.$$

Say wlog that $(p_0, p_1) \Vdash \sigma_0 \setminus n \subseteq \sigma_1$. Note that whenever $\exists r_0 \leq p_0(p_0 \Vdash m \in \sigma)$ for some $m \geq n$ then $p_1 \Vdash m \in \sigma$. Suppose this was not the case. Then there is $r_1 \leq p_1$ so that $r_1 \Vdash m \notin \sigma$. But then $(r_0, r_1) \Vdash \exists m \geq n(m \in \sigma_0 \wedge m \notin \sigma_1)$ which is a contradiction to $(r_0, r_1) \leq (p_0, p_1)$. □

Still in L, let $\langle \mathbb{P}_\xi, p_\xi, \sigma_\xi : \xi < \kappa \rangle$ enumerate all triples $\langle \mathbb{P}, p, \sigma \rangle$, where $\mathbb{P} \in H(\kappa)$, $p \in \mathbb{P}$ and $\sigma \in H(\kappa)$ is a $\mathbb{P}$ name so that $p \Vdash \varphi(\sigma)$. This is possible as $|H(\kappa)| = \kappa$.

For every $\xi < \kappa$ we find $p^0_\xi, p^1_\xi \leq p_\xi$ in $\mathbb{P}_\xi$ and $n \in \omega$ so that for every $m \geq n$

$$\exists r \leq p^0_\xi (r \Vdash m \in \sigma_\xi) \rightarrow p^1_\xi \Vdash m \in \sigma_\xi.$$

Let $x_\xi = \{m \in \omega : p^1_\xi \Vdash m \in \sigma_\xi\}$ for every $\xi < \kappa$.

Claim. *$\{x_\xi : \xi < \kappa\}$ has the finite intersection property.*

Proof of Claim. Suppose $x_{\xi_0}, \dots x_{\xi_{k-1}}$ are such that $\bigcap_{i<k} x_{\xi_i}$ is finite, say $\bigcap_{i<k} x_{\xi_i} \subseteq n$. Consider the poset $\mathbb{Q} = \prod_{i<k} \mathbb{P}_{\xi_i} \in H(\kappa)$, $(p^0_{\xi_0}, \dots, p^0_{\xi_{k-1}}) \in \mathbb{Q}$ and for every $i < k$, σ_i the $\mathbb{Q}$ name so that whenever $(G_0, \dots, G_{k-1})$ is $\mathbb{Q}$ generic then $\sigma_i[G_0 \times \dots \times G_{k-1}] = \sigma_{\xi_i}[G_i]$.

We have that $(p^0_{\xi_0}, \dots, p^0_{\xi_{k-1}}) \Vdash \varphi(\sigma_0) \wedge \dots \wedge \varphi(\sigma_{k-1})$ and thus, as X has the finite intersection property, there is $m \geq n$ and $(r_0, \dots, r_{k-1}) \leq (p^0_{\xi_0}, \dots, p^0_{\xi_{k-1}})$ so that

$$(r_0, \dots, r_{k-1}) \Vdash m \in \bigcap_{i<k} \sigma_i.$$

But this means that $r_i \Vdash m \in \sigma_i$ and thus $m \in x_{\xi_i}$ for each individual i. This contradicts $\bigcap_{i<k} x_{\xi_i} \subseteq n$ as $m \geq n$. □

Let $\mathcal{F}$ be the filter generated by $\{x_\xi : \xi < \kappa\}$. We have that $\mathcal{F} \in \mathcal{P}([\omega]^\omega)$ and thus $\mathcal{F} \in H(\kappa)$. Moreover we have that $\mathbb{M}(\mathcal{F}) \in H(\kappa)$. Thus in W there is $y \in [\omega]^\omega$ a $\mathbb{M}(\mathcal{F})$ generic real over L.

Claim. *For every $x \in X$, $y \subseteq^* x$. In particular X is not an ilt.*

Proof of Claim. Let $x \in X$ be arbitrary. Then we have in L a poset $\mathbb{P} \in H(\kappa)$ and a $\mathbb{P}$ name σ so that there is in W a $\mathbb{P}$ generic G over V so that $x = \sigma[G]$. Moreover there is $p \in G$ so that $p \Vdash \varphi(\sigma)$.

It suffices to show that there is some $\xi < \kappa$ and $q \in G$ so that $q \Vdash x_\xi \subseteq^* \sigma$. To see this we simply show that the set of conditions $q \in \mathbb{P}$ so that $\exists \xi < \kappa(q \Vdash x_\xi \subseteq^* \sigma)$ is dense below p. To show this fix $p' \leq p$ arbitrary. Let ξ be such that $\langle \mathbb{P}, p', \sigma \rangle = \langle \mathbb{P}_\xi, p_\xi, \sigma_\xi \rangle$. But then $p^1_\xi \leq p_\xi$ and $p^1_\xi \Vdash x_\xi \subseteq^* \sigma_\xi$. □

This finishes the proof of Theorem 6.1.

7. $\mathbf{\Sigma}^1_2$ vs $\mathbf{\Pi}^1_1$

Theorem 7.1. *Any $\Pi^1_1(x)$ ilt contains a $\Pi^1_1(x)$ maximal tower.*

Proof. We are going to prove the statement only for lightface Π^1_1 as everything will relativize. So let X be a Π^1_1 ilt.

Claim. *$X \cap L$ is cofinal in X (where L is the constructible universe).*

Proof. By Theorem 5.3 we have that $\omega_1 = \omega_1^L$ must be the case. Thus X can be written as a union $\bigcup_{\xi<\omega_1} X_\xi$ of analytic sets X_ξ which are coded in L (see the proof of Lemma 5.2). Note that $X \cap L$ is an ilt in L by a downwards absoluteness argument. This implies that for every $\xi < \omega_1$ there is $x \in L \cap X$ which is a pseudointersection of X_ξ. The statement "x is a pseudointersection of X_ξ" is absolute. Thus $X \cap L$ is indeed cofinal in X. □

We may now work entirely in L, assume $X \in L$ and construct a Π^1_1 tower that is cofinal in X (which implies that it is cofinal in X as interpreted in V).

Recall that $C_1 = \{x \in 2^\omega : x \in L_{\omega_1^x}\}$ is the largest thin Π^1_1 set and for any y, $C_1(y) = \{x \in 2^\omega : x \in L_{\omega_1^x}[y]\}$ is the largest thin $\Pi^1_1(y)$ set (see e.g [34]).

Claim. *$C_1 \cap X$ is cofinal in X.*

Proof. Suppose not, i.e there is $y \in X$ so that $C_1 \cap X \subseteq \{x \in [\omega]^\omega : y \subseteq^* x\}$. The set $Y := \{x \in X : x \subseteq^* y\}$ is $\Pi^1_1(y)$. We distinguish between two cases.

- Case 1: Y is thin. Then $Y \subseteq C_1(y)$. Moreover $\{\omega_1^z : z \in Y\}$ is unbounded in $\omega_1 = \omega_1^L$ (if r is recursive such that $X = \{x : r(x) \in \mathrm{WO}\}$ and δ bounds $\{\omega_1^z : z \in Y\}$, then $\{x : \|r(x)\| < \delta\}$ is a Borel subset of X that is cofinal in X which is impossible). But note that there is $\alpha < \omega_1$ large enough so that $L_\alpha[y] = L_\alpha$ and further $L_\beta[y] = L_\beta$ for any $\beta \geq \alpha$. So if $\omega_1^z > \alpha$ and $z \in Y$ then $z \in L_{\omega_1^z}[y] = L_{\omega_1^z}$. This is a contradiction to our assumption.
- Case 2: Y contains a perfect set P. By a theorem of Martin and Friedman (see [30]), P contains reals in any Δ^1_1 degree above the degree of some $d \in 2^\omega$. The set C_1 is unbounded in the Δ^1_1 degrees of L (see [34]) and is closed under Δ^1_1 bi-reducibility. Thus there is $z \in P$ so that $z \in C_1$. Again we get a contradiction to our assumption.

□

From now on we may assume that $X \subseteq C_1$. In the next step we will thin out X even further. For each $x \in X$ let $\alpha_x < \omega_1^x$ be such that $x \in L_{\alpha_x}$ (this is possible as $x \in L_{\omega_1^x}$). Further let $r_x \colon [\omega]^\omega \to 2^\omega$ be recursive so that $\alpha_x = \|r_x(x)\|$. As there are only countably many recursive functions, there is one r so that the set $\{x \in X : r_x = r\}$ is cofinal in X. Fix such an r. Let

$$Y := \{x \in X : x \in L_{\|r(x)\|}\}.$$

Y is a Π^1_1 cofinal subset of X. Thus let $s \colon [\omega]^\omega \to 2^\omega$ be a recursive function such that $Y = \{x \in [\omega]^\omega : s(x) \in \mathrm{WO}\}$. We define the following well-order $\lhd$ on Y:

$$x \lhd y \leftrightarrow \|s(x)\| < \|s(y)\| \vee (\|s(x)\| = \|s(y)\| \wedge x <_L y)$$

Let $\varphi_0(w, v)$ be a Σ^1_1 formula expressing that (ω, E_w) is properly embedable into (ω, E_v) and let $\varphi_1(w, v)$ be a Σ^1_1 formula expressing that (ω, E_w) is isomorphic to (ω, E_v). Moreover let $\psi(x, y)$ be a Σ^1_1 formula so that whenever $y \in Y$ and x is arbitrary then $\psi(x, y)$ is equivalent to $x \in L_{\|r(y)\|} \wedge L_{\|r(y)\|} \models x <_L y$. Let $\chi(x, y)$ be the Σ^1_1 formula

$$\varphi_0(s(x), s(y)) \vee (\varphi_1(s(x), s(y)) \wedge \psi(x, y)).$$

We see that when $y \in Y$ then

$$\chi(x, y) \leftrightarrow x \lhd y.$$

In particular, when $y \in Y$ then $\chi(x, y)$ implies that $x \in Y$.

Finally we define $T := \{y \in Y : \forall x \lhd y (y \subseteq^* x)\}$. We have that T is Π^1_1 as $y \in T$ iff

$$y \in Y \wedge \forall x(\neg\chi(x,y) \vee y \subseteq^* x).$$

T is obviously a tower as the order ${}^* \supseteq$ on T coincides with $\lhd$. T is cofinal in Y as for any $x \in Y$ if we let y be $\lhd$ least in Y so that $y \subseteq^* x$ then $y \in T$. □

Theorem 7.2. *The existence of a $\Sigma^1_2(x)$ ilt implies the existence of a $\Pi^1_1(x)$ ilt.*

Proof. Again we are going to prove the statement only for lightface Σ^1_2 ilt, as the proof will relativize. Let X be a Σ^1_2 ilt. As in the proof of Theorem 7.1 we can show that $X \cap L$ is cofinal in X (and this uses $\omega_1 = \omega_1^L$). So as $[\omega]^\omega \cap L$ is Σ^1_2 we may just assume that $X \subseteq L$. Let $\varphi(x,w)$ be Π^1_1 such that $x \in X$ iff $\exists w \varphi(x,w)$. Using Π^1_1 uniformization we can further assume that $x \in X$ iff $\exists! w \varphi(x,w)$.

The idea will now be to get a linearly ordered tower that basically consists of $x \in X$ together with their unique witness w. To do this we have to introduce some notation.

- For $y \subseteq [\omega \times \omega]^\omega$, we write y_n for y's n'th vertical section.
- For $x \in [\omega]^\omega$, we write $x(n)$ for the n'th element of x.

We now define the new ilt Y which lives on $\omega \times \omega$. A set $y \in [\omega \times \omega]^\omega$ is in Y iff the following are satisfied:

(1) For every $n \geq 1$, $y_n = y_0 \setminus y_0(n)$ or $y_n = y_0 \setminus y_0(n+1)$.

(2) If $w \in 2^\omega$ is such that $w(n) = \begin{cases} 0 \text{ if } y_{n+1} = y_0 \setminus y_0(n+1) \\ 1 \text{ if } y_{n+1} = y_0 \setminus y_0(n+2) \end{cases}$ then $\varphi(y_0, w)$ and in particular $y_0 \in X$.

Claim. *Y is Π^1_1 ilt.*

Proof. (i) Checking whether $y \in [\omega \times \omega]^\omega$ is as described in (1) is Δ^1_1. Checking whether for the function $w \in 2^\omega$ as in (2), $\varphi(y_0, w)$ holds is Π^1_1.

(ii) Y is linearly ordered by $\subseteq^*$: Let us note first that whenever $x \subsetneq^* y$ then eventually $x(n) > y(n)$. Why is this the case? As $x \subsetneq^* y$ (so $x \neq^* y$), there is a big enough $n \in \omega$ so that $\forall m \geq n(|y \cap x(m)| > m)$. But this means that $x(m) > y(m)$ for all $m \geq n$.

Now let's assume that $x \neq y \in Y$ and without loss of generality that $x_0 \subsetneq^* y_0$. By the observation above there is an n so that for every $m \geq n$, $x_0(m) > y_0(m)$ and $x_0(m) \in y_0$. But this also means that $\forall m \geq n$,

$$x_m \subseteq x_0 \setminus x_0(m) \subseteq y_0 \setminus y_0(m+1) \subseteq y_m.$$

In particular $x_m \subseteq y_m$ for $m \geq n$. For $k < n$ we have that $x_k \subsetneq^* y_k$. Thus all together we have that $x \subsetneq^* y$.

(iii) Y has no pseudointersection: Suppose z is a pseudointersection of Y. If there is $n \in \omega$ so that $|z_n| = \omega$, then z_n is a pseudointersection of X. Else let $x = \{\min z_n : n \in \omega \wedge z_n \neq \emptyset\}$. It is easy to see that x must be infinite (else z would not be $\subseteq^*$ below any member of Y). We claim that x is a pseudointersection of X. Namely let $y_0 \in X$ be arbitrary where $y \in Y$. As $z \subseteq^* y$, there is an n so that $\forall m \geq n(z_m \neq \emptyset \to (m, \min z_m) \in y)$. This means in particular that $\forall m \geq n(z_m \neq \emptyset \to \min z_m \in y_0)$. □

□

Theorem 7.3. *The following are equivalent:*

(1) There is a $\Sigma^1_2(x)$ ilt.
(2) There is a $\Pi^1_1(x)$ ilt.
(3) There is a $\Pi^1_1(x)$ maximal tower.
(4) There is a $\Sigma^1_2(x)$ maximal tower.

Proof. We have shown above that (1) → (2) → (3). (3) → (4) → (1) are trivial from the definitions. □

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