Symmetric Complex Banach Manifolds

and

Hermitian Jordan Triple Systems

Diplomarbeit

von

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Erklärung

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1. Introduction

The categorical equivalence between J^*-triple systems and simply-connected symmetric complex Banach manifolds with base point is done by W.Kaup and presented in his article "Algebraic Characterization of Symmetric Complex Banach Manifolds".

In chapter 2, following [8] we introduce the notion of a J^*-triple system and give a geometric characterization in terms of polynomial vector fields. Due to the last, we obtain and study the properties of certain Banach Lie algebras connected with the J^*-triple systems, which play an important role in the construction of the categorical equivalence.

In chapter 3 we give the notion of a symmetric Banach manifold as it is presented in [8]. An important step in giving the categorical equivalence is the fact that the group of all biholomorphic isometries of a Banach manifolds carries the structure of a real Banach Lie group [16]. Furthermore following [8] we prove that the existence of a point in the manifold satisfying 3.10.i and 3.10.ii, implies the existence of a symmetry at each point of the manifold. In connection with the same article [8], Vigué proved that for a connected normed Banach manifold, the converse implication also holds (compare [18]) and so is obtained the modern definition of a symmetric Banach manifold. We give also a construction of a J^*-triple system from a symmetric Banach manifold, as it is presented in [8] and show that it induces a functor $\mathfrak{F}$ from the category of symmetric complex Banach manifolds with a base point into the category of J^*-triple systems.

In chapter 4, we give a construction of a simply-connected symmetric complex Banach manifold with base point from a J^*-triple system. We use also the notion of algebraic groups in infinite dimensions and the fact that every algebraic group in infinite dimensions is a Banach Lie group in the norm topology, compare [6]. The construction of the canonical chart about the base point follows [17].

In chapter 5, we give the proof of the following statement, which is the main result of our work.

Every morphism of J^*-triple systems determines a morphism of the corresponding simply-connected symmetric complex Banach manifolds.

Hence the construction given in chapter 4, determines a functor $\mathfrak{J}$ from the category of J^*-triple system, into the category of simply-connected symmetric complex Banach manifolds with base point and so is obtained the categorical equivalence between the both considered categories.

In chapter 6, we give some further notions connected with the theory of hermitian Jordan triple systems and consider Cartan factors of type I - VI.

In this place I would like to express my gratitude to Prof. Dr. W. Kaup for his constant help. Thanks to him this work was possible. Thanks are due and to PD Dr. D. Zaitsev.

Notation: For every complex Banach space U, $L(U)$ denotes the Banach algebra of all endomorphisms of U and $L^k(U)$ the Banach space of all continuous homogeneous polynomials $U \to U$ of degree k. With $\mathfrak{P}$ we denote the Banach Lie algebra of all polynomial vector fields on U. For every $\nu \in \{-1, 0, 1, \cdots\}$, $\mathfrak{P}_\nu$ is the Banach subspace of all polynomial vector fields of the form $p_\nu \, {\partial}/{\partial z}$, where p_ν is a homogeneous polynomial of degree $\nu + 1$. Note that $\mathfrak{P} = \oplus_{\nu=-1}^{\infty} \mathfrak{P}_\nu$. In the following $\mathbb{K}$ is the field of real or complex numbers. For complex Banach spaces U and V, and a holomorphic mapping $h: U \to V$, we denote by $dh(a)$ the differential of h at the point $a \in U$.

2. Hermitian Jordan Triple Systems

Suppose A is a normed algebra over $\mathbb{K}$ with a unit element e, such that $\|e\| = 1$. Denote by A' the dual space of A, i.e. the space of all continuous linear functionals on A. Let

$$S(A) = \{x \in A\colon \ \|x\| = 1\} .$$

For every fixed $x \in S(A)$ define

$$D(A,x) = \{f \in A'\colon \ f(x) = 1 = \|f\|\} .$$

By the Hahn-Banach theorem, $D(A,x)$ is non-empty for every $x \in S(A)$.

2.1 Definition. Suppose $a \in A$ and $x \in S(A)$. Define

$$V(A,a,x) = \{f(ax)\colon \ f \in D(A,x)\}$$

and

$$V(A,a) = \bigcup \{V(A,a,x)\colon \ x \in S(A)\} .$$

The set $V(A,a)$ is called the *numerical range of a*. The real number $\nu(a)$, defined by

$$\nu(a) = \sup\{ \, |\lambda|\colon \ \lambda \in V(A,a)\}$$

is called the *numerical radius of a*.

2.2 Definition. Suppose A is a complex unital Banach algebra and $h \in A$. We say that h is *hermitian* if

$$V(A,h) \subset \mathbb{R} .$$

The set of all hermitian elements of A is denoted by $H(A)$.

In the case $\lambda \in L(U)$ for some complex Banach space U, λ is hermitian if and only if $\exp(it\lambda) \in \mathsf{GL}(U)$ is an isometry for every $t \in \mathbb{R}$. The subset $H(U)$ of all hermitian operators in $L(U)$ is a real Banach space.

Suppose U is a complex Banach space. Consider the Banach space $L^2(U)$ of all continuous homogeneous quadratic polynomials $q : U \to U$. For every $q \in L^2(U)$ define

$$\{wqz\} := \frac{1}{2}\big(q(w+z) - q(w) - q(z)\big) . \tag{2.3}$$

Obviously the mapping $\{wqz\}$ is symmetric, bilinear in $(w,z) \in U^2$. Furthermore, for every $\alpha \in U$ and $q \in L^2(U)$ define

$$\lambda(z) = \{\alpha q z\} .$$

Then $\lambda(z) \in L(U)$. We use the following notation $\lambda = \alpha \,\square\, q$.

2.4 Definition. The pair $(U,*)$ is called a *hermitian Jordan triple system* (or simply a *J^*-triple*) if and only if

(i) $* : U \to L^2(U)$ is a conjugate linear (continuous) mapping (we write α^* instead of $*(\alpha)$ for every $\alpha \in U$).

(ii) $\{\{v\alpha^* w\}\beta^* z\} + \{\{v\alpha^* z\}\beta^* w\} - \{v\alpha^*\{w\beta^* z\}\} = \{w\{\alpha v^* \beta\}^* z\}$ for all $v, \alpha, w, \beta, z \in U$.

(iii) The operator $\alpha \,\square\, \alpha^* \in L(U)$ is hermitian for every $\alpha \in U$.

2.5 Definition. *A morphism* of hermitian Jordan triple systems $(U,*) \to (V,*)$ is a continuous linear mapping $\lambda : U \to V$ such that

(iv) $\lambda\{z\alpha^* w\} = \{(\lambda z)(\lambda\alpha)^*(\lambda w)\}$ for all $z, \alpha, w \in U$.

Every morphism of J^*-triple systems is called a J^*-triple morphism.

2.6 Definition. Suppose λ is a J^*-triple morphism. The mapping λ is called a *metric J^*-morphism*, if $\|\lambda\| \leq 1$.

The class of all hermitian Jordan triple systems is a category. If we consider only metric J^*-morphisms, we get a subcategory.

2.7 Proposition. *Let $(U,*)$ be a J^*-triple system. The group of all J^*-automorphisms, denoted by $\mathsf{GL}(U,*)$, is a real Banach Lie group in the norm topology. The Lie algebra of $\mathsf{GL}(U,*)$ is the set $\mathsf{Der}(U,*)$ of all $\lambda \in L(U)$ satisfying*

$$\lambda(\{z\alpha^* w\}) = \{(\lambda z)\alpha^* w\} + \{z(\lambda\alpha)^* w\} + \{z\alpha^*(\lambda w)\} \tag{2.8}$$

for all $z, \alpha, w \in U$.

Proof. For every $z, \alpha, w \in U$ consider the mapping

$$p_{z,\alpha,w}(\lambda, \lambda^{-1}) := \lambda\{z(\lambda^{-1}(\alpha))^* w\} - \{\lambda(z)\alpha^*\lambda(w)\} ,$$

where $\lambda \in \mathsf{GL}(U)$. Then $p_{z,\alpha,w}$ is a homogeneous polynomial of degree 2 (over $\mathbb{R}$) in the sense of [6]. Furthermore consider the set

$$P := \left\{p_{z,\alpha,w} : \quad z, \alpha, w \in U\right\} .$$

Then we have

$$\mathsf{GL}(U,*) = \left\{\lambda \in \mathsf{GL}(U) : p(\lambda, \lambda^{-1}) = 0 \text{ for all } p \in P\right\} ,$$

and in particular $\mathsf{GL}(U,*)$ is an algebraic subgroup of $\mathsf{GL}(U)$ in the sense of [6]. Therefore ([6], Th.1) implies that $\mathsf{GL}(U,*)$ is a real Banach Lie group in the norm topology and has a Banach Lie algebra

$$\mathfrak{g} = \left\{\lambda \in L(U) : \ \exp(t\lambda) \in \mathsf{GL}(U,*) \text{ for all } t \in \mathbb{R}\right\} .$$

Consider an arbitrary element $\lambda \in \mathfrak{g}$. We shall prove that $\lambda \in \mathsf{Der}(U,*)$. Suppose that z, α, w are arbitrary elements in U. We have

$$\exp(t\lambda)(\{z\alpha^* w\}) = \{\exp(t\lambda)(z)(\exp(t\lambda)(\alpha))^*(\exp(t\lambda)(w))\}$$

for all $t \in \mathbb{R}$. Then differentiating at $t = 0$ we obtain

$$\begin{aligned}\frac{d}{dt}\exp(t\lambda)(\{z\alpha^* w\})_{|t=0} = \{(\frac{d}{dt}\exp(t\lambda)(z)_{|t=0})\alpha^* w\} + \{z(\frac{d}{dt}\exp(t\lambda)(\alpha)_{|t=0})^* w\} \\ + \{z\alpha^*(\frac{d}{dt}\exp(t\lambda)(w)_{|t=0}\}) ,\end{aligned}$$

which implies

$$\lambda(\{z\alpha^* w\}) = \{(\lambda z)\alpha^* w\} + \{z(\lambda\alpha)^* w\} + \{z\alpha^*(\lambda w)\} .$$

Hence $\lambda \in \mathsf{Der}(U,*)$.

Now suppose $\lambda \in \mathsf{Der}(U,*)$. We shall prove that $\lambda \in \mathfrak{g}$, i.e. $\exp(t\lambda) \in \mathsf{GL}(U,*)$. Fix arbitrary elements $z, \alpha, w \in U$ and consider the differential equation

$$\frac{d}{dt} g(t)_{|t=0} = \lambda(\{z\alpha^* w\}), \quad g(0) = \{z\alpha^* w\} . \tag{2.9}$$

The mapping $y(t) := \exp(t\lambda)$ for all $t \in \mathbb{R}$ is a solution of 2.9. Furthermore consider the mapping

$$h(t) := \{(\exp(t\lambda)(z))(\exp(t\lambda)(\alpha))^*(\exp(t\lambda)(z))\}$$

for all $t \in \mathbb{R}$. We have

$$\frac{d}{dt} h(t)_{|t=0} = \{\lambda(z)\alpha^* w\} + \{z(\lambda(\alpha))^* w\} + \{z\alpha^*(\lambda(w))\} ,$$

and using the assumption that $\lambda \in \mathsf{Der}(U,*)$ we obtain

$$\frac{d}{dt} h(t)_{|t=0} = \lambda\{z\alpha^* w\} .$$

Also we have $h(0) = \{z\alpha^* w\}$. Therefore $h(t)$ is a solution of 2.9. Since 2.9 has a unique solution, we have obtained

$$\exp(t\lambda)(\{z\alpha^* w\}) = \{(\exp(t\lambda)(z))(\exp(t\lambda)(\alpha))^*(\exp(t\lambda)(w))\}$$

for all $t \in \mathbb{R}$. Since $z, \alpha, w \in U$ were arbitrary, the last equation implies $\exp(t\lambda) \in \mathsf{GL}(U,*)$ for all $t \in \mathbb{R}$.

□

We shall give a characterization of hermitian Jordan triple systems in terms of polynomial vector fields.

Suppose U is a complex Banach space and $* : U \to L^2(U)$ is a conjugate linear, continuous mapping. For every $\alpha \in U$ denote by $q_\alpha : U \times U \to U$ the uniquely determined bounded, symmetric, bilinear mapping such that $q_\alpha(z,z) = \alpha^*(z)$ for all $z \in U$.

The following basic equalities associated with the triple product will be of constant use.

2.10 Lemma. *Suppose U is a complex Banach space and $*: U \to L^2(U)$ is a conjugate-linear continuous mapping. Then the following equalities hold.*

(i) *$\{\alpha\beta^*\gamma\} = q_\beta(\alpha,\gamma)$ for all elements α, β, γ in U.*

(ii) *$d\alpha^*(z)(y) = 2q_\alpha(z,y)$ for all α, z, y in U, where $d\alpha^*(z)$ is the differential of α^* in the point $z \in U$.*

Proof. Equality (i) is a direct conclusion of the definition of the triple product, (compare 2.3). To prove (ii) suppose q is a homogeneous polynomial in $L^2(U)$ and denote by $\tilde{q}$ the corresponding bilinear symmetric mapping $\tilde{q}: U \times U \to U$, such that $\tilde{q}(x,x) = q(x)$ for all $x \in U$. Then

$$dq(z)(y) = 2\tilde{q}(z,y)$$

for all z, y in U. □

The following sets of polynomial vector fields on U will be of main importance for the construction of the functorial correspondences $\mathfrak{F}$ and $\mathfrak{J}$, (compare chapter 1). Define

$$\mathfrak{p} := \left\{(\alpha - \alpha^*)\, \partial/\partial z \,:\, \alpha \in U\right\} \subset \mathfrak{P}_{-1} \oplus \mathfrak{P}_1 , \tag{2.11}$$

and

$$\mathfrak{k} := \left\{ i\lambda\, {\partial}/{\partial z} : \lambda \in H(U) \text{ and } [i\lambda\, {\partial}/{\partial z}, \mathfrak{p}] \subset \mathfrak{p} \right\} \subset \mathfrak{P}_0 . \tag{2.12}$$

Note that $\mathfrak{p}$ is a closed $\mathbb{R}$-linear vector subspace of $\mathfrak{P}_{-1} \oplus \mathfrak{P}_1$.

2.13 Lemma. *The vector space $\mathfrak{k}$ is a closed real Lie subalgebra of $\mathfrak{P}_0$.*

Proof. The set $\mathfrak{k}$ is a real vector subspace of $\mathfrak{P}_0$. We shall prove that it is closed under the bracket product. Notice that

$$\left[i\lambda\, {\partial}/{\partial z}, i\mu\, {\partial}/{\partial z} \right] = \left[i\lambda, i\mu \right] {\partial}/{\partial z} = i\Big(i[\lambda, \mu] \Big) {\partial}/{\partial z} \in \mathfrak{k} ,$$

since for every $\lambda, \mu \in H(U)$ we have $i(\lambda\mu - \mu\lambda) \in H(U)$ (compare [2], p.47), and

$$\Big[[i\lambda\, {\partial}/{\partial z}, i\mu\, {\partial}/{\partial z}], \mathfrak{p} \Big] = -\Big[[i\mu\, {\partial}/{\partial z}, \mathfrak{p}], i\lambda\, {\partial}/{\partial z} \Big] - \Big[[i\lambda\, {\partial}/{\partial z}, \mathfrak{p}], i\mu\, {\partial}/{\partial z} \Big] .$$

Consequently for every $i\lambda\, {\partial}/{\partial z}, i\mu\, {\partial}/{\partial z} \in \mathfrak{k}$ we have

$$\left[i\lambda\, {\partial}/{\partial z}, i\mu\, {\partial}/{\partial z} \right] \in \mathfrak{k} .$$

□

2.14 Lemma. *Let $(U, *)$ be a J^*-triple system. Then $[\mathfrak{p}, \mathfrak{p}] \subset \mathfrak{k}$.*

Proof. Consider arbitrary elements α, β in the space U, and the corresponding vector field

$$\mu\, {\partial}/{\partial z} := [(\alpha - \alpha^*)\, {\partial}/{\partial z}, (\beta - \beta^*)\, {\partial}/{\partial z}] .$$

For the holomorphic mapping $\mu(z)$ we obtain

$$\mu(z) = -d\alpha^*(z)((\beta - \beta^*)(z)) + d\beta^*(z)((\alpha - \alpha^*)(z)) . \tag{2.15}$$

Using the notation q_α (resp. q_β) for the symmetric, bilinear, bounded mappings associated with the homogeneous polynomials $\alpha^* \in L^2(U)$ (resp. $\beta^* \in L^2(U)$), we obtain

$$\mu(z) = -2q_\alpha(z, (\beta - \beta^*)(z)) + 2q_\beta(z, (\alpha - \alpha^*)(z)) . \tag{2.16}$$

Then 2.16 and 2.10.i imply

$$\mu(z) = -2\{z\alpha^*\beta\} + 2\{z\beta^*\alpha\} + 2\{z\alpha^*\{z\beta^*z\}\} - 2\{z\beta^*\{z\alpha^*z\}\} . \tag{2.17}$$

Applying property 2.4.ii of the triple product for $w = v = z$ we obtain the following equalities

$$\{\{z\beta^*z\}\alpha^*z\} + \{\{z\beta^*z\}\alpha^*z\} - \{z\beta^*\{z\alpha^*z\}\} = \{z\{\beta z^*\alpha\}^*z\} ,$$

$$\{\{z\alpha^*z\}\beta^*z\} + \{\{z\alpha^*z\}\beta^*z\} - \{z\alpha^*\{z\beta^*z\}\} = \{z\{\alpha z^*\beta\}^*z\} .$$

Subtracting we get

$$\begin{aligned} 2\{\{z\beta^*z\}\alpha^*z\} &- 2\{\{z\alpha^*z\}\beta^*z\} \\ &= (\{z\beta^*\{z\alpha^*z\}\} + \{z\{\beta z^*\alpha\}^*z\}) - (\{z\{\alpha z^*\beta\}^*z\} + \{z\alpha^*\{z\beta^*z\}\}) \\ &= \{\{z\alpha^*z\}\beta^*z\} - \{\{z\beta^*z\}\alpha^*z\} . \end{aligned}$$

Therefore

(2.18) $$\{\{z\beta^* z\}\alpha^* z\} = \{\{z\alpha^* z\}\beta^* z\}\ .$$

By 2.18 and 2.17 we get

$$\mu(z) = -2\{z\alpha^*\beta\} + 2\{z\beta^*\alpha\}\ .$$

Claim: There exists a hermitian element $\lambda \in L(U)$ such that $\mu = i\lambda$.

Using the conjugate linearity of $*$ we obtain

$$\begin{aligned} -\{z\alpha^*\beta\} + \{z\beta^*\alpha\} &= -i^2\{\alpha\beta^* z\} + i^2\{\beta\alpha^* z\} \\ &= -i\{(i\alpha)\beta^* z\} - i\{\beta(i\alpha)^* z\} \\ &= -i(\{(i\alpha)\beta^* z\} - \{\beta(i\alpha)^* z\})\ . \end{aligned}$$

But,

$$\begin{aligned} \{(i\alpha)\beta^* z\} - \{\beta(i\alpha)^* z\} &= \{(i\alpha+\beta)\beta^* z\} - \{\beta\beta^* z\} + \{(\beta+i\alpha)(i\alpha)^* z\} - \{(i\alpha)(i\alpha)^* z\} \\ &= \{(i\alpha+\beta)(i\alpha+\beta)^* z\} - \{\beta\beta^* z\} - \{(i\alpha)(i\alpha)^* z\} \\ &= ((i\alpha+\beta)\,\square\,(i\alpha+\beta)^* - (i\alpha)\,\square\,(i\alpha)^* - \beta\,\square\,\beta^*)(z)\ . \end{aligned}$$

Therefore the mapping $\mu(z)$ admits the following representation,

$$\mu(z) = -2i\Big((i\alpha+\beta)\,\square\,(i\alpha+\beta)^* - (i\alpha)\,\square\,(i\alpha)^* - \beta\,\square\,\beta^*\Big)(z)\ .$$

Hence $\mu(z) = i\lambda(z)$, where

$$\lambda(z) := -2\Big((i\alpha+\beta)\,\square\,(i\alpha+\beta)^* - (i\alpha)\,\square\,(i\alpha)^* - \beta\,\square\,\beta^*\Big)(z)\ .$$

From 2.4.iii and the fact that the set of all hermitian elements in $L(U)$ is real vector subspace, follows that λ is a hermitian element. The claim is proved.

Claim: The vector field $\mu(z)\,\partial/\partial z$ satisfies $[\mu\,\partial/\partial z\,, \mathfrak{p}] \subset \mathfrak{p}$.

Consider an arbitrary element $\gamma \in U$, and denote by $\chi(z)\,\partial/\partial z$ the bracket product of the vector fields $\mu(z)\,\partial/\partial z$ and $(\gamma - \gamma^*)(z)\,\partial/\partial z$ (taken in this order). Using the same arguments like above and 2.4.ii we obtain

$$\begin{aligned} \chi(z) &= d\mu(z)((\gamma-\gamma^*)(z)) - d(\gamma-\gamma^*)(z)(\mu(z)) \\ &= -2\{\gamma\alpha^*\beta\} + 2\{\{z\gamma^* z\}\alpha^*\beta\} + 2\{\gamma\beta^*\alpha\} - 2\{\{z\gamma^* z\}\beta^*\alpha\} \\ &\qquad - 4\{z\gamma^*\{z\alpha^*\beta\}\} + 4\{z\gamma^*\{z\beta^*\alpha\}\} \\ &= -2\{\gamma\alpha^*\beta\} + 2\{\gamma\beta^*\alpha\} - 2\big(2\{\{\beta\alpha^* z\}\gamma^* z\} - \{\beta\alpha^*\{z\gamma^* z\}\}\big) \\ &\qquad + 2\big(2\{\{\alpha\beta^* z\}\gamma^* z\} - \{\alpha\beta^*\{z\gamma^* z\}\}\big) \\ &= -2\{\gamma\alpha^*\beta\} + 2\{\gamma\beta^*\alpha\} - 2\big(\{z\{\alpha\beta^*\gamma\}^* z\} - \{z\{\beta\alpha^*\gamma\}^* z\}\big)\ . \end{aligned}$$

Therefore

$$\chi(z) = 2\Big((\{\gamma\beta^*\alpha\} - \{\gamma\alpha^*\beta\}) - (\{\gamma\beta^*\alpha\} - \{\gamma\alpha^*\beta\})^*\Big)(z)\ ,$$

or $\chi(z) = 2(x - x^*)(z)$, where the element x is defined as follows

$$x := \{\gamma\beta^*\alpha\} - \{\gamma\alpha^*\beta\}\ ,$$

i.e. $\chi(z)\,\partial/\partial z \in \mathfrak{p}$. The claim is proved.

As a result of the preceding two claims we obtain $[\mathfrak{p}, \mathfrak{p}] \subset \mathfrak{k}$. $\square$

2.19 Lemma. *Suppose U is a complex Banach space and $*: U \to L^2(U)$ is a continuous, conjugate-linear mapping. Furthermore suppose that the inclusion $[\mathfrak{p},\mathfrak{p}] \subset \mathfrak{k}$ holds. Then the pair $(U,*)$ is a J^*-triple system.*

Proof. Consider arbitrary elements α, β in the Banach space U. Denote by $f_{\alpha,\beta}(z)\, \partial/\partial z$ the bracket product of the vector fields $(\alpha-\alpha^*)(z)\,\partial/\partial z$ and $(\beta-\beta^*)(z)\,\partial/\partial z$. For the holomorphic mapping $f_{\alpha,\beta}(z)$ we obtain

$$\begin{aligned} f_{\alpha,\beta}(z) &= d(\alpha-\alpha^*)(z)((\beta-\beta^*)(z)) - d(\beta-\beta^*)(z)((\alpha-\alpha^*)(z)) \\ &= -d\alpha^*(z)((\beta-\beta^*)(z)) + d\beta^*(z)((\alpha-\alpha^*)(z)) \,. \end{aligned}$$

By the properties of the differential of a homogeneous polynomial we get

$$\begin{aligned} f_{\alpha,\beta}(z) &= -2q_\alpha(z,\beta) + 2q_\alpha(z,\beta^*(z)) + 2q_\beta(z,\alpha) - 2q_\beta(z,\alpha^*(z)) \\ &= -2\{z\alpha^*\beta\} + 2\{z\beta^*\alpha\} + 2\{z\alpha^*\{z\beta^*z\}\} - 2\{z\{\beta^*\{z\alpha^*z\}\} \,. \end{aligned}$$

The hypothesis $[\mathfrak{p},\mathfrak{p}] \subset \mathfrak{k}$ implies that $f_{\alpha,\beta}(z)$ is a linear mapping. Therefore the terms in the last sum of order higher than 1 are identically zero, which is equivalent to

$$\{z\alpha^*\{z\beta^*z\}\} = \{z\beta^*\{z\alpha^*z\}\} \,.$$

In particular we have obtained the following representation

$$f_{\alpha,\beta}(z) = -2\beta\,\square\,\alpha^*(z) + 2\alpha\,\square\,\beta^*(z) \,. \tag{2.20}$$

Fix an arbitrary element $\alpha \in U$ and apply the above results to the vector field corresponding to the elements α and $i\alpha$. The conjugate-linearity of the mapping $*$ and representation 2.20 imply

$$f_{\alpha,i\alpha} = -2(i\alpha)\,\square\,\alpha^* + 2\alpha\,\square\,(i\alpha)^* = -i2\alpha\,\square\,\alpha^* - 2i\alpha\,\square\,\alpha^* = -4i(\alpha\,\square\,\alpha^*) \,.$$

Since $f_{\alpha,i\alpha}(z)\,\partial/\partial z$ lies in $\mathfrak{k}$ and $H(U)$ is a real vector space, we obtain that the linear operator $\alpha\,\square\,\alpha^*$ is hermitian. This result does not depend on the choice of α and therefore condition (iii) in the definition of a J^*-triple system holds.

Consider arbitrary elements $\alpha,\beta,\gamma \in U$. Denote by $f_{\alpha,\beta,\gamma}\,\partial/\partial z$ the bracket product of the vector fields

$$f_{\alpha,\beta}\,\partial/\partial z = [(\alpha-\alpha^*)\,\partial/\partial z, (\beta-\beta^*)\,\partial/\partial z]$$

and

$$f_\gamma\,\partial/\partial z = (\gamma-\gamma^*)\,\partial/\partial z \,,$$

i.e.

$$f_{\alpha,\beta,\gamma}\,\partial/\partial z = [f_{\alpha,\beta}\,\partial/\partial z, f_\gamma\,\partial/\partial z] \,.$$

Using representation 2.20 we obtain

$$\begin{aligned} f_{\alpha,\beta,\gamma}(z) &= d(f_{\alpha,\beta})(z)(f_\gamma(z)) - d(f_\gamma)(z)(f_{\alpha,\beta}(z)) \\ &= -2\{\gamma\alpha^*\beta\} + 2\{\gamma\beta^*\alpha\} + 2\{\{z\gamma^*z\}\alpha^*\beta\} - 2\{\{z\gamma^*z\}\beta^*\alpha\} \\ &\qquad - 4\{z\gamma^*\{z\alpha^*\beta\}\} + 4\{z\gamma^*\{z\beta^*\alpha\}\} \\ &= 2(\{\gamma\beta^*\alpha\} - \{\gamma\alpha^*\beta\}) - 2(2\{z\gamma^*\{z\alpha^*\beta\}\} - \{\{z\gamma^*z\}\alpha^*\beta\}) \\ &\qquad + 2(2\{z\gamma^*\{z\beta^*\alpha\}\} - \{\{z\gamma^*z\}\beta^*\alpha\}) \,. \end{aligned} \tag{2.21}$$

By the definition of $\mathfrak{k}$ we have $[\mathfrak{k},\mathfrak{p}] \subset \mathfrak{p}$. The vector field $f_{\alpha,\beta}\,{\partial}/{\partial z}$ lies in $\mathfrak{k}$ and $f_\gamma\,{\partial}/{\partial z}$ lies in $\mathfrak{p}$. Therefore $f_{\alpha,\beta,\gamma} \in \mathfrak{p}$. The representation 2.21 and property 2.4.ii imply

$$f_{\alpha,\beta,\gamma}(z) = 2\big((\{\gamma\beta^*\alpha\} - \{\gamma\alpha^*\beta\}) - (\{\gamma\beta^*\alpha\} - \{\gamma\alpha^*\beta\})^*\big)\,. \tag{2.22}$$

Applying 2.21 and 2.22 we obtain

$$\big(2\{z\gamma^*\{z\alpha^*\beta\}\} - \{\{z\gamma^*z\}\alpha^*\beta\}\big) - \big(2\{z\gamma^*\{z\beta^*\alpha\}\} - \{\{z\gamma^*z\}\beta^*\alpha\}\big) =$$
$$\{z\{\gamma\beta^*\alpha\}^*z\} - \{z\{\gamma\alpha^*\beta\}^*z\}\,.$$

Consider the case when $\beta := i\alpha$. Then the last equality implies

$$\{z\{\gamma\alpha^*\alpha\}^*z\} = 2\{\{\alpha\alpha^*z\}\gamma^*z\} - \{\alpha\alpha^*\{z\gamma^*z\}\}\,.$$

Using polarization we get that 2.4.iii is satisfied and hence $(U,*)$ is a J^*-triple. □

2.23 Proposition. *Suppose U is a complex Banach space and $* : U \to L^2(U)$ is a continuous, conjugate-linear mapping. The pair $(U,*)$ is a hermitian Jordan triple system if and only if the real vector space $\mathfrak{l} = \mathfrak{k} \oplus \mathfrak{p}$ is a real Lie subalgebra of $\mathfrak{P}$.*

Proof. Consider arbitrary vector fields X, Y in $\mathfrak{l}$. Then $X = X_1 + X_2$, $Y = Y_1 + Y_2$, where X_1, Y_1 are vector fields in $\mathfrak{k}$ and X_2, Y_2 are vector fields in $\mathfrak{p}$. Furthermore consider the bracket product

$$[X,Y] = [X_1,Y_1] + [X_1,Y_2] + [X_2,Y_1] + [X_2,Y_2]\,.$$

The first term of the above sum is in $\mathfrak{k}$, since $\mathfrak{k}$ is a real Lie algebra. The second and the third terms are in $\mathfrak{p}$, by the definition of $\mathfrak{k}$. Since the last term is a sum of polynomial vector fields of first and third order, we may conclude

$$[X,Y] \in \mathfrak{l} \quad \text{if and only if} \quad [X_2,Y_2] \in \mathfrak{k}\,.$$

Therefore the real vector space $\mathfrak{l}$ is a real Lie subalgebra of $\mathfrak{P}$ if and only if $[\mathfrak{p},\mathfrak{p}] \subset \mathfrak{k}$. □

Suppose $(U,*)$ is a J^*-triple system. Denote by $\check{\mathfrak{k}}$ the smallest closed real Lie subalgebra of $\mathfrak{P}_0$ containing $[\mathfrak{p},\mathfrak{p}]$ and the element $i\partial = iz\,{\partial}/{\partial z} \in \mathfrak{P}_0$. Furthermore define

$$\widehat{\mathfrak{k}} := \Big\{X \in \mathfrak{P}_0 : [X,\mathfrak{p}] \subset \mathfrak{p}\Big\}\,.$$

Since $\mathfrak{p}$ is a closed real vector space, $\widehat{\mathfrak{k}}$ is a closed vector subspace of $\mathfrak{P}_0$. The Jacobi identity implies that $\widehat{\mathfrak{k}}$ is closed under bracket products and therefore is a real Banach Lie subalgebra of $\mathfrak{P}_0$.

2.24 Proposition. *The real Banach Lie algebras $\check{\mathfrak{k}}, \widehat{\mathfrak{k}}$ and $\mathfrak{k}$, associated with a J^*-triple $(U,*)$, satisfy $\check{\mathfrak{k}} \subset \mathfrak{k} \subset \widehat{\mathfrak{k}}$.*

Proof. The second of the two claimed inclusions follows directly from the definitions of the algebras $\mathfrak{k}$ and $\widehat{\mathfrak{k}}$. By Lemma 2.14 we have $[\mathfrak{p},\mathfrak{p}] \subset \mathfrak{k}$ and therefore in order to obtain the second one it is sufficient to prove that $i\partial \in \mathfrak{k}$. Consider the vector field $iz\,{\partial}/{\partial z}$. Obviously id is a Hermitian element of $L(U)$. Suppose α is an arbitrary element of the vector space U. Denote by $f_{i,\alpha}\,{\partial}/{\partial z}$ the bracket product of $iz\,{\partial}/{\partial z}$ and

$(\alpha - \alpha^*)\,\partial/\partial z$ (taken in this order). Using the notation q_α for the symmetric bilinear mapping corresponding to $\alpha^* \in L^2(U)$, we obtain

$$f_{i,\alpha}(z) = i\alpha - i\alpha^*(z) + 2q_\alpha(z, iz) ,$$

for all $z \in U$. The conjugate linearity of $*$ and 2.10.i imply

$$f_{i,\alpha} = i\alpha - i\alpha^*(z) + 2i\alpha^*(z) = (i\alpha - (i\alpha)^*)(z) ,$$

for all $z \in U$. Since U is a complex Banach space, the vector field $f_{i,\alpha}\,\partial/\partial z \in \mathfrak{p}$ and therefore $i\partial \in \mathfrak{k}$. □

By proposition 2.14 we get that

$$\check{\mathfrak{l}} = \check{\mathfrak{k}} \oplus \mathfrak{p} \quad \text{and} \quad \widehat{\mathfrak{l}} = \widehat{\mathfrak{k}} \oplus \mathfrak{p}.$$

are closed real Lie subalgebras of $\mathfrak{P}$.

2.25 Proposition. *Suppose $(U, *)$ is a J^*-triple. Then we have:*

(i) $\check{\mathfrak{l}} \subset \mathfrak{l} \subset \widehat{\mathfrak{l}}$ *are real Banach Lie algebras without center and* $\check{\mathfrak{l}}$ *is an ideal in* $\widehat{\mathfrak{l}}$.

(ii) $\widehat{\mathfrak{k}} = \{\lambda\,\partial/\partial z \colon \lambda \in \mathsf{Der}(U, *)\}$.

(iii) $\mathfrak{l} \cap i\mathfrak{l} = \{\alpha\,\partial/\partial z \in \mathfrak{P}_{-1} \colon \alpha^* = 0\}$ *and the sum* $\mathfrak{k} + i\mathfrak{k}$ *in* $\mathfrak{P}_0$ *is topologically direct.*

(iv) *Every (continuous) derivation of* $\widehat{\mathfrak{l}}$ *is inner if*

$$*\colon U \to U^*$$

is homeomorphism, where $U^* := \{\alpha^* : \alpha \in U\}$.

Proof.

(i) Consider an arbitrary vector field X in the center $\mathrm{Z}(\check{\mathfrak{l}})$ of $\check{\mathfrak{l}}$. The definition of $\check{\mathfrak{l}}$ implies that it has a representation of the form $X = \lambda\,\partial/\partial z + (\alpha - \alpha^*)\,\partial/\partial z$ for some vector fields $\lambda\,\partial/\partial z \in \check{\mathfrak{k}}$ and $(\alpha - \alpha^*)\,\partial/\partial z \in \mathfrak{p}$. Denote by $Y = g(z)\,\partial/\partial z$ the bracket product of X and $iz\,\partial/\partial z$. Then by

$$\mathrm{Z}(\check{\mathfrak{l}}) = \{X \in \check{\mathfrak{l}} : [X, Y] = 0 \text{ for all } Y \in \check{\mathfrak{l}}\} , \tag{2.26}$$

we have

$$\begin{aligned} g(z) &= d(\lambda + (\alpha - \alpha^*))(z)(iz) - d(iz)(z)(\lambda + (\alpha - \alpha^*)(z)) \\ &= \lambda(iz) - 2q_\alpha(z, iz) - i\lambda - i\alpha + i\alpha^*(z) = -2i\alpha^*(z) - i\alpha + i\alpha^*(z) \\ &= ((i\alpha)^* - (i\alpha))(z) = -((i\alpha) - (i\alpha)^*)(z) = 0 \end{aligned}$$

for all z in U. Therefore $(i\alpha) = (i\alpha)^*(z)$ for all $z \in U$. But $\alpha^*(0) = 0$, and therefore $\alpha = 0$. Then the vector field X is of the form $X = \lambda\,\partial/\partial z \in \mathfrak{P}_0$.

Using the same arguments we find an explicit representation of the vector field $Y = h_\alpha(z)\,\partial/\partial z$, which denotes the bracket product of the vector fields X and $(\alpha - \alpha^*)\,\partial/\partial z$. Then

$$\begin{aligned} h_\alpha(z) &= d\lambda(z)((\alpha - \alpha^*)(z)) - d(\alpha - \alpha^*)(z)(\lambda(z)) \\ &= \lambda((\alpha - \alpha^*)(z)) + 2q_\alpha(z, \lambda(z)) \\ &= \lambda(\alpha) - \lambda(\alpha^*(z)) + 2\{z\alpha^*\lambda(z)\} \\ &= \lambda(\alpha) - \lambda\{z\alpha^* z\} + 2\{z\alpha^*\lambda(z)\} = 0 \end{aligned}$$

for all z in U. In particular $h_\alpha(0) = \lambda(\alpha) = 0$. Since X is in the center of $\check{\mathfrak{l}}$, the last equation holds for all $\alpha \in U$, i.e. λ is identically zero. Therefore $X \equiv 0$, and hence the Banach Lie algebra $\check{\mathfrak{l}}$ is without center. Since $Z(\widehat{\mathfrak{l}}) \subseteq Z(\mathfrak{l}) \subseteq Z(\check{\mathfrak{l}})$, the Banach Lie algebras $\mathfrak{l}$ and $\widehat{\mathfrak{l}}$ are without center. The Jacobi identity implies that $\check{\mathfrak{l}}$ is an ideal in $\widehat{\mathfrak{l}}$.

(ii) Consider an arbitrary vector field X in the Banach Lie algebra $\widehat{\mathfrak{k}}$. It has a representation of the form $X = \lambda(z)\, \partial/\partial z$, where $\lambda(z) \in L(U)$.

Fix an arbitrary element $\alpha \in U$. Denote by $f_{\lambda,\alpha}(z)\, \partial/\partial z$ the bracket product of X and $(\alpha - \alpha^*)\, \partial/\partial z$ (taken in this order). In particular for the holomorphic mapping $f_{\lambda,\alpha}(z)$ we obtain

$$
\begin{aligned}
f_{\lambda,\alpha}(z) &= d\lambda(z)((\alpha - \alpha^*)(z)) - d((\alpha - \alpha^*)(z))(\lambda(z)) \\
&= \lambda((\alpha - \alpha^*)(z)) + d\alpha^*(z)(\lambda(z)) \\
&= \lambda(\alpha) - \lambda(\alpha^*(z)) + 2q_\alpha(z, \lambda(z)) \\
&= \lambda(\alpha) - \lambda\{z\alpha^* z\} + 2\{z\alpha^*(\lambda(z))\}
\end{aligned}
\tag{2.27}
$$

for all $z \in U$. The definition of $\mathfrak{k}$ implies that $f_{\lambda,\alpha}(z)\, \partial/\partial z \in \mathfrak{p}$. Representation 2.27 implies

$$
f_{\lambda,\alpha}(z) = (\lambda(\alpha) - \lambda(\alpha)^*)(z) \quad \text{for all} \quad z \in U\,. \tag{2.28}
$$

As a direct consequence of 2.27 and 2.28 we obtain

$$
(\lambda(\alpha))^*(z) = \{z(\lambda(\alpha))^* z\} = \lambda\{z\alpha^* z\} - 2\{z\alpha^*(\lambda z)\} \quad \text{for all} \quad z \in U\,,
$$

and therefore

$$
\lambda\{z\alpha^* z\} = \{(\lambda z)\alpha^* z\} + \{z(\lambda(\alpha))^* z\} + \{z\alpha^* \lambda(z)\} \quad \text{for all} \quad z \in U\,. \tag{2.29}
$$

Since α was arbitrary element of U, the last equation holds for all α in U. Consider arbitrary elements w, α, z in U. Applying 2.29 to the fixed elements we obtain the equalities

$$
\begin{aligned}
\lambda\{z\alpha^* z\} &= \{(\lambda z)\alpha^* z\} + \{z(\lambda\alpha)^* z\} + \{z\alpha^*(\lambda z)\} \\
\lambda\{(w+z)\alpha^*(w+z)\} &= \{(\lambda(w+z))\alpha^*(w+z)\} \\
&\quad + \{(w+z)(\lambda\alpha)^*(w+z)\} + \{(w+z)\alpha^*(\lambda(w+z))\} \\
\lambda\{w\alpha^* w\} &= \{(\lambda w)\alpha^* w\} + \{w(\lambda\alpha)^* w\} + \{w\alpha^*(\lambda w)\}\,.
\end{aligned}
$$

Therefore

$$
\begin{aligned}
\lambda\{w\alpha^* z\} &= \lambda\Big(\frac{1}{2}\big(\alpha^*(w+z) - \alpha^*(w) - \alpha^*(z)\big)\Big) \\
&= \frac{1}{2}\big(\lambda(\{(w+z)\alpha^*(w+z)\}) - \lambda(\{w\alpha^* w\}) - \lambda(\{z\alpha^* z\})\big) \\
&= \{\lambda(w)\alpha^* z\} + \{w(\lambda(\alpha))^* z\} + \{z\alpha^*(\lambda(w))\}\,.
\end{aligned}
$$

Suppose $\lambda \in \mathsf{Der}(U, *)$ and α is an arbitrary element of the vector space U. Then 2.27 implies

$$
[\lambda(z)\, \partial/\partial z, (\alpha - \alpha^*)\, \partial/\partial z] = (\lambda(\alpha) - (\lambda(\alpha))^*)\, \partial/\partial z \in \mathfrak{p}\,.
$$

Hence $\lambda\, \partial/_{\partial z}$ lies in $\widehat{\mathfrak{k}}$.

(iii) Suppose the vector field X lies in $\mathfrak{l} \cap i\mathfrak{l}$. Since X lies in $\mathfrak{l} = \mathfrak{k} \oplus \mathfrak{p}$, X admits the representation

$$X = i\lambda\, \partial/_{\partial z} + (\alpha - \alpha^*)\, \partial/_{\partial z} = (i\lambda + (\alpha - \alpha^*))\, \partial/_{\partial z} ,$$

where $\lambda \in H(U)$ and $\alpha \in U$. Since X lies in $i\mathfrak{l}$, we have that $X = iX_1$ for some vector field $X_1 \in \mathfrak{l}$. The above arguments show that $X_1 = i\lambda_1\, \partial/_{\partial z} + (\beta - \beta^*)\, \partial/_{\partial z}$ for some hermitian element $\lambda_1 \in H(U)$ and $\beta \in U$. Using the conjugate-linearity of $*$ we obtain

$$\begin{aligned}(i\lambda + (\alpha - \alpha^*))\, \partial/_{\partial z} &= i(\lambda + (-i\alpha + i\alpha^*))\, \partial/_{\partial z} \\ &= i(\lambda + (-i\alpha + (-i\alpha)^*))\, \partial/_{\partial z} .\end{aligned}$$

Therefore X_1 admits the representation $X_1 = (\lambda + (-i\alpha + i\alpha^*))\, \partial/_{\partial z} \in \mathfrak{l}$. As a consequence we obtain that $\lambda = i\lambda_1$. From $-i\alpha + (-i\alpha)^*\, \partial/_{\partial z} \in \mathfrak{p}$ follows $(-i\alpha)^* = -(-i\alpha)^*$. Hence $i\alpha^* = -i\alpha^*$, i.e. $2i\alpha^* = 0$. Therefore $\alpha^* = 0$. Then the considered vector field X is of the form $X = (\lambda(z) + \alpha)\, \partial/_{\partial z}$. Since $\lambda = i\lambda_1$ and λ, λ_1 are hermitian elements for the numerical range of λ we obtain:

$$V(L(U), \lambda) = V(L(U), i\lambda_1) = iV(L(U), \lambda) \subset i\mathbb{R} ,$$

since $V(L(U), \lambda_1) \subset \mathbb{R}$. But $V(L(U), \lambda) \subset \mathbb{R}$, since λ is hermitian. So we have obtained $V(L(U), \lambda) = 0$. The algebra $L(U)$ is a complex Banach algebra, and therefore for the numerical radius $\nu(\lambda)$ the following inequality holds $\nu(\lambda) \geq \frac{1}{e}\|\lambda\|$ (compare [2], p. 34). This implies $\|\lambda\| = 0$ and hence $\lambda = 0$. Therefore the considered vector field X is of the form $X = \alpha\, \partial/_{\partial z}$, where $\alpha^* = 0$.

Obviously every vector field of the form $\gamma\, \partial/_{\partial z}$, for some γ in U, such that $\gamma^* = 0$ lies in $\mathfrak{l} \cap i\mathfrak{l}$.

The sum $H(U) + iH(U)$ is topologically direct in $L(U)$ by ([2], p. 50).

(iv) Suppose D is an arbitrary derivation of $\widehat{\mathfrak{l}}$, i.e. $D: \widehat{\mathfrak{l}} \to \mathfrak{l}$ is a continuous linear mapping satisfying

$$D([X, Y]) = [D(X), Y] + [X, D(Y)]$$

for all X, Y in $\widehat{\mathfrak{l}}$. Consider the image $D(i\partial)$. We show that without loss of generality we may assume that $D(i\partial)$ lies in $\widehat{\mathfrak{k}}$. Suppose that $D(i\partial) = \lambda\, \partial/_{\partial z} + X_\alpha$ for some linear mapping $\lambda \in \mathsf{Der}(U, *)$ and vector field $X_\alpha := (\alpha - \alpha^*)\, \partial/_{\partial z} \in \mathfrak{p}$. Then $\eta := D - \mathsf{ad}(X_{i\alpha})$, where $X_{i\alpha} := (i\alpha - (i\alpha^*))\, \partial/_{\partial z} \in \mathfrak{p}$ is a derivation of $\widehat{\mathfrak{l}}$, such that $\eta(i\partial) \in \widehat{\mathfrak{k}}$. Now if we prove that η is an inner derivation, we can conclude directly that D is an inner derivation as a sum of η and $\mathsf{ad}(X_{i\alpha})$. So we assume that $D(i\partial) = \lambda\, \partial/_{\partial z}$ lies in $\widehat{\mathfrak{k}}$. Furthermore consider the equality

$$D(0) = 0 = D([i\partial, \widehat{\mathfrak{k}}]) = [D(i\partial, \widehat{\mathfrak{k}})] + [i\partial, D(\widehat{\mathfrak{k}})] .$$

So we obtain

$$[D(X), i\partial] = [D(i\partial), X]$$

for every vector field $X \in \widehat{\mathfrak{k}}$. But $[\lambda\, \partial/_{\partial z}, X] \in \widehat{\mathfrak{k}}$ for every $X \in \widehat{\mathfrak{k}}$ and so we obtain also $[D(X), i\partial] \in \widehat{\mathfrak{k}}$ for all $X \in \widehat{\mathfrak{k}}$, i.e. $[D(\widehat{\mathfrak{k}}), i\partial] \subset \widehat{\mathfrak{k}}$. Therefore $D(\widehat{\mathfrak{k}}) \subset \widehat{\mathfrak{k}}$. Analogously we obtain that $D(\mathfrak{p}) \subset \mathfrak{p}$. Now consider the complexification

$$\mathfrak{g} := \widehat{\mathfrak{l}} \oplus i\widehat{\mathfrak{l}} = \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1$$

of $\widehat{\mathfrak{l}}$, where $\mathfrak{g}_\nu \subset \mathfrak{p}_v$ for every $\nu \in \{-1,0,1\}$. The mapping D extends to a continuous linear derivation of $\mathfrak{g}$. Furthermore $D(\mathfrak{g}_{-1}) \subset \mathfrak{g}_{-1}$ for every $\nu \in \{-1,0,1\}$. Since $D(\mathfrak{g}_{-1}) \subset \mathfrak{g}_{-1}$ and $\mathfrak{g}_1 \cong U$, the restriction $D_{\mathfrak{g}_{-1}}$ determines a continuous linear mapping $\lambda: U \to U$. Define $\lambda(\alpha) := \beta$, if $D(\alpha\, \partial/\partial z) = \beta\, \partial/\partial z$. So is defined also a linear vector field

$$X = \lambda\, \partial/\partial z \in \mathfrak{P}_0 \,.$$

As a direct conclusion of the Jacobi identity, we have

$$[Y,[X,Z]] = [X,[Y,Z]] - [[X,Y],Z]$$

for all vector fields Y, Z in $\mathfrak{g}$. For every vector field $Y = \alpha\, \partial/\partial z \in \mathfrak{g}_{-1}$ we have that $[X,Y] = \lambda(\alpha)\, \partial/\partial z$ lies in $\mathfrak{g}_{-1}$, and so

$$[X,Y] = D(Y) \quad \text{for all} \quad Y \in \mathfrak{g}_1 \,.$$

Now consider arbitrary vector fields $Y \in \mathfrak{g}_{-1}$ and $Z \in \mathfrak{g}_0$. Then $[Y,Z]$ lies in $\mathfrak{g}_{-1}$ and the above result implies

$$\begin{aligned} [Y,[X,Z]] &= D([Y,Z]) - [D(Y),Z] \\ &= [D(Y),Z] + [Y,D(Z)] - [D(Y),Z] \\ &= [Y,D(Z)] \,. \end{aligned}$$

Then

$$D(Z) = [X,Z] \quad \text{for all} \quad Z \in \mathfrak{g}_0 \,.$$

Now we apply the same argument to arbitrary vector fields $Y \in \mathfrak{g}_{-1}$ and $Z \in \mathfrak{g}_1$. Then the vector field $[Y,Z]$ lies in $\mathfrak{g}_0$ and by the above result $[X,[Y,Z]] = D([Y,Z])$. Therefore

$$\begin{aligned} [Y,[X,Z]] &= [D(Y),Z] + [Y,D(Z)] - [D(Y),Z] \\ &= [Y,D(Z)] \,. \end{aligned}$$

Then

$$D(Z) = [X,Z] \quad \text{for all} \quad Z \in \mathfrak{g}_1 \,.$$

In particular we obtain that X lies in $\widehat{\mathfrak{k}}$. Consequently $D = \mathsf{ad}(X)$ is an inner derivation of $\widehat{\mathfrak{l}}$. □

2.30 Definition. Suppose $(U,*)$ is a J^*-triple system. A closed $\mathbb{C}$-linear subspace $V \subset U$ is called a *subsystem* if $\alpha^*(V) \subset V$ for all $\alpha \in V$. If $V \subset U$ is only a closed $\mathbb{R}$-linear subspace with this property V is called *a real subsystem.*

For every real subsystem V of $(U,*)$ the closure $V^{\mathbb{C}}$ of $V + iV$ in U is a complex subsystem.

2.31 Definition. Let V be a real subsystem of the J^*-triple $(U,*)$. Then

(i) V is called *associative* if

$$\{\{\alpha\beta^*\gamma\}\delta^*z\} = \{\alpha\beta^*\{\gamma\delta^*z\}\}$$

for all $\alpha, \beta, \gamma, \delta, z \in V$

(ii) V is called *flat* if

$$\{\alpha\beta^*z\} = \{\beta\alpha^*z\}$$

for all $\alpha, \beta, z \in V$.

Every flat subsystem is associative. For every associative subsystem the expression in 2.31.i also coincides with $\{\alpha\{\beta\gamma^*\delta\}^*z\}$.

2.32 Proposition. *Suppose $(U, *)$ is a J^*-triple system, and $V \subset U$ is an associative subsystem such that $U = V + iV$.*

(i) *The closed $\mathbb{R}$-linear subspace A of $L(U)$ generated by $V \square V^*$ is a commutative Banach algebra.*

(ii) *V is flat if and only if*

$$\{(\alpha - \alpha^*)\, {\partial}/{\partial z} \colon \alpha \in V\} \subset \mathfrak{p}$$

is a commutative Lie subalgebra.

Proof. Consider the set $V \square V^* \subset L(U)$.

Claim: $V \square V^*$ is closed under multiplication in $L(U)$.

Suppose $\alpha \square \beta^*,\ \gamma \square \delta^* \in V \square V^*$. Then

$$(\alpha \square \beta^* \circ \gamma \square \delta^*)(z) = \{\alpha \beta^* \{\gamma \delta^* z\}\} = \{\alpha \{\beta \gamma^* \delta\}^* z\} = (\alpha \square \{\beta \gamma^* \delta\}^*)(z)\ ,$$

where we have used that U is associative. By the definition of the triple product we have

$$\{\beta \gamma^* \delta\} = \frac{1}{2}(\gamma^*(\beta + \delta) - \gamma^*(\beta) - \gamma^*(\delta))\ . \tag{2.33}$$

The space V is a flat subsystem. Therefore $\alpha^*(V) \subset V$ for all $\alpha \in V$, and hence by 2.33 we have $\{\beta \gamma^* \delta\} \in V$. Therefore $\alpha \square \{\beta \gamma^* \delta\}^* \in V \square V^*$, or $V \square V^*$ is closed under multiplication in $L(U)$.

Claim: The product in $V \square V^*$ commutes.

Suppose $\alpha \square \beta^*,\ \gamma \square \delta^* \in V \square V^*$. Then

$$\begin{aligned}(\alpha \square \beta^* \circ \gamma \square \delta^*)(z) &= \{\alpha \beta^* \{\gamma \delta^* z\}\} = \{\alpha \beta^* \{z \delta^* \gamma\}\} \\ &= \{\{\alpha \beta^* z\} \delta^* \gamma\} = \{\gamma \delta^* \{\alpha \beta^* z\}\} \\ &= (\gamma \square \delta^* \circ \alpha \square \beta^*)(z)\ ,\end{aligned}$$

where we have used the symmetry of the triple product on the outer arguments, and the fact that U is associative.

The above claims imply that A is a commutative Banach subalgebra of $L(U)$, i.e. (i) is proved.

In order to obtain (ii) consider arbitrary elements α, β in V. Then for the bracket product $f_{\alpha,\beta}\, {\partial}/{\partial z}$ of the corresponding vector fields $(\alpha - \alpha^*)\, {\partial}/{\partial z}$ and $(\beta - \beta^*)\, {\partial}/{\partial z}$ we get

$$\begin{aligned}f_{\alpha,\beta}(z) &= d(\alpha - \alpha^*)(z)((\beta - \beta^*)(z)) - d(\beta - \beta^*(z))(\alpha - \alpha^*(z)) \\ &= -d\alpha^*(z)((\beta - \beta^*)(z)) + d\beta^*(z)((\alpha - \alpha^*)(z))\end{aligned}$$

Then using the notation q_α for the bilinear mapping corresponding to the homogeneous polynomial α^*, we get

$$\begin{aligned}f_{\alpha,\beta}(z) &= -2q_\alpha(z, \beta) + 2q_\alpha(z, \beta^*(z)) + 2q_\beta(z, \alpha) - 2q_\beta(z, \alpha^*(z)) \\ &= -2\{z\alpha^*\beta\} + 2\{z\alpha^*\{z\beta^* z\}\} + 2\{z\beta^*\alpha\} - 2\{z\beta^*\{z\alpha^* z\}\}\ ,\end{aligned}$$

and by 2.18

$$f_{\alpha,\beta}(z) = -2\{z\alpha^*\beta\} + 2\{z\beta^*\alpha\}.$$

Then obviously V is flat if and only if

$$\{(\alpha - \alpha^*)\, {\partial}/{\partial z}\ :\ \alpha \in V\}$$

is a commutative Lie algebra. □

2.34 Example. Let S be a locally compact topological space and let $c: S \to \mathbb{R}$ a bounded continuous function. Then $U = C_0(S, \mathbb{C})$ is a complex Banach space. For all $\alpha, z \in U$ define $\alpha^*(z) = c\overline{\alpha}z^2$. Then $(U, *)$ is a J^*-triple system, which we denote by $(C_0(S), c)$. If $c \equiv 1$, we write $C_0(S)$.

2.35 Proposition. *Suppose S is a locally compact topological space. Then*
(i) *The J^*-triple system $(C_0(S), c)$ is associative.*
(ii) *The real subsystem $C_0(S, \mathbb{R})$ in $(C_0(S), c)$ is flat.*

Proof.
(i) Simple computations imply

$$\{\{\alpha\beta^*\gamma\}\delta^*z\} = c\overline{\delta}zc\overline{\beta}\alpha\gamma$$

and

$$\{\alpha\beta^*\{\gamma\delta^*z\}\} = c\overline{\beta}\alpha c\overline{\delta}\gamma z \ .$$

Therefore the system is associative.
(ii) Suppose α, β, z are arbitrary elements of $C_0(S, \mathbb{R})$. Then

$$\begin{aligned}\{\alpha\beta^*z\} &= \frac{1}{2}(\beta^*(\alpha + z) - \beta^*(\alpha) - \beta^*(z)) \\ &= \frac{1}{2}(c\overline{\beta}(\alpha + z)^2 - c\overline{\alpha}\overline{\beta} - c\overline{\alpha}z^2) = c\overline{\beta}\alpha z = c\beta\alpha z \ .\end{aligned}$$

Equivalent computations imply

$$\begin{aligned}\{\beta\alpha^*z\} &= \frac{1}{2}(\alpha^*(\beta + z) - \alpha^*(\beta) - \alpha^*(z)) \\ &= \frac{1}{2}(c\overline{\alpha}(\beta + z)^2 - c\overline{\alpha}\overline{\beta} - c\overline{\alpha}z^2) = c\overline{\alpha}\beta z = c\alpha\beta z \ .\end{aligned}$$

Hence $C_0(S, \mathbb{R})$ is a flat subsystem. □

2.36 Example. Suppose $\alpha \in U$ is a J^*-triple system. Consider an arbitrary element $\alpha \in U$. Denote by U_α the smallest closed, $\mathbb{R}$-linear, $\alpha \square \alpha^*$-invariant subspace of U containing α. Then U_α is a flat subsystem of $(U, *)$ and the complex subsystem ${U_\alpha}^{\mathbb{C}}$ is associative.

3. Symmetric Banach Manifolds

3.1 Definition. Suppose D is a Hausdorff topological space. A triple (V, p, E) is called a *chart* of D if V is an open subset of D, E is a Banach space over $\mathbb{K}$ and $p: V \to E$ is a homeomorphism onto an open subset of E. If $a \in V$ is a point satisfying $p(a) = 0$, (V, p, E) is called a chart about a. The charts (V, p, E) and (W, q, F) are said to be $\mathbb{K}$-*analytically compatible* if the homeomorphism

$$p \circ q^{-1}: p(V \cap W) \to q(V \cap W)$$

between the open subsets of E and F, respectively, is bianalytic. In this case , E and F are isomorphic provided $V \cap W \neq \emptyset$. An *atlas* of D is a collection of pairwise compatible charts covering D. A maximal atlas (under inclusion) endows D with the structure of a Banach manifold over $\mathbb{K}$. In case that $\mathbb{K} = \mathbb{C}$ we refer to D as a complex Banach manifold.

For every complex Banach manifold we denote by $\mathsf{Aut}(D)$ the group of all biholomorphic automorphisms and by $\mathrm{aut}(D)$ the set of all complete holomorphic vector fields. We denote by T_aD the tangent space at the point $a \in D$, and by Tah the differential at the point a of every holomorphic mapping $h: D \to D'$, where D' is also a complex Banach manifold.

3.2 Definition. Let D be a complex Banach manifold and

$$\nu : TD \to \mathbb{R}$$

a lower semi-continuous function. Then ν is called a *norm* on TD if the restriction of ν to every tangent space T_x, $x \in D$, is a norm on T_x with the following property: There is a neighborhood U of $x \in D$ biholomorphically equivalent to a domain in a complex Banach space E such that

$$c\|a\| \leq \nu(u,a) \leq C\|a\| \qquad \text{for all} \qquad (u,a) \in U \times E \cong T(U)$$

and suitable constants $0 < c \leq C$.

3.3 Definition. A complex Banach manifold D together with a fixed norm ν on the tangent bundle TD is called a *normed complex Banach manifold.*

3.4 Definition. Let (D, ν) and $(\tilde{D}, \tilde{\nu})$ be normed complex Banach manifolds. A holomorphic mapping $\varphi : D \to \tilde{D}$ is called *contracting* if $\tilde{\nu} \circ T\varphi \leq \nu$, and an *isometry* if equality holds.

For a normed complex Banach manifold D the length of every piecewise smooth curve $\gamma: [0,1] \to D$ is well defined:

$$L_\nu(\gamma) = \inf\Big\{ \int_0^1 h(t)\, dt : h \text{ integrable },\ h(t) \geq \nu \circ \dot{\gamma}(t) \text{ for all } t \in I \Big\}\ .$$

Suppose D is a connected normed complex Banach manifold. Then we define the (depending on the norm ν) distance $d(x,y)$ for all $x, y \in D$ the following way,

$$d(x,y) = \inf\Big\{ L_\nu(\gamma) :\ \gamma(0) = x,\ \gamma(1) = y \Big\}\ .$$

Furthermore for every connected normed Banach manifold D the metric d is compatible with the topology of D (compare [17], p. 201, Pr. 12.22).

3.5 Example. Suppose D is a complex Banach manifold biholomorphically equivalent to a bounded domain in a complex Banach space. Denote by $\Delta = \{z \in \mathbb{C}\colon |z| < 1\}$ and by $\mathfrak{F}$ the set of all holomorphic functions defined on D with values in Δ. Further let

$$\tau \colon \Delta \times \mathbb{C} \to \mathbb{R}$$

be the norm on $T\Delta \cong \Delta \times \mathbb{C}$ defined by $\tau(t,a) = |a|$. Then

$$\nu := \sup\{\tau \circ Tf\colon\ f \in \mathfrak{F}\}$$

is a norm on the tangent bundle TD (known as the *Carathéodory norm on* D). Every $g \in \mathsf{Aut}(D)$ is an isometry with respect to this norm.

The following statement is due to Upmeier (compare [16]).

3.6 Proposition. *Let D be a connected normed complex Banach manifold. Denote by L the group of all biholomorphic isometries $g : D \to D$. There exists a topology $\mathcal{T}_a$ on L such that $(L, \mathcal{T}_a)$ is a real Banach Lie group with the properties:*

(i) *The mapping $L \times D \to D$ defined by $(g, x) \mapsto gx$ is real analytic.*

(ii) *A homomorphism $\lambda : \mathbb{R} \to L$ is analytic if and only if the mapping $\mathbb{R} \times D \to D$ defined by $(t, x) \mapsto \lambda(t)x$ is analytic.*

For the following L is always endowed with the topology $\mathcal{T}_a$. Every isometry $g \in L$ is also an isometry with respect to the metric d.

3.7 Definition. A closed subspace E of a Banach space L over $\mathbb{K}$ is called *split* (or *direct subspace*) if E has a topological complement in L, i.e. a closed subspace $F \subset L$ such that $L = E \oplus F$ is the topological direct sum. Any such decomposition is called a *splitting* of L.

3.8 Definition. Suppose $g: D \to \tilde{D}$ is an analytic mapping between Banach manifolds and let $T_a g: T_a D \to T_{ga}\tilde{D}$ be the differential of g at $a \in D$. The mapping g is called a *submersion at a* if the following conditions hold:

$T_a g$ is surjective and the null-space $\operatorname{Ker} T_a g$ *is a split subspace of $T_a D$.*

The mapping g is called a *submersion*, if it is a submersion at each point of $a \in D$.

3.9 Definition. Suppose $g: D \to \tilde{D}$ is an analytic mapping between Banach manifolds and let $T_a g: T_a D \to T_g a\tilde{D}$ be the differential of g at $a \in D$. The mapping g is called an *immersion* at a it the following condition holds

$T_a g$ is injective and the image space $T_a g(T_a D)$ is a split (closed) subspace of $T_{g(a)}\tilde{D}$.

The mapping g is called an *immersion*, if it is an immersion at each point of $a \in D$.

3.10 Definition. Let D be a connected normed complex Banach manifold and L the Lie group of all biholomorphic isometries of D. Then D is called *symmetric* if there is a point $a \in D$ such that:

(i) There is an involution $s \in L$ with a as isolated fixed point.

(ii) The mapping $L \to D$ defined by $g \mapsto ga$ is a submersion.

For the following D will denote a symmetric complex Banach manifold and L the real Banach Lie group of all biholomorphic isometries on D.

We use the following statement, known as the Surjective Mapping Theorem.

3.11 Proposition. *(Graves). Let U be open in a Banach space E. Let $f: U \to F$ be a C^1 map into a Banach space F. Let $x_0 \in E$. If $df(x_0)$ is surjective, then f is locally open in a neighborhood of x_0. More precisely, there exists an open neighborhood V of x_0 contained in U having the following property. For each $x \in V$ and open ball B_x centered at x, contained in V, the image $f(B_x)$ contains an open neighborhood of $f(x)$.*

Proof. ([13], p. 193).

3.12 Proposition. *Suppose D is a symmetric Banach manifold and L is the Lie group of all biholomorphic isometries of D. Then condition* 3.10.*ii implies that the group L is transitive on D.*

Proof. Suppose $\chi: L \to D$, is defined by $\chi(g) := g(a)$ for all $g \in L$. The orbit $L(a)$ is equal to the image $\chi(L) \subset D$. Using 3.11 we prove that $\chi(L)$ is open in D. Fix a point $b \in \chi(L)$ and $g \in L$, such that $b = \chi^{-1}(b)$, and consider charts (U, φ) and (V, ψ) about

g and b respectively. Consider the diagram

$$\begin{array}{ccc} L & \xrightarrow{\chi} & D \\ \downarrow{\scriptstyle\varphi} & & \downarrow{\scriptstyle\psi} \\ \varphi(U) & \xrightarrow{\psi\circ\chi\circ\varphi^{-1}} & \psi(V) . \end{array}$$

By the definition of symmetric manifold we conclude that $d(\psi\circ\chi\circ\varphi^{-1})(g)$ is surjective. Apply 3.11 to the points $\varphi(g) \in \varphi(U)$ and $\psi(b) \in \psi(V)$. Therefore there exist an open neighborhood $W \subset \varphi(U)$ of $\varphi(g)$, such that $(\psi \circ \chi \circ \varphi^{-1})(W) \subset \psi(V)$ is an open neighborhood of $\psi(b)$. This implies that $\psi^{-1}((\psi\circ\chi\circ\varphi^{-1})(W))$ is an open neighborhood of b in D. Moreover $\psi^{-1}((\psi \circ \chi \circ \varphi^{-1})(W)) \subset \chi(L)$. Then $L(a)$ is open in D.

The orbit $L(a)$ is closed in D. Denote by $U := L(a)$. Assume the contrary. Then $\overline{U}\backslash U \neq \emptyset$, or there exists a point $c \in \partial U = \overline{U} \cap \overline{D\backslash U}$. Since U is open, $a \in U$, and the metric d is compatible with the topology of D, there exists a positive number r such that the ball

$$B(a,r) = \{x \in D\colon\ d(x,a) < r\} \subset U\ .$$

Since $c \in \overline{U}$, every open set containing c has nonempty intersection with U. In particular for the ball with radius r and center c we obtain $B(c,r) \cap U \neq \emptyset$, which is equivalent to the existence of a point $b \in B(c,r) \cap U$. From $b \in U$ follows the existence of an element $g \in L$, such that $g(b) = a$. Using the fact that every isometry in L is also an isometry with respect to the metric d, we obtain

$$d(a,g(c)) = d(g(b),g(c)) = d(b,c) < r\ .$$

This implies $g(c) \subset U$. Therefore $c \in U$, which contradicts the choice of the point c. Hence $U = \overline{U}$ and hence the orbit $L(a)$ is closed in D.

Since D is connected, the proof is complete. □

The following proposition is known as Cartan Uniqueness Theorem.

3.13 Proposition. *Suppose D is a bounded domain in a complex Banach space U. A holomorphic mapping $g\colon D \to D$ is the identity on D, if there exists a point $a \in D$ such that $g(a) = a$ and $dg(a) = \mathrm{id} \in L(U)$.*

Proof. ([11]) □

3.14 Proposition. *Suppose D is a symmetric Banach manifold. Then*
(iii) *To every $x \in D$ there exists a uniquely determined involution $s_x \in L$ with x as isolated fixed point.*

Proof. Consider an arbitrary element x of D. Since L acts transitively on D there exists a biholomorphic isometry $g \in L$ such that $g(a) = x$. Consider the mapping $s_x := g \circ s \circ g^{-1}$. Obviously $s_x \in L$ is involutive, and x is a fixed point of s_x.

Claim: x is an isolated fixed point of s_x.

Since a is isolated fixed point of s there exists a neighborhood V of a such that $s(y) \neq y$ for all $y \in V\backslash\{a\}$. The mapping g is in particular a bijective homeomorphism and therefore is open. Hence $V_1 := g(V)$ is an open neighborhood of $g(a) = x$. We shall prove that $s_x(y) \neq y$ for all $y \in V_1\backslash\{x\}$, and therefore x is an isolated fixed point of s_x. Assume the contrary. Then there exists $y \in V_1\backslash\{x\}$ such that $s_x(y) = y$. Therefore $y = (g \circ s \circ g^{-1})(y) = (g \circ s)(g^{-1}(y))$, or equivalently $g^{-1}(y) = s(g^{-1}(y))$. Since $y \in V_1\backslash\{x\}$

and g is bijective, we have $g^{-1}(y) \in g^{-1}(V_1 \backslash \{x\}) = V \backslash \{g^{-1}(x)\} = V \backslash \{a\}$. So we have obtained that s has a different from a fixed point in V which is a contradiction.

Claim: The involution s_x is uniquely determined.

Assume the contrary. Then there exists an involution $\tau_x \in L$ with x as isolated fixed point, $\tau_x \neq s_x$. There exists a chart (W, φ, E) about x, such that $\varphi(W)$ is a bounded domain in the complex Banach space E. Consider the mappings $\tau'_x := \varphi \circ \tau \circ \varphi^{-1}$ and $s'_x := \varphi \circ s \circ \varphi^{-1}$. Since $\tau'_x, s'_x \in \mathsf{Aut}(D)$ are involutions, we obtain $d\tau'_x = ds'_x = -id$ on $\varphi(W)$ as well as $\tau'_x(\varphi(x)) = s'_x(\varphi(x))$. By 3.13 we obtain $\tau'_x = s'_x$. Since φ is a bijection, $\tau_{|W} = s_{|W}$. By the uniqueness of the analytic continuation on W we obtain $\tau_x = s_x$, which is the desired contradiction. □

So we have obtained that for every connected normed complex Banach manifold D the existence of a point $a \in D$ satisfying the conditions 3.10.i and 3.10.ii imply 3.14.iii. For the inverse implication compare [18].

For every symmetric Banach manifold M and every $x \in M$, we denote by s_x the symmetry at the point x.

3.15 Definition. Let D and $\tilde{D}$ be symmetric complex Banach manifolds. A holomorphic mapping $h : D \to \tilde{D}$ is called a *morphism of symmetric manifolds* if

$$h \circ s_x = s_{hx} \circ h$$

for all $x \in D$.

3.16 Definition. A morphism of symmetric manifolds which is in addition a contraction is called a *metric morphism*.

Consider a symmetric Banach manifold D. Denote by L the Banach Lie group of all biholomorphic isometries of D. The group L acts analytically, transitively and faithfully on the manifold D (3.6, 3.12, 3.10). Therefore there is a uniquely determined analytic and faithful action of the Banach Lie algebra $\mathfrak{l} = \mathrm{Lie}(L)$ on the manifold D (compare [17], p. 99, Pr. 6.12). In particular there is a uniquely determined mapping ρ such that the diagram

$$\begin{array}{ccc} L & \xrightarrow{\ \mathsf{id}\ } & \mathsf{Aut}(D) \\ \uparrow{\scriptstyle \exp} & & \uparrow{\scriptstyle \exp} \\ \mathfrak{l} & \xrightarrow{\ \rho\ } & \mathrm{aut(D)} \end{array} \tag{3.17}$$

commutes. Since ρ is injective, the Banach Lie algebra $\mathfrak{l}$ can be considered as a Banach algebra of holomorphic vector fields on D, i.e. $\mathfrak{l} \cong \rho(\mathfrak{l}) \subset \mathrm{aut(D)}$. Further we have that the mapping

(3.18) $\qquad \rho \colon \mathfrak{l} \times D \to TD \qquad$ defined by $\qquad (X, a) \mapsto (\rho X)a$

is analytic. As a consequence of 3.10.ii we obtain that the evaluation mapping ρ_a associated with ρ and each point $a \in D$,

(3.19) $\qquad \rho_a \colon \mathfrak{l} \to T_a D \qquad$ defined by $\qquad X \mapsto (\rho X)a$

is an analytic surjection.

Fix a point a in D called *base point* in the following. Denote by

$$K := \{g \in L \colon ga = 0\}$$

the isotropy subgroup at a. We denote the symmetry at the point a by $s = s_a$.

3.20 Lemma. *The symmetry s is in the center of K.*

Proof. Consider an arbitrary element $g \in K$. Using the same ideas as in the proof of 3.14, we obtain that $\sigma := g \circ s \circ g^{-1}$ is involutive isometry in L, with $a = g^{-1}(a)$ as isolated fixed point. The uniqueness of the symmetry at each point implies $\sigma = s$. Hence we have $s \circ \sigma = g \circ s$. □

For every Banach Lie group G denote by $\mathsf{Aut}(G)$ the group of all (analytic) Lie automorphisms of G. For every Banach Lie algebra $\mathfrak{g}$ denote by $\mathsf{Aut}(\mathfrak{g})$ the group of all (continuous) Lie automorphisms of $\mathfrak{g}$. For every $g \in G$

$$\text{Int}(g)h := ghg^{-1} ,$$

defines a bianalytic group automorphisms $\text{Int}(g)$ of G called the *inner automorphism* induced by g.

Suppose $\mathfrak{g}$ is the Banach Lie algebra of a Banach Lie group G. Then there exists a uniquely determined mapping $\mathsf{Ad}(g) \in \mathsf{Aut}(\mathfrak{g})$ such that the diagram

$$\begin{array}{ccc} G & \xrightarrow{\text{Int(g)}} & G \\ \uparrow{\scriptstyle \exp} & & \uparrow{\scriptstyle \exp} \\ \mathfrak{g} & \xrightarrow{\mathsf{Ad}(g)} & \mathfrak{g} \end{array} \tag{3.21}$$

commutes. The mapping

$$\mathsf{Ad}\colon G \to \mathsf{Aut}(\mathfrak{g})$$

is a homomorphism and defines an analytic action of G on $\mathfrak{g}$ called *the adjoint action* of G on $\mathfrak{g}$.

Denote by $\mathfrak{l}$ the Banach Lie algebra of the Banach Lie group L. Since Ad is a homomorphism, we have

$$[\mathsf{Ad}(s)]^2 = \mathsf{Ad}(s^2) = \mathsf{Ad}(\mathsf{id}) = \mathsf{id} \ .$$

Hence the Banach Lie algebra $\mathfrak{l}$ admits the representation

$$\mathfrak{l} = \mathfrak{k} \oplus \mathfrak{p} ,$$

where

$$\mathfrak{k} := \{X \in \mathfrak{l}\colon \mathsf{Ad}(s)X = X\} \qquad \mathfrak{p} := \{X \in \mathfrak{l}\colon \mathsf{Ad}(s)X = -X\} , \tag{3.22}$$

i.e. $\mathfrak{k}$ and $\mathfrak{p}$ are the (+1) and (-1)eigenspaces of $\mathsf{Ad}(s)$ in $\mathfrak{l}$ respectively.

3.23 Proposition. *The Lie algebra $\mathfrak{k}$ can be identified with the Lie algebra of the Banach Lie group K.*

Proof. ([17], p. 289, Cor. 17.18). □

3.24 Lemma. *The Lie algebra $\mathfrak{k}$ consists of all vector fields in $\mathfrak{l}$ vanishing at a.*

Proof. Let X be a vector field in $\mathfrak{k}$ and $g_t(a)$ denote the corresponding analytic flow. By 3.23 we have $g_t(a) = a$ for all $t \in \mathbb{R}$. Then it is sufficient to notice that

$$X_a = 0 \iff g_t(a) = a \text{ for all } \ t \in \mathbb{R} \ .$$

Let $g_t(a) = a$ for all $t \in \mathbb{R}$. Then:

$$X_a = \left.\frac{\partial g_t(a)}{\partial t}\right|_{t=0} = \left.\frac{\partial a}{\partial t}\right|_{t=0} = 0 \,.$$

Let $X_a = 0$. The system

$$X_{g_t(a)} = \frac{\partial g_t(a)}{\partial t}, \; g_0(a) = a \,,$$

has a unique solution $g_t(a) = a$ for all $t \in \mathbb{R}$. □

In order to obtain a J^*- triple system from a symmetric Banach manifold consider the tangent space to D at the base point a. Define $U := T_aD$ (U is a complex Banach space with respect to $\nu_{|T_a}$, where ν is the norm on TD) and denote by W an open, K-invariant local coordinate neighborhood of a. We may assume that $W \subset U$ and $a = 0$. Furthermore consider the mapping

$$\sigma \colon W \to U$$

defined by

$$\sigma(z) := \frac{1}{2}(z - s(z))$$

for all $z \in W$. Note that

$$\sigma(s(z)) = \frac{1}{2}(s(z) - s^2(z)) = \frac{1}{2}(s(z) - z) = -\sigma(z) \tag{3.25}$$

for all $z \in W$. Also we have $\sigma(0) = 0$ and $d\sigma(0) = \mathrm{id}$. Hence by the Implicit Function Theorem there exist open neighborhoods $W_1 \subset W$ and $W_2 \subset U$ such that

$$\sigma_{|W_1} \colon W_1 \to W_2$$

is biholomorphic. We consider the following open neighborhoods of 0,

$$V_1 := W_1 \cap W \quad \text{and} \quad V_2 := \sigma(V_1) \,.$$

Since W is K-invariant, V_1 is K-invariant. In particular $s(V_1) = V_1$. Then we have the commuting diagram

$$\begin{array}{ccc} V_1 & \xrightarrow{\;s\;} & V_1 \\ \downarrow{\scriptstyle\sigma} & & \downarrow{\scriptstyle\sigma} \\ V_2 & \xrightarrow{\sigma\circ s\circ\sigma^{-1}} & V_2 \end{array} \,.$$

So we have obtained that under the biholomorphic transformation σ, the symmetry s transforms to the involutive isometry $\rho := \sigma \circ s \circ \sigma^{-1}$. Note that a is an isolated fixed point of ρ. We show also that $\rho(z) = -z$. Really, consider an arbitrary point $z \in V_2$ and its uniquely determined preimage under $\sigma_{|V_1}$, i.e. the point $z_1 \in V_1$ such that $\sigma(z_1) = z$. Then by 3.25 we have

$$\rho(z) = (\sigma \circ s \circ \sigma^{-1})(z) = (\sigma \circ s)(z_1) = -\sigma(z_1) = -z \,.$$

Therefore for every $z \in V_2$ we have $\rho(z) = -z$. Hence we may assume that

$$s(z) = -z \tag{3.26}$$

for all $z \in W$.

3.27 Proposition. *Every vector field X in the $(+1)$-eigenspace ((-1)-eigenspace) of* $\mathsf{Ad}(s)$ *in* $\mathfrak{l}$ *respectively has a local representation in W of the form $f(z)\,{}^{\partial}\!/_{\partial z}$, where f is odd (even).*

Proof. Consider an arbitrary vector field $X \in \mathfrak{k}$. By 3.23 and 3.20 we obtain

$$s \circ \exp(tX) = \exp(tX) \circ s \tag{3.28}$$

for all $t \in \mathbb{R}$. From the assumption $s(z) = -z$ for all $z \in W$ follows

$$s(z + tXz + \frac{t^2}{2}X^2z + \cdots) = -z - tXz - \frac{t^2}{2}X^2z + \cdots \tag{3.29}$$

for all $t \in \mathbb{R}$ and $z \in W$. Therefore 3.28 and 3.29 imply

$$-z - tXz - \frac{t^2}{2}X^2z + \cdots = -z + tX(-z) + \frac{t^2}{2}X^2(-z) + \cdots$$

Hence $f(-z) = -f(z)$.

Consider arbitrary vector field $X \in \mathfrak{p}$. Then 3.22 and 3.21 imply

$$\mathrm{Int}(s)(\exp(tX)) = \exp(\mathsf{Ad}(s)tX) = \exp(-tX)$$

for all $t \in \mathbb{R}$. Therefore

$$s \circ \exp(tX) \circ s^{-1} = \exp(-tX) \tag{3.30}$$

for all $t \in \mathbb{R}$. Since we have $s(z) = -z$ for all $z \in W$, 3.30 imply

$$(s \circ \exp(tX))(z) = (\exp(-tX) \circ s)(z) = \exp(-tX)(-z)$$

for all $t \in \mathbb{R}$ and $z \in U$. Expanding in power series we obtain

$$s(z + tXz + \frac{t^2}{2}X^2z + \cdots) = -z - tXz - \frac{t^2}{2}X^2z + \cdots$$

for all $t \in \mathbb{R}$ and $z \in W$. Hence

$$-z - tXz - \frac{t^2}{2}X^2z + \cdots = -z + t(-X)(-z) + \frac{t^2}{2}(-X)^2(-z) + \cdots$$

for all $t \in \mathbb{R}$ and $z \in W$. Therefore $f(-z) = f(z)$. □

3.31 Lemma. *The space $\mathfrak{p}$ is* $\mathsf{Ad}(K)$*-invariant.*

Proof. Consider an arbitrary elements $X \in \mathfrak{p}$ and $g \in K$. Ad is a homomorphism and the symmetry s of a is in the center of K (3.20). Then

$$\begin{aligned}\mathsf{Ad}(s)\big(\mathsf{Ad}(g)X\big) &= \big(\mathsf{Ad}(s) \circ \mathsf{Ad}(g)\big)X = \mathsf{Ad}(s \circ g)X \\ &= \mathsf{Ad}(g \circ s)X = \mathsf{Ad}(g)\big(\mathsf{Ad}(s)X\big) \\ &= \mathsf{Ad}(g)(-X) = -\mathsf{Ad}(g)X\ .\end{aligned}$$

Therefore $\mathsf{Ad}(g)$ lies in the (-1)-eigenspace of $\mathsf{Ad}(s)$ and the proof is complete. □

3.32 Proposition. *For every element $\alpha \in U$ there exists a uniquely determined vector field X^α in $\mathfrak{p}$ such that $X^\alpha_a = \alpha$. The local representation of $X^\alpha = f_\alpha(z)\,\partial/\partial z$ in W depends real-analytically on α and complex-analytically on z.*

Proof. Define $\Phi\colon \mathfrak{p} \to T_aD$ by $\Phi := \rho_{a|\mathfrak{p}}$. Proposition 3.24 implies that $\mathsf{Ker}\,\rho_a = \mathfrak{k}$ and Φ is a real-linear bijection. In particular it is a bianalytic mapping. Therefore the vector field X^α defined by $X^\alpha := (\Phi)^{-1}(\alpha)$ for every $\alpha \in U$, depends real-analytically on α. In particular this holds for the local representation $X^\alpha = f_\alpha(z)\,\partial/\partial z$ in W. □

The mapping $\Phi\colon U \to \mathfrak{p}$ defined in 3.32 is $\mathbb{R}$-linear, i.e. $\Phi(tX) = tX$ for all $t \in \mathbb{R}$. For the vector fields in $\mathfrak{p}$ this implies $tX^\alpha = X^{t\alpha}$ for all $t \in \mathbb{R}$ and $\alpha \in U$. For every $\alpha \in U$ define the vector field $iX^\alpha := X^{i\alpha}$ and the vector space

$$i\mathfrak{p} := \{iX^\alpha\colon\ \alpha \in U\} = \{iX\colon\ X \in \mathfrak{p}\}\ .$$

In the terms of the notation used in 3.32 this is equivalent to $i\Phi(\alpha) := \Phi(i\alpha)$. This way we have obtained a $\mathbb{C}$-linear mapping $\Phi\colon U \to \mathfrak{p} + i\mathfrak{p}$. For every $g \in K$ extend the $\mathbb{R}$-linear mapping $\mathsf{Ad}(g)\colon \mathfrak{p} \to \mathfrak{p}$ on $\mathfrak{p} + i\mathfrak{p}$ by

$$\mathsf{Ad}(g)(iX^\alpha) := i\,\mathsf{Ad}(g)X^\alpha \tag{3.33}$$

(equivalent to $\mathsf{Ad}(g)(X^{i\alpha}) := i\,\mathsf{Ad}(g)X^\alpha$). As a direct conclusion of 3.31 we obtain,

3.34 Lemma. *The space $\mathfrak{p} + i\mathfrak{p}$ is $\mathsf{Ad}(K)$-invariant.*

Proof. Consider an arbitrary elements $X \in \mathfrak{p} + i\mathfrak{p}$ and $g \in K$. The vector field X has a representation of the form $X = X_1 + iX_2$, for some X_1 and X_2 in $\mathfrak{p}$. The 3.31 implies that

$$\mathsf{Ad}(g)(X) = \mathsf{Ad}(g)(X_1 + iX_2) = \mathsf{Ad}(g)(X_1) + \mathsf{Ad}(g)(iX_2) = \mathsf{Ad}(g)X_1 + i\,\mathsf{Ad}(g)X_2$$

lies in $\mathfrak{p} + i\mathfrak{p}$. □

Consider the vector field

$$Y^\alpha := \frac{1}{2}(X^\alpha - iX^{i\alpha})\ \in \mathfrak{p} + i\mathfrak{p}\ .$$

For a small neighborhood $\tilde{W}$ of $0 \in U$ the mapping

$$\tau\colon \tilde{W} \to W$$

$$\tau(\alpha) = \exp(Y^\alpha)(0)$$

is well defined and is of the form

$$\tau(\alpha) = \alpha + \text{terms in } \alpha \text{ of degree} \geq 3\ .$$

Therefore we may assume that

$$\tau\colon \tilde{W} \to W$$

is biholomorphic.

3.35 Lemma. *For every isometry $g \in K$ there exists a uniquely determined mapping $\tilde{g} \in \mathsf{GL}(U)$, such that $\mathsf{Ad}(g)X^\alpha = X^{\tilde{g}(\alpha)}$ for all α in U.*

Proof. Define $\tilde{g} := \Phi^{-1} \circ \mathsf{Ad}(g) \circ \Phi$. Then

$$X^{\tilde{g}(\alpha)} = \Phi(\tilde{g}(\alpha)) = \mathsf{Ad}(g)(\Phi(\alpha)) = \mathsf{Ad}(g)X^\alpha .$$

3.36 Lemma. *For every $g \in K$ and every $\alpha \in \tilde{W}$ the following equation holds,*

$$g(\tau(\alpha)) = \tau(\tilde{g}(\alpha)) . \tag{3.37}$$

Proof. Consider an arbitrary element $g \in K$ and $\alpha \in W$. Using the definitions of τ and Y^α we obtain

$$\tau(\tilde{g}(\alpha)) = \exp\big(Y^{\tilde{g}(\alpha)}\big)(0) = \exp\big(\frac{1}{2}(X^{\tilde{g}\alpha} - iX^{i\tilde{g}\alpha})\big)(0) .$$

But $i\tilde{g}(\alpha) = \tilde{g}(i\alpha)$. Using the $\mathbb{R}$-linearity of $\mathsf{Ad}(s)$ and 3.33 we obtain that $\tau(\tilde{g}(\alpha))$ is equal to

$$\exp\big(\frac{1}{2}(\mathsf{Ad}(g)X^\alpha - iX^{\tilde{g}(i\alpha)})\big)(0) = \exp\big(\mathsf{Ad}(g)(\frac{1}{2}(X^\alpha - iX^{i\alpha}))\big)(0) .$$

The definition of Y^α and 3.21 imply

$$\exp\big(\mathsf{Ad}(g)Y^\alpha\big)(0) = \mathrm{Int}(g)(\exp Y^\alpha)(0) = g((\exp Y^\alpha)(0)) = g(\tau(\alpha)) .$$

Therefore 3.37 is proved. □

For every $\alpha \in U$ consider also the vector field:

$$\tilde{Y}^\alpha := \mathsf{Ad}(\tau^{-1})Y^\alpha \quad \text{on } \tilde{W} .$$

Using the same arguments as in the proof of 3.36 we obtain

$$\begin{aligned} \exp(\tilde{Y}^\alpha)(0) &= \exp\big(\mathsf{Ad}(\tau^{-1})Y^\alpha\big)(0) = \big(\tau^{-1} \circ \exp(Y^\alpha) \circ \tau\big)(0) \\ &= \tau^{-1}\big(\exp(Y^\alpha)(0)\big) = \tau^{-1}(\tau(\alpha)) = \alpha . \end{aligned}$$

Replace the local coordinate W by $\tilde{W}$ via τ. For every $g \in K$ we get a mapping f, such that the diagram

$$\begin{array}{ccc} W & \xrightarrow{\;g\;} & W \\ \big\downarrow{\scriptstyle \tau^{-1}} & & \big\downarrow{\scriptstyle \tau^{-1}} \\ \tilde{W} & \xrightarrow{\;f\;} & \tilde{W} \end{array}$$

commutes, i.e. $g(x) = (\tau \circ f \circ \tau^{-1})(x)$ for every $x \in W$. Then $\tau^{-1} \circ g \circ \tau = f$. But $\tau^{-1} \circ g \circ \tau = \tilde{g}$. Consequently the isometry g transforms to $\tilde{g}$. Hence for every element g of the isotropy subgroup K we may assume that $g_{|W}$ is linear.

In this way for every $\alpha \in U$ we obtain that the vector field Y^α transforms to a vector field Z such that the diagram

$$\begin{array}{ccc} W & \xrightarrow{\;Y^\alpha\;} & TW \cong W \times U \\ \big\downarrow{\scriptstyle \tau^{-1}} & & \big\downarrow{\scriptstyle \tau^{-1}} \\ \tilde{W} & \xrightarrow{\;Z\;} & T\tilde{W} \cong \tilde{W} \times U , \end{array}$$

commutes. From $Z = \mathsf{Ad}(\tau^{-1})Y^\alpha$ we obtain that $Z = \tilde{Y}^\alpha$. Therefore we may assume that $\alpha = \exp(Y^\alpha)(0)$ for all $\alpha \in W \subset U$.

3.38 Proposition. $\mathfrak{k} = \left\{ i\lambda\, {}^{\partial}\!/\!_{\partial z} \, : \, \lambda \in H(U) \text{ and } [i\lambda\, {}^{\partial}\!/\!_{\partial z}\, , \mathfrak{p}] \subset \mathfrak{p} \right\}$

Proof. Consider an arbitrary vector field X in $\mathfrak{k}$. By 3.23 we have $\exp(tX) \in K$ for every $t \in \mathbb{R}$. Hence X has a local representation on W of the form $X = i\lambda\, {}^{\partial}\!/\!_{\partial z}$ for some $\lambda \in H(U)$. Since $\mathsf{Ad}(s)$ is a homomorphism of Lie algebras we have

$$\mathsf{Ad}(s)([i\lambda\, {}^{\partial}\!/\!_{\partial z}\, , \mathfrak{p}]) = [\mathsf{Ad}(s) i\lambda\, {}^{\partial}\!/\!_{\partial z}\, , \mathsf{Ad}(s)\mathfrak{p}] = -[i\lambda\, {}^{\partial}\!/\!_{\partial z}\, , \mathfrak{p}]$$

and therefore $[i\lambda\, {}^{\partial}\!/\!_{\partial z}\, , \mathfrak{p}] \subset \mathfrak{p}$. Therefore

$$\mathfrak{k} \subset \left\{ i\lambda\, {}^{\partial}\!/\!_{\partial z} \, : \, \lambda \in H(U) \text{ and } [i\lambda\, {}^{\partial}\!/\!_{\partial z}\, , \mathfrak{p}] \subset \mathfrak{p} \right\} .$$

Consider an arbitrary vector field X in $\mathfrak{l}$ satisfying $\mathsf{Ad}(s)[X, Y] = -[X, Y]$ for all $Y \in \mathfrak{p}$. Then $\mathsf{Ad}(s)X = X$, and therefore $X \in \mathfrak{k}$. □

3.39 Lemma. $\mathfrak{k} \cap i\mathfrak{k} = 0$

Proof. Suppose $X \in \mathfrak{k} \cap i\mathfrak{k}$. By 3.38 we obtain the existence of a hermitian element $\lambda \in H(U)$ such that $X = i\lambda\, {}^{\partial}\!/\!_{\partial z}$. Since $X \in i\mathfrak{k}$, there is a vector field $X_1 \in \mathfrak{p}$ such that $X = iX_1$. Applying 3.38 we get the existence of a hermitian element $\lambda_1 \in H(U)$ such that $X_1 = i\lambda_1\, {}^{\partial}\!/\!_{\partial z}$. In particular $\lambda = i\lambda_1$. Since λ is hermitian the numerical range $V(L(U), \lambda) \subset \mathbb{R}$. Using the same argument for λ_1 we obtain $V(L(U), \lambda_1) \subset \mathbb{R}$. The basic properties of the numerical range (compare [2], p. 15) imply $V(L(U), \lambda) = V(L(U), i\lambda_1) = iV(L(U), \lambda) \subset i\mathbb{R}$. Therefore $V(L(U), \lambda) \subset \mathbb{R} \cap i\mathbb{R}$. So we have obtained $V(L(U), \lambda) = 0$. But for the numerical radius of λ the following inequality holds $\nu(\lambda) \geq \frac{1}{\mathrm{e}}\|\lambda\|$ (compare [2], p. 34, Th.1), and therefore $\|\lambda\| = 0$. That is $\lambda = 0$. □

3.40 Proposition. *There exists a continuous, conjugate-linear mapping* $*\colon U \to L^2(U)$ *such that* $\mathfrak{p} = \{(\alpha - \alpha^*)\, {}^{\partial}\!/\!_{\partial z} \colon \alpha \in U\}$

Proof. Consider a sufficiently small neighborhood of 0. Suppose X^α is a vector field in the (-1)-eigenspace of $\mathsf{Ad}(s)$. By 3.32 and 3.27 the local representation of X^α, $X^\alpha = f_\alpha(z)\, {}^{\partial}\!/\!_{\partial z}$ in W depends real-analytically on $\alpha \in U$ and $f_\alpha(z)$ is an even mapping. Therefore for every α in U,

$$X^\alpha = \Big(\alpha + \sum_{k=1}^{\infty} p_k(\alpha)\Big)\, {}^{\partial}\!/\!_{\partial z} \in \mathfrak{p}$$

and

$$Y^\alpha = \Big(\alpha + \sum_{k=1}^{\infty} q_k(\alpha)\Big)\, {}^{\partial}\!/\!_{\partial z} \in \mathfrak{p} + i\mathfrak{p} \, .$$

where $p_k(\alpha), q_k(\alpha) \in L^{2k}(U)$ and $q_k(\alpha)$ depends $\mathbb{C}$-linearly on α. For every α and β in U for the bracket product $[Y^\alpha, Y^\beta]$ we have

$$[Y^\alpha, Y^\beta] = \frac{1}{4}([X^\alpha, X^\beta] - i[X^\alpha, X^{i\beta}] - i[X^{i\alpha}, X^\beta] - [X^{i\alpha}, X^{i\beta}]) \, .$$

But

$$[X^{i\alpha}, X^\beta] + [X^\alpha, X^{i\beta}] = i[X^\alpha, X^\beta] - i[X^{i\alpha}, X^{i\beta}]$$

is in $\mathfrak{k} \cap i\mathfrak{k}$ and therefore vanishes. Hence the vector fields Y^α, Y^β commute. Therefore for all $z \in W$ we get

$$\exp(tY^\alpha)(z) = \exp(Y^{t\alpha}) \exp(Y^z)(0) = \exp(Y^{t\alpha + z})(0) = t\alpha + z \, ,$$

i.e.

$$Y^\alpha = \alpha\,{}^{\partial}\!/\!_{\partial z} \in \mathfrak{P}_{-1} \quad \text{for all} \quad \alpha \in U.$$

In particular $q_1(\alpha) = 0$ and therefore $* := -p_1$ is a conjugate linear mapping.

Since $[X^\alpha, Y^\beta] \in \mathfrak{P}_0$ for all $\beta \in U$(by the properties of $\mathsf{Ad}(s)$) we obtain

$$X^\alpha = (\alpha - \alpha^*)\,{}^{\partial}\!/\!_{\partial z} \in \mathfrak{P}_{-1} \oplus \mathfrak{P}_1 \quad \text{for all} \quad \alpha \in U\,.$$

Hence we have obtained

$$\mathfrak{p} \subset \{(\alpha - \alpha^*)\,{}^{\partial}\!/\!_{\partial z} \colon \alpha \in U\}\,.$$

Since $\Phi\colon U \to \mathfrak{p}$ is a bijection the proof is complete. □

Proposition 3.40 and 2.23 imply that $(U, *)$ is a hermitian Jordan triple system.

3.41 Proposition. *Suppose D and $\tilde{D}$ are symmetric complex Banach manifolds with base points a and $\tilde{a}$ respectively and $h\colon D \to \tilde{D}$ is a morphism with $h(a) = \tilde{a}$. Denote by $(U, *)$ and $(\tilde{U}, *)$ the corresponding J^*-triple systems. Then the differential of h at the base point a of D is a J^*-morphism.*

Proof. Denote by λ the differential of h at the base point a. Suppose $g \in P := \exp(\mathfrak{p}) \subset L$. Obviously $g(a)$ is a fixed point of $g \circ s \circ g^{-1}$.

Since a is isolated fixed point of s, there exists an open neighborhood $V \subset D$ of a with the property $s(x) \neq x$ for all $x \in V \backslash \{a\}$. The mapping g is open and therefore $g(V)$ is an open neighborhood of $g(a)$ in D. Assume there is a point $x \in g(V) \backslash \{a\}$ such that $gsg^{-1}(x) = x$. Then $g^{-1}(x) \in V \backslash \{a\}$ is a fixed point of s which is a contradiction with the choice of the neighborhood V. Hence $g(a)$ is an isolated fixed point of the involution gsg^{-1}.

By 3.14 we obtain

$$s_{ga} = g \circ s \circ g^{-1} = g^2 \circ s\,. \tag{3.42}$$

By the definition of a morphism of symmetric Banach manifolds, we have $s_{\tilde{a}} \circ h = s_{ha} \circ h = h \circ s_a$. Then 3.42 implies

$$s_{hga} \circ s_{\tilde{a}} \circ h = s_{hga} \circ h \circ s_a = h \circ s_{ga} \circ s_a = h \circ g^2 \circ s \circ s = h \circ g^2\,. \tag{3.43}$$

Consider an arbitrary vector field $X \in \mathfrak{p}$, $X = (\alpha - \alpha^*)\,{}^{\partial}\!/\!_{\partial z}$ for some $\alpha \in U$. In particular for every $t \in \mathbb{R}$ we have $g_t := \exp(tX) \in P$. Furthermore define

$$\tilde{g}_t := s_{hg_{\frac{t}{2}}a} \circ s_{\tilde{a}} \quad \text{for all } t \in \mathbb{R}\,.$$

Then the equality 3.43 implies

$$\tilde{g}_t \circ h = h \circ g_t$$

for all $t \in \mathbb{R}$. There is a uniquely determined vector field $Y \in \tilde{\mathfrak{p}}$, where $\tilde{\mathfrak{p}}$ is defined for $\tilde{U}$ the same way as $\mathfrak{p}$ for U, such that $\exp(tY) = \tilde{g}_t$ for all $t \in \mathbb{R}$. For some $\beta \in \tilde{U}$ we have $Y = (\beta - \beta^*)\,{}^{\partial}\!/\!_{\partial z}$. Hence

$$h \circ \exp(tX) = \exp(tY) \circ h \quad \text{for all} \quad t \in \mathbb{R}\,.$$

We may assume that the local coordinate neighborhoods W of $a \in D$ and $\tilde{W}$ of $\tilde{a} \in \tilde{D}$ satisfy $h(W) \subset \tilde{W}$. Differentiating at $t = 0$ we obtain

$$\frac{\partial h(\exp(tX)(z))}{\partial t}\Big|_{t=0} = \frac{\partial \exp(tY)(h(z))}{\partial z}\Big|_{t=0} .$$

That is

$$h'(z)(\alpha - \alpha^*(z)) = (\beta - \beta^*(h(z))) \quad \text{for all} \quad z \in W . \tag{3.44}$$

For $z = 0$ this implies $h'(0)(\alpha) = \beta$, which is equivalent to $\lambda(\alpha) = \beta$. Therefore for every fixed $z \in W$ the mapping $h'(z)(\alpha) - \beta$ depends conjugate-linearly on α. Since $h'(z)(\alpha) - \beta$ depends also $\mathbb{C}$-linearly on α we obtain $h'(z)(\alpha) - \beta = 0$, or $h'(z)(\alpha) = \lambda(\alpha) = h'(0)(\alpha)$ for every $z \in W$. In particular we have obtained $h(z) = \lambda(z)$ for all $z \in W$. Then 3.44 implies

$$\lambda(\alpha^*(z)) = (\lambda(\alpha))^*(h(z)) ,$$

which is equivalent to

$$\lambda(\{z\alpha^*z\}) = \{\lambda(z)(\lambda(\alpha))^*(\lambda(z))\}$$

for all $z \in W$. Therefore λ is a J^*-morphism. □

Proposition 3.41 implies that the given construction is a functor $\mathfrak{F}$ from the category of symmetric complex Banach manifolds with base point into the category of hermitian Jordan triple systems.

4. Construction of a Symmetric Complex Banach Manifold from a Hermitian Jordan Triple System

Suppose $(U, *)$ is a J^*-triple system. As we have shown in chapter 2, conditions (ii) and (iii) of the definition of a hermitian Jordan triple system imply $[\mathfrak{p}, \mathfrak{p}] \subset \mathfrak{k}$ and hence

$$\check{\mathfrak{l}} = \check{\mathfrak{k}} \oplus \mathfrak{p} \subset \mathfrak{l} = \mathfrak{k} \oplus \mathfrak{p} \subset \widehat{\mathfrak{l}} = \widehat{\mathfrak{k}} \oplus \mathfrak{p}$$

are real Banach Lie algebras. Let V denote the closure (uniform closure) of U^* in $L^2(U)$, i.e. $V = \overline{U^*}$. Consider the following complex Banach subspaces of $\mathfrak{P}$,

$$\begin{aligned} \mathfrak{g}_{-1} &:= \mathfrak{P}_{-1} , \\ \mathfrak{g}_1 &:= \{q\,\partial/_{\partial z} : q \in V\} \subset \mathfrak{P}_1 , \\ \mathfrak{g}_0 &:= \{X \in \mathfrak{P}_0 : [X, \mathfrak{g}_1] \subset \mathfrak{g}_1\} \subset \mathfrak{P}_0 . \end{aligned}$$

4.1 Lemma. *The direct sum $\mathfrak{g} := \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1$ is a graded complex Banach Lie subalgebra of the Banach Lie algebra $\mathfrak{P}$ of all polynomial vector fields associated with U.*

Proof. Obviously $\mathfrak{g}$ is a complex Banach space as a direct sum of complex Banach spaces. Notice that the following relations hold,

$$[\mathfrak{g}_{-1}, \mathfrak{g}_{-1}] = 0, \quad [\mathfrak{g}_{-1}, \mathfrak{g}_0] \subset \mathfrak{g}_{-1}, \quad [\mathfrak{g}_0, \mathfrak{g}_0] \subset \mathfrak{g}_0 \quad \text{and} \quad [\mathfrak{g}_1, \mathfrak{g}_0] \subset \mathfrak{g}_1; \tag{4.2}$$

(i) We shall prove that $[\mathfrak{g}_{-1}, \mathfrak{g}_1] \subset \mathfrak{g}_0$.

Consider arbitrary vector fields X in $\mathfrak{g}_{-1}$ and Y in $\mathfrak{g}_1$. Notice that they have representations of the form $X = \alpha\,\partial/\partial z$ for some element $\alpha \in U$ and $Y = q\,\partial/\partial z$ for some homogeneous polynomial $q \in V$. Denote by Z their bracket product, $[X,Y]$. Then Z has a representation of the form, $Z = g(z)\,\partial/\partial z$ for some holomorphic mapping $g(z)$.

1. Suppose $q = \beta^*$ for some element $\beta \in U$. Using the basic properties of the triple product we obtain

$$g(z) = d\alpha(z)(\beta^*(z)) - d\beta^*(z)(\alpha) = -2q_\beta(z,\alpha) = -2\{z\beta^*\alpha\} = -2(\alpha\,\square\,\beta^*)(z)\,.$$

Therefore Z lies in $\mathfrak{P}_0$.

Therefore it is sufficient to show that the inclusion $[Z,\mathfrak{g}_1] \subset \mathfrak{g}_1$ holds. Consider a vector field $p\,\partial/\partial z$ in $\mathfrak{g}_1$.

1.1 Suppose $p = \gamma^*\,\partial/\partial z$ for some γ in U. Then for the bracket product $h(z)\,\partial/\partial z$ of the fields Z and $p\,\partial/\partial z$, we obtain

$$\begin{aligned} h(z) &= d(-2\{z\alpha^*\beta\})(z)(\gamma^*(z)) - d\gamma^*(z)((-2\beta\,\square\,\alpha^*)(z)) \\ &= -2\{\{z\gamma^*z\}\alpha^*\beta\} - 2q_\gamma(z,-2\{z\alpha^*\beta\}) \\ &= -2\{\{z\gamma^*z\}\alpha^*\beta\} + 4\{z\gamma^*\{z\alpha^*\beta\}\} \\ &= 2\{z\{\alpha\beta^*\gamma\}^*z\}\,. \end{aligned}$$

That is $h(z) = 2\{\alpha\beta^*\gamma\}^*(z)$ and hence the vector filed Z lies in $\mathfrak{g}_1$.

1.2 Suppose $p = \lim_{n\to\infty}\gamma_n{}^*$ for some sequence $(\gamma_n) \subset U$. Using 1.1 we obtain

$$h(z) = \lim_{n\to\infty} 2\{z\{\alpha\beta^*\gamma_n\}^*z\} = 2\lim_{n\to\infty}(\{\alpha\beta^*\gamma_n\}^*)(z)\,.$$

Since the space $\mathfrak{g}_1$ is closed, we have obtained $Z \in \mathfrak{g}_1$.

The considered cases 1.1 and 1.2 imply $[\mathfrak{g}_{-1},\mathfrak{g}_1] \subset \mathfrak{g}_0$.

2. Suppose $q = \lim_{n\to\infty}\beta_n$ for some sequence (β_n) in U. The case 1 and the uniform continuity imply $Z \in \mathfrak{g}_1$.

(ii) We shall prove that $[\mathfrak{g}_1,\mathfrak{g}_1] = 0$. Suppose $X = q\,\partial/\partial z$ and $Y = p\,\partial/\partial z$ are vector fields in $\mathfrak{g}_1$. Denote by $Z = g(z)\,\partial/\partial z$ the bracket product $[X,Y]$.

1. Suppose $q = \alpha^*$ and $p = \beta^*$ for some $\alpha,\beta \in U$. Then

$$\begin{aligned} g(z) &= d\alpha^*(z)(\beta^*(z)) - d\beta^*(z)(\alpha^*(z)) \\ &= 2q_\alpha(z,\beta^*(z)) - 2q_\beta(z)(\alpha^*(z)) \\ &= \{z\alpha^*\{z\beta^*z\}\} - \{z\beta^*\{z\alpha^*z\}\} \\ &= 0 \end{aligned}$$

because of 2.18. Therefore the vector field Z vanishes.

2. Suppose $q = \lim_{n\to\infty}\alpha_n{}^*$ and $\gamma = \beta^*$ for some sequence (α_n) in U and some element β in U. The considered case 1. implies that the vector field Z vanishes.

3. Suppose $q = \lim_{n\to\infty}\alpha_n{}^*$ and $\gamma = \lim_{n\to\infty}\beta_n{}^*$ for some sequences (α_n) and (β_n) in the Banach space U. Then the cases 1. and 2. imply that the vector field Z vanishes.

So we have obtained $[\mathfrak{g}_1,\mathfrak{g}_1] = 0$.

The two considered cases (i) and (ii), as well as 4.2 imply that $\mathfrak{g}$ is a graded complex Banach Lie algebra. □

4.3 Lemma. *The complex Banach Lie algebra $\mathfrak{g}$ has no center.*

Proof. Suppose X is a vector field, which lies in the center of $\mathfrak{g}$. It has a representation of the form

$$X = \alpha\, \partial/\partial z + \lambda\, \partial/\partial z + x^*\, \partial/\partial z\,, \text{ where } \alpha \in U,\ \ x^* \in V\ \ \lambda\, \partial/\partial z \in \mathfrak{g}_0\,.$$

For every vector field $Y = z\, \partial/\partial z$ in $\mathfrak{g}$, the product $[X, Y]$ vanishes. In particular $Y = z\, \partial/\partial z$ implies:

$$[X, z\, \partial/\partial z] = [\alpha\, \partial/\partial z\,, z\, \partial/\partial z] + [\lambda\, \partial/\partial z\,, z\, \partial/\partial z] + [x^*\, \partial/\partial z] = h(z)\, \partial/\partial z\,,$$

where

$$\begin{aligned} h(z) &= -\alpha + dx^*(z)(z) - x^*(z) \\ &= -\alpha + 2q_x(z, z) - x^*(z) \\ &= -\alpha + x^*(z) = -\alpha + x^*(z) = 0 \end{aligned}$$

for all z in U. Then $x^*(z) = \alpha$ for all $z \in U$. But $z = 0$ implies $\alpha = 0$ and therefore $x^* \equiv 0$. As a consequence we obtain $X = \lambda\, \partial/\partial z \in \mathfrak{g}_0$. Consider an arbitrary element $\alpha \in U$. Then

$$[\lambda\, \partial/\partial z\,, (\alpha - \alpha^*)\, \partial/\partial z] = \big(\lambda(\alpha) - \lambda(\{z\alpha^* z\}) - 2\{z\alpha^*(\lambda(z))\}\big)\, \partial/\partial z = 0$$

implies $\lambda(\alpha) = 0$. But the last holds for all $\alpha \in U$, therefore $\lambda \equiv 0$. Hence X vanishes and the proof is complete. □

4.4 Lemma. *Every derivation of $\mathfrak{g}$ is inner.*

Proof. Consider an arbitrary derivation D of $\mathfrak{g}$, i.e. continuous $\mathbb{C}$-linear mapping $D\colon \mathfrak{g} \to \mathfrak{g}$, such that

$$D([X, Y]) = [D(X), Y] + [X, D(Y)]$$

for all X, Y in $\mathfrak{g}$. Suppose that $D(z\, \partial/\partial z) = X_{-1} + X_0 + X_1$, where $X_\nu \in \mathfrak{g}_\nu$ for every $\nu \in \{-1, 0, 1\}$. Then the mapping $\eta := D - \mathsf{ad}(X_1 - X_{-1})$ is a derivation of $\mathfrak{g}$, such that $\eta(z\, \partial/\partial z) = X_0$ lies in $\mathfrak{g}_0$. Here we have used that $[z\, \partial/\partial z\,, X_{-1}] = X_{-1}$ and $[X_1, z\, \partial/\partial z] = X_1$. If we prove that η is an inner derivation of $\mathfrak{g}$, then D is an inner derivation of $\mathfrak{g}$ as a sum of η and $\mathsf{ad}(-X_{-1} + X_1)$, where the vector fields $X_{-1} \in \mathfrak{g}_{-1}$ and $X_1 \in \mathfrak{g}_1$ are uniquely determined from the image $D(z\, \partial/\partial z)$. Therefore without loss of generality we may assume that $D(z\, \partial/\partial z) \in \mathfrak{g}_0$. This implies that $D(\mathfrak{g}_\nu)$ lies in $\mathfrak{g}_\nu$ for all $\nu = \{-1, 0, 1\}$. In particular for $\nu = -1$ we obtain

$$D_{|\mathfrak{g}_{-1}}\colon \mathfrak{g}_{-1} \to \mathfrak{g}_{-1}$$

is a continuous linear mapping and therefore (since $\mathfrak{g}_{-1} \cong U$) uniquely determines a continuous linear mapping $\lambda\colon U \to U$, and hence a linear vector field $X = \lambda\, \partial/\partial z \in \mathfrak{P}_0$ (define $\lambda(\alpha) := \beta$, when $D(\alpha\, \partial/\partial z = \beta\, \partial/\partial z)$). Then for all vector fields $Y = \alpha\, \partial/\partial z$ in $\mathfrak{g}_{-1}$ we get $[X, Y] = \lambda(\alpha)\, \partial/\partial z$. This implies

$$D(Y) = [X, Y] \quad \text{for all} \quad Y \in \mathfrak{g}_{-1}\,.$$

For all constant vector fields $Y \in \mathfrak{g}_{-1}$ and all linear vector fields $Z \in \mathfrak{g}_0$ we have that the vector field $[Y, Z]$ lies in $\mathfrak{g}_{-1}$, and hence the above result implies $[X, [Y, Z]] = D([Y, Z])$. So we obtain

$$\begin{aligned} [Y, [X, Z]] &= [X, [Y, Z]] - [[X, Y], Z] \\ &= D([Y, Z]) - [D(Y), Z] \\ &= [D(Y), Z] + [Y, D(Z)] - [D(Y), Z] \\ &= [Y, D(Z)]\,. \end{aligned}$$

Then

$$D(Z) = [X, Z] \quad \text{for all} \quad Z \in \mathfrak{g}_0 .$$

Now we apply the same argument to all constant vector fields $Y \in \mathfrak{g}_{-1}$ and all quadratic vector fields $Z \in \mathfrak{g}_1$. In this case the vector field $[Y, Z]$ lies in $\mathfrak{g}_0$. So using the the result that $[X, [Y, Z]] = D([Y, Z])$, we obtain

$$D(Z) = [X, Z] \quad \text{for all} \quad Z \in \mathfrak{g}_1 .$$

In particular the above result implies that the vector field $[X, Z]$ lies in $\mathfrak{g}_1$ for all $Z \in \mathfrak{g}_1$ (we have that $D(\mathfrak{g}_1)$ is a subset of $\mathfrak{g}_1$) and hence X lies in $\mathfrak{g}_0$. Therefore $D = \mathsf{ad}(X)$ is an inner derivation of $\mathfrak{g}$. □

4.5 Proposition. *The Banach Lie algebra* $\mathfrak{g}$ *is isomorphic to the Banach Lie algebra* $\mathsf{Der}(\mathfrak{g})$ *of all derivations of* $\mathfrak{g}$ *via* ad.

Proof. Consider the continuous Lie algebra homomorphism $\mathsf{ad}\colon \mathfrak{g} \to \mathsf{Der}(\mathfrak{g})$. Proposition 4.4 implies that ad is surjective. Since $\mathsf{Ker}(\mathsf{ad})$ is equal to the center of $\mathfrak{g}$, proposition 4.3 implies that ad is injective. Therefore ad is a topological isomorphism of Lie algebras. □

4.6 Proposition. *The group* $\mathsf{Aut}(\mathfrak{g})$ *of all Lie automorphisms of* $\mathfrak{g}$ *is a complex Lie subgroup of* $\mathsf{GL}(\mathfrak{g})$. *Its Lie algebra is the set*

$$\mathfrak{w} := \{g \in L(\mathfrak{g}) : \exp(tg) \in \mathsf{Aut}(\mathfrak{g}) \quad \textit{for all} \quad t \in \mathbb{R}\} .$$

Proof. For every $g \in \mathsf{GL}(\mathfrak{g})$ the following holds

$$g \in \mathsf{Aut}(\mathfrak{g}) \iff g([X, Y]) = [g(X), g(Y)] \text{ for all } X, Y \in \mathfrak{g} .$$

For all vector fields X, Y in $\mathfrak{g}$ define

$$P_{X,Y}(g) := [g(X), g(Y)] - g([X, Y]) \quad \text{for all} \quad g \in \mathsf{GL}(\mathfrak{g}) .$$

Note that $P_{X,Y}\colon L(\mathfrak{g}) \times L(\mathfrak{g}) \to \mathfrak{g}$ is a homogeneous polynomial of deg ≤ 2. Then

$$\mathsf{Aut}(\mathfrak{g}) = \{g \in \mathsf{GL}(\mathfrak{g}) : P_{X,Y}(g) = 0 \text{ for all } X, Y \in \mathfrak{g}\}$$

is algebraic subgroup of $\mathsf{GL}(\mathfrak{g})$ of deg ≤ 2 in the sense of [6] and consequently is a Lie subgroup of $\mathsf{GL}(\mathfrak{g})$ with Lie algebra $\mathfrak{w}$. □

4.7 Lemma. $\mathsf{Der}(\mathfrak{g}) = \mathfrak{w}$

Proof. We shall prove that $\mathfrak{w} \subset \mathsf{Der}(\mathfrak{g})$. Consider an arbitrary element $\theta \in \mathfrak{w}$. Then the mapping $\theta\colon \mathfrak{g} \to \mathfrak{g}$ is linear and $\exp(t\theta) \in \mathsf{Aut}(\mathfrak{g})$ for every $t \in \mathbb{R}$, which implies

$$\exp(t\theta)[X, Y] = [\exp(t\theta)X, \exp(t\theta)Y]$$

for all $t \in \mathbb{R}$ and all $X, Y \in \mathfrak{g}$. In particular then we obtain

$$\begin{aligned}
\theta([X, Y]) &= \frac{d}{dt}\big(\exp(t\theta)[X, Y]\big)_{|t=0} \\
&= \Big[\frac{d}{dt}\big(\exp(t\theta)X\big)_{|t=0}, Y\Big] + \Big[X, \frac{d}{dt}\big(\exp(t\theta)Y\big)_{|t=0}\Big] \\
&= [\theta(X), Y] + [X, \theta(Y)] .
\end{aligned}$$

Therefore $\theta \in \mathsf{Der}(\mathfrak{g})$.

Consider arbitrary element $\theta \in \mathsf{Der}(\mathfrak{g})$. We shall prove that $\exp(t\theta) \in \mathsf{Aut}(\mathfrak{g})$ for every $t \in \mathbb{R}$.

Fix arbitrary vector fields $X, Y \in \mathfrak{g}$ and consider the differential equation

$$\frac{d}{dt} g(t)_{|t=0} = \theta([X,Y]) \ , \quad g(0) = [X,Y] \ . \tag{4.8}$$

Obviously the mapping $y(t) = \exp(t\theta)[X,Y]$ is a solution of 4.8. Furthermore consider the mapping $h(t) = \big[\exp(t\theta)X, \exp(t\theta)Y\big]$. We have

$$\begin{aligned} \frac{d}{dt} h(t)_{|t=0} &= \Big[\frac{d}{dt}\big(\exp(t\theta)X\big)_{|t=0}, Y\Big] + \Big[X, \frac{d}{dt}\big(\exp(t\theta)Y\big)_{|t=0}\Big] \\ &= \big[\theta(X), Y\big] + \big[X, \theta(Y)\big] \\ &= \theta\big([X,Y]\big) \ , \end{aligned}$$

since $\theta \in \mathsf{Der}(\mathfrak{g})$. Note also that $h(0) = [X,Y]$. Since the differential equation 4.8 has a unique solution, we have obtained $h(t) = y(t)$ for all $t \in \mathbb{R}$, which is exactly

$$\exp(t\theta)[X,Y] = [\exp(t\theta)X, \exp(t\theta)Y]$$

for every $t \in \mathbb{R}$. Since the vector fields X and Y were arbitrary, we have obtained $\exp(t\theta) \in \mathsf{Aut}(\mathfrak{g})$ for every $t \in \mathbb{R}$. □

Denote by G the connected identity component of $\mathsf{Aut}(\mathfrak{g})$. Then G is a complex Banach Lie group with Lie algebra $\mathsf{Der}(\mathfrak{g})$, isomorphic to $\mathfrak{g}$ via ad. Consider the closed complex Banach Lie subalgebra $\mathfrak{h} := \mathfrak{g}_0 \oplus \mathfrak{g}_1$ of $\mathfrak{g}$, and the closed subgroup $H := \{g \in G\colon\ g(\mathfrak{h}) = \mathfrak{h}\}$ of G.

4.9 Proposition. *H is a complex Banach Lie subgroup of G with Lie algebra isomorphic to the Banach Lie algebra $\mathfrak{h}$.*

Proof. ([17], p. 147). □

Denote by $Q := G/H$ the quotient space. Using Godement's theorem we shall prove that there exists a unique complex Banach manifold structure on Q such that the canonical projection is a holomorphic submersion. For convenience we give the definition of a submanifold and Godement's Theorem, see also [17].

4.10 Definition. A subset N of a Banach manifold M is called a (direct) submanifold if for every point $o \in N$, there exist a chart (P, p, Z) of M about o and a split subspace W of Z such that

$$p(N \cap P) = W \cap p(P) \ .$$

4.11 Theorem. *Let R be an equivalence relation on a Banach manifold M. Then the following conditions are equivalent:*

(i) *R is a closed submanifold of $M \times M$ and the projections $\pi_k\colon R \to M$ are analytic submersions.*

(ii) *R is a (Hausdorff) Banach manifold such that the projection $\pi_R\colon M \to M/R$ is an analytic submersion.*

In this case the manifold structure on M/R is uniquely determined by the condition that π_R is an analytic submersion.

We divide the proof in several claims.

Claim: There exist an equivalence relation R in G such that $Q = G/R$.

Proof: Consider the holomorphic mapping $\Psi: G \times H \to G \times G$ defined by $\Psi(g,h) := (g, gh)$. Define $R := \Psi(G \times H)$.

(i) R is reflexive: For every element $g \in G$ we have $(g, ge) = \Psi(g,e) \in R$.

(ii) R is transitive: Suppose $(g_1, g_2) \in R$ and $(g_2, g_3) \in R$. Then there exist elements $h_1, h_2 \in H$ such that $g_2 = g_1 h_1$ and $g_3 = g_2 h_2$. Therefore $g_3 = g_1 h_1 h_2$, or $(g_1, g_3) = \Psi(g_1, h_1 h_2) \in R$.

(iii) R is symmetric: Suppose $(g_1, g_2) \in R$. Then there exists an element $h \in H$ with the property $g_2 = g_1 h$. Then $g_1 = g_2 h^{-1}$, or $(g_2, g_1) = \Psi(g_2, h^{-1}) \in R$.

Therefore R is an equivalence relation in G. Furthermore for any two elements $g_1, g_2 \in G$ we have $(g_1, g_2) \in R$ if and only if there exist an element $h \in H$ such that $g_2 = g_1 h$. The last is equivalent to $g_1^{-1} g_2 = h \in H$, which holds if and only if $g_1 H = g_2 H$. Therefore $Q = G/R$. □

Claim: R is a closed submanifold of $G \times G$.

Proof: R is closed in $G \times G$, since H is closed in G.

Consider the differential of Ψ at $(e,e) \in G \times G$, i.e. the mapping

$$T_{(e,e)}\Psi: T_eG \times T_eH \to T_eG \times T_eG.$$

Then $T_{(e,e)}\Psi(v,u) = (v, v+u)$ for all $(v,u) \in T_eG \times T_eH$ and therefore $T_{(e,e)}\Psi$ is injective. Since $\mathfrak{h}$ is a direct subspace of $\mathfrak{g}$, we have

$$T_{(e,e)}(T_eG \times T_eH) = \{(v,w) \in T_eG \times T_eG \colon\ w - v \in T_eH\}$$

is a split subspace of $T_eG \times T_eG$. Therefore Ψ is an immersion at (e,e). Using the fact that the diagram

$$\begin{array}{ccc} G \times H & \xrightarrow{\ \Psi\ } & G \times G \\ \big\downarrow{\scriptstyle L_g \times R_h} & & \big\downarrow{\scriptstyle L_g \times L_g R_h} \\ G \times H & \xrightarrow{\ F\ } & G \times G \end{array}$$

commutes for all $g \in G$ and $h \in H$, we conclude that Ψ is an immersion. (Here L_g and R_h denote left and right multiplication in G, with g and h respectively).

Consider an arbitrary point $a \in G \times H$. Since Ψ is a holomorphic immersion at a, there exist charts (P, p, E) of $G \times H$ about a and (T, q, F) of $G \times G$ about $\Psi(a)$ such that $\Psi(P) \subset T$, $F = E \oplus V$ for some complex Banach space V, and the diagram

$$\begin{array}{ccc} P & \xrightarrow{\ \Psi\ } & T \\ \big\downarrow{\scriptstyle p} & & \big\downarrow{\scriptstyle q} \\ E & \xrightarrow{\ \mathrm{id}\ } & F \end{array}$$

commutes. Since Ψ is a homeomorphism onto R, we may assume that $\Psi(P) = R \cap T$. Then $q(R \cap T) = q(\Psi(P)) = p(P)$. Denote by π the continuous projection from F onto E, and define

$$T' := \{b \in T \colon\ \pi(q(b)) \in p(P)\}\ .$$

T' is an open neighborhood of $\Psi(a)$ and $q(R \cap T') = p(P) = E \cap q(T')$. Therefore R is a submanifold of $G \times G$. □

Claim: The projections $\pi_k: R \to G$, for $k = 1, 2$, are holomorphic submersions.

Proof: We prove that the mapping Ψ is biholomorphic. Suppose m is an arbitrary point in $G \times H$. The tangent space at m satisfies

$$\begin{aligned}(T_{\Psi(m)}q)(T_{\Psi(m)}R) &= E = (T_m p)(T_m(G \times H)) \\ &= T_m(q \circ \Psi)T_m(G \times H) \\ &= (T_{\Psi(m)}q \circ T_m\Psi)(T_m(G \times H)) \,.\end{aligned}$$

Therefore $T_m\Psi: T_m(G \times H) \to T_{\Psi(m)}R$ is an isomorphism for every $m \in G \times H$. By the inverse mapping theorem we obtain that Ψ is biholomorphic.

Denote by $\pi_1: G \times G \to G$ the projection onto the first factor. We have

$$\pi_{1|R} \circ \Psi = \pi_{1|G \times H}.$$

Therefore $\pi_{1|R}: R \to G$ is a holomorphic submersion. □

Hence by Godement's Theorem we have obtained the following proposition.

4.12 Proposition. *The quotient space $Q := G/H$ can be endowed with a unique complex structure such that Q is a complex Banach manifold and the canonical projection $\pi: G \to Q$ is a holomorphic submersion.*

4.13 Lemma. *The left translation action r of the group G on the complex Banach manifold Q is holomorphic.*

Proof. Denote by r_G the product mapping in G and by $\pi: G \to Q$ the canonical projection. We have the commuting diagram

$$\begin{array}{ccc} G \times G & \xrightarrow{r_G} & G \\ \Big\downarrow{\scriptstyle \mathrm{id} \times \pi} & & \Big\downarrow{\scriptstyle \pi} \\ G \times Q & \xrightarrow{r} & Q \,. \end{array}$$

Since $\mathrm{id} \times \pi$ is a surjective holomorphic submersion, the mapping r is holomorphic. □

For the following r will denote the left translation action of the group G on the manifold Q and ρ the corresponding holomorphic action of the Banach Lie algebra $\mathfrak{g}$ on Q. Recall that given r, the mapping ρ is uniquely determined as the differential of r, i.e. the uniquely determined holomorphic mapping ρ such that the diagram

$$\begin{array}{ccc} G & \xrightarrow{r} & \mathrm{Aut}(Q) \\ \Big\uparrow{\scriptstyle \exp} & & \Big\uparrow{\scriptstyle \exp} \\ \mathfrak{g} & \xrightarrow{\rho} & \mathrm{aut}(Q) \end{array}$$

commutes. Denote the point H in Q by a.

4.14 Proposition. *There exist an open neighborhood P of a in Q and a biholomorphic mapping $p: P \to \mathfrak{g}_{-1}$ such that $p(a) = 0$ and*

$$(p \circ r_a \circ \exp)(\alpha\, \partial/\partial z) = \alpha \quad \textit{for all} \quad \alpha \in U \,.$$

In the following we will refer to the triple (P, p, U) as the canonical chart about a in Q.

Proof. Consider the holomorphic mapping $\varphi\colon \mathfrak{g} \to G$ defined by

$$\varphi(X+Y) = \exp(X)\exp(Y)$$

for all X, Y in $\mathfrak{g}$. Denote by α the canonical projection of $\mathfrak{g}$ onto $\mathfrak{g}_{-1}$ and by π the canonical projection of G onto Q. Then for the holomorphic mapping ψ defined by

$$\psi(X) := (r_a \circ \exp)X$$

for all $X \in \mathfrak{g}_{-1}$, the diagram

$$\begin{array}{ccc} \mathfrak{g} & \xrightarrow{\varphi} & G \\ \downarrow{\scriptstyle\alpha} & & \downarrow{\scriptstyle\pi} \\ \mathfrak{g}_{-1} & \xrightarrow{\psi} & Q \end{array}$$

commutes. Taking differentials we obtain the commuting diagram

$$\begin{array}{ccc} \mathfrak{g} & \xrightarrow{\rho_e} & T_eG \\ \downarrow{\scriptstyle\alpha} & & \downarrow{\scriptstyle T_e\pi} \\ \mathfrak{g}_{-1} & \xrightarrow{T_0\psi} & T_aQ\,. \end{array}$$

For the group H we have $H = \pi^{-1}(a)$. Since π is a holomorphic submersion (4.12), by ([17] Cor. 8.9), we obtain $T_eH = \operatorname{Ker} T_e\pi$, or $\operatorname{Ker} T_e\pi = \rho_e(\mathfrak{h})$. Therefore $T_0\psi$ is an isomorphism and by the inverse mapping theorem the mapping ψ is locally-biholomorphic at $0 \in \mathfrak{g}_{-1}$.

For every vector field $\beta\,{}^{\partial}\!/_{\partial z} \in \mathfrak{g}_{-1}$ consider the diagram

$$\begin{array}{ccc} \mathfrak{g}_{-1} & \xrightarrow{\psi} & Q \\ \downarrow{\scriptstyle L_\beta} & & \downarrow{\scriptstyle r(g)} \\ \mathfrak{g}_{-1} & \xrightarrow{\psi} & Q\,, \end{array}$$

where $L_\beta(\alpha\,{}^{\partial}\!/_{\partial z}) := (\alpha+\beta)\,{}^{\partial}\!/_{\partial z}$ for all $\alpha\,{}^{\partial}\!/_{\partial z} \in \mathfrak{g}_{-1}$, $g := \exp(\beta\,{}^{\partial}\!/_{\partial z}) \in G$ and $r(g)(kH) := gkH$ for all $kH \in Q$. Then for every vector field $\alpha\,{}^{\partial}\!/_{\partial z} \in \mathfrak{g}_{-1}$ we have

$$\begin{aligned} \big(r(g)\circ\psi\big)(\alpha\,{}^{\partial}\!/_{\partial z}) &= r(g)\big((r_a\circ\exp)(\alpha\,{}^{\partial}\!/_{\partial z})\big) \\ &= r(g)\big(r_a(\exp(\alpha\,{}^{\partial}\!/_{\partial z}))\big) \\ &= r_a\big(\exp(\beta\,{}^{\partial}\!/_{\partial z})\exp(\alpha\,{}^{\partial}\!/_{\partial z})\big) \\ &= r_a\big(\exp(\alpha+\beta)\,{}^{\partial}\!/_{\partial z}\big) \\ &= \big(r_a\circ\exp\circ L_\beta\big)(\alpha\,{}^{\partial}\!/_{\partial z}) \\ &= \big(\psi\circ L_\beta\big)(\alpha\,{}^{\partial}\!/_{\partial z})\,. \end{aligned}$$

and hence the above diagram commutes.

Therefore ψ is locally biholomorphic on $\mathfrak{g}_{-1}$. Then ψ is locally biholomorphic on $\mathfrak{g}_{-1}$. Then $P := \psi(\mathfrak{g}_{-1})$ is an open subset on Q. For all $\alpha, \beta \in U$ the mapping

$$g := \exp((\alpha-\beta)\,{}^{\partial}\!/_{\partial z}) \in G$$

satisfies $g(I) = I + (\beta - \alpha)\,{}^{\partial}\!/_{\partial z}$. Therefore $g \in H$ if and only if $\alpha = \beta$. Then ψ is injective on $\mathfrak{g}_{-1}$. In particular $\psi\colon \mathfrak{g}_{-1} \to P$ is biholomorphic. Define $p := \psi^{-1}$. Then

$$(p \circ r_a \circ \exp)(X) = (\psi^{-1} \circ \psi)(X) = X$$

for all $X \in \mathfrak{g}_{-1}$.

Since U and $\mathfrak{g}_{-1}$ are isomorphic (via $\alpha \mapsto \alpha\,{}^{\partial}\!/_{\partial z}$ for all $\alpha \in U$), we may assume that ψ is defined on U, i.e. $\psi\colon U \to P$ and

$$\psi(\alpha) = (r_a \circ \exp)(\alpha\,{}^{\partial}\!/_{\partial z})$$

for all $\alpha \in U$. Then 4.12 holds. Also $p\colon P \to U$ and the triple (P, p, U) is the desired canonical chart about a in Q. □

4.15 Lemma. *For every vector field $X \in \mathfrak{g}$ the following equation holds*

$$p_*(\rho X) = X\ ,$$

where (P, p, U) is the canonical chart of Q about a.

Proof. ([17], p. 154, Th. 9.21). □

4.16 Proposition. *Let L be a subgroup of G containing $\exp(\check{\mathfrak{l}})$. Suppose $\alpha \in U$ satisfies $(1 - \alpha\,\square\,\alpha^*) \in \mathsf{GL}(U)$. Then the orbit $L(m)$ is open in Q, where $m := p^{-1}(\alpha)$ and (P, p, U) is the canonical chart of Q about $a := H$.*

Proof. For every $\beta \in U$ define $X_\beta := (\beta - \beta^*)\,{}^{\partial}\!/_{\partial z}$ and $X^\beta := X_\beta + \frac{1}{2}[X_\alpha, X_\beta] \in \check{\mathfrak{l}}$. Furthermore consider the mapping $\psi\colon U \to \check{\mathfrak{l}}$ defined by $\psi(\beta) := X^\beta$ for all $\beta \in U$. In particular ψ is $\mathbb{R}$-linear and continuous.

Consider the mapping $\varphi\colon U \to Q$ defined by $\varphi := r_m \circ \exp \circ \psi$. Then φ is real-analytic. We claim that $T_0\varphi\colon U \to T_m Q$ is an isomorphism. Hence by the inverse mapping theorem we get that φ is real-bianalytic on a neighborhood of $0 \in U$. Since $\varphi(U) \subset L(m)$, the orbit $L(m)$ is a neighborhood of $m \in Q$ and is therefore open.

Note that $(T_0\varphi)(\beta) = \rho_m(X^\beta)$ for all $\beta \in U$. Furthermore since $p\colon P \to U$ is a biholomorphic mapping, it induces an isomorphism of Banach Lie algebras such that the diagram

$$\begin{array}{ccc} P & \xrightarrow{\;p\;} & U \\ \big\downarrow{\scriptstyle \rho X} & & \big\downarrow{\scriptstyle p_*(\rho X^\beta)} \\ TP & \xrightarrow{\;Tp\;} & TU \end{array}$$

commutes. Therefore

$$\begin{aligned}(T_m p \circ T_0\varphi)(\beta) &= (T_m p)((T_0\varphi)(\beta)) = (T_m p)(\rho_m(X^\beta)) \\ &= (T_m p)((\rho X^\beta)_m) = (p_*(\rho X^\beta))_\alpha\ .\end{aligned}$$

Proposition 4.15 implies

$$(p_*(\rho X^\beta))_\alpha = {X^\beta}_\alpha = (\mathrm{id}_U - \alpha\,\square\,\alpha^*)(\beta)\ .$$

We have obtained

$$(T_m p \circ T_0\varphi)(\beta) = \psi_\alpha(\beta) \quad \text{for all} \quad \beta \in U\ ,$$

where $\psi_\alpha := (\mathrm{id}_U - \alpha \square \alpha^*) \in \mathsf{GL}(U)$. Since $T_m p$ is an isomorphism, the mapping $T_0\varphi$ is an isomorphism. So our claim is proved. □

For the following we consider the Banach Lie groups

$$\check{L} := \langle \exp \rho(\check{\mathfrak{l}}) \rangle \subset L := \langle \exp \rho(\mathfrak{l}) \rangle \subset \widehat{L} := \langle \exp \rho(\widehat{\mathfrak{l}}) \rangle \subset \mathsf{Aut}(Q)$$

with Lie algebras $\check{\mathfrak{l}}, \mathfrak{l}$ and $\widehat{\mathfrak{l}}$ respectively acting analytically on Q with differentials $\rho_{|\check{\mathfrak{l}}}$, $\rho_{|\mathfrak{l}}$ and $\rho_{|\widehat{\mathfrak{l}}}$.

4.17 Proposition. *The orbit $N := r_a(L) = L(a)$ is an open connected submanifold of Q on which the group L acts transitively.*

Proof. Applying 4.16 to $\alpha = 0$ we obtain that $\widehat{L}(a)$ is an open connected submanifold of Q containing $\check{L}(a)$ as an open subset. Since $\check{\mathfrak{l}}$ is an ideal of $\widehat{\mathfrak{l}}$, for all vector fields $X \in \widehat{\mathfrak{l}}$ and $Y \in \check{\mathfrak{l}}$ we get:

$$\exp(X)\exp(Y)\exp(-X) = \exp(\exp(\mathsf{ad}\, X)Y).$$

Therefore $\check{L}$ is a normal subgroup of $\widehat{L}$.

Consider an arbitrary element $n \in \check{L}(a)$. There exist a mapping $g \in \check{L}$ such that $n = g(a)$. Then

$$\check{L}(n) = \check{L}(g(a)) = g(g^{-1}\check{L}g(a)) = g(\check{L}(a))$$

is open in $\widehat{L}(a)$. Then $\check{L}(n)$ is both closed and open in $\widehat{L}(a)$. Since $\widehat{L}(a)$ is connected we obtain $\check{L}(a) = L(a) = N = \widehat{L}(a)$. □

In particular we have obtained that the action $\rho\colon \mathfrak{l} \to \mathrm{aut}(N)$ is analytic and faithful.

The manifold N is connected, as well as locally connected and locally simply-connected. Therefore N admits a universal covering $\mu\colon D \to N$, where D is a connected and simply-connected topological space and μ is a covering projection. Furthermore there exists a uniquely determined complex Banach manifold structure on D such that μ is locally biholomorphic.

Fix a point a in D over $a \in N$, i.e. a point $a \in D$ satisfying $\mu^{-1}(a) = a \in D$.

Denote by ρ_1 the lifted action of $\mathfrak{l}$ on D. It is analytic and faithful, since ρ is analytic and faithful. Define

$$L_1 := \langle \exp \rho_1(\mathfrak{l}) \rangle \subset \mathsf{Aut}(D)\ . \tag{4.18}$$

The group L_1 is a connected Banach Lie group, with Lie algebra $\mathfrak{l}$ acting analytically on D with differential ρ_1. Recall the following definition

4.19 Definition. An analytic action r of a Banach Lie group G on a Banach manifold M is called locally transitive at $o \in M$ if the evaluation mapping

$$r_o\colon G \to M$$

is an analytic submersion at $e \in G$. The action r is called locally transitive, if it is locally transitive at every point $m \in M$.

4.20 Proposition. *The canonical action of L_1 on D, denoted by r_1, is locally transitive and hence transitive.*

Proof. The proof is based mainly on the following

Claim: The canonical action r of the Banach Lie group L on N is locally transitive.

Proof: We shall prove that $r_a: L \to N$ is a submersion, which is equivalent to prove that $\rho_a: \mathfrak{l} \to T_a N$ is surjective and has a split null-space. Consider the canonical chart $(P \cap N, p, U)$ about a in N. Note that $T_a p: T_a N \to U$ is an isomorphism. Further consider an arbitrary vector field $X \in \mathfrak{l}$. The mapping p induces an isomorphism of Banach Lie algebras, such that the diagram

$$\begin{array}{ccc} P & \xrightarrow{\ p\ } & U \\ \Big\downarrow X & & \Big\downarrow p_* X \\ TP & \xrightarrow{\ Tp\ } & TU \end{array}$$

commutes. Therefore

$$(T_a p)(\rho_a X) = (T_a p)((\rho X)a) = p_*(\rho X)_{p(a)} = p_*(\rho X)_0.$$

Proposition 4.15 implies

$$p_*(\rho X)_0 = X_0 = \psi(X),$$

where $\psi(X) := X_0$ for all $X \in \mathfrak{l}$. This implies

$$(T_a p) \circ \rho_a = \psi$$

and therefore $\mathsf{Ker}(\rho_a) = \mathsf{Ker}(\psi) = \mathfrak{k}$. We have obtained that $\mathsf{Ker}(\rho_a)$ is a split subspace of $\mathfrak{l}$. Furthermore ρ_a is surjective, since $T_a p$ is an isomorphism and ψ is surjective. The group L acts transitively on N and hence the claim is proved.

As a consequence of the above claim we obtain that the lifted action $\tilde{r}$ of the covering group $\tilde{L}$ of L, on D is locally transitive. But $\tilde{r}$ and the action r_1 can be identified. Therefore the action of L_1 on D is locally-transitive. □

4.21 Proposition. *There exists a chart (P_1, p_1, U) of D about a such that*

$$p_{1*}(\rho_1 X) = X \quad \textit{for all} \quad X \in \mathfrak{l}.$$

The isotropy subgroup at a, $K_1 := \{g \in L_1 : r_1(g, a) = a\}$ is a connected Banach Lie subgroup of L_1 with Lie algebra $\mathfrak{k}$. The canonical mapping $\varphi: L_1/K_1 \to D$ defined by $\varphi(gK_1) := r_1(g, a)$ is real-bianalytic. In the following we refer to the chart (P_1, p_1, U) as the canonical chart of D about a.

Proof. By 4.20 the action r_1 of L_1 on D is locally-transitive. In particular the evaluation mapping r_{1a} at the point a, is a submersion at $e \in L_1$. Then the diagram

$$\begin{array}{ccc} L_1 & \xrightarrow{\ r_{1a}\ } & D \\ \Big\downarrow L_g & & \Big\downarrow r_1(g) \\ L_1 & \xrightarrow{\ r_{1a}\ } & D \end{array}$$

commutes for all $g \in L_1$, where L_g is the left multiplication with $g \in L_1$ and $r_1(g)(b) := r_1(g,b)$ for all $b \in D$. Therefore $r_{1a}: L_1 \to D$ is a submersion. By ([17], Cor. 8.9, p.128) the group $K_1 = {r_{1a}}^{-1}(a)$ is a closed submanifold and hence a Banach Lie subgroup of L_1. Furthermore

$$T_e K_1 = \operatorname{Ker} T_e r_{1a} = \rho_{1e}(\operatorname{Ker} \rho_{1a}) .$$

Then for the Banach Lie algebra of K_1, denoted by $\mathfrak{L}(K_1)$, we have

$$\mathfrak{L}(K_1) = \operatorname{Ker} \rho_{1a} .$$

Denote by $\pi: L_1 \to L_1/K_1$ the canonical projection and note that

$$\varphi \circ \pi = r_{1a} .$$

Since r_{1a} and π are analytic submersions, the mapping φ is bianalytic. Therefore L_1/K_1 is simply-connected in the quotient topology. By ([4], p.59, Cor.1) the group K_1 is connected (since L_1 is connected). Let $(P \cap N, p, U)$ denote the canonical chart of N about a. Since D is the universal covering of N, there exists an open neighborhood P_1 of a in D such that $\mu_{|P_1}$ is a homeomorphism onto an open subset of $P \cap N$. Define $p_1 := p \circ \mu_{|P_1}$. The triple (P_1, p_1, U) is a chart of D about a satisfying

$$p_{1*}(\rho_1 X) = X$$

for all $X \in \mathfrak{l}$. Therefore $(T_a p_1 \circ \rho_{1a})(X) = X_0$ for all $X \in \mathfrak{l}$. Hence $\operatorname{Ker}(\rho_{1a}) = \mathfrak{k}$. That is $\mathfrak{L}(K_1) = \mathfrak{k}$. □

4.22 Lemma. *Suppose X is a vector field $\mathfrak{k}$ and define $g := \exp(X)$. Then the diagram*

$$\begin{array}{ccc} r_1(g)^{-1} \cap P_1 & \xrightarrow{r_1(g)} & P_1 \\ \downarrow{\scriptstyle p_1} & & \downarrow{\scriptstyle p_1} \\ U & \xrightarrow{\exp(X)} & U \end{array} \tag{4.23}$$

commutes.

Proof. According to proposition 4.15

$$X = p_{1*}(\rho_1 X) .$$

Then

$$\exp(X) \circ p_1 = \exp(p_{1*}(\rho_1 X)) \circ p_1 .$$

Therefore by ([17], Pr. 5.16, p.80) we have

$$(\exp(p_{1*}(\rho_1 X)) \circ p_1 = p_1 \circ \exp(\rho_1 X) \circ {p_1}^{-1} \circ p_1 = p_1 \circ \exp(\rho_1 X) .$$

Since ρ_1 is the differential of r_1 we have

$$\exp(\rho_1 X) = (r_1 \circ \exp)(X) = r_1(g) .$$

Therefore

$$\exp(X) \circ p_1 = p_1 \circ r_1(g) ,$$

or the diagram above commutes. □

4.24 Proposition. *The simply-connected complex Banach manifold D associated with the J^*-triple system is a circular, symmetric complex Banach manifold.*

Proof. Consider the canonical chart (P_1, p_1, U) about a in D. The mapping p_1 is locally bianalytic in a neighborhood of 0 in P_1 and therefore induces an isomorphism $T_a p_1 : T_a D \to U$. Endow $T_a D$ with a compatible norm such that $T_a p_1$ becomes an isometry. Note that $K_1 = \langle \exp(\mathfrak{k}) \rangle$. Therefore using proposition 4.22 and passing to the differentials in 4.23 we obtain the commuting diagram

$$\begin{array}{ccc} T_a D & \xrightarrow{T_a r_1(g)} & T_a D \\ \Big\downarrow T_a p_1 & & \Big\downarrow T_a p_1 \\ U & \xrightarrow{\exp(X)} & U\,. \end{array}$$

Therefore $T_a r_1(g)$ is a linear isometry for every $g \in K_1$ ($\exp(X)$ is a linear isometry for every $X \in \mathfrak{k}$). Hence the norm on $T_a D$ is invariant under K_1.

For every $m \in D$ define $\nu_m : T_m D \to \mathbb{R}_+$ by

$$\nu_m(v) := |T_m(r_1(h))v| \,,$$

where $h \in L_1$ is such that $r_1(h, m) = a$. Then

$$\nu : TD \to \mathbb{R}_+, \quad \nu_{|T_m D} := \nu_m$$

is well-defined norm on the tangent bundle TD. ν is invariant under L_1.

The evaluation mapping $r_{1a} : L_1 \to D$ is an analytic submersion. Therefore there exists a chart (Γ, γ, Z) of D about a and a real-analytic mapping $\chi : \Gamma \to L_1$ such that $\chi(a) = e \in L_1$ and $r_{1a} \circ \chi = \mathrm{id}_\Gamma$. In particular ν is continuous.

Consider the mapping $\zeta : \Gamma \to L(Z)$, defined by

$$\zeta(m) := T_a\gamma \circ T_m(r_1(\chi(m))) \circ (T_m\gamma)^{-1}.$$

The mapping ζ is real-analytic. Therefore we may assume that

$$|\zeta(m) - \mathrm{id}_Z| \leq \frac{1}{2},$$

for all $m \in \Gamma$, where Z carries the norm induced by $T_a\gamma : T_a D \to Z$. Then we have

$$|T_a\gamma T_m(r_1(\chi(m)))v - (T_m\gamma)v| \leq \frac{1}{2}|(T_m\gamma)v| \,.$$

Hence

$$\frac{1}{2}|(T_m\gamma)(v)| \leq \nu_m(v) \leq \frac{3}{2}|(T_m\gamma)(v)| \quad \text{for all} \quad v \in T_m D.$$

Therefore (D, ν) is a connected normed Banach manifold.

Note that L_1 is the Banach Lie group of all biholomorphic isometries of D. Furthermore the vector field $iI := iz\frac{\partial}{\partial z}$ is in $\mathfrak{k}$ and $p_{1*}(\rho_1(iI)) = iI$, according to 4.21. Therefore $t \mapsto \exp(t\rho_1(iI))$ defines an isometric circle action on D with fixed point a. Hence D is circular about a and $s_a := \exp(\pi\rho_1(iI))$ is a symmetry of D about a. So we have obtained that the simply-connected symmetric complex Banach manifold associated with the J^*-triple system $(U, *)$ is a symmetric Banach manifold. □

5. Morphisms

For every complex Banach manifold M denote by $\mathfrak{V}(M)$ the Lie algebra of all holomorphic vector fields on M. For every open subset $V \subset M$ denote by $\mathcal{H}(V)$ the space of all complex valued holomorphic functions defined on V.

Note that for all Banach manifolds M and M' the space $\mathfrak{V}(M) \times \mathfrak{V}(M')$ carries the structure of a Lie algebra, where the bracket product is defined componentwise,

$$\big[(X,X'),(Y,Y')\big] := \big([X,Y],[X',Y']\big) .$$

Suppose that M and M' are complex Banach manifolds and that $\varphi\colon M \to M'$ is a holomorphic mapping. We say that the vector fields $X \in \mathfrak{V}(M)$ and $X' \in \mathfrak{V}(M')$ are *φ-related* if

$$(5.1) \qquad (T_a\varphi)(X_a) = X'_{\varphi(a)}$$

for all a in M. If we consider the vector fields X and X' as differential operators condition 5.1 is equivalent to

$$X(f \circ \varphi_{|V}) = (X'f) \circ \varphi_{|V}$$

for all open $V \subset M$, $V' \subset M'$ such that $\varphi(V) \subset V'$ and every function $f \in \mathcal{H}(V')$.

5.2 Proposition. *Suppose M and M' are complex Banach manifolds and $\varphi\colon M \to M'$ is a holomorphic mapping. Then the vector space*

$$\widetilde{\mathfrak{V}} := \big\{(X,X') \in \mathfrak{V}(M) \times \mathfrak{V}(M') : \ X \text{ and } X' \text{ are } \varphi-\text{related}\,\big\}$$

is closed under the bracket product and therefore is a Lie subalgebra of $\mathfrak{V}(M) \times \mathfrak{V}(M')$.

Proof. Consider arbitrary pairs of vector fields (X,X'), (Y,Y') in $\widetilde{\mathfrak{V}}$ and their bracket product $\big([X,Y],[X',Y']\big)$. We shall prove that $[X,Y]$ and $[X',Y']$ are φ-related.

Suppose V and V' are open subsets of M and M' respectively such that $\varphi(V) \subset V'$. Furthermore consider an arbitrary function $f \in \mathcal{H}(V')$. Then we have

$$\begin{aligned}\big([X',Y']f\big) \circ \varphi_{|V} &= \big[Y'(X'f) - X'(Y'f)\big] \circ \varphi_{|V} \\ &= Y'(X'f) \circ \varphi_{|V} - X'(Y'f) \circ \varphi_{|V} .\end{aligned}$$

Since X and X' are φ-related, we have

$$X(g \circ \varphi_{|V}) = X'g \circ \varphi_{|V}$$

for all $g \in \mathcal{H}(V')$. By the same reasons, we have

$$Y(g \circ \varphi_{|V}) = Y'g \circ \varphi_{|V}$$

for all $g \in \mathcal{H}(V')$. Therefore

$$X'(Y'f) \circ \varphi_{|V} = X(Y'f \circ \varphi_{|V}) = X\big(Y(f \circ \varphi_{|V})\big) ,$$

$$Y'(X'f) \circ \varphi_{|V} = Y(X'f \circ \varphi_{|V}) = Y(X(f \circ \varphi_{|V})) .$$

So we obtain

$$\begin{aligned}([X',Y']f)\circ\varphi_{|V} &= Y(X(f\circ\varphi_{|V})) - (X(Y(f\circ\varphi_{|V})))\\ &= [X,Y](f\circ\varphi_{|V})\ ,\end{aligned}$$

which implies that the vector fields $[X,Y]$ and $[X',Y']$ are φ-related and therefore the pair $\big([X,Y],[X',Y']\big)$ lies in $\widetilde{\mathfrak{V}}$. □

Suppose $(U,*)$ and $(U',*)$ are J^*-triple systems. Denote by (D,a) and (D',a') the simply-connected, symmetric complex Banach manifolds associated with them according to the construction given in chapter 4. Furthermore suppose $\lambda\colon (U,*)\to(U',*)$ is a J^*-morphism. We shall prove that λ uniquely determines a morphism $\Lambda\colon (D,a)\to(D',a')$ of the corresponding symmetric Banach manifolds, such that $d\Lambda(a)=\lambda$ (recall the properties of the canonical charts about a and a', in D and D' respectively).

Let L_1 and $L_1{}'$ be the connected real Banach Lie groups with Banach Lie algebras $\mathfrak{l}$ and $\mathfrak{l}'$, acting analytically and transitively on D and D' respectively, introduced also in chapter 4, compare 4.18. Denote by L and L' their universal covering groups. Then L and L' are simply-connected Banach Lie groups with Banach Lie algebras $\mathfrak{l}$ and $\mathfrak{l}'$, acting analytically and transitively on D and D' respectively. Denote the action of L on D by r, and the action of L' on D' by r'. Furthermore denote by K and K' the isotropy subgroups at a and a' respectively, i.e.

$$K:=\{g\in L : r(g,a)=a\}\quad\text{and}\quad K':=\{g'\in L' : r'(g',a')=a'\}\ .$$

Note that K and K' are Banach Lie subgroups of L and L' respectively. The canonical projections $\pi\colon L\to L/K$ and $\pi'\colon L'\to L'/K'$ are analytic (submersions). The canonical bijections

$$\varphi\colon L/K\to D,\ \varphi(gK):=r(g,a)=r(g)(a)$$

and

$$\varphi'\colon L'/K'\to D',\ \varphi'(g'K'):=r'(g',a')=r'(g')(a')$$

are bianalytic. Note that $\varphi\circ\pi=r_a$ and $\varphi'\circ\pi'=r'_{a'}$. In the following we identify D with L/K and D' with L'/K' via φ and φ' respectively. In this case r_a is identified with π and $r'_{a'}$ with π'.

According to proposition 5.2 the subspace $\tilde{\mathfrak{l}}$ of $\mathfrak{l}\oplus\mathfrak{l}'$, consisting of all pairs of vector fields (X,X') such that X and X' are λ-related is a (real) Banach Lie subalgebra. Since the action of $\mathfrak{l}$ on D and the action of $\mathfrak{l}'$ on D' are faithful, there exists a simply-connected Banach Lie group $\tilde{L}$ with Banach Lie algebra $\tilde{\mathfrak{l}}$ (compare [17], Th.7.5).

Denote by θ and θ' the restrictions on $\tilde{\mathfrak{l}}$, of the canonical projections of $\mathfrak{l}\oplus\mathfrak{l}'$ onto the first and the second factor respectively. Since the groups L, L' and $\tilde{L}$ are simply-connected, θ and θ' uniquely determine Banach Lie group homomorphisms Θ and Θ' such that the diagrams

$$(5.3)\qquad \begin{array}{ccc} \tilde{\mathfrak{l}} & \xrightarrow{\ \theta\ } & \mathfrak{l}\\ \big\downarrow{\scriptstyle\exp} & & \big\downarrow{\scriptstyle\exp}\\ \tilde{L} & \xrightarrow{\ \Theta\ } & L \end{array}\qquad\qquad \begin{array}{ccc} \tilde{\mathfrak{l}} & \xrightarrow{\ \theta'\ } & \mathfrak{l}'\\ \big\downarrow{\scriptstyle\exp} & & \big\downarrow{\scriptstyle\exp}\\ \tilde{L} & \xrightarrow{\ \Theta'\ } & L' \end{array}$$

commute. Denote by $\tilde{K}$ the Banach Lie group $\Theta^{-1}(K)$. The space $\tilde{L}/\tilde{K}$ carries uniquely determined Banach manifold structure such that the canonical projection $\tilde{\pi}\colon\tilde{L}\to\tilde{L}/\tilde{K}$

is an analytic submersion. The mapping $\tilde{r}(\tilde{g}_1, \tilde{g}_2\tilde{K}) := \tilde{g}_1\tilde{g}_2\tilde{K}$ defines an analytic action of $\tilde{L}$ on $\tilde{L}/\tilde{K}$. Note also that $\tilde{r}_{\tilde{a}} = \tilde{\pi}$, where $\tilde{a} := \tilde{K} \in \tilde{L}/\tilde{K}$.

In the following we consider the analytic mappings

$$\Psi\colon \tilde{L}/\tilde{K} \to L/K,\ \Psi(\tilde{g}\tilde{K}) := \Theta(\tilde{g})K$$

and

$$\Psi'\colon \tilde{L}/\tilde{K} \to L'/K',\ \Psi'(\tilde{g}\tilde{K}) := \Theta'(\tilde{g})K'$$

Note that by the definitions of Ψ and Ψ' we have the commuting diagrams

$$(5.4)\qquad \begin{array}{ccc} \tilde{L} & \xrightarrow{\Theta} & L \\ \downarrow{\scriptstyle \tilde{\pi}} & & \downarrow{\scriptstyle \pi} \\ \tilde{L}/\tilde{K} & \xrightarrow{\Psi} & L/K \end{array} \qquad \begin{array}{ccc} \tilde{L} & \xrightarrow{\Theta'} & L' \\ \downarrow{\scriptstyle \tilde{\pi}} & & \downarrow{\scriptstyle \pi'} \\ \tilde{L}/\tilde{K} & \xrightarrow{\Psi'} & L'/K'\,. \end{array}$$

Furthermore since $\Theta(\tilde{K}) \subset K$ the mapping Ψ is injective. As a direct conclusion of the definition of $\tilde{\mathfrak{l}}$ we have that $\mathfrak{p} \subset \mu(\tilde{\mathfrak{l}})$ (recall that $\mathfrak{p} = \{(\alpha - \alpha^*)\,{}^{\partial}\!/\!{}_{\partial z} : \alpha \in U\ \}$) and hence Ψ is surjective. So we have obtained that the mapping Ψ is bijective, which implies that Ψ is bianalytic.

Identifying L/K with D and L'/L' with D' via φ and φ' respectively we obtain the commuting diagram 1.

$$\begin{array}{ccccc} L & \xleftarrow{\Theta} & \tilde{L} & \xrightarrow{\Theta'} & L' \\ \downarrow{\scriptstyle r_a} & & \downarrow{\scriptstyle \tilde{\pi}} & & \downarrow{\scriptstyle r'_{a'}} \\ D & \xleftarrow{\Psi} & \tilde{L}/\tilde{K} & \xrightarrow{\Psi'} & D' \end{array}$$

Diagram 1

We shall prove that

$$(5.5)\qquad \Lambda\colon D \to D',\ \Lambda := \Psi' \circ \Psi^{-1}$$

is a morphism of symmetric manifolds. Obviously $\Lambda(a) = a'$.

5.6 Lemma. *For a sufficiently small open neighborhood W of a in D and up to a biholomorphic transformation of W the equality $\Lambda = \lambda$ holds.*

Proof. Passing to the differentials of diagram 1 we obtain

$$\begin{array}{ccccc} \mathfrak{l} & \xleftarrow{\theta} & \tilde{\mathfrak{l}} & \xrightarrow{\theta'} & \mathfrak{l}' \\ \downarrow{\scriptstyle \rho_a} & & \downarrow{\scriptstyle T_{\tilde{e}}(\tilde{\pi})} & & \downarrow{\scriptstyle \rho'_{a'}} \\ D & \xleftarrow{T_{\tilde{a}}\Psi} & T_{\tilde{a}}(\tilde{L}/\tilde{K}) & \xrightarrow{T_{\tilde{a}}\Psi'} & D' \end{array}$$

Diagram 2

where we have used that $T_e(r_a) = \rho_a$ and $T_{e'}(r'_{a'}) = \rho'_{a'}$.

Consider an arbitrary pair of vector fields (X, X') in $\tilde{\mathfrak{l}}$. Considering diagram 2 we obtain

$$\begin{aligned}(T_a\Lambda)\big((\rho_a \circ \theta)(X, X')\big) &= (T_{\tilde{a}}\Lambda)\big((T_{\tilde{a}}\Psi \circ T_{\tilde{e}}(\tilde{\pi}))(X, X')\big)\\ &= \big(T_{\tilde{a}}\Psi' \circ T_{\tilde{a}}(\Psi^{-1}) \circ T_{\tilde{a}}\Psi \circ T_{\tilde{e}(\tilde{\pi})}\big)(X, X')\\ &= (T_{\tilde{a}}\Psi' \circ T_{\tilde{e}}(\tilde{\pi}))(X, X')\\ &= \big(\rho'_{a'} \circ \theta'\big)(X, X')\ ,\end{aligned}$$

where we have used that the lefthand side and the righthand side of diagram 2 commutes, as well as the definition of Λ. So we have obtained

$$(T_a\Lambda)\big((\rho X)_a\big) = (\rho' X')_{a'}\ , \tag{5.7}$$

for every pair of λ-related vector fields in $\mathfrak{l} \oplus \mathfrak{l}'$.

There exist charts (P, p, U) and (P', p', U') about a and a' in D and D' respectively such that p and p' are bianalytic and

$$p_*(\rho Y)_0 = Y_0 \quad \text{and} \quad p'_*(\rho' Y')_0 = Y'_0\ , \tag{5.8}$$

for all vector fields $Y \in \mathfrak{l}$ and $Y' \in \mathfrak{l}'$.

In particular equality 5.7 implies

$$\big(T_a\Lambda \circ (T_a p)^{-1} \circ T_a p\big)\big((\rho X)_a\big) = \big((T_{a'}p')^{-1} \circ (T_{a'}p')\big)\big((\rho' X')_{a'}\big)\ .$$

Since p and p' induce homomorphisms of Banach Lie algebras we obtain

$$(T_a p)\big((\rho X)_a\big) = p_*(\rho X)_{p(a)} = p_*(\rho X)_0$$

$$(T_{a'}p')\big((\rho' X')_{a'}\big) = p'_*(\rho' X')_{p'(a')} = p'_*(\rho' X')_0\ .$$

Therefore we have

$$\big((T_a\Lambda) \circ (T_a p)^{-1}\big)\big(p_*(\rho X)_0\big) = (T_{a'}p')^{-1}\big(p'_*(\rho' X')_0\big)\ .$$

Using 5.8 and the definition of $\tilde{\mathfrak{l}}$ we obtain

$$\big((T_{a'}p' \circ (T_a\Lambda) \circ (T_a p)^{-1})\big)(X_0) = X'_0 = \lambda(X_0)\ . \tag{5.9}$$

Suppose α is an arbitrary element of U and consider the pair of vector fields (X, X') in $\tilde{\mathfrak{l}}$ where

$$X^\alpha := (\alpha - \alpha^*)\,\partial/\partial z \in \mathfrak{l} \quad \text{and} \quad X^{\lambda(\alpha)} := (\lambda(\alpha) - \lambda(\alpha^*))\,\partial/\partial z\ .$$

Then 5.9 implies

$$\big((T_{a'}p') \circ (T_a\Lambda) \circ (T_a p)^{-1}\big)(\alpha) = \lambda(\alpha)\ .$$

Hence for a sufficiently small open neighborhood W of a in D we have $\Lambda = \lambda$.

5.10 Lemma. *For every $\tilde{\sigma} \in \tilde{L}$ the equalities*

$$\Psi \circ \tilde{\sigma} = \Theta(\tilde{\sigma}) \circ \Psi \quad \text{and} \quad \Psi' \circ \tilde{\sigma} = \Theta'(\tilde{\sigma}) \circ \Psi' \tag{5.11}$$

hold.

Proof. Since the mapping Ψ is bianalyitc we may identify $\tilde{L}/\tilde{K}$ with D. Then $\tilde{a}$ is identified with a. So we may assume that $\Psi \in \mathrm{Aut}(D)$ and obtain the commuting diagrams

$$\begin{array}{ccc} \tilde{L} & \xrightarrow{\Theta} & L \\ \downarrow{\scriptstyle \tilde{r}_a} & & \downarrow{\scriptstyle r_a} \\ D & \xrightarrow{\Psi} & D \end{array} \qquad \begin{array}{ccc} \tilde{L} & \xrightarrow{\Theta'} & L' \\ \downarrow{\scriptstyle \tilde{r}_a} & & \downarrow{\scriptstyle r'_{a'}} \\ D & \xrightarrow{\Psi'} & L'/K' \end{array}$$

Since $\tilde{L}$ and L act transitively on D, as well as L' acts transitively on D' we get

$$\begin{array}{ccc} \tilde{L} & \xrightarrow{\Theta} & L \\ \downarrow{\scriptstyle \tilde{r}_x} & & \downarrow{\scriptstyle r_{\Psi(x)}} \\ D & \xrightarrow{\Psi} & D \end{array} \qquad \begin{array}{ccc} \tilde{L} & \xrightarrow{\Theta'} & L' \\ \downarrow{\scriptstyle \tilde{r}_x} & & \downarrow{\scriptstyle r'_{\Psi'(x)}} \\ D & \xrightarrow{\Psi'} & L'/K' \end{array}$$

for all $x \in D$. The last diagrams are exactly

$$r_{\Psi(x)} \circ \Theta = \Psi \circ \tilde{r}_x \quad \text{and} \quad r'_{\Psi'(x)} \circ \Theta' = \Psi' \circ \tilde{r}_x$$

for every $x \in D$.

Consider arbitrary element $\tilde{\sigma} \in \tilde{L}$. We have

$$(r_{\Psi(x)} \circ \Theta)(\tilde{\sigma}) = r_{\Psi(x)}(\Theta(\tilde{\sigma})) = \Theta(\tilde{\sigma})(\Psi(x)) = (\Theta(\tilde{\sigma}) \circ \Psi)(x)$$

$$(\Psi \circ \tilde{r}_x)(\tilde{\sigma}) = (\Psi \circ \tilde{\sigma})(x) \ ,$$

which implies

$$\Theta(\tilde{\sigma}) \circ \Psi = \Psi \circ \tilde{\sigma} \ .$$

Analogously we have

$$(r'_{\Psi'(x)} \circ \Theta')(\tilde{\sigma}) = (\Theta'(\tilde{\sigma}) \circ \Psi')(x)$$

$$(\Psi' \circ \tilde{r}_x)(\tilde{\sigma}) = \Psi'(\tilde{\sigma}(x)) = \Psi' \circ \tilde{\sigma} \ ,$$

which implies

$$\Theta'(\tilde{\sigma}) \circ \Psi' = \Psi' \circ \tilde{\sigma} \ .$$

So we have obtained the desired equalities. □

5.12 Lemma. *For every $g \in \tilde{L}$ the following equality holds*

$$\Lambda \circ \Theta(g) = \Theta'(g) \circ \Lambda \ .$$

Proof. Suppose $g \in \tilde{L}$ is an arbitrary element. By 5.11 we have

$$\Psi \circ g = \Theta(g) \circ \Psi \ .$$

Therefore

$$\Theta(g) = \Psi \circ g \circ \Psi^{-1} .$$

Hence

$$\Lambda \circ \Theta(g) = (\Psi' \circ \Psi^{-1}) \circ (\Psi \circ g \circ \Psi^{-1}) = \Psi' \circ g \circ \Psi^{-1} .$$

By 5.11 we have also

$$\Psi' \circ g = \Theta'(g) \circ \Psi' .$$

Therefore we have

$$\Lambda \circ \Theta(g) = \Theta'(g) \circ \Psi' \circ \Psi^{-1} = \Theta'(g) \circ \Lambda ,$$

which proves the claim. □

5.13 Proposition. *The mapping Λ is a morphism of symmetric Banach manifolds.*

Proof. As we have seen in chapter 3, we have $s_a = -\mathrm{id}$ and $s_{a'} = -\mathrm{id}$ on sufficiently small neighborhoods of a in D and a' in D' respectively (compare 3.26). But $\lambda \circ (-\mathrm{id}) = -\mathrm{id} \circ \lambda$, since λ is a linear mapping. Then by lemma 5.6 and by the uniqueness of the analytic continuations we obtain

$$\Lambda \circ s_a = s_{a'} \circ \Lambda .$$

We shall prove that

$$\Lambda \circ s_x = s_{\Lambda(x)} \circ \Lambda$$

for every $x \in D$.

For convenience consider diagram 1. Note that there exists an element $\sigma \in \tilde{L}$ such that

$$\Theta(\sigma) = s_a \quad \text{and} \quad \Theta'(\sigma) = s_{a'} .$$

Suppose $x \in D$ is an arbitrary element. Since Ψ is bijective there exists uniquely determined preimage $\tilde{\sigma}_\pi \in \tilde{L}/\tilde{K}$ of x, i.e. an element $\tilde{\sigma}_\pi \in \tilde{L}/\tilde{K}$ such that $\Psi(\tilde{\sigma}_\pi) = x$. Let $\tilde{\sigma} \in (\tilde{\pi})^{-1}(\tilde{\sigma}_\pi)$. The lefthand side of diagram 1, implies that

$$(r_a \circ \Theta)(\tilde{\sigma}) = x ,$$

which is equivalent to

$$\Theta(\tilde{\sigma}) = g ,$$

where g is an element of L such that $r_a(g) = g(a) = x$.

Also we have $\Lambda(x) = \Psi'(\tilde{\sigma}_\pi)$. The righthand side of diagram 1 implies

$$({r'}_{a'} \circ \Theta')(\tilde{\sigma}) = \Psi'(\tilde{\sigma}_\pi) = \Lambda(x)$$

which is equivalent to

$$\Theta'(\tilde{\sigma}) = g' ,$$

where g' is an element of L' such that $r'_{a'}(g') = g'(a') = \Lambda(x)$.

Consider the symmetries s_x and $s_{\Lambda(x)}$ at x and $\Lambda(x)$ respectively, and recall that

$$s_x = g \circ s_a \circ g^{-1} \quad \text{and} \quad s_{\Lambda(x)} = g' \circ s_{a'} \circ g^{-1} .$$

The above representation of g and g' implies

$$s_x = \Theta(\tilde{\sigma}) \circ \Theta(\sigma) \circ (\Theta(\tilde{\sigma}))^{-1} = \Theta(\tilde{\sigma} \circ \sigma \circ \tilde{\sigma}^{-1}) ,$$

$$s_{\Lambda(x)} = \Theta'(\tilde{\sigma}) \circ \Theta'(\sigma) \circ (\Theta'(\tilde{\sigma}))^{-1} = \Theta'(\tilde{\sigma} \circ \sigma \circ \tilde{\sigma}^{-1}) ,$$

where we have used that Θ and Θ' are group homomorphisms.

Now applying 5.12 and using the fact that Θ and Θ' are group homomorphisms we obtain

$$\begin{aligned}
\Lambda \circ s_x &= \big(\Lambda \circ \Theta(\tilde{\sigma})\big) \circ \Theta(\sigma) \circ \Theta(\tilde{\sigma})^{-1} \\
&= \Theta'(\tilde{\sigma}) \circ \big(\Lambda \circ \Theta(\sigma)\big) \circ \Theta(\tilde{\sigma})^{-1} \\
&= \Theta'(\tilde{\sigma}) \circ \Theta'(\sigma) \circ \big(\Lambda \circ \Theta(\tilde{\sigma})^{-1}\big) \\
&= \big(\Theta'(\tilde{\sigma}) \circ \Theta'(\sigma) \circ \Theta'(\tilde{\sigma})^{-1}\big) \circ \Lambda \\
&= s_{\Lambda(x)} \circ \Lambda .
\end{aligned}$$

Therefore Λ is a morphism of symmetric manifolds. □

So we have proved the following proposition.

5.14 Proposition. *Suppose $(U, *)$ and $(U', *)$ are J^*-triple systems. Denote by (D, a) and (D', a') the corresponding simply-connected symmetric complex Banach manifolds. Then every J^*-morphism $\lambda: U \to U'$ determines a morphism $\Lambda: D \to D'$ of symmetric complex Banach manifolds.*

In diagram 3 we give the spaces and the connections between them used in the proof of proposition (5.14).

$$\begin{array}{ccccc}
\mathfrak{l} & \xleftarrow{\ \theta\ } & \tilde{\mathfrak{l}} & \xrightarrow{\ \theta'\ } & \mathfrak{l}' \\
\big\downarrow \exp & & \big\downarrow \exp & & \big\downarrow \exp \\
L & \xleftarrow{\ \Theta\ } & \tilde{L} & \xrightarrow{\ \Theta'\ } & L' \\
\big\downarrow \pi & & \big\downarrow \tilde{\pi} & & \big\downarrow \pi' \\
L/K & \xleftarrow{\ \Phi\ } & \tilde{L}/\tilde{K} & \xrightarrow{\ \Phi'\ } & L'/K' \\
\big\downarrow \varphi & & & & \big\downarrow \varphi' \\
D & & \xrightarrow{\qquad\quad \Lambda' \qquad\quad} & & D'
\end{array}$$

Diagram 3

Here the mappings Φ and Φ' are defined the same way as Ψ and Ψ', compare 5.4. Note that $\Lambda' := \varphi' \circ (\Phi' \circ \Phi^{-1}) \circ \varphi^{-1}$. Actually diagram 1 is obtained from diagram 3, through the identification of the following spaces and mappings. The space L/K with D via φ and the space L'/K' with D' via the the mapping φ'. Then obviously $\varphi \circ \Phi$ is identified with Ψ (in diagram 1) and analogously $\varphi' \circ \Phi'$ with Ψ' (in diagram 1). Then Λ' is identified with Λ (compare 5.5).

Having proposition 5.14 we may conclude that the construction of a symmetric complex Banach manifold from a J^*-triple system given in chapter 3, determines a functor $\mathfrak{J}$ from the category of J^*-triples into the category of simply-connected, symmetric complex Banach manifolds with base point.

The functors $\mathfrak{F}$ and $\mathfrak{J}$ induce an equivalence of the categories of hermitian Jordan triple systems and simply-connected symmetric complex Banach manifolds with base point, which is the main result of [8].

It should be noted that in the case of metric morphisms, we obtain equivalence of subcategories.

6. Examples

Suppose $(U, *)$ is a J^*-triple system. For every element $x \in U$ define the mapping $Q(x)\colon U \to U$ by $Q(x)(y) := \{xy^*x\}$. Obviously, $Q(x)$ is an antilinear operator. We refer to it as *the quadratic representation* associated with x. It satisfies the *fundamental formula*

$$Q(Q(x)y) = Q(x)Q(y)Q(x)$$

for all $x, y \in U$, compare [17].

In the following we give some general notions of the theory of hermitian Jordan triple systems and some examples.

6.1 Definition. An element x of a hermitian Jordan triple system $(U, *)$ is called *trivial*, if the quadratic representation associated with x vanishes, i.e. $Q(x) = 0$. The element x is called *invertible*, if the quadratic representation associated with x is a bijection.

6.2 Definition. A hermitian Jordan triple system $(U, *)$ is called *trivial*, if every element of U is trivial.

6.3 Definition. A linear subspace W (not necessarily closed) of a hermitian Jordan triple system $(U, *)$ is called *a subtriple*, if $Q(W)W \subset W$, and *an inner ideal*, if $Q(W)U \subset U$.

As a direct conclusion of the fundamental formula we obtain that for every $x \in U$, the space $W = Q(x)U$ is an inner ideal, called *the principle ideal* generated by x.

6.4 Definition. A linear subspace W of a hermitian Jordan triple system U is called an ideal, if

$$\{WU^*U\} + \{UW^*U\} \subset W\,.$$

6.5 Definition. Suppose $(U, *)$ is a hermitian Jordan triple system and A, B are arbitrary subsets of U. Then A and B are called *orthogonal* if $A \,\square\, B^* = 0$ (and hence also $B \,\square\, A^* = (A \,\square\, B^*)^* = 0$).

6.6 Definition. A closed ideal W of a hermitian Jordan triple system $(U, *)$ is called *direct*, if there exists a closed ideal $W' \subset U$ such that $U = W \oplus W'$.

6.7 Definition. A non-trivial hermitian Jordan triple system $(U, *)$ is called *algebraically simple* (resp. *simple*, resp. *invertible*), if the only ideals (resp. closed ideals, resp. direct ideals) in U are U and $\{0\}$.

6.8 Definition. A hermitian Jordan triple system $(U, *)$ is called *prime*, if for every two ideals $V, W \subset U$ such that $Q(V)W = 0$ holds $V = 0$ or $W = 0$.

A prime hermitian Jordan triple system does not contain orthogonal ideals different from 0. Furthermore it can be shown that the following implications hold

$$\textit{algebraically simple} \Rightarrow \textit{simple} \Rightarrow \textit{prime} \Rightarrow \textit{indivisible}$$

and that none of them is invertible, compare [10].

Suppose $(U, *)$ is a J^*-triple system. For every real number t we can define a new triple product ${}^t\{\ \}$ in U by

$${}^t\{xy^*z\} := t\{xy^*z\}$$

where x, y, z are arbitrary elements in U, and hence a new hermitian Jordan triple system, which we denote by ${}^tU$. The triple systems U and ${}^tU$ are called *proportional*. In the case that t is a positive real number, the mapping $x \mapsto \sqrt{t}x$ determines an J^*-isomorphism of U and ${}^tU$. The J^*-triple system ${}^-U :={}^{-1} U$ is called *the dual of* U. In general U and ${}^-U$ are not isomorphic.

6.9 Definition. Suppose $(U, *)$ is a J^*-triple system. An element $e \neq 0$ in U is called a *tripotent*, if $\{ee^*e\} = \sigma e$ for $\sigma \in \{-1, 1\}$. The coefficient $\sigma = \sigma(e)$ is called the *sign* of e. In the case that $\sigma = +1$ the element e is called *a positive tripotent*, and in the case that $\sigma = -1$ the element e is called *a negative tripotent.*

Obviously when passing to the dual of a triple system, all tripotnets change their sign. If e is a tripotent, the element $c := \sigma(e)e$ is also a tripotent, called the *tripotent associated with* e. Furthermore for every J^*-triple system $(U, *)$ and every tripotent e in U, the product

$$xy := \{xc^*y\} \tag{6.10}$$

determines a Jordan algebra structure on U, according to which the element e is an idempotent. Consider also the corresponding Peirce decomposition of U (compare [1])

$$U = U_1 \oplus U_{\frac{1}{2}} \oplus U_0 \ ,$$

where for every $\nu \in \mathbb{R}$, $U_\nu := U_\nu(e) := \{x \in U : ex = \nu x\}$ denotes the ν-eigenspace of $e \square c^*$. Since $\{U_i U_j{}^* U_k\} \subset U_{i-j+k}$, every U_ν is a subtriple. The space $U_1 = Q(e)U$ is a complex Jordan algebra (considered with the product 6.10) with unit element e and involution $x \mapsto x^* := \{ex^*c\}$. In particular

$$V := \{x \in U_1 : x^* = x\}$$

is a real Jordan algebra with unit element e and complexification $V^{\mathbb{C}} := V \oplus iV = U_1$. If e is a positive tripotent, we can represent the triple product in U_1 through the Jordan product and the involution the following way

$$\{xy^*z\} = (xy^*)z + x(y^*z) - (xz)y^*$$

for all $x, y, z \in U_1$.

6.11 Definition. A tripotent $e \neq 0$ of a hermitian Jordan triple system $(U, *)$ is called *minimal*, if $U_1(e) = \mathbb{C}e$. The tripotent e is called *complete*, if $U_0(e) = 0$.

6.12 Definition. A subset A of a hermitian Jordan triple system $(U,*)$ is called *complete*, if the only element in U orthogonal to A is the zero element, i.e. if $A \square x^* = 0$ implies that $x = 0$.

6.13 Definition. A subset $\mathcal{E}$ of a hermitian Jordan triple system $(U,*)$ is called *an orthogonal system*, if the zero element does not belong to $\mathcal{E}$ and any two elements of $\mathcal{E}$ are orthogonal.

Obviously every orthogonal system is contained in a flat subsystem.

6.14 Definition. A hermitian Jordan triple system $(U,*)$ is called *atomic*, if there exists a complete orthogonal system $\mathcal{E} \subset U$ of minimal tripotents.

Suppose n and m are cardinal numbers, and that H and K are complex Hilbert spaces of dimension n, m. Consider the Banach space of all bounded linear operators $L(H,K)$. For convenience we denote the operator norm $\| \ \|_\infty$ by $\| \ \|$. For every $\lambda \in L(H,K)$ we denote by λ^* the corresponding adjoint operator. For every $x \in K$ and $y \in H$ the mapping $z \mapsto (z|y)x$ determines an operator

$$x \otimes y^* \in L(H,K) .$$

We have

$$\|x \otimes y^*\| = \|x\|.\|y\| \quad \text{and} \quad (x \otimes y^*)^* = y \otimes x^* .$$

Identifying the Hilbert spaces H and K in a natural way with $L(\mathbb{C},H)$ and $L(\mathbb{C},K)$ the operator $x \otimes y^*$ can be written in the form xy^*. We consider the following examples of atomic hermitian Jordan triple systems.

6.15 Example. Suppose $U = L(H,K)$ and define a triple product

$$\{xy^*z\} := \frac{1}{2}(xy^*z + zy^*x) \tag{6.16}$$

for all $x, y, z \in U$. A subset $\mathcal{E} \subset U$ is a complete orthogonal system of minimal tripotents if and only if the equality

$$\mathcal{E} = \{x_i \otimes {y_i}^* : i \in I\}$$

holds, where $\{x_i : i \in I\} \subset K$, $\{y_i : i \in I\} \subset H$ are orthogonal subsets and at least one of them is complete. We refer to $(U,*)$ as a *Cartan factor of type I* and we write $U = I_{n,m}$. Since $I_{n,m}$ and $I_{m,n}$ are isometrically isomorphic, we can always consider that $n \leq m$.

6.17 Example. Suppose $x \mapsto \overline{x}$ is a conjugation of the complex Hilbert space H (i.e. an isometric , antilinear, involutive mapping of H into H). Then through

$$z'(x) := \overline{z^*(\overline{x})}$$

we define a $\mathbb{C}$-linear transposition $z \mapsto z'$ in $L(H)$. Furthermore define $x \otimes y' := x \otimes \overline{y}^*$ for all $x, y \in H$. Then we have $(x \otimes y')' = y \otimes x'$. Consider the space

$$U := \{z \in L(H) : z' = -z\} .$$

Then U considered with the norm of $L(H)$ and the triple product 6.16 is a J^*-triple system. We refer to U as a *Cartan factor of type II* and we write $U = II_n$. A subset

$\mathcal{E} \subset U$ is a complete subsystem of minimal tripotents in U if and only if there exist disjoint orthogonal subsets $\mathcal{A}$ and $\mathcal{B}$ of H and a bijective mapping $f: \mathcal{A} \to \mathcal{B}$ such that the equality

$$\mathcal{E} = \{x \otimes y' - y \otimes x' : x \in \mathcal{A},\ y = f(x)\} ,$$

holds, and the set $\mathcal{A} \cup \mathcal{B}$ is an orthonormal basis of H, or an orthonormal basis of a closed hyperplane in H.

6.18 Example. With the notation used in the previous example, consider the space

$$U := \{z \in L(H) : z' = z\} .$$

Then $(U, *)$ is a J^*-triple system, known as a *Cartan factor of type III*. We write $U = III_n$. A subset $\mathcal{E} \subset U$ is a complete orthogonal system of minimal tripotents, if $\mathcal{E} = \{x \otimes x' : x \in \mathcal{A}\}$ where $\mathcal{A}$ is an orthonormal basis of H.

6.19 Example. Suppose again that $x \mapsto \overline{x}$ is a conjugation of H and $n \geq 3$. Then $U = H$ is a J^*-triple system according to the triple product

$$\{xy^*z\} = \frac{1}{2}\big((x|y)z + (z|y)x - (x|\overline{z})\overline{y}\big) .$$

A subset $\mathcal{E} \subset U$ is a complete orthogonal system of minimal tripotents if and only if $\mathcal{E} = \{\alpha, \beta\}$, where $\alpha, \beta \in U$ are unit vectors with $(\alpha|\beta) = 0$ and $\alpha \wedge \overline{\beta} = 0$. Furthermore

$$\|z\|_\infty{}^2 := \frac{1}{2}\big((z|z) + \sqrt{(z|z)^2 - |(z|\overline{z})|^2}\big) ,$$

is an equivalent norm in U. The space U considered with the norm $\|\ \|_\infty$ is called also *a complex Spin factor* or *a Cartan factor of type IV*. We write $U = IV_n$. If $\mathcal{E} = \{\alpha, \beta\}$ is a complete orthogonal system of minimal tripotents, then $e = \alpha + \beta$ is an unitary element in U (that is $e \square e^* = \mathrm{id}_U$). The real Jordan algebra

$$V = \{x \in U : x^* = Q(e)x = x\}$$

is called *a (real) spin factor of dimension n*.

6.20 Example. Suppose U is a triple system, associated with the bounded symmetric exceptional domains in $\mathbb{C}^{16}$ (resp. $\mathbb{C}^{27}$) - compare [14]. According to the spectral norm $\|\ \|_\infty$ (compare [14]), U is a J^*-triple system, called *a Cartan factor of type V (resp. VI)*.

We give also the notion of a JB^*-triple system, which is introduced by W.Kaup in his article [9] and which indicates part of the contemporary development of the theory of hermitian Jordan triple systems.

6.21 Definition. A hermitian Jordan triple system $(U, *)$ is called a JB^*-triple if the following conditions hold:

(i) $\sigma(\alpha \square \alpha^*) \geq 0$ for all $\alpha \in U$

(ii) $\|\alpha \square \alpha^*\| = \|\alpha\|^2$ for all $\alpha \in U$.

(Here $\sigma(\alpha \square \alpha^*)$ denotes the spectrum of the hermitian element $\alpha \square \alpha^* \in L(U)$).

Furthermore it is proved that there exists a categorical equivalence between the category of JB^*-triples and the category of bounded symmetric domains with base point, compare [9]. Every Cartan factor of type $I - VI$ is a JB^*-triple.

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