Open Covers and Symmetric Operators

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Abstract: We show that despite the infinitary nature of discreteness, the theorem of Stone-Michael can be proved entirely outside the topological context as an essentially finitary and combinatorial theorem.

Key Words: open covers, symmetric operators, discrete and disjoint refinements

1. Introduction

We show that each open cover in a topological space gives rise naturally to a symmetric operator on the underlying set. If each open cover has a star refinement, then in a natural way the induced symmetric operator also admits star refinements. Surprisingly the existence of a σ-discrete refinement of an open cover then corresponds to the existence of a σ-disjoint refinement of the induced symmetric operators. Thus the classical theorem proved by Stone and Michael that *if every open cover has a star refinement which is an open cover, then every open cover has a σ-disjoint open refinement which covers*, has a finitary combinatorial counterpart in this language of symmetric operators. That is we show that despite the infinitary nature of discreteness the theorem of Stone-Michael can be proved entirely outside the topological context, as an essentially finitary and combinatorial theorem: *If a class of symmetric operators admits star refinements, then every operator in that class has a σ-disjoint refinement which covers in that class.* The original topological theorem can then still be derived as a special case.

2. Preliminaries

2.1 Definition. Suppose $\mathcal{A}$ and $\mathcal{B}$ are subfamilies of $\mathcal{P}(X)$. Then $\mathcal{B}$ is said to be a refinement of $\mathcal{A}$ if for every $B \in \mathcal{B}$ there exists a $A \in \mathcal{A}$ such that $B \subset A$. The family $\mathcal{B}$ is said to be a precise refinement of $\mathcal{A}$ if $\mathcal{B}$ and $\mathcal{A}$ are indexed by the same index set S and for every $s \in S$ $B_s \subset A_s$.

2.2 Definition. Let $\mathcal{A} = \{A_s : s \in S\}$ be a subfamily of $\mathcal{P}(X)$. The star of a set $Y \subset X$ with respect to $\mathcal{A}$ is the set $\mathsf{St}(Y, \mathcal{A}) = \cup\{A_s : Y \cap A_s \neq \emptyset\}$.

2.3 Definition. The family $\mathcal{B} = \{B_t : T \in T\}$ is said to be a star refinement of the family $\mathcal{A} = \{A_s : s \in S\}$ if for every $t \in T$ there exists $s \in S$ such that $\mathsf{St}(B_t, \mathcal{B}) \subset \mathrm{A}_s$

2.4 Definition. The family $\mathcal{B}$ is said to be barycentric refinement of the family $\mathcal{A} = \{A_s : s \in S\}$ if for every $x \in X$ there exists $s \in S$ such that $\mathsf{St}(x, \mathcal{B}) \subset \mathrm{A}_s$.

We introduce the following definitions and notation.

2.5 Definition. Any mapping $\mathrm{G}\colon \mathcal{P}(X) \to \mathcal{P}(X)$ will be called *an operator* on X. The set of all operators on X, will be denoted $\mathcal{O}(X)$.

2.6 Definition. An operator $\mathrm{G} \in \mathcal{O}(X)$ will be called *monotone* if $\mathrm{G}(A) \subset \mathrm{G}(B)$ for any $A, B \in \mathcal{P}(X)$ such that $A \subset B$. The set of all monotone operators on a given set X will be denoted $\mathcal{M}(X)$.

2.7 Definition. An operator $\mathrm{G} \in \mathcal{O}(X)$ will be called *pointwise* if it preserves unions, i.e. $\mathrm{G}(A) = \cup\{\mathrm{G}(a) : a \in A\}$ for any set $A \in \mathcal{P}(X)$. The set of all pointwise operators on a given set X will be denoted $\mathcal{P}t(X)$.

In the following we fix certain notation, which will be of constant use.

• For any operator $\mathrm{G} \in \mathcal{O}(X)$ and any family $\mathcal{A} \in \mathcal{P}^2(X)$ we denote by $\mathrm{G}(\mathcal{A})$ or simply $\mathrm{G}\mathcal{A}$ the family $\{\mathrm{G}(A) : A \in \mathcal{A}\}$. In particular for any $\mathrm{G} \in \mathcal{O}(X)$ we denote by G also the family $\{\mathrm{G}(x) : x \in X\}$.

• For any two operators $\mathrm{G}, \mathrm{H} \in \mathcal{O}(X)$ and any two families $\mathcal{A}, \mathcal{B} \in \mathcal{P}^2(X)$ denote by $\mathrm{G}(\mathcal{A}) < \mathrm{G}(\mathcal{B})$ the fact that $\mathrm{G}(\mathcal{A})$ is a precise refinement of $\mathrm{G}(\mathcal{B})$. In further applications we often refer to the above notation as *inequality* between the considered families.

• For any two operators $\mathrm{G}, \mathrm{H} \in \mathcal{O}(X)$ and any two families $\mathcal{A}, \mathcal{B} \in \mathcal{P}^2(X)$ denote by $\mathrm{G}(\mathcal{A}) \prec \mathrm{G}(\mathcal{B})$ the fact that $\mathrm{G}(\mathcal{A})$ is a refinement of $\mathrm{G}(\mathcal{B})$.

• For any two families $\mathcal{A}, \mathcal{B} \in \mathcal{P}^2(X)$ indexed by the same index set S, we denote by $\mathcal{A}_{-\mathcal{B}}$ the family $\{(A_{-B})_s : s \in S\}$ where $(A_{-B})_s = A_s \setminus \cup_{t<s} B_t$ and $<$ is a well order on the index set S.

3. Pointwise Operators and Pointwise Inverse

3.1 Definition. The operator G is called *inverse to* the operator H if $x \in \mathrm{G}(y)$ if and only if $y \in \mathrm{H}(x)$ for any two points x and y in the underlying set. In case that G is inverse to H and G is pointwise, we refer to it as the *pointwise inverse* of H.

3.2 Remark. Note that the pointwise inverse always exists and is unique among the pointwise operators.

3.3 Definition. For any operator G define by $\mathrm{G}^-(x) = \{y \in X : x \in \mathrm{G}(y)\}$ a pointwise operator G^-, and by $\mathrm{G}^+(A) = \{x \in X : \mathrm{G}(x) \subset A\}$ where $A \in \mathcal{P}(X)$ an operator G^+.

3.4 Remark. Note that G^- is the (unique) pointwise inverse of G and that G^+ is not necessarily pointwise, but is monotone. Any pointwise operator is also monotone.

3.5 Proposition. *For any two operators* G *and* H *we have* $(\mathrm{GH})^- = \mathrm{H}^-\mathrm{G}^-$.

Proof. Directly from the definition of $(\mathrm{GH})^-$. □

3.6 Definition. An operator G will be called *symmetric* if $\mathrm{G} = \mathrm{G}^-$.

Note that a symmetric operator is necessarily pointwise. The set of all symmetric operators on a given set will be denoted $\mathcal{S}m(X)$.

3.7 Proposition. $\mathcal{S}m(X) \subset \mathcal{P}t(X) \subset \mathcal{M}(X) \subset \mathcal{O}(X)$

3.8 Proposition. *For any pointwise operator* G *the operators* GG^- *and* $\mathrm{G}^-\mathrm{G}$ *are symmetric.*

Proof. For pointwise operators $(\mathrm{G}^-)^- = \mathrm{G}$. □

3.9 Lemma. *For any pointwise operator* G *and any two subsets* A *and* B *of the underlying set the intersection* $A \cap \mathrm{GG}^-(B)$ *is nonempty if and only if the intersection* $\mathrm{GG}^-(A) \cap B$ *is nonempty if and only if* $\mathrm{G}^-(A) \cap \mathrm{G}^-(B)$ *is nonempty.*

Proof. The intersection $\mathrm{G}^-(A) \cap \mathrm{G}^-(B)$ is nonempty if and only if $\exists a \in A \exists b \in B \exists x$ such that $x \in \mathrm{G}^-(a) \cap \mathrm{G}^-(b)$. But this is equivalent to $\exists a \in A \exists b \in B \exists x$ such that $a \in \mathrm{G}(x)$ and $x \in \mathrm{G}^{-1}(b)$, i.e. $\exists a \in A \exists b \in B$ such that $a \in \mathrm{GG}^{-1}(b)$, i.e. the intersection $A \cap \mathrm{GG}^{-1}(B)$ is nonempty. □

3.10 Remark. Suppose G is a monotone operator such that $\mathrm{G}^-(A) \cap \mathrm{G}^-(B) \neq \emptyset$ for some A and B. Then $\mathrm{GG}^-(A) \cap B \neq \emptyset$ and $A \cap \mathrm{GG}^-(B) \neq \emptyset$.

3.11 Proposition. *For any operator* G *the inequality* $1 < \mathrm{G}$ *implies that* $1 < \mathrm{G}^-$ *and* $\mathrm{G}^+ < 1$.

Proof. For any x we have $x \in \mathrm{G}(x)$. Then by the definition of inverse operator we obtain $x \in \mathrm{G}^-(x)$, that is $1 < \mathrm{G}^-$. To obtain the other inequality suppose that $x \in \mathrm{G}^+(A)$ for some subset A of X. By definition that is $\mathrm{G}(x) \subset A$. But $x \in \mathrm{G}(x)$ and so $x \in A$. □

3.12 Proposition. *For any pointwise operator* G *and any subset* A *of the underlying set the identities* $\mathrm{G}^-(A^c) = \mathrm{G}^+(A)^c$ *and* $(\mathrm{G}^-)^+(A^c) = \mathrm{G}(A)^c$ *hold.*

Proof. The point x belongs to $\mathrm{G}^-(A^c)$ if and only if the intersection $\mathrm{G}(x) \cap A^c$ is nonempty, which is equivalent to $x \in \mathrm{G}^+(A)^c$. The point x belongs to $(\mathrm{G}^-)^+(A^c)$ if and only if the intersection $\mathrm{G}^-(x) \cap A$ is empty, which is equivalent to $x \notin \mathrm{G}(A)$. □

4. Cross Multiplication

4.1 Proposition. *For any pointwise operator* G *and any two subsets* A *and* B *of the underlying set* $\mathrm{G}(A) \subset B$ *if and only if* $A \subset \mathrm{G}^+(B)$.

Proof. Note that by definition $\mathrm{G}(a) \subset B$ if and only if $a \in \mathrm{G}^+(B)$ and use the fact that G is pointwise. □

We refer to the above Proposition as *cross multiplication* of G and G^+.

4.2 Corollary. *For any pointwise operator* G *we have* $\mathrm{GG}^+ < 1$ *and* $1 < \mathrm{G}^+\mathrm{G}$.

4.3 Proposition. *Suppose* G *and* H *are pointwise operators. Then* $(\mathrm{GH})^+ = \mathrm{H}^+\mathrm{G}^+$.

Proof. For any subset A of the underlying set we have $x \in (\mathrm{GH})^+(A)$ if and only if $\mathrm{GH}(x) \subset A$, which by cross multiplication holds if and only if $\mathrm{H}(x) \subset \mathrm{G}^+(A)$ and again by cross multiplication this holds, if and only if $x \in \mathrm{H}^+\mathrm{G}^+(A)$. □

4.4 Corollary. *Suppose* A *and* B *are pointwise operators such that* $A^n < B$ *for some* $n \in \mathbb{N}$. *Then* $A^m B^+ < A^{+p}$ *for any* $m, p \in \mathbb{N}$ *such that* $m + p = n$.

Proof. By Corollary 4.2 we have $BB^+ < 1$ and so $A^n B^+ < 1$. Apply p-times cross multiplication of A and A^+ to obtain the desired inequality. □

4.5 Proposition. *For any pointwise operator* G *and any monotone operator* H *such that* $\mathrm{G} \prec \mathrm{H}$ *we have* $\mathrm{GG}^- < \mathrm{HH}^-$.

Proof. Consider arbitrary point z and $\omega \in \mathrm{GG}^-(z)$. There exists a point $x \in \mathrm{G}^-(z)$ such that $\omega \in \mathrm{G}(x)$. Since $\mathrm{G} \prec \mathrm{H}$, there exists a point y such that $\mathrm{G}(x) \subset \mathrm{H}(y)$. But

$z \in \mathrm{G}(x)$ and so $z \in \mathrm{H}(y)$. Then in particular $y \in \mathrm{H}^-(z)$. By monotonicity we obtain $\mathrm{H}(y) \subset \mathrm{HH}^-(z)$. But $\omega \in \mathrm{H}(y)$ and so $\omega \in \mathrm{HH}^-(z)$. □

4.6 Proposition. *Suppose* G *is a pointwise operator,* $\mathcal{A}$ *and* $\mathcal{B}$ *families such that* $\mathrm{G}^-(\mathcal{A}) < \mathrm{G}^+(\mathcal{B})$. *Then* $\mathrm{G}^-(\mathcal{A}_{-\mathcal{B}})$ *is disjoint.*

Proof. Consider arbitrary elements $(A_{-B})_s$ and $(A_{-B})_t$ of $\mathcal{A}_{-\mathcal{B}}$. We can assume that $s < t$, where $<$ denotes the total ordering relation on the index set. Since $\mathrm{GG}^-(A_s) \subset B_s$, we have $\mathrm{GG}^-(A_{-B})_s \cap (A_{-B})_t$ is empty, which implies by Lemma 3.9 that $\mathrm{G}^-(\mathcal{A}_{-\mathcal{B}})$ is disjoint. □

4.7 Lemma. *Suppose* $\{\mathcal{A}_i\}_{i=1}^{\infty}$ *is a sequence in* $\mathcal{P}^2(X)$ *such that* $\cup \cup \mathcal{A}_i = X$. *Then* $\cup(\mathcal{A}_{i-\mathcal{A}_{i+1}}) = X$.

Proof. For any $x \in X$ choose the smallest index $s(x)$ such that $x \in \mathcal{A}_{i,s(x)}$ for some i. Then $x \in (A_{i-A_{i+1}})_{s(x)}$. □

5. Star and σ-disjoint refinements

5.1 Definition. A class of operators $\Theta(X)$ is said to admit star refinements, if for every $\mathrm{S} \in \Theta(X)$ there exists $\mathrm{S}' \in \Theta(X)$ such that $(\mathrm{S}')^2 < \mathrm{S}$.

5.2 Proposition. *Suppose* $\Theta(X)$ *is a class of monotone operators* > 1 *which admits star refinements. Then for every natural number* $n \geq 2$ *and every* $\mathrm{S} \in \Theta(X)$ *there exists* $\mathrm{S}' \in \theta(X)$ *such that* $(\mathrm{S}')^n < \mathrm{S}$.

Proof. Consider any $\mathrm{S}_0 = \mathrm{S} \in \Theta(X)$ and for every $i = 0, \ldots, n-1$ choose $\mathrm{S}_{i+1} \in \Theta(X)$ such that $\mathrm{S}_{i+1}^2 < \mathrm{S}_i$. Then by monotonicity $\mathrm{S}_n^{2^n} < \mathrm{S}$ and since $\mathrm{S} > 1$, $\mathrm{S}_n^n < \mathrm{S}$. □

5.3 Remark. Note that the above proposition holds also for monotone operators which are < 1.

5.4 Definition. A class of operators $\Theta(X)$ is said to admit σ-disjoint refinements, if for every $\mathrm{S} \in \Theta(X)$ there exists a cover P of X, which decomposes as $P = \cup P_i$ and a family $\{\mathrm{S}_i : i \in \omega\}$ in $\Theta(X)$ such that for each $i \in \omega$ $\mathrm{S}_{i+1}(P_i)$ is a disjoint refinement of S.

5.5 Theorem. *Suppose* $\Theta(X)$ *is a class of symmetric operators* > 1 *which admits star refinements, then for every* $\mathrm{S} \in \Theta(X)$ *and every natural number* $n \geq 1$ *there is a cover* P*, which decomposes as* $P = \cup P_i$ *and a family* $\{S_i : i \in \omega\}$ *in* $\Theta(X)$ *such that for each* i, n $\mathrm{S}_{i+1}^n(P_i)$ *is a disjoint refinement of* S. *Thus* $\cup \mathrm{S}_{i+1}^n(P_i)$ *is a* σ*-disjoint refinement of* S, *which is a cover.*

Proof. Consider any $\mathrm{S} = \mathrm{S}_0 \in \Theta(X)$ and for every $i \geq 0$ choose $\mathrm{S}_{i+1} \in \Theta(X)$ such that $\mathrm{S}_{i+1}^{(2n+1)} < \mathrm{S}_i$. By Theorem 4.4 $\mathrm{S}_{i+1}^n(\mathrm{S}_i^+\mathrm{S}) < (S_{i+1}^n)^+(\mathrm{S}_{i+1}^+\mathrm{S})$. Let $P_i = \mathrm{S}_i^+ S_{-S_{i+1}^+}\mathrm{S}$. Then by Lemma 4.7 (applied for $\mathcal{A}_i = \mathrm{S}_i^+\mathrm{S}$) $P = \cup P_i$ is a cover, since $\cup \mathcal{A}_0 = \cup S_0^+\mathrm{S} = X$. Furthermore by 4.6 the family $\mathrm{S}_{i+1}^n(P_i)$ is disjoint. Since $\mathrm{S}_{i+1}^+ < 1$, $\mathrm{S}_{i+1}^n(P_i)$ is a refinement of S. □

6. Barycentric Refinements

6.1 Definition. For any class $\Theta(X)$ of pointwise operators let $\mathrm{S}_\Theta(X)$ be the class of operators of the form GG^- for G in $\Theta(X)$.

6.2 Remark. Note that $\mathrm{S}_\Theta(X)$ consists of symmetric operators.

6.3 Definition. A class of pointwise operators $\Theta(X)$ is said to admit barycentric refinements if for every $\mathrm{S} \in \Theta(X)$, there exists $\mathrm{T} \in S_\Theta(X)$ such that $\mathrm{T} \prec S$.

6.4 Theorem. *Suppose $\Theta(X)$ is is a class of pointwise operators > 1 which admits barycentric refinements. Then:*
(i) $\mathrm{S}_\Theta(X)$ *admits star refinements.*
(ii) *For every* $\mathrm{S} \in \Theta(X)$ *and every natural number* $n \geq 2$ *there exists* $\mathrm{S}' \in \mathrm{S}_\Theta(X)$ *such that* $(\mathrm{S}')^n \prec \mathrm{S}$.
(iii) *For every* $\mathrm{S} \in \Theta(X)$ *and every natural number* $n \geq 2$ *there exists a cover P which decomposes as* $\cup P_i$ *and a family of operators* $\{S_i : i \in \omega\}$ *in* $\mathrm{S}_{\Theta(X)}$ *such that* $S_{i+1}^n(P_i)$ *is disjoint refinement of* S.

Proof. Let $\mathrm{S} \in \mathrm{S}_\Theta(X)$. Then $\mathrm{S} = \mathrm{GG}^-$ for some $\mathrm{G} \in \Theta(X)$. Since $\Theta(X)$ admits barycentric refinements there exists an $\mathrm{S}' \in \mathrm{S}_\Theta(X)$ such that $\mathrm{S}' \prec \mathrm{G}$. Then by Proposition 4.5 $(\mathrm{S}')^2 < \mathrm{S}$ and so $\mathrm{S}_\Theta(X)$ admits star refinements. For part ii conider any $\mathrm{S} \in \Theta(X)$. Since $\Theta(X)$ admits barycentric refinements there exists a $\mathrm{T} \in \mathrm{S}_\Theta(X)$ such that $\mathrm{T} \prec \mathrm{S}$. But by part i and Proposition 5.2 there exists $\mathrm{T}' \in \mathrm{S}_\Theta(X)$ such that $(\mathrm{T}')^n < \mathrm{T}$, and so $(\mathrm{T}')^n \prec \mathrm{S}$. Part iii follows by theorem 5.5 applied to $S_\Theta(X)$. □

6.5 Remark. Note that 6.4.ii (for $n = 2$) is weaker than star refinements - 5.1.

7. Open Operators

7.1 Definition. For any topological space X a pointwise operator G on X will be called *open* if $1 < \mathrm{G}$ and $\mathrm{G}(x)$ is open for any $x \in X$. The set of all open operators on X will be denoted by $\mathcal{T}(X)$.

7.2 Theorem. *A topological space X has the property that every open cover of X has an open barycentric refinement which covers if and only if $\mathcal{T}(X)$ admits barycentric refinements.*

Proof. Suppose $\mathcal{A}$ is an open cover of X and $\mathcal{B}$ is a given open barycentric refienement of $\mathcal{A}$ which covers. Let B be an open operator such that $\mathrm{B}(x) \in \mathcal{B}$ for every $x \in X$. Then $\mathrm{BB}^- \prec \mathcal{A}$. □

7.3 Corollary. *If X is a topological space such that every open cover of X has an open star refinement, then every open cover has an open symmetric star refinement.*

Proof. Apply Theorem 6.4.ii to $\mathcal{T}(X)$ for $n = 3$, and note that for symmetric open operators S, $\mathrm{S}^3(x) = \mathsf{St}(\mathrm{S}(x), \mathrm{S})$. □

7.4 Proposition. *An open family $\mathcal{A}$ is discrete (resp. locally finite) if and only if there exists an open operator* G *such that* $\mathrm{G}^-\mathcal{A}$ *is disjoint (resp. point finite).*

Proof. The family $\mathcal{A}$ is not discrete (resp. not locally finite) if and only if for every open operator G there exists a point x and a set of indexes I of cardinality 2 (resp. $\geq \omega$) such that $\mathrm{G}(x) \cap A_i \neq \emptyset$ for every $i \in I$, which by cross multiplication holds if and only if $x \in \mathrm{G}^-(A_i)$ for every $i \in I$, i.e. for every open operator G the family $\mathrm{G}^-\mathcal{A}$ is not disjoint (resp. not point finite). □

7.5 Theorem. *Suppose X is a topological space such that every open cover has an open star refinement. Then every open cover has an open σ-discrete refinement, which is a cover.*

Proof. By Theorem 7.2 $\mathcal{T}(X)$ admits barycentric refinements, and so by Theorem 6.4.iii for every $S \in \mathcal{T}(X)$ there exists a cover $P = \cup P_i$ and a family $\{\mathrm{S}_i : i \in \omega\}$ in $\mathrm{S}_{\mathcal{T}}(X)$ such that $\mathrm{S}^2_{i+1}(P_i)$ is a disjoint refinement of S. But by Proposition 7.4 $\mathrm{S}_{i+1}(P_i)$ is a discrete refinement of S, and so $\cup \mathrm{S}_{i+1}(P_i)$ is σ-discrete and covers. □

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