Bounding, splitting and almost disjointness

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Abstract

This bachelor thesis provides an introduction to set theory, cardinal characteristics of the continuum and the generalized cardinal characteristics on arbitrary regular cardinals. It covers the well known *ZFC* relations of the bounding, the dominating, the almost disjointness and the splitting number as well as some more recent results about their generalizations to regular uncountable cardinals. We also present an overview on the currently known consistency results and formulate open questions.

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Introduction

Since it was proven that the continuum hypothesis ($2^{\aleph_0} = \aleph_1$) is independent of the axioms of *ZFC*, the natural question arises what value $2^{\aleph_0}$ could take and it was shown that $2^{\aleph_0}$ could be consistently nearly anything. In some model it can be $\aleph_1$, it can be $\aleph_2$, $\aleph_{37}$, $\aleph_{\omega+1}$ but for example not $\aleph_\omega$ (it must have uncountable cofinality by Königs Theorem, see Theorem 1.2.29). $2^{\aleph_0}$ is the cardinality of all functions $\omega \to 2$, or equally the cardinality of the functions $\omega \to \omega$, the subsets of ω, or the set of real numbers $\mathbb{R}$. We will denote this cardinal by $\mathfrak{c}$, called the "continuum".

A cardinal characteristic of the continuum, sometimes also called "cardinal invariant", is an infinite cardinal that lies somewhere between $\aleph_0$ (strictly above) and $\mathfrak{c}$ and that is usually defined as the least size a set of functions or of subsets must have to satisfy some property. As with $\mathfrak{c}$, the value of these cardinal characteristics can vary among models of *ZFC*, but many inequalities or other relationships can be proven between them. These cardinals are often closely related to (set theoretical) ideals on the real number line, such as the Lebesgue-null ideal or the meagre ideal which play an important role in analysis, measure theory and general topology. The study of such cardinal numbers is part of a field called "infinitary combinatorics".

After giving a survey on mathematical logic and some basic concepts of set theory (this section can be skipped by those who are familiar with it) we will go on to introduce four of the main cardinal characteristics of the continuum under current research and prove some relations between them. First we are going to focus on the bounding and the dominating number. For $f, g \in \omega^\omega$, we say that f dominates g (strictly) iff the set of $n \in \omega$ where $f(n) > g(n)$ has finite complement, written as $g <^* f$. The bounding number $\mathfrak{b}$ is then the least size of a family of functions in ω^ω not all dominated at once by one function f. The dominating number $\mathfrak{d}$ is the least size of a family of functions in ω^ω that provides for any $g \in \omega^\omega$ a function f so that $g <^* f$. After $\mathfrak{b}$ and $\mathfrak{d}$ we will go over to the splitting and the almost-disjointness number. For X and S infinite subsets of ω, we say that S splits X iff infinitely many elements of X lie in S and infinitely many elements of X lie outside of S. The splitting number $\mathfrak{s}$ is the least size of a family of infinite subsets of ω that contains for any $X \in [\omega]^\omega$ a set S that splits X. A family of infinite subsets of ω is called almost-disjoint family iff their elements have pairwise finite intersection. The almost-disjointness number $\mathfrak{a}$ is the least size of an infinite almost-disjoint family not properly included in any other one. The main relations that we are going to prove are summed up in the following diagram of inequalities:

$$\begin{tikzcd}
\mathfrak{s} \arrow[r] & \mathfrak{d} \arrow[r] & \mathfrak{c} \\
\aleph_1 \arrow[u] \arrow[r] & \mathfrak{b} \arrow[u] \arrow[r] & \mathfrak{a} \arrow[u]
\end{tikzcd}$$

Diagram 1

No other inequality between any two of the cardinals in the diagram can be proven from *ZFC* alone. We will provide references for all the independence results that led to this observation.

The next step is then to generalize the cardinals so defined for arbitrary cardinals κ

(mainly regular ones) instead of ω. Diagram 1 only changes slightly then:

$$\begin{tikzcd}
\mathfrak{s}(\kappa) \arrow[r] & \mathfrak{d}(\kappa) \arrow[r] & 2^{\kappa} \\
\kappa^{+} \arrow[r] & \mathfrak{b}(\kappa) \arrow[r] \arrow[u] & \mathfrak{a}(\kappa) \arrow[u]
\end{tikzcd}$$

Diagram 2

In order to make the diagram more complete we provide a review on the very recent proof (May 2015) from D. Raghavan and S. Shelah of the following theorem:

Theorem 3.2.1. *Let $\kappa > \aleph_0$ be a regular cardinal. Then $\mathfrak{s}(\kappa) \leq \mathfrak{b}(\kappa)$.*

Another main result that we will prove is the following theorem from Blass, Hyttinen and Zhang:

Theorem 3.3.1. *Let $\kappa > \aleph_0$ be a regular cardinal, then $\mathfrak{d}(\kappa) = \kappa^{+} \to \mathfrak{a}(\kappa) = \kappa^{+}$.*

Some questions related to Diagram 2 still stay open. They are discussed in the end.

1 Prerequisites

1.1 Model Theory/Predicate Logic

Though not completely necessary for reading this thesis, it is good to know some basic notions of model theory and at least to understand what models of set theory are and what consistency and independence are. For an introductory book on mathematical logic (model theory, recursion theory) I recommend [10]. [11] or [9] provide an introduction to set theory. Nevertheless I am stating some of the required notions here:

Definition 1.1.1 (Predicate Logic)**.** The symbols of predicate logic are: $\neg$, $\wedge$, (,), $\exists$, $\doteq$.

Definition 1.1.2 (Language)**.** A language $\mathcal{L}$ for predicate logic is a set of constant, function and relation symbols with some given arity (the number of arguments).

We can now write formulas or sentences (i.e. universal closures of formulas) in some given language, like: $\exists x(x \doteq x)$ (any language) or $\forall x \exists y(x < y)$ (language $\{<\}$).

Notice the following abbreviations: $\forall x\varphi$ for $\neg\exists x\neg\varphi$, $\varphi \vee \psi$ for $\neg(\neg\varphi \wedge \neg\psi)$, $\varphi \to \psi$ for $\neg\varphi \vee \psi$ and $\varphi \leftrightarrow \psi$ for $\varphi \to \psi \wedge \psi \to \varphi$

Definition 1.1.3 (Structure)**.** Let $\mathcal{L}$ be a language. Then a $\mathcal{L}$ structure is a non-empty set $\mathcal{M}$ together with interpretations for the symbols of $\mathcal{L}$. So for example a constant symbol c in $\mathcal{L}$ is interpreted as a constant $c^{\mathcal{M}}$ in $\mathcal{M}$, a function symbol f with arity n is interpreted as a function $f^{\mathcal{M}} : \mathcal{M}^n \to \mathcal{M}$, etc.

A structure can or cannot satisfy a sentence. For example $\forall x \exists y(x \doteq y \cdot y)$ is satisfied by $(\mathbb{R}, +, \cdot)$, the field of real numbers, but not by $(\mathbb{Q}, +, \cdot)$. We then write $(\mathbb{R}, +, \cdot) \models \forall x \exists y(x \doteq y \cdot y)$. We can also make assignments for variables, as in: $(\mathbb{R}, >) \models \exists x(x > y)[y/\pi]$.

Definition 1.1.4 (Formal proof)**.** Let Σ be a set of sentences in some language $\mathcal{L}$, φ another $\mathcal{L}$ sentence. A formal proof of φ from Σ is a finite sequence $(\varphi_i)_{i \leqslant n}$ such that $\varphi_n = \varphi$ and for every φ_i:

- $\varphi_i \in \Sigma$
- or φ_i is a tautology or a logical axiom[1]
- or there are $l, k < i \leqslant n$ with $\varphi_l = \varphi_k \to \varphi_i$ (this is called Modus Ponens)

We write $\Sigma \vdash \varphi$ iff a formal proof of φ from Σ exists.

Definition 1.1.5 (Consistency)**.** Let Σ be a set of sentences in some language $\mathcal{L}$. Then Σ is said to be inconsistent iff there is an $\mathcal{L}$ sentence φ such that $\Sigma \vdash \varphi \wedge \neg\varphi$. Consistent then means not inconsistent. We often write $\mathrm{Con}(\Sigma)$ as abbreviation for "Σ is consistent".

A structure that satisfies every sentence in Σ is generally called a model for Σ. By the Completeness Theorem, there is a model for Σ iff Σ is consistent or equivalently: every model for Σ satisfies φ iff $\Sigma \vdash \varphi$. For a complete proof of the Completeness Theorem see [10, Section II.12].

Definition 1.1.6. The language of set theory is $\{\in\}$, that is, it contains one binary relation symbol.

[1] A logical axiom is for example $\forall x \forall y(x \doteq y \to y \doteq x)$. For a complete list see [10, Section II.10]

A model for set theory is then a $\{\in\}$ structure that satisfies *ZFC*. And generally everything we do in mathematics can be done in *ZFC*. Every mathematical theorem can be stated as a theorem in *ZFC*. Even the definitions we just made can be seen as definitions in *ZFC*, which is then used as our metatheory[2] (the theory or the logic used to define predicate logic). If *ZFC* is consistent then it is not provable from *ZFC* that *ZFC* is consistent (or equivalently that *ZFC* has a model). This is called Gödel's Second Incompleteness Theorem (See for example [10, Theorem IV.5.32]). Working in *ZFC*, consistency is often assumed to be true to gain results of interest (if it wasn't consistent, then just everything would be provable).

Throughout the thesis we will generally want to prove theorems of *ZFC* but we won't write formal proofs in the sense of our Definition 1.1.4. Finding formal proofs of theorems of *ZFC* is practically impossible (though they must exist!). But out of our exact definition of a formal proof we can develop "proving rules" (e.g. proof by contradiction), that justify the usual mathematical reasoning we know, so that we really do not need to write an exact formal proof of every theorem. This is very well explained in [10, Section II.11]. Also we won't only write sentences that contain just $\in$ and the symbols of predicate logic, because they would become very long. We can actually add symbols to our language that stand for well defined notions that we can develop in *ZFC* as: $\emptyset$, $\cap$, $\cup$, $\subseteq$, $\mathcal{P}(\cdot)$ and all the other symbols used in naive set theory. Then $A \subseteq B$ is an abbreviation for $\forall X(X \in A \to X \in B)$, and $\mathcal{P}(A)$ stands for the unique set that contains all subsets of A. Recall: $\mathcal{P}(A)$ exists and is unique because *ZFC* says so (see Power Set Axiom, Comprehension and Extensionality) and that is why we can use a symbol for it. Every sentence we write with these symbols is then equivalent to a sentence that only uses $\in$ (equivalent in the sense of the definition of these symbols).

Now only two more definitions:

Definition 1.1.7 (Relative Consistency)**.** Let φ be a $\{\in\}$ sentence. Then φ is said to be relatively consistent (or just consistent) with *ZFC* iff Con(*ZFC*) implies Con($ZFC + \varphi$)[3]. Equivalently: If there is a model of *ZFC*, then there is one that satisfies φ. Or still equivalently: if *ZFC* is consistent then *ZFC* does not prove $\neg\varphi$.

Definition 1.1.8 (Relative Independence)**.** Let φ be a $\{\in\}$ sentence. Then φ is said to be relatively independent (or just independent) of *ZFC* iff φ is consistent and $\neg\varphi$ is consistent with *ZFC*.

The most common example of an independence result is probably *CH* (Continuum Hypothesis). *CH* is the statement that every infinite subset of $\mathbb{R}$ is either bijective to $\mathbb{N}$ or to $\mathbb{R}$, or more formally $2^{\aleph_0} = \aleph_1$. Kurt Gödel showed (1938) that if there is a model of *ZFC* then we can extract a subclass (the so called constructible universe) that is also a model of *ZFC* and that satisfies *CH*. In the 1960's Paul Cohen used his method of forcing (which became an important tool in set theory) to construct an extension of some model where *CH* fails to be true.

1.2 Ordinals/Cardinals

We will now have to develop the theory of ordinals and cardinals, which play a main role in this paper. Simply said, an ordinal is a set that represents some type of well-ordering. A

[2] The circularity we are facing here is a philosophical issue, that is worth being discussed, but won't be part of this thesis.

[3] here $ZFC + \varphi$ means the axioms of *ZFC* together with the additional axiom φ

cardinal is an ordinal that represents the size of ordinals. By the *Axiom of Choice* (AC) there is a well-order on every set, this one is then isomorphic to some ordinal, which again is bijective to a cardinal. The result is: Every set is bijective to some cardinal that represents the size of this set.

We will not prove every result in this section. The proofs are mostly covered by [10] and [11].

1.2.1 Ordinals

Definition 1.2.1 (Well-ordering)**.** A set A is well ordered by a relation $<$ iff the following hold:

- $<$ is irreflexive: $\forall x \in A \neg(x < x)$
- $<$ is transitive: $\forall x, y, z \in A(x < y \land y < z \to x < z)$
- $<$ satisfies trichotomy: $\forall x, y \in A(x = y \lor x < y \lor y < x))$
- $<$ is well-founded: $\forall X(\neq \emptyset) \subseteq A \exists x \in X \neg \exists y \in X(y < x)$

So a well-ordering is a total order that is well-founded (every non-empty subset has a least element).

Definition 1.2.2 (Transitive Set)**.** A set A is transitive iff $\forall x \in A(x \subseteq A)$.

We can view a transitive set as a set that really contains every set that it is built on. For example $\{\{\{\emptyset\}\}\}$ is not transitive. It does neither contain $\{\emptyset\}$ nor $\emptyset$.

Definition 1.2.3 (Ordinal)**.** An ordinal (or ordinal number or *von Neumann* ordinal) is a set that is transitive and is well-ordered by $\in$.

Definition 1.2.4 (Successor)**.** The successor of a set A is defined as:

$$S(A) := A \cup \{A\}$$

It contains all the elements of A and A itself.

We can now make some easy examples of ordinals: $0 := \emptyset$, $1 := S(0) = \{\emptyset\}$, $2 := S(1) = \{\emptyset, \{\emptyset\}\} = \{0, 1\}$, $3 := S(2) = \{0, 1, 2\}$, 4, 5, ... It is easy to see that these sets are ordinals. Notice that $0 \in 1 \in 2 \in 3 \ldots$.
An easy first result about ordinals is the following:

Lemma 1.2.5. *Every ordinal α that is not 0 contains 0.*

Proof. $\alpha \neq \emptyset$ has a least element β as it is well ordered. This least element can only be 0. Indeed, if not then there is a $\gamma \in \beta$ and by transitivity γ is also in α, but then β wouldn't be least. $\square$

Hereby we see that 0 really lies under all other ordinals. Analogously one can show that every ordinal with at least two members contains 1, every ordinal with at least three members contains 2, etc ...
We have seen that out of 0, we can get another ordinal $S(0)$, which then leads to a further one $S(S(0))$ ... This generalizes to the following:

Lemma 1.2.6. *Let α be an ordinal. Then $S(\alpha)$ is also an ordinal.* $\square$

Lemma 1.2.7. *The class of ordinals is a transitive class. That is, every element of an ordinal is also an ordinal.* $\square$

We now get to the main theorem about ordinals:

Theorem 1.2.8. *The class of ordinals is well-ordered by $\in$. That is:*
- $\in$ *is irreflexive on the ordinals:* $\forall\alpha(\alpha \notin \alpha)$
- $\in$ *is transitive on the ordinals:* $\forall\alpha\beta\gamma(\alpha \in \beta \wedge \beta \in \gamma \to \alpha \in \gamma)$
- $\in$ *satisfies trichotomy on the ordinals:* $\forall\alpha\beta(\alpha = \beta \vee \alpha \in \beta \vee \beta \in \alpha)$
- $\in$ *is well-founded on the ordinals: Every non-empty set of ordinals has a ($\in$-) least element*

Greek letters "range" over the ordinals here. □

The class of ordinals is really a proper class. This means, there is no set that contains all ordinals. In some sense the collection of all ordinals is just too big. Assume there was a set ON that contains all ordinals. By Lemma 1.2.7 and Theorem 1.2.8 it is easy to see that ON would also be an ordinal, so $ON \in ON$. But this is not possible because $\in$ is irreflexive on ordinals.

Instead of using the symbol $\in$ for the order relation on the ordinals, we will now write $<$ (or $\leq$).

Lemma 1.2.9. *Let X be a non empty set of ordinals. Then $\bigcap X$ and $\bigcup X$ are ordinals and $\bigcap X = \min X$, $\bigcup X = \sup X$.* □

We now define some important subclasses of ordinals:

Definition 1.2.10. An ordinal α is called:
- a successor ordinal iff $\alpha = S(\beta)$ for some β
- a limit ordinal iff $\alpha \neq 0$ and α is not a successor ordinal
- a natural number iff $\forall\beta \leq \alpha$: β is 0 or a successor ordinal

0, 1, 2, 3, ... are clearly natural numbers. Also: if n is a natural number, then $S(n)$ is one as well. Until now we have only seen ordinals that are natural numbers, so that one might think there are no other ones. Also natural numbers are all successor ordinals (except 0), so we may ask if there are any limit ordinals. But with the natural numbers we can get a new ordinal ω, the set of all natural numbers (this one exists by the *Axiom of Infinity* and *Comprehension*). It is easy to check that ω is an ordinal and in particular that it is the least limit ordinal (if it wasn't then it must be a natural number and would contain itself ...) . Now we can even go further: $S(\omega)$, $S(S(\omega))$, $S(S(S(\omega)))$, ...

All of this suggests the following representation of ordinals:

$0 < 1 < 2 < 3 < 4 < \cdots < \omega < \omega+1 < \omega+2 < \cdots < \omega+\omega = \omega\cdot 2 < \omega\cdot 3 < \cdots < \omega\cdot\omega = \omega^2 < \omega^3 < \cdots < \omega^\omega < \omega^{\omega^\omega} < \ldots$

On ordinals we can define functions by recursion, as the factorial function: $0! = 1$, $n+1! = n!\cdot(n+1)$. This is justified by:

Theorem 1.2.11 (Ordinary Recursion)**.** *Let f be a function symbol of our language. Let α be an ordinal. Then there is a unique function F on α with the following properties:*
- $F(0) = f(0)$
- $F(\beta) = f(F \restriction \beta)$ *for* $\beta < \alpha$

□

In fact we haven't said yet what a function is. A function F is a set of ordered pairs such that $(x,y) \in F \wedge (x,y') \in F \to y = y'$. $F(x)$, $F \restriction X$, $dom(F)$, $ran(F)$, injective, surjective and bijective are defined in the obvious way.

Definition 1.2.12 (Order-isomorphism)**.** Let A, A' be sets with order-relations $<, <'$. Then a function $f : A \to A'$ is called an order-isomorphism (with respect to $(A, <)$ and $(A', <)$) iff f is bijective and $\forall x, y \in A(x < y \leftrightarrow f(x) < f(y))$.

Theorem 1.2.13 (Order Type)**.** *Let $(A, <)$ be a well-order. Then there is a unique ordinal α and a unique order-isomorphism $f : A \to \alpha$. This unique α is called the order type of $(A, <)$ and is denoted as* $\mathrm{type}(A)$. □

1.2.2 Cardinals

Definition 1.2.14.
- $A \preccurlyeq B$ iff there is an injection from A to B.
- $A \approx B$ iff there is a bijection from A to B.
- $A \prec B$ iff $A \preccurlyeq B$ and $A \not\approx B$.

It is easy to see that $\approx$ is an equivalence relation.

Theorem 1.2.15 (Schröder-Bernstein)**.** *$A \approx B$ iff $A \preccurlyeq B$ and $B \preccurlyeq A$.* □

A direct implication of Schröder-Bernstein is: $A \prec B$ iff $A \preccurlyeq B$ and $B \not\preccurlyeq A$.

Definition 1.2.16.
- A is called finite iff there is $n \in \omega$ with $A \preccurlyeq n$.
- Infinite means not finite.
- A is called countably infinite iff $A \approx \omega$.
- A is called uncountable iff $\omega \prec A$.

Theorem 1.2.17 (Cantor)**.** *$A \prec \mathcal{P}(A)$.* □

We can produce our first example of an uncountable set: $P(\omega)$.

Definition 1.2.18 (Cardinal)**.** A cardinal (or cardinal number or *von Neumann* cardinal) is an ordinal κ such that $\forall \alpha < \kappa(\alpha \prec \kappa)$.

ω is a cardinal but $\omega + 1$ is not because $\omega < \omega + 1$ but $\omega \approx \omega + 1$.

Definition 1.2.19. For a set A, $|A|$ is the unique cardinal such that $A \approx |A|$.

Justification 1.2.20. By the *Axiom of Choice* A is well-orderable, thus isomorphic to an ordinal α. In particular $A \approx \alpha$. Then $|A| = \min\{\beta \leq \alpha : \beta \approx \alpha\}$. There can't be any other such cardinal because assume there were two of them: κ, λ, then $\kappa \approx \lambda$ but then one of them wouldn't be a cardinal (wlog $\kappa < \lambda \to \kappa \prec \lambda \to \kappa \not\approx \lambda$).

Definition 1.2.21. For any cardinal κ, κ^+ is the least cardinal greater than κ.

Justification 1.2.22. We know that $\kappa \prec \mathcal{P}(\kappa)$. Then $\kappa^+ = \min\{\alpha \leq |\mathcal{P}(\kappa)| : \kappa \prec \alpha\}$.

κ^+ can also be defined without using $|\mathcal{P}(\kappa)|$ and especially without using AC (*Hartogs* 1915). We now define the $\aleph$-cardinals by recursion:

Definition 1.2.23.
- $\aleph_0 = \omega_0 := \omega$
- $\aleph_{\alpha+1} = \omega_{\alpha+1} := \aleph_\alpha^+$
- $\aleph_\eta = \omega_\eta := \sup\{\aleph_\xi : \xi < \eta\}$

Note that $\sup\{\aleph_\xi : \xi < \eta\}$ is really a cardinal, in particular it is the least cardinal greater then all $\aleph_\xi$ for $\xi < \eta$. The $\aleph$-cardinals really form all infinite cardinals.
Now some cardinal arithmetic:

Definition 1.2.24. Let κ, λ be cardinals.

- $\kappa + \lambda := |\kappa \times \{0\} \cup \lambda \times \{1\}|$
- $\kappa \cdot \lambda := |\kappa \times \lambda|$
- $\kappa^\lambda := |\{f | f : \lambda \to \kappa\}|$

Lemma 1.2.25. *Let κ, λ, θ be infinite cardinals.*

- $\kappa + \lambda = \max\{\kappa, \lambda\}$
- $\kappa \cdot \lambda = \max\{\kappa, \lambda\}$
- $(\kappa^\lambda)^\theta = \kappa^{\lambda \cdot \theta}$
- *If $\kappa \leq 2^\lambda$, then $\kappa^\lambda = 2^\lambda = |\mathcal{P}(\lambda)|$* □

We have already seen *CH*. Now *GCH* (*General Continuum Hypothesis*) is a generalization of *CH*. *GCH* is the statement that $2^{\aleph_\alpha} = \aleph_{\alpha+1}$ for any ordinal α, equivalently: $|\mathcal{P}(\aleph_\alpha)| = \aleph_{\alpha+1}$. *GCH* is also independent of *ZFC*. In general the values of cardinal exponentiation can vary strongly among models of *ZFC*. But under *GCH* they become very easy to compute. For example (assuming *GCH*): ${\aleph_{10}}^{\aleph_\omega} = 2^{\aleph_\omega} = \aleph_{\omega+1}$.

1.2.3 Regular Cardinals

Definition 1.2.26 (Cofinality)**.** Let γ be a limit ordinal, then the cofinality of γ is:

$$\mathrm{cf}(\gamma) := \min\{\mathrm{type}(X) : X \subseteq \gamma \wedge \sup X = \gamma\}$$

γ is called regular iff $\mathrm{cf}(\gamma) = \gamma$

Lemma 1.2.27. *Let γ be a limit ordinal, then:*

- *If $X \subseteq \gamma$ is unbounded in γ, then* $\mathrm{cf}(\gamma) = \mathrm{cf}(\mathrm{type}(X))$
- $\mathrm{cf}(\gamma)$ *is regular*
- $\mathrm{cf}(\gamma) \leq |\gamma|$
- *If γ is a successor cardinal ($\gamma = \aleph_{\alpha+1}$) then γ is regular*
- *If γ is a limit cardinal ($\gamma = \aleph_\eta$) then* $\mathrm{cf}(\gamma) = \mathrm{cf}(\eta)$

□

Lemma 1.2.28. *Let κ be a regular cardinal and let $\mathcal{A}$ be a set of size $\lambda < \kappa$ where $\forall A \in \mathcal{A}(|A| < \kappa)$. Then $|\bigcup \mathcal{A}| < \kappa$* □

Theorem 1.2.29 (König)**.** *Let $\kappa \geq 2$ and λ an infinite cardinal, then* $\mathrm{cf}(\kappa^\lambda) > \lambda$. □

2 Cardinal Characteristics of the continuum

2.1 The bounding and the dominating number

We will denote by ω^ω the set of functions $\omega \to \omega$, not to be confused with cardinal or ordinal arithmetic where ω^ω means something else.

Definition 2.1.1. Let $f, g \in \omega^\omega$, then $f <^* g$ iff $\{n \in \omega : g(n) \leq f(n)\}$ is finite or equivalently $\{n \in \omega : f(n) < g(n)\}$ has finite complement or still equivalently $\{n \in \omega : g(n) \leq f(n)\}$ is bounded. We then say that g (eventually) dominates f.

Lemma 2.1.2. *$<^*$ is irreflexive and transitive.*

Proof.
- Irreflexive: $\{n \in \omega : f(n) < f(n)\} = \emptyset$
- Transitive: If $f <^* g$ and $g <^* h$ then $\{n \in \omega : f(n) < h(n)\} \supseteq \{n \in \omega : f(n) < g(n) < h(n)\} = \{n \in \omega : f(n) < g(n)\} \cap \{n \in \omega : g(n) < h(n)\}$ and as both $\{n \in \omega : f(n) < g(n)\}$ and $\{n \in \omega : g(n) < h(n)\}$ have finite complement, also $\{n \in \omega : f(n) < g(n)\} \cap \{n \in \omega : g(n) < h(n)\}$ and furthermore $\{n \in \omega : f(n) < h(n)\}$ have finite complement, so $f <^* h$.

□

Note that $<^*$ is not a total order, because it does not satisfy trichotomy. For example let $f(n)$ be 0 for even n and 1 for odd n and let $g(n)$ be 0 for odd n and 1 for even n. Then $f \neq g$ and neither f dominates g nor g dominates f.

Definition 2.1.3 (Unbounded family)**.** Let $\mathcal{B} \subseteq \omega^\omega$, then $\mathcal{B}$ is said to be *unbounded* iff it is unbounded with respect to $<^*$, that is $\neg\exists f \in \omega^\omega \forall b \in \mathcal{B}(b <^* f)$.

Definition 2.1.4 (Bounding number)**.** The *bounding number* is the least size of an unbounded family of functions $\omega \to \omega$:

$$\mathfrak{b} := \min\{|\mathcal{B}| : \mathcal{B} \subseteq \omega^\omega \wedge \neg\exists f \in \omega^\omega \forall b \in \mathcal{B}(b <^* f)\}$$

In order for the bounding number to be well-defined, we must of course first investigate if any unbounded families exist. But this is very easy to see: ω^ω itself is unbounded, because a bound for ω^ω would have to dominate itself, which is impossible by Lemma 2.1.2. It is also clear that the bounding number is at most $\mathfrak{c}$. The next result will give us a lower bound on $\mathfrak{b}$:

Lemma 2.1.5. $\aleph_0 < \mathfrak{b}$

Proof. Clearly the empty set is not unbounded. So let $\{b_i : i \in \omega\}$ be an at most countable family of functions $\omega \to \omega$, then we can define a function $b \in \omega^\omega$ as follows: for all $n \in \omega$, $b(n) := \max\{b_i(n) : i \leq n\} + 1$. Then b is a bound for $\{b_i : i \in \omega\}$, because take any $k \in \omega$, then $b(n) = \max\{b_i(n) : i \leq n\} + 1 > b_k(n)$ for every $n \geq k$, so in particular the set $\{n \in \omega : b_k(n) \geq b(n)\}$ is bounded by k and thus $b_k <^* b$.

We have thus shown that an at most countable family can't be unbounded, so the least size of such a family $\mathfrak{b}$ must be strictly greater than $\aleph_0$. □

Definition 2.1.6 (Dominating family)**.** Let $\mathcal{D} \subseteq \omega^\omega$, then $\mathcal{D}$ is called a *dominating family* iff $\forall f \in \omega^\omega \exists d \in \mathcal{D}(f <^* d)$.

Definition 2.1.7 (Dominating number)**.** The *dominating number* is the least size of a dominating family:

$$\mathfrak{d} := \min\{|\mathcal{D}| : \mathcal{D} \subseteq \omega^\omega \wedge \forall f \in \omega^\omega \exists d \in \mathcal{D}(f <^* d)\}$$

Again, ω^ω is a dominating family itself, because let $f \in \omega^\omega$, then it is dominated by $d \in \omega^\omega$ where $d(n) := f(n) + 1$ for every $n \in \omega$. We clearly have $\mathfrak{d} \leq \mathfrak{c}$.

Lemma 2.1.8. $\aleph_1 \leq \mathfrak{b} \leq \mathfrak{d} \leq \mathfrak{c}$

Proof. $\aleph_1 \leq \mathfrak{b}$ is due to Lemma 2.1.5. It remains to show $\mathfrak{b} \leq \mathfrak{d}$, which is easy to see from the fact that any dominating family is also an unbounded family. Let $\mathcal{D}$ be dominating and assume there is some g that bounds $\mathcal{D}$. As $\mathcal{D}$ is dominating, there is some $d \in \mathcal{D}$ with $g <^* d$, but also $d <^* g$ which leads to $g <^* g$. We have a contradiction. Now it is clear that the least size of an unbounded family will not be greater than the least size of a dominating family. □

If we assume *CH*, then of course all these values become trivial and we have $\mathfrak{b} = \mathfrak{d} = \mathfrak{c} = \aleph_1$. So it is natural to ask if we can have (i.e. if it is consistent to have) $\mathfrak{b} < \mathfrak{d}$. The answer is yes. In the original Cohen forcing extension that proved Con($\neg$*CH*) we have that $\mathfrak{b} = \aleph_1 < \mathfrak{d} = \aleph_2$ (See [11]). Furthermore one can construct with standard techniques (from [11] for example) a model where $\mathfrak{b} = \lambda < \mathfrak{d} = \kappa$ for arbitrary (suitable) cardinals λ, κ.

Lemma 2.1.9. $\mathfrak{b}$ *is regular.*

Proof. Let $\{b_i : i < \mathfrak{b}\}$ be an unbounded family of size $\mathfrak{b}$. Now construct a new unbounded family $\{b'_i : i < \mathfrak{b}\}$ inductively as follows: for $i \in \mathfrak{b}$ we chose b'_i to be a bound for $\{b'_j : j < i\} \cup \{b_i\}$ (this is possible because $|\{b'_j : j < i\} \cup \{b_i\}| < \mathfrak{b}$).

It is clear that $\{b'_i : i < \mathfrak{b}\}$ is also unbounded because, assume there is a g with $\forall i < \mathfrak{b}(b'_i <^* g)$ then also $\forall i < \mathfrak{b}(b_i <^* g)$, which is a contradiction. Also by our construction $\{b'_i : i < \mathfrak{b}\}$ is ordered increasingly with respect to $<^*$, that is $\forall i, j < \mathfrak{b}(i < j \to b'_i < b'_j)$.

Now assume $\mathrm{cf}(\mathfrak{b}) < \mathfrak{b}$, that is $\mathfrak{b}$ is not regular. Then there is an unbounded subset $X \subseteq \mathfrak{b}$ with $|X| < \mathfrak{b}$ and especially $\{b'_i : i \in X\}$ is an unbounded family, because assume there is a g with $\forall i \in X(b'_i <^* g)$ then also $\forall j \in \mathfrak{b}(b'_j <^* g)$ because for every $j \in \mathfrak{b}$ there is a $j < i \in X$ as X is unbounded and so $b'_j <^* b'_i <^* g$. As $|\{b'_i : i \in X\}| < \mathfrak{b}$ this is a contradiction of the minimality of $\mathfrak{b}$. □

Lemma 2.1.10. $\mathrm{cf}(\mathfrak{d}) \geq \mathfrak{b}$

Proof. Assume $\mathfrak{d} \geq \mathfrak{b} > \mathrm{cf}(\mathfrak{d})$. Let $\mathcal{D} = \{d_i : i < \mathfrak{d}\}$ be a dominating family of size $\mathfrak{d}$. Let $X \subseteq \mathfrak{d}$, such that $\sup X = \mathfrak{d}$ and $|X| = \mathrm{cf}(\mathfrak{d})$. For every $j \in X$ define $\mathcal{D}_j := \{d_i : i < j\}$. Every $\mathcal{D}_j$ is then not a dominating family as $j < \mathfrak{d}$, so there is a function f_j that is not dominated by any function in $\mathcal{D}_j$. Fix all those f_j, then $|\{f_j : j \in X\}| \leq |X| < \mathfrak{b}$ so $\{f_j : j \in X\}$ is bounded by some function f. f is not dominated by any function in any of the $\mathcal{D}_j$'s, because let $i < j \in X$ then $f_j <^* f <^* f_i$ is a contradiction of the choice of f_j. Clearly $\bigcup_{j \in X} \mathcal{D}_j = \mathcal{D}$, so no function in $\mathcal{D}$ dominates f – a contradiction. □

All inequalities that we just proved can be summed up to the following theorem:

Theorem 2.1.11. $\aleph_1 \leq \mathfrak{b} = \mathrm{cf}(\mathfrak{b}) \leq \mathrm{cf}(\mathfrak{d}) \leq \mathfrak{d} \leq \mathfrak{c}$

It was shown that all values that do not contradict this theorem and $\mathrm{cf}(\mathfrak{c}) > \aleph_0$ are possible for $\mathfrak{b}$, $\mathfrak{d}$ and $\mathfrak{c}$ (See [2, Theorem 2.5]).

Very often in the literature the bounding and the dominating numbers are defined with respect to $\leq^*$ (that is $f \leq^* g$ iff $\{n \in \omega : f(n) > g(n)\}$ is finite). It is very easy to see that these definitions are equivalent to our official ones.

2.2 Splitting and mad families

Definition 2.2.1 (Almost included)**.** Let $A, B \subseteq \omega$, then we say that A is *almost included* in B ($A \subseteq^* B$) iff $B \setminus A$ is finite.

Definition 2.2.2 (Almost disjoint)**.** Let $A, B \subseteq \omega$, then A and B are *almost disjoint* (or *ad*) iff $A \cap B$ is finite.

For any set X and a cardinal κ, we denote by $[X]^\kappa$ the subsets of X with cardinality κ and by $[X]^{<\kappa}$ the subsets of X with cardinality smaller than κ.

Definition 2.2.3 (Almost disjoint family)**.** A set $\mathcal{A} \subseteq [\omega]^\omega$ is called an *almost disjoint family* iff for any distinct $A, B \in \mathcal{A}$, A and B are almost disjoint.

Definition 2.2.4 (Maximal almost disjoint family)**.** An almost disjoint family $\mathcal{A} \subseteq [\omega]^\omega$ is called a *maximal almost disjoint* family (or *mad* family) iff it is not properly included in any other ad family, that is $\forall X \in [\omega]^\omega \exists A \in \mathcal{A}(|A \cap X| = \aleph_0)$.

Definition 2.2.5 (Almost disjointness number)**.** The *almost disjointness number* is the least size of an infinite mad family, that is:

$$\mathfrak{a} := \min\{|\mathcal{A}| : \mathcal{A} \text{ is an infinite mad family}\}$$

The reason why we only look at infinite mad families is because there are trivial examples of finite mad families and $\mathfrak{a}$ would turn out to be simply 1 ($\{\omega\} \subseteq [\omega]^\omega$ is already a mad family with cardinality 1).

By taking any almost disjoint family and applying the Lemma of Zorn one can see that maximal almost disjoint families exist and our cardinal is well defined. Trivially we have the upper bound $\mathfrak{a} \leq \mathfrak{c}$. Once again, as with $\mathfrak{b}$, a diagonalization argument shows that this cardinal is uncountable:

Lemma 2.2.6. $\aleph_0 < \mathfrak{a}$

Proof. Let $\{A_i : i \in \omega\} \subseteq [\omega]^\omega$ be a countable almost disjoint family. For any $i \in \omega$ let $a_i = \min A_i \setminus \bigcup_{j<i} A_j$. This is possible because A_i shares only finitely many elements with each A_j where $j < i$ so its intersection with $\bigcup_{j<i} A_j$ is also only finite and as A_i is infinite $A_i \setminus \bigcup_{j<i} A_j$ is non empty.
Let $A := \{a_i : i \in \omega\}$. Then $\{A_i : i \in \omega\} \cup \{A\}$ is almost disjoint, because for any i, $A \cap A_i \subseteq \{a_j : j \leq i\}$, so $A \cap A_i$ is finite. In particular $A \neq A_i$ for any i, so we really got a new almost disjoint family. We have thus shown that no countable almost disjoint family can be maximal, so $\aleph_0 < \mathfrak{a}$. $\square$

Theorem 2.2.7. $\aleph_1 \leq \mathfrak{b} \leq \mathfrak{a} \leq \mathfrak{c}$

Proof. Assume $\aleph_0 < \mathfrak{a} < \mathfrak{b}$ and let $\{A_i : i < \mathfrak{a}\}$ be a mad family of size $\mathfrak{a}$. For every i with $\omega \leq i < \mathfrak{a}$ we define $f_i : \omega \to \omega$ as follows: $f_i(n) := \min \omega \setminus (A_i \cap A_n)$. It is clear that these minima exist as $A_i \cap A_n$ is finite. The values of $f_i(n)$ are bounds for the intersections $A_i \cap A_n$. Now as we assumed $\mathfrak{a} < \mathfrak{b}$, we have that $\{f_i : \omega \leq i < \mathfrak{a}\}$ is bounded by some function $f : \omega \to \omega$.

As in the proof of Lemma 2.2.6 a very similar diagonalization shows that our family is not maximal: For every $n \in \omega$ let $a_n := \min A_n \setminus (\bigcup_{k<n} A_k \cup f(n))$. Again we have that $A_n \setminus \bigcup_{k<n} A_k$ is unbounded as well as $A_n \setminus (\bigcup_{k<n} A_k \cup f(n))$ so the a_n's are well defined and if we let $A := \{a_n : n \in \omega\}$ then A is almost disjoint from all A_n for $n \in \omega$. Furthermore for any i with $\omega \leq i < \mathfrak{a}$, $A \cap A_i \subseteq \{a_n : f_i(n) \geq f(n)\}$ because if $f(n) > f_i(n)$ then $a_n > f_i(n)$ but as $a_n \in A_n$ and $f_i(n)$ is a bound for $A_n \cap A_i$, a_n cannot be in A_i. As f dominates f_i, $\{a_n : f_i(n) \geq f(n)\}$ is finite and so is $A \cap A_i$. Of course this also shows that $A \neq A_i$ for all i and our family is not maximal – we have a contradiction. □

Again we may ask if it possible to have $\mathfrak{a} < \mathfrak{b}$. Shelah proved the consistency of $\mathfrak{b} = \aleph_1 < \mathfrak{a} = \aleph_2$ in [14]. The more general case $\mathfrak{b} = \kappa < \mathfrak{a} = \kappa^+$ was proven to be consistent by Brendle in [5].

Definition 2.2.8. Let $S, X \subseteq \omega$. We say that S *splits* X iff $X \cap S$ and $X \setminus S$ are infinite.

It is clear from the definition that in order for S to split X, S and X must be infinite.

Definition 2.2.9 (Splitting family)**.** Let $\mathcal{S} \subseteq [\omega]^\omega$, then $\mathcal{S}$ is a *splitting family* iff for all $X \in [\omega]^\omega$ there is a $S \in \mathcal{S}$ so that S splits X.

Definition 2.2.10 (Splitting number)**.** The *splitting number* is the least size of a splitting family:

$$\mathfrak{s} := \min\{|\mathcal{S}| : \mathcal{S} \subseteq [\omega]^\omega \text{ is a splitting family}\}$$

Note that $[\omega]^\omega$ is a splitting family and that we have the trivial upper bound $\mathfrak{s} \leq \mathfrak{c}$.

Lemma 2.2.11. $\aleph_0 < \mathfrak{s}$

Proof. Clearly the empty set is not a splitting family. So let $\{S_n : n \in \omega\} \subseteq [\omega]^\omega$ be an at most countable family of unbounded subsets of ω. We will show that it cannot be a splitting family.

We will define a sequence of sets $\langle X_n \rangle_{n\in\omega}$ as follows: Let $X_0 := S_0$. X_0 is infinite so $X_0 \cap S_1$ or $X_0 \cap (\omega \setminus S_1)$ is infinite, so chose X_1 among them such that it is infinite. Inductively proceed as follows: For X_{n+1} chose an infinite set among $X_n \cap S_{n+1}$ and $X_n \cap (\omega \setminus S_{n+1})$.
For all n, X_n is infinite so for all n chose $x_n \in X_n$, so that all x_n are pairwise distinct. Then $X := \{x_n : n \in \omega\} \in [\omega]^\omega$ is not split by any S_n, because for any n we have that $X \subseteq^* S_n$ or $X \subseteq^* \omega \setminus S_n$ as all x_i for $i \geq n$ are all either in S_n or in $\omega \setminus S_n$.

We conclude that $\{S_n : n \in \omega\}$ is not a splitting family. □

Theorem 2.2.12. $\aleph_1 \leq \mathfrak{s} \leq \mathfrak{d} \leq \mathfrak{c}$

Proof. We have to prove $\mathfrak{s} \leq \mathfrak{d}$ and we are going to do that by showing that there is a splitting family of size $\mathfrak{d}$.

Let $\{d_i : i \in \mathfrak{d}\}$ be a dominating family of size $\mathfrak{d}$. For every $i \in \mathfrak{d}$ we define a strictly increasing sequence $\langle i_n \rangle_{n\in\omega}$ as follows: $i_0 := 0$, $i_{n+1} = \max(d_i{''}(i_n + 1)) + 1$. Note that we have $k \leq i_n \to d_i(k) < i_{n+1}$. Then $\{\bigcup_{n\in\omega}[i_{2n}, i_{2n+1}) : i \in \mathfrak{d}\}$ is a splitting family.

Let $X \in [\omega]^\omega$. We enumerate X by an increasing sequence $\langle x_n \rangle_{n\in\omega}$. Define a function $f : \omega \to \omega$ with $f(k) = x_{n+1}$ iff $k \in [x_n, x_{n+1})$. Then f is dominated by some d_i which implies that for some $x \in X$ we have that $f(k) < d_i(k)$ for all $k \geq x$. But then any interval $[i_l, i_{l+1})$ with $i_l > x$ intersects X, because assume we have $x \leq x_n < i_l < i_{l+1} \leq x_{n+1}$ for some n, then $d_i(x_n) < i_{l+1}$ as $x_n \leq i_l$ but also $f(x_n) = x_{n+1}$ by definition of f. So we have that $d_i(x_n) < i_{l+1} \leq f(x_n)$ – a contradiction.

Now it is clear that $\bigcup_{n\in\omega}[i_{2n}, i_{2n+1})$ intersects X in an unbounded set as well as $\omega \setminus \bigcup_{n\in\omega}[i_{2n}, i_{2n+1}) = \bigcup_{n\in\omega}[i_{2n+1}, i_{2n+2})$, so X is split by $\bigcup_{n\in\omega}[i_{2n}, i_{2n+1})$. □

Again there is a positive answer for the consistency of the strict inequality. In the so called Hechler model we have that $\mathfrak{s} < \mathfrak{d} = \mathfrak{b}$ as pointed out by J. Baumgartner and P. Dordal in [1].

We can summarize this whole section with the following diagram:

$$\begin{array}{ccccc} \mathfrak{s} & \longrightarrow & \mathfrak{d} & \longrightarrow & \mathfrak{c} \\ \uparrow & & \uparrow & & \uparrow \\ \aleph_1 & \longrightarrow & \mathfrak{b} & \longrightarrow & \mathfrak{a} \end{array}$$

Diagram 1

Each arrow (or each path in the diagram) stands for a provable inequality and what is remarkable is that these are the only inequalities between the cardinals $\mathfrak{b}, \mathfrak{d}, \mathfrak{a}, \mathfrak{s}$ and $\mathfrak{c}$ provable from *ZFC* (of course, always keep in your mind: if *ZFC* is consistent!). For example $\mathfrak{b}$ and $\mathfrak{s}$ are independent, this means all possibilities ($\mathfrak{s} < \mathfrak{b}$, $\mathfrak{b} < \mathfrak{s}$, $\mathfrak{b} = \mathfrak{s}$) are consistent. We have the same for $\mathfrak{a}$, $\mathfrak{s}$ and $\mathfrak{a}$, $\mathfrak{d}$. The consistency of $\mathfrak{b} = \mathfrak{s} = \mathfrak{a} = \mathfrak{d}$ is clear from the consistency of *CH*. What about the strict inequalities?

- Independence of $\mathfrak{s}$ and $\mathfrak{b}$: The consistency of $\mathfrak{b} = \aleph_1 < \mathfrak{s} = \aleph_2$ is due to S. Shelah and comes from the same paper ([14]) where Con($\mathfrak{b} = \aleph_1 < \mathfrak{a} = \aleph_2$) is proved. The more general Con($\mathfrak{b} = \kappa < \mathfrak{a} = \kappa^+$) for a suitable κ (namely κ regular) is due to V. Fischer and J. Steprans' article [8]. Baumgartner and Dordal's paper [1] has a proof for Con($\mathfrak{s} < \mathfrak{d} = \mathfrak{b}$) as already mentioned.
- Independece of $\mathfrak{a}$ and $\mathfrak{d}$: Shelah proved Con($\aleph_2 \leq \mathfrak{d} < \mathfrak{a} = \mathfrak{c}$) in his article [16]. The consistency of $\aleph_2 \leq \mathfrak{d} < \mathfrak{a} < \mathfrak{c} + \mathrm{cf}(\mathfrak{a}) = \omega$ is due to Brendle, [4]. Note that the two last results show that $\mathfrak{a}$ can have countable but also uncountable cofinality. The consistency of $\mathfrak{d} = \aleph_1 < \mathfrak{a} = \aleph_2$ is still open. $\mathfrak{a} < \mathfrak{d}$ holds in the Cohen model.
- Independence of $\mathfrak{a}$ and $\mathfrak{s}$: The consistency of $\mathfrak{b} = \mathfrak{a} = \kappa < \mathfrak{s} = \lambda$ is due to Fischer and Brendle. See [6] for their paper. For $\mathfrak{s} < \mathfrak{a}$ remember that $\mathfrak{s} < \mathfrak{b}$ was consistent and as $\mathfrak{b} \leq \mathfrak{a}$ we get Con($\mathfrak{s} < \mathfrak{a}$).

3 Cardinal Characteristics on regular uncountable cardinals

3.1 Generalized bounding, splitting and almost disjointness

The cardinal characteristics we defined until now all dealt with functions $\omega \to \omega$ or with subsets of ω. What about replacing ω with an arbitrary cardinal κ?

In this section we are going to redefine the cardinal characteristics and their related notions ($<^*$, almost disjoint, splitting) for arbitrary cardinals κ. In particular we are going to look at regular cardinals. The premise of being regular will allow us to translate many of the proofs of Section 2 to the cardinal characteristics on κ.

Definition 3.1.1. Let κ be an infinite cardinal, $f, g : \kappa \to \kappa$, $A, B \subseteq \kappa$.

- We say that f dominates g ($g <^* f$) iff $|\{\beta < \kappa : g(\beta) \geq f(\beta)\}| < \kappa$.
- We say that A and B are (κ-) almost disjoint iff $|A \cap B| < \kappa$.
- We say that A splits B iff $|A \cap B| = \kappa$ and $|B \setminus A| = \kappa$

Note that if κ is regular: $g <^* f$ is equivalent to saying that there is a δ so that $f(\beta) > g(\beta)$ for all $\beta > \delta$, A and B are almost disjoint iff $A \cap B$ is bounded in κ and A splits B iff $A \cap B$ and $B \setminus A$ are unbounded in κ.

Definition 3.1.2. Let κ be a regular infinite cardinal.

- $\mathfrak{b}(\kappa)$ is the least size of an unbounded subset of κ^κ
- $\mathfrak{d}(\kappa)$ is the least size of a dominating family
- $\mathfrak{a}(\kappa)$ is the least size of a (κ-)mad family of size at least κ
- $\mathfrak{s}(\kappa)$ is the least size of a splitting family

As for $\aleph_0$ where we have proven $\mathfrak{b} > \aleph_0$ we can prove that $\mathfrak{b}(\kappa) > \kappa$ for an arbitrary regular κ. In fact we are going to prove a stronger statement which also includes singular κ's:

Lemma 3.1.3. *Let κ be an infinite cardinal, then $\mathfrak{b}(\kappa) > \mathrm{cf}(\kappa)$*

Proof. Let $\{b_i : i < \mathrm{cf}(\kappa)\} \subseteq \kappa^\kappa$. Let $\langle c_j \rangle_{j<\mathrm{cf}(\kappa)}$ be an increasing cofinal sequence in κ. Then define $b(\alpha) := \sup\{b_i(\alpha) : i < \min\{j : c_j > \alpha\}\} + 1$. Note that $b(\alpha) < \kappa$ because $b(\alpha)$ is the supremum of a sequence of type less than $\mathrm{cf}(\kappa)$ (plus 1), so that b is really a function $\kappa \to \kappa$. Then $b_i <^* b$ for all i. So $\mathfrak{b}(\kappa) > \mathrm{cf}(\kappa)$. □

In the case κ regular, Lemma 3.1.3 translates to $\mathfrak{b}(\kappa) > \kappa$.

Lemma 3.1.4. $\mathfrak{b}(\kappa)$ *is regular,* $\mathrm{cf}(\mathfrak{d}(\kappa)) \geq \mathfrak{b}(\kappa)$ □

The proof is exactly the same as for Lemma 2.1.9 and Lemma 2.1.10. Note also that we don't need regularity of κ here.

Theorem 3.1.5. $\mathrm{cf}(\kappa)^+ \leq \mathfrak{b}(\kappa) \leq \mathfrak{d}(\kappa) \leq 2^\kappa$*; if κ is regular,* $\kappa^+ \leq \mathfrak{b}(\kappa) \leq \mathfrak{d}(\kappa) \leq 2^\kappa$ □

For $\mathfrak{a}(\kappa)$, the reason why we restrict ourselves to mad families of size at least κ is again that we have trivial examples of mad families of size less than κ (partitions for example). In fact if we allow κ to be singular these trivial examples are mad families of size $\mathrm{cf}(\kappa)$ (a partition of size more than $\mathrm{cf}(\kappa)$ does not have to be maximal). For the following Lemma we extend the definition of $\mathfrak{a}(\kappa)$ to singular κ by allowing mad families of size at least $\mathrm{cf}(\kappa)$.

Lemma 3.1.6. $\mathfrak{a}(\kappa) > \mathrm{cf}(\kappa)$

Proof. Let $\langle c_i \rangle_{i<\mathrm{cf}(\kappa)}$ be a cofinal sequence in κ. Let $\{A_i : i < \mathrm{cf}(\kappa)\}$ be an almost disjoint family of size $\mathrm{cf}(\kappa)$ (we want the A_i's to be pairwise distinct). For every $i < \mathrm{cf}(\kappa)$ we have that $|A_i \setminus (\bigcup_{j<i} A_j)| = \kappa$ so chose a set $X_i \subseteq A_i \setminus (\bigcup_{j<i} A_j)$ of size $|c_i|$. Then we have that $X := \bigcup_{i<\mathrm{cf}(\kappa)} X_i$ is almost disjoint from every A_i and $|X| = \kappa$. $\square$

For κ regular this yields $\mathfrak{a}(\kappa) > \kappa$.

Theorem 3.1.7. *Let κ be regular, then $\kappa^+ \leq \mathfrak{b}(\kappa) \leq \mathfrak{a}(\kappa) \leq 2^\kappa$.* $\square$

The proof is essentially the same as the proof for Theorem 2.2.7, replacing ω by κ and using regularity. It is still open if we can separate $\mathfrak{b}(\kappa)$ and $\mathfrak{a}(\kappa)$, that is, if $\mathfrak{b}(\kappa) < \mathfrak{a}(\kappa)$ is consistent.

We now for the first time encounter an inequality which is not transferable to the generalized cardinal characteristics. For the splitting number we had the inequality $\mathfrak{s} > \aleph_0$. The situation is not the same for $\mathfrak{s}(\kappa)$. In fact it was shown in [12] that $\mathfrak{s}(\kappa) \geq \kappa \leftrightarrow \kappa$ is strongly inaccessible and furthermore, in [17], that $\mathfrak{s}(\kappa) \geq \kappa^+ \leftrightarrow \kappa$ is weakly compakt. Strong inaccessible or weakly compact cardinals are some sorts of "large cardinal numbers". For instance κ strongly inaccessible means uncountable regular and for any cardinal $\lambda < \kappa$, $2^\lambda < \kappa$. It is consistent with *ZFC* that such cardinals do not exist.

We have a positive answer for $\mathfrak{s}(\kappa) \leq \mathfrak{d}(\kappa)$ though.

Theorem 3.1.8. *Let κ be regular, then $\mathfrak{s}(\kappa) \leq \mathfrak{d}(\kappa) \leq 2^\kappa$.* $\square$

The proof of Theorem 2.2.12 generalizes to this case.

We now get the following diagram for regular κ:

$$\begin{tikzcd}
\mathfrak{s}(\kappa) \arrow[r] & \mathfrak{d}(\kappa) \arrow[r] & 2^\kappa \\
\kappa^+ \arrow[r] & \mathfrak{b}(\kappa) \arrow[r] \arrow[u] & \mathfrak{a}(\kappa) \arrow[u]
\end{tikzcd}$$

Diagram 2

Are these all provable inequalities? What other inequalities are possible?

- The consistency of $\mathfrak{a}(\kappa) < \mathfrak{d}(\kappa)$ is known. See for example [7]. However, it is unknown if $\mathfrak{d}(\kappa) < \mathfrak{a}(\kappa)$ is possible.
- $\mathfrak{s}(\kappa) < \kappa$ is true in a model with no inaccessibles, thus is consistent. The possible consistencies regarding $\mathfrak{s}(\kappa)$ (for example: is it consistent that $\mathfrak{s}(\kappa) \geq \kappa$) depend on various large cardinal assumption which represent higher consistency strengths than *ZFC* itself. For example it is consistent with the metatheory (which is usually *ZFC* itself) that no model for *ZFC* has inaccessibles, meaning *ZFC* proves $\mathfrak{s}(\kappa) < \kappa$ (by Gödel's Completeness Theorem).
- The next subsection will provide a proof for $\mathfrak{s}(\kappa) \leq \mathfrak{b}(\kappa)$, showing that $\mathfrak{s}(\kappa), \mathfrak{b}(\kappa)$ and $\mathfrak{s}(\kappa), \mathfrak{a}(\kappa)$ are not independent. So not only it is (relatively) consistent that *ZFC* proves $\mathfrak{s}(\kappa) < \kappa < \mathfrak{b}(\kappa)$, but our metatheory (*ZFC*) proves that there is a proof (we give it explicitly) of $\mathfrak{s}(\kappa) \leq \mathfrak{b}(\kappa)$.

3.2 An unexpected inequality between the bounding and splitting numbers

In this subsection we are going to review a recent result (May 2015) of Dilip Raghavan and Saharon Shelah that deals with the bounding and splitting number at regular uncountable cardinals. See [13] for their paper. As mentioned in the chapter about cardinal characteristics of the continuum the bounding number $\mathfrak{b}$ and the splitting number $\mathfrak{s}$ at ω are independent. Shelah proved (1984) the consistency of $\mathfrak{b} < \mathfrak{s}$, Baumgartner and Dordal proved (1985) the consistency of $\mathfrak{s} < \mathfrak{b}$. At uncountable regular cardinals the situation is different:

Theorem 3.2.1. *Let $\kappa > \aleph_0$ be a regular cardinal. Then $\mathfrak{s}(\kappa) \leq \mathfrak{b}(\kappa)$.*

Before getting to the proof we have to review an important concept in model theory that is crucial in the proof: Elementary Submodels.

3.2.1 Elementary Submodels

The proofs can all be found in [10, II.16].

Definition 3.2.2 (Submodel)**.** Let $\mathcal{L}$ be a language and $\mathfrak{A}$, $\mathfrak{B}$ be $\mathcal{L}$-structures. Then $\mathfrak{A}$ is a submodel of $\mathfrak{B}$ ($\mathfrak{A} \subseteq \mathfrak{B}$) iff:

- $A \subseteq B$
- for any function symbol $f \in \mathcal{L}$ with arity n: $f^{\mathfrak{A}} = f^{\mathfrak{B}} \restriction A^n$
- for any constant symbol $c \in \mathcal{L}$: $c^{\mathfrak{A}} = c^{\mathfrak{B}}$
- for any relation symbol $R \in \mathcal{L}$ with arity n: $R^{\mathfrak{A}} = R^{\mathfrak{B}} \cap A^n$

Definition 3.2.3 (Elementary Submodel)**.** Let $\mathcal{L}$ be a language and $\mathfrak{A}$, $\mathfrak{B}$ be $\mathcal{L}$-structures with $\mathfrak{A}$ a submodel of $\mathfrak{B}$. Then $\mathfrak{A}$ is called an elementary submodel of $\mathfrak{B}$ ($\mathfrak{A} \preccurlyeq \mathfrak{B}$) iff for any $\mathcal{L}$-formula φ and any assignement $\vec{a}$ in A we have $\mathfrak{A} \models \varphi[\vec{a}] \leftrightarrow \mathfrak{B} \models \varphi[\vec{a}]$.

Lemma 3.2.4 (Tarski-Vaught Criterion)**.** *The following are equivalent:*

- $\mathfrak{A} \preccurlyeq \mathfrak{B}$
- *For all existential formulas (formulas of the form $\exists y \varphi(y, \vec{x})$) and all assignments $\vec{a}$ in A: if $\mathfrak{B} \models \exists y \varphi(y, \vec{x})[\vec{a}]$ then there is a $b \in A$ such that $\mathfrak{B} \models \varphi(y, \vec{x})[b, \vec{a}]$.*

□

If something is definable within $\mathfrak{A}$ and belongs to $\mathfrak{B}$ then it (or some witness) will also belong to $\mathfrak{A}$. An arbitrary example: $(\mathbb{Q}, \cdot)$ is not an elementary submodel of $(\mathbb{R}, \cdot)$. Though $(\mathbb{R}, \cdot) \models \exists y (x = y \cdot y)[2]$, $(\mathbb{Q}, \cdot)$ does not.

Theorem 3.2.5 (Downward Löwenheim-Skolem-Tarski)**.** *Let $\mathfrak{B}$ be a $\mathcal{L}$ structure, λ a cardinal with $\max(|\mathcal{L}|, \aleph_0) \leq \lambda < |B|$ and $S \subseteq B$ with $|S| \leq \lambda$. Then there is an elementary submodel $\mathfrak{A}$ of $\mathfrak{B}$ such that $S \subseteq A$ and $|A| = \lambda$.* □

Definition 3.2.6. For any x define a sequence $\langle x_n \rangle_{n \in \omega}$inductively as follows: $x_0 := x$, $x_{n+1} := x_n \cup \bigcup x_n$. Then $\operatorname{trcl}(x) := \bigcup_{n \in \omega} x_n$ is called the transitive closure of x.

Note that $\operatorname{trcl}(x)$ is a transitive set. Moreover it is the smallest transitive set having x as a subset.

Definition 3.2.7. Let θ be an infinite cardinal. $H(\theta) := \{x : |\operatorname{trcl}(x)| < \theta\}$.

Intuitively $H(\theta)$ consists of all sets in that less than θ other sets occur (all the elements, elements of elements ...). It will contain many different sets (namely $\sup_{\xi<\theta} 2^\xi$ many ones) like all the subsets of any $\xi < \theta$, functions $\xi \to \xi$, definitely $\emptyset$, natural numbers ...

What is often done, and will be done here when we prove Theorem 3.2.1, is to view $H(\theta)$ as a structure defined in *ZFC* with a relation that coincides with membership ($\in$) in *ZFC*. That is we have a language $\{\in'\}$ and the set $H(\theta)$ and define the structure where $\in'$ is interpreted as $\in'^{H(\theta)} = \{(x, y) \in H(\theta) \times H(\theta) : x \in y\}$. Within *ZFC* we define what predicate logic is and what relations like $\models$ mean. The point is $H(\theta)$ will satisfy some of *ZFC* (*ZFC* now written down as formulas in the predicate logic we just defined). $H(\theta)$ is transitive ($y \in x \in H(\theta) \to y \in H(\theta)$) and by this it will satisfy *Extensionality* (Two sets are the same iff they have the same elements) which is essential for being able to define sets unique inside the structure. The formula $\neg\exists y(y \in' x)$ must not define a unique empty set; if *Extensionality* does not hold there could be more then one empty set. Also *Pairing* will hold, *Comprehension*, ... Also by transitivity, many important notions of set theory will be absolute for $H(\theta)$. This means for example that if some element x in $H(\theta)$ has the property $\varphi(x) :=$ " x is an ordinal" in $H(\theta)$, then it "really" is an ordinal. This holds in general for formulas $\varphi(x)$ in which quantifiers are bounded (that is, of the form $\forall x_1 \in x_2$ or $\exists x_1 \in x_2$). These formulas are called Δ_0. For more on absoluteness of transitive models see [11, I.16].

It is not directly clear that $H(\theta)$ really is a set, but it is. See [11, II.2] for more on that.

3.2.2 Filters and Ideals

Another notion that we need for the proof of Theorem 3.2.1 is that of a filter which we will now discuss. We also introduce the dual concept of an ideal for the sake of completeness and because it very much fits into the topic.

Definition 3.2.8 (Filter)**.** Let X be a non empty set, then a non empty subset $\mathcal{F} \subseteq \mathcal{P}(X)$ is called a filter iff:

- $\emptyset \notin \mathcal{F}$
- $\forall A, B \in \mathcal{F}[A \cap B \in \mathcal{F}]$
- $\forall A, B \subseteq X[A \subseteq B \wedge A \in \mathcal{F} \to B \in \mathcal{F}]$

Definition 3.2.9 (κ-complete filter)**.** Let X be a non empty set with a filter $\mathcal{F}$, let κ be a cardinal. Then $\mathcal{F}$ is said to be κ-complete iff for any subset $\mathcal{E} \subseteq \mathcal{F}$ with $|\mathcal{E}| < \kappa$, $\bigcap \mathcal{E} \in \mathcal{F}$.

By definition a filter is closed by intersections and it is easy to show by induction that a filter is closed by any finite intersections. So a filter is always at least ω-complete. Intuitively we may think of a filter on X as a set that consists of "large" subsets of X. For example a filter on X always contains X itself ($\mathcal{F}$ is non empty so there is some $A \in \mathcal{F}$, but as $A \subseteq X$, also $X \in \mathcal{F}$). The intersection of two large sets is still large and a superset of a large set is also large. The dual notion of this is an ideal:

Definition 3.2.10 (Ideal)**.** Let X be a non empty set, then a non empty subset $\mathcal{I} \subseteq \mathcal{P}(X)$ is called an ideal iff:

- $X \notin \mathcal{I}$
- $\forall A, B \in \mathcal{I}[A \cup B \in \mathcal{I}]$
- $\forall A, B \subseteq X[A \subseteq B \wedge B \in \mathcal{I} \to A \in \mathcal{I}]$

An ideal contains what we would call "small" or "negligible" sets. A lot of such filters or ideals occur naturally in mathematics, especially when studying the structure of the real number line: The set of meagre sets $\mathcal{M}$ or the Lebesgue null sets $\mathcal{N}$ are ideals on $\mathbb{R}$. Filters play also an important role in general topology. The duality of these two notions is pointed out by the following lemma:

Lemma 3.2.11. *If $\mathcal{F}$ is a filter on X, then $\{X \setminus F : F \in \mathcal{F}\}$ is an ideal on X. If $\mathcal{I}$ is an ideal on X, then $\{X \setminus I : I \in \mathcal{I}\}$ is a filter on X.* $\square$

Definition 3.2.12. Let X be a set and $\mathcal{F}$ be a filter on X. Then for any $f, g \in X^X$ we define $f =_{\mathcal{F}} g$ iff $\{x \in X : f(x) = g(x)\} \in \mathcal{F}$.

Informally $f =_{\mathcal{F}} g$ iff f is nearly the same as g, that is they coincide on a very large set and differ only on a negligible part of X.

Lemma 3.2.13. *Let X be a set and $\mathcal{F}$ be a filter on X, then $=_{\mathcal{F}}$ is an equivalence relation on X^X.*

Proof. It is easy to check the properties of an equivalence relation:

- Reflexive: $f =_{\mathcal{F}} f$ because $\{x \in X : f(x) = f(x)\} = X \in \mathcal{F}$
- Symmetric: $f =_{\mathcal{F}} g \leftrightarrow g =_{\mathcal{F}} f$ as $\{x \in X : f(x) = g(x)\} = \{x \in X : g(x) = f(x)\}$
- Transitive: If $f =_{\mathcal{F}} g$ and $g =_{\mathcal{F}} h$ then $\{x \in X : f(x) = g(x)\} \in \mathcal{F}$ and $\{x \in X : g(x) = h(x)\} \in \mathcal{F}$ so $\{x \in X : f(x) = h(x)\} \supseteq \{x \in X : f(x) = g(x)\} \cap \{x \in X : g(x) = h(x)\} \in \mathcal{F}$ so $\{x \in X : f(x) = h(x)\} \in \mathcal{F}$ and $f =_{\mathcal{F}} h$.

$\square$

3.2.3 Proof of Theorem 3.2.1 ($\mathfrak{s}(\kappa) \leq \mathfrak{b}(\kappa)$)

Throughout the proof we fix a regular uncountable cardinal κ. We want to show that $\mathfrak{s}(\kappa) \leq \mathfrak{b}(\kappa)$ and we are going to do that by proving $\mathfrak{b}(\kappa) \not< \mathfrak{s}(\kappa)$. The assertion is trivial if $\mathfrak{s}(\kappa) \leq \kappa^+$ because $\kappa^+ \leq \mathfrak{b}(\kappa)$ so we will consider the case where $\kappa < \lambda < \mathfrak{s}(\kappa)$ for some λ and show that a subset of κ^κ with cardinality λ can't be unbounded, respectively, is bounded.

Now let $\{f_\xi : \xi < \lambda\}$ be a set of functions $\kappa \to \kappa$. Let $\theta := (2^{2^{\mathfrak{s}(\kappa)}})^+$ and $M \preccurlyeq H(\theta)$ (elementary submodel) with $\lambda \cup \{f_\xi : \xi < \lambda\} \subseteq M$ and $|M| = \lambda$. This is possible by Theorem 3.2.5. So M is now a set that contains our functions f_ξ, all $\alpha < \lambda$ and all definable sets (definable in M) that belong to $H(\theta)$ and has cardinality λ. This set M is the starting point for the whole proof.

Definition 3.2.14. As $|M| < \mathfrak{s}(\kappa)$, $M \cap \mathcal{P}(\kappa)$ is not a splitting family, so let A_* be a subset of κ that is not split by any $x \in M \cap \mathcal{P}(\kappa)$.

Definition 3.2.15. Let

$$D := \{x \in \mathcal{P}(\kappa) : A_* \subseteq^* x\}$$

Lemma 3.2.16. *D is a κ-complete filter.*

Proof. We can easily check the properties of a κ-complete filter:

- $\emptyset \notin D$: clearly $A_* \not\subseteq^* \emptyset$ ($|A_* \setminus \emptyset| = \kappa$)
- Let $\delta < \kappa$ and $(B_i)_{i<\delta}$ be a collection of sets in D. Then $\bigcap_{i<\delta} B_i \in D$: $B_i \in D \leftrightarrow A_* \subseteq^* B_i \leftrightarrow |A_* \cap (\kappa \setminus B_i)| < \kappa$. We have that $|A_* \cap (\kappa \setminus \bigcap_{i<\delta} B_i)| = |A_* \cap (\bigcup_{i<\delta} \kappa \setminus B_i)| = |\bigcup_{i<\delta} A_* \cap (\kappa \setminus B_i)| < \kappa$, because κ is regular and $\bigcup_{i<\delta} A_* \cap (\kappa \setminus B_i)$ is the union of δ ($< \kappa$) many sets of cardinality less than κ.

- Let $x \in D$ and $x \subseteq y$, then $y \in D$: Clearly $A_* \subseteq^* x \subseteq y$ implies that $A_* \subseteq^* y$.

$\square$

Lemma 3.2.17. *Let $\delta < \kappa$. For any partition $(X_\alpha)_{\alpha<\delta} \subset M$ of κ, there is a unique α with $X_\alpha \in D$.*

Proof. Recall that we have chosen A_* not to be split by any $X \in M \cap \mathcal{P}(\kappa)$. Let $X \in M \cap \mathcal{P}(\kappa)$, then either $X \in D$ or $\kappa \setminus X \in D$, because assume $A_* \not\subseteq^* X \wedge A_* \not\subseteq^* \kappa \setminus X$, then $|A_* \setminus X| = \kappa$ and $|A_* \setminus (\kappa \setminus X)| = |A_* \cap X| = \kappa$, and that means exactly that X splits A_*.

- Existence: Assume that there is no $\alpha < \delta$ with $X_\alpha \in D$, then $\forall \alpha < \delta$ we have $\kappa \setminus X_\alpha \in D$ and by κ-completeness of D, $\bigcap_{\alpha<\delta}(\kappa \setminus X_\alpha) \in D$. But also $\bigcap_{\alpha<\delta}(\kappa \setminus X_\alpha) = \kappa \setminus \bigcup_{\alpha<\delta} X_\alpha = \emptyset \notin D$.
- Uniqueness: Assume $X_\alpha, X_\beta \in D$, then also $X_\alpha \cap X_\beta = \emptyset \in D$.

$\square$

We now define an equivalence between functions in $M \cap \kappa^\kappa$ modulo our filter D:

Definition 3.2.18. For $f, g \in M \cap \kappa^\kappa$, $f \approx_D g$ iff $\{\alpha < \kappa : f(\alpha) = g(\alpha)\} \in D$.

By Lemma 3.2.13 $\approx_D$ is really an equivalence relation.

Definition 3.2.19. For $f \in M \cap \kappa^\kappa$, $[f]$ is the equivalence class of f. We let $L := \{[f] : f \in M \cap \kappa^\kappa\}$.

Definition 3.2.20. For $f, g \in M \cap \kappa^\kappa$, $[f] <_D [g]$ iff $\{\alpha < \kappa : f(\alpha) < g(\alpha)\} \in D$.

Justification 3.2.21. In order to be well-defined, $[f] <_D [g]$ needs to be independent of the choice of a representative. This is easily shown:

Let $f' \in [f]$, then $\{\alpha < \kappa : f'(\alpha) < g(\alpha)\} \supseteq \{\alpha < \kappa : f'(\alpha) = f(\alpha)\} \cap \{\alpha < \kappa : f(\alpha) < g(\alpha)\} \in D$. Analogous when $g' \in [g]$ then $\{\alpha < \kappa : f(\alpha) < g'(\alpha)\} \in D$.

Lemma 3.2.22. *$(L, <_D)$ is a linear order.*

Proof. Let us check the defining properties of a linear order:

- Irreflexive: $\forall f \in M \cap \kappa^\kappa$ we have $\{\alpha < \kappa : f(\alpha) < f(\alpha)\} = \emptyset \notin D$.
- Transitive: Let $[f] <_D [g]$, $[g] <_D [h]$, then $\{\alpha < \kappa : f(\alpha) < h(\alpha)\} \supseteq \{\alpha < \kappa : f(\alpha) < g(\alpha)\} \cap \{\alpha < \kappa : g(\alpha) < h(\alpha)\} \in D$.
- Trichotomy: The sets $\{\alpha < \kappa : f(\alpha) < g(\alpha)\}$, $\{\alpha < \kappa : f(\alpha) = g(\alpha)\}$, $\{\alpha < \kappa : f(\alpha) > g(\alpha)\}$ make a partition of κ in M (they are all definable out of elements of M and $H(\theta)$ satisfies their existence), so exactly one of them is in D by Lemma 3.2.17.

$\square$

We will now denote by c_α the constant function $\kappa \to \kappa$ with $c_\alpha(\beta) = \alpha$ for all β and by i the identity function on κ. Note that these functions belong to M: they clearly belong to $H(\theta)$ and they are definable within M (by a formula of the kind $\varphi(f) := \forall x((x \in' \operatorname{dom}(f) \to f(x) = x) \wedge \operatorname{dom}(f) = \kappa)$ or $\varphi(f) := \forall x((x \in' \operatorname{dom}(f) \to f(x) = \alpha) \wedge \operatorname{dom}(f) = \kappa)$).

Lemma 3.2.23. *The set $\{[c_\alpha] : \alpha < \kappa\} \subseteq L$ has a least upper bound in L.*

Proof. Note first that $[i]$ bounds $\{[c_\alpha] : \alpha < \kappa\}$:

$\{\beta < \kappa : c_\alpha(\beta) < i(\beta)\} = \{\beta < \kappa : \alpha < \beta\}$ and $A_* \subseteq^* \{\beta < \kappa : \alpha < \beta\}$ because $|A_* \cap (\kappa \setminus \{\beta < \kappa : \alpha < \beta\})| = |A_* \cap \{\beta < \kappa : \beta \leq \alpha\}| = |A_* \cap S(\alpha)| < |S(\alpha)| < \kappa$. So for all α we have $\{\beta < \kappa : c_\alpha(\beta) < i(\beta)\} \in D$ and $[c_\alpha] <_D [i]$.

Now assume there is no least upper bound. Then for any bound (as for example $[i]$) there is a smaller one. We can define a sequence $(f_n)_{n\in\omega}$ on ω by recursion letting $f_0 := i$ and $f_{n+1} = f_n^-$, where f^- denotes a function so that $[f^-] <_D [f]$ and $[f^-]$ is a bound (one can easily define $()^-$ on the set of functions f where $[f]$ is bounding). Now as we have that $\forall n \in \omega([f_{n+1}] <_D [f_n])$ which means $\forall n \in \omega(\{\alpha < \kappa : f_{n+1}(\alpha) < f_n(\alpha)\} \in D)$, also the set $\bigcap_{n\in\omega}\{\alpha < \kappa : f_{n+1}(\alpha) < f_n(\alpha)\}$ is in D because of κ-completeness and κ is uncountable. In particular $\bigcap_{n\in\omega}\{\alpha < \kappa : f_{n+1}(\alpha) < f_n(\alpha)\} \neq \emptyset$, so $\exists\beta$ such that $\forall n \in \omega$ we have that $f_{n+1}(\beta) < f_n(\beta)$, but then the set $\{f_n(\beta) : n \in \omega\} \neq \emptyset$ has no least element which is a contradiction (every non-empty set of ordinals has a least element, see Theorem 1.2.8). □

Definition 3.2.24. Fix a function $f_* \in M \cap \kappa^\kappa$, such that $[f_*]$ is a (the) least upper bound of $\{[c_\alpha] : \alpha < \kappa\}$.

Lemma 3.2.25. *$f_*(A_*)$ is unbounded in κ.*

Proof. Let $\alpha < \kappa$. Then $\{\beta < \kappa : \alpha < f_*(\beta)\} \in D$ because $[c_\alpha] <_D [f_*]$ and $A_* \subseteq^* \{\beta < \kappa : \alpha < f_*(\beta)\}$. So in particular $A_* \cap \{\beta < \kappa : \alpha < f_*(\beta)\} \neq \emptyset$ ($|A_*| = \kappa$) and $\exists\beta \in A_*$ so that $\alpha < f_*(\beta) \in f_*(A_*)$. □

Definition 3.2.26. A set $C \subseteq \kappa$ is called a club set (closed unbounded set) in κ iff $\sup C = \kappa$ and C is closed in κ, that is, $\sup c \in C$ for all $c \subseteq C$ that are bounded in κ.

Proposition 3.2.27. *If $C \in M \cap \mathcal{P}(\kappa)$ is a club in κ, then $f_*(A_*) \subseteq^* C$.*

Proof. We are going to prove $f_*^{-1}(C) \in D$ which implies that $f_*(A_*) \subseteq^* C$:

We know that f_*, C, κ all belong to M so we can easily write a formula that defines $f_*^{-1}(C)$ and $\kappa \setminus f_*^{-1}(C) = f_*^{-1}(\kappa \setminus C)$ in M and as they are in $H(\theta)$ they also belong to M. Now $f_*^{-1}(C)$ and $f_*^{-1}(\kappa \setminus C)$ partition κ so exactly one of them is in D. Assume $f_*^{-1}(C) \notin D$, so $X_0 := f_*^{-1}(\kappa \setminus C)$ must be in D. Also $X_1 := \{\beta < \kappa : 0 < f_*(\beta)\} \in D$ because $[c_0] <_D [f_*]$, so $X := X_0 \cap X_1 \in D$. Furthermore $X \in M$ because again X_0, X_1 and hereby X are definable using sets in M. Let $f : \kappa \to \kappa$ be defined as follows:

$$f(\alpha) = \begin{cases} \sup(C \cap f_*(\alpha)) & \text{if } \alpha \in X \\ 0 & \text{otherwise} \end{cases}$$

Here again $f \in M$ by the same argumentation as always. So $[f] \in L$.
Note that if $\alpha \in X$ then $f(\alpha) = \sup(C \cap f_*(\alpha)) < f_*(\alpha)$: Clearly $\sup(C \cap f_*(\alpha)) \leq f_*(\alpha)$ so consider the case where $f_*(\alpha) = \sup(C \cap f_*(\alpha))$. If $\sup(C \cap f_*(\alpha))$ is a successor ordinal then $\sup(C \cap f_*(\alpha)) < f_*(\alpha)$ is clear. If $\sup(C \cap f_*(\alpha)) = 0$ then as $f_*(\alpha) \neq 0 (\alpha \in X_1)$ $\sup(C \cap f_*(\alpha)) < f_*(\alpha)$. If $f_*(\alpha) = \sup(C \cap f_*(\alpha))$ is a limit ordinal then C has to be unbounded in $f_*(\alpha)$ and as C is closed this would mean that $f_*(\alpha) \in C$. But $\alpha \in X_0 = f_*^{-1}(\kappa \setminus C)$ so $f_*(\alpha) \notin C$.

We have hereby shown that $[f] <_D [f_*]$ because $X \subseteq \{\alpha < \kappa : f(\alpha) < f_*(\alpha)\}$ and $X \in D$.

Now for any $\alpha < \kappa$ there is a $\delta \in C$ with $\alpha < \delta$ because C is unbounded in κ. We know that $[c_\delta] <_D [f_*]$ so $Y := \{\beta < \kappa : \delta < f_*(\beta)\} \in D$. Now if $\beta \in X \cap Y$ then $c_\alpha(\beta) = \alpha < \delta \leq \sup(C \cap f_*(\beta)) = f(\beta)$. As $X, Y \in D$, also $X \cap Y \in D$ so $[c_\alpha] <_D [f]$.

We now know that for any α we have $[c_\alpha] <_D [f] <_D [f_*]$, which contradicts the choice of f_*. □

Lemma 3.2.28. *If $f \in M \cap \kappa^\kappa$, then $C_f := \{\alpha < \kappa : f$ is closed under $\alpha\} \in M$ is a club in κ.*

Proof.

- Closed: Let $(\alpha_i)_{i<\delta}$ be a sequence of elements in C_f. Then f is also closed under $\bigcup_{i<\delta} \alpha_i$, because if $\beta \in \bigcup_{i<\delta} \alpha_i$ then there is some $i < \delta$ with $\beta \in \alpha_i$ and so $f(\beta) \in \alpha_i \subseteq \bigcup_{i<\delta} \alpha_i$.
- Unbounded: Let $\alpha < \kappa$ and assume there is no $\beta \in C_f, \alpha < \beta$. Then f is not closed under $\alpha+1$ so that $\alpha+1 < \sup f''(\alpha+1)$. Again f is not closed under $\sup f''(\alpha+1)$ … So let us define a sequence $(\alpha_n)_{n\in\omega}$ with: $\alpha_0 := \alpha+1$, $\alpha_{n+1} = \sup f''(\alpha_n)+1$. Then $\alpha < \sup_{n\in\omega} \alpha_n < \kappa$ because κ is regular and uncountable (no $(<\kappa)$-sequence can be unbounded) and f is closed under $\sup_{n\in\omega} \alpha_n$: Let $\beta \in \sup_{n\in\omega} \alpha_n$, then $\beta \in \alpha_n$ for some $n \in \omega$ and $f(\beta) < \sup f''(\alpha_n) + 1 = \alpha_{n+1} < \sup_{n\in\omega} \alpha_n$.

□

Proposition 3.2.29. *$M \cap \kappa^\kappa$ is bounded.*

Proof. By Lemma 3.2.25 $f_*(A_*)$ is unbounded in κ. Let us define a function $h : \kappa \to f_*(A_*)$ that assigns to any $\alpha < \kappa$ an element $h(\alpha) \in f_*(A_*)$ strictly greater than α (this is possible because $f_*(A_*)$ is unbounded). We will now show that h bounds $M \cap \kappa^\kappa$:

Let $f \in M \cap \kappa^\kappa$. As $C_f \in M$ is a club, by Proposition 3.2.27, $f_*(A_*) \subseteq^* C_f$. So $|f_*(A_*) \setminus C_f| < \kappa$ and in particular $f_*(A_*) \setminus C_f$ is bounded (because of regularity of κ) by some $\delta < \kappa$. So $f_*(A_*) \setminus \delta \subseteq C_f$.

Now fix some α with $\delta \leq \alpha < \kappa$. Then $\delta \leq \alpha < h(\alpha)$, so $h(\alpha) \in f_*(A_*) \setminus \delta$ and in particular $h(\alpha) \in C_f$ which means that f is closed under $h(\alpha)$. Now, as $\alpha < h(\alpha)$, $f(\alpha) < h(\alpha)$. α was chosen arbitrarily so $\forall \alpha < \kappa$ we have $[\delta \leq \alpha \to f(\alpha) < h(\alpha)]$ and $|\{\alpha < \kappa : h(\alpha) \leq f(\alpha)\}| \leq |\delta| < \kappa$ which means exactly that $f <^* h$. We conclude that $\forall f \in M \cap \kappa^\kappa (f <^* h)$. □

Now it is clear that also $\{f_\xi : \xi < \lambda\} \subseteq M \cap \kappa^\kappa$ is bounded and we are finished. □

For many uncountable κ (namely if $\kappa = \kappa^{<\kappa}$) it is possible to separate $\mathfrak{b}(\kappa)$ and $\mathfrak{s}(\kappa)$.

3.3 A relation between generalized dominating and mad families

In this section we are going to prove the following theorem from Andreas Blass, Tapani Hyttinen and Yi Zhang (see [3]):

Theorem 3.3.1. *Let $\kappa > \aleph_0$ be a regular cardinal, then $\mathfrak{d}(\kappa) = \kappa^+ \to \mathfrak{a}(\kappa) = \kappa^+$.*

Proof. We assume $\mathfrak{d}(\kappa) = \kappa^+$ which means that there is a dominating family of cardinality κ^+ and our goal will then be to construct a mad family of cardinality κ^+ which implies $\mathfrak{a}(\kappa) = \kappa^+$ as $\mathfrak{a}(\kappa) \geq \kappa^+$. In fact, for the construction we will need a dominating family that satisfies special properties. Start with any dominating family $\{h_i : i < \kappa^+\}$:

- We can then get a dominating family that contains only strictly increasing functions: This is possible because we can replace every function h_i by a function g_i, defined recursively as follows: $g_i(0) := h_i(0)$, $g_i(\alpha) := \max(\sup\{g_i(\beta) : \beta < \alpha\} + 1, h_i(\alpha))$.

Note that $\sup\{g_i(\beta) : \beta < \alpha\}$ and furthermore $\sup\{g_i(\beta) : \beta < \alpha\} + 1$ are really in κ because κ is regular ($\{g_i(\beta) : \beta < \alpha\}$ has cardinality $< \kappa$) and so g_i is really a function $\kappa \to \kappa$. Then g_i is clearly increasing and we get a dominating family $\{g_i : i < \kappa^+\}$.

- Furthermore we can get a scale. That is a dominating family $\{f_i : i < \kappa^+\}$ enumerated increasingly with respect to $<^*$: $\forall j < i < \kappa^+[f_j <^* f_i]$.
  This is possible by a similar argument: Define a new dominating family $\{f_i : i < \kappa^+\}$ recursively as follows: Let $f_0 := g_0$. For $i < \kappa^+$, the set $\{f_j : j < i\} \cup \{g_i\}$ has cardinality less than κ^+ and thus is bounded by some function b_i ($\kappa^+ \leq \mathfrak{b}(\kappa)$ see Theorem 3.1.3). As $\{g_i : i < \kappa^+\}$ is a dominating family there is some g in this family with $b_i <^* g$. Let $f_i := g$. Then $\{f_i : i < \kappa^+\}$ is a dominating family because for any i we have $g_i <^* b_i <^* f_i$ (and thus if some function is dominated by g_i it is also dominated by f_i). And for any $j < i < \kappa^+$ we have $f_j <^* b_i <^* f_i$ so the functions are ordered increasingly. Our new family is a subset of $\{g_i : i < \kappa^+\}$, so its elements are still increasing functions.

We want to start our construction with an almost-disjoint family $\{S_j : j < \kappa\}$ of size κ and we will then add inductively for each $i \in [\kappa, \kappa^+)$ a new set S_i almost-disjoint from all the S_j's, $j < i$, before. For example we may take a partition of κ into κ many unbounded sets (There is a bijection $g : \kappa \times \kappa \to \kappa$, so the sets $S_j := \{g(\alpha, j) : \alpha < \kappa\}$ may serve as this family).

Now before we can really start the construction and prove that the resulting family is mad, we need some more ingredients:

- At each step $i \in [\kappa, \kappa^+)$ we will want to re-index the previous S_j's in type κ. So for every $i \in [\kappa, \kappa^+)$ we fix a bijection $G_i : \kappa \to i$. Our construction will depend on these bijections.

- We denote by $[\kappa, \kappa^+) \cap cof(\omega)$ the set $\{i : i \in [\kappa, \kappa^+) \wedge \mathrm{cf}(i) = \omega\}$, that is the set of i's in $[\kappa, \kappa^+)$ with cofinality ω. This is of course an abuse of notation as $cof(\omega)$ is a proper class.

  Lemma 3.3.2 (Club-guessing sequence)**.** *There is a sequence $\langle C_i \rangle$ (called club-guessing sequence) that assigns to each $i \in [\kappa, \kappa^+) \cap cof(\omega)$ a set C_i of order-type ω with $\sup(C_i) = i$ and such that for each club C in κ^+ there is an i with $C_i \subseteq C$.* □

  A proof for this lemma can be found in [15, Chapter III]. We will need such a club-guessing sequence, so fix the sets C_i.

- For each $i \in [\kappa, \kappa^+) \cap cof(\omega)$ we now define

$$D_i := \{\alpha < \kappa : \forall j \in C_i(G_i{}''\alpha \cap j = G_j{}''\alpha)\}$$

  Lemma 3.3.3. *For every $i \in [\kappa, \kappa^+) \cap cof(\omega)$, D_i is unbounded in κ.*

  Proof. Let $\delta < \kappa$. We will show that there is some $\alpha \in D_i$ with $\alpha > \delta$:
  First we are going to index the elements of C_i which is of type ω with a sequence $c_1, c_2, c_3, \ldots$ Then define recursively a strictly increasing $\omega \times \omega$-sequence as follows:
  – $\alpha_{(0,0)} := \delta$

– $\alpha_{(k,n+1)} := \min\{\beta > \alpha_{(k,n)} : G_i{}''\alpha_{(k,n)} \cap c_{n+1} \subseteq G_{c_{n+1}}{}''\beta\}$
– $\alpha_{(k+1,0)} := \min\{\beta > \sup_{n\in\omega} \alpha_{(k,n)} : \bigcup_{n\in\omega} G_{c_n}{}''\alpha_{(k,n)} \subseteq G_i{}''\beta\}$

Note that the elements of the sequence are always smaller than κ, in particular $\delta < \alpha := \sup_{(n,k)\in\omega\times\omega} \alpha_{(n,k)} < \kappa$. Then $\alpha \in D_i$, because take any $l > 0$, then:

$$\begin{aligned}\beta \in G_i{}''\alpha \cap c_l &\Rightarrow \exists(k,n)[\beta \in G_i{}''\alpha_{(k,n)} \cap c_l]\\ &\Rightarrow \beta \in G_i{}''\alpha_{(k+1,0)} \cap c_l \Rightarrow \beta \in G_{c_l}{}''\alpha_{(k+1,l)} \Rightarrow \beta \in G_{c_l}{}''\alpha\end{aligned}$$

and

$$\beta \in G_{c_l}{}''\alpha \Rightarrow \exists(k,n)[\beta \in G_{c_l}{}''\alpha_{(k,n)}] \Rightarrow \beta \in G_i{}''\alpha_{(k+1,0)} \cap c_l \Rightarrow \beta \in G_i{}''\alpha \cap c_l$$

so $G_i{}''\alpha \cap c_l = G_{c_l}{}''\alpha$. □

We will denote by $\mathrm{next}(D_i, \delta)$ the least element of D_i greater than δ.

Recapitulate what we have so far: a dominating family $\{f_i : i < \kappa^+\}$ with the conditions stated above, an almost-disjoint family $\{S_j : j < \kappa\}$ of cardinality κ, bijections $G_i : \kappa \to i$, a club-guessing sequence and the sets D_i.

We are ready for the construction: For $i \in [\kappa, \kappa^+)$ we define recursively:

$$S_i := \begin{cases}\emptyset & \text{if } \mathrm{cf}(i) \neq \omega\\ \{\gamma < \kappa : \forall j < i[f_i(\mathrm{next}(D_i, G_i^{-1}(j))) < \gamma \to \gamma \notin S_j]\} & \text{else}\end{cases}$$

Let $\mathcal{S} := \{S_i : i < \kappa^+\}$. Then $\mathcal{S}$ is an almost-disjoint family: Let $j < i < \kappa^+$, where $i \geq \kappa$ then $S_i \cap S_j$ is bounded by $f_i(\mathrm{next}(D_i, G_i^{-1}(j)))$ because if there is some $\gamma \in S_i$ with $\gamma > f_i(\mathrm{next}(D_i, G_i^{-1}(j)))$ then by definition of S_i we have that $\gamma \notin S_j$. If we let $j < i < \kappa$ then clearly $|S_i \cap S_j| < \kappa$ because we began with an almost-disjoint family.

It is usually required that a mad family contains only unbounded subsets of κ. Of course $\mathcal{S}$ may contain many bounded subsets (It will for instance contain $\emptyset$ as $\mathrm{cf}(\kappa) \neq \omega$), but this won't bother us: If we show that $\mathcal{S}$ is maximal (in that sense that if we add an unbounded subset of κ, our family will fail to be almost-disjoint) we will also have shown that $\mathcal{S} \cap [\kappa]^\kappa$ is maximal (bounded subsets are almost-disjoint with any subset of κ).

Now assume there is some unbounded set $X \subseteq \kappa$ such that $|S_i \cap X| < \kappa$ for all $i < \kappa^+$. This is going to produce a contradiction:

Let us define a sequence $(\alpha_i)_{i<\kappa^+}$ on $[\kappa, \kappa^+)$ recursively as follows:

- $\alpha_0 := \kappa$
- $\alpha_\eta := \bigcup_{i<\eta} \alpha_i$ for η a limit
- α_{i+1}: We define a function $h_{i+1} : \kappa \to \kappa$ as follows:

$$h_{i+1}(\beta) := \sup(X \cap S_{G_{\alpha_i}(\beta)})$$

Then we chose α_{i+1} with $\alpha_i < \alpha_{i+1}$, $\mathrm{cf}(\alpha_{i+1}) = \omega$ and $h_{i+1} <^* f_{\alpha_{i+1}}$. This is possible: We can first chose an α''_{i+1} with $h_{i+1} <^* f_{\alpha''_{i+1}}$ because we have a dominating family, then if $\alpha''_{i+1} \leq \alpha_i$ we can chose an $\alpha'_{i+1} > \alpha_i$ and we still have $h_{i+1} <^* f_{\alpha''_{i+1}} <^* f_{\alpha'_{i+1}}$. And at last we just let $\alpha_{i+1} := \alpha'_{i+1} + \omega$ and all the conditions hold.

Lemma 3.3.4. *The set $C := \{\alpha_i : i < \kappa^+\}$ is a club in κ^+.*

Proof.

- Closed: This is clear because we have $\bigcup_{i<\eta} \alpha_i = \alpha_\eta$ for limits η.
- Unbounded: C has cardinality κ^+ (as (α_i) is strictly increasing) so it can only be unbounded.

□

Now we can fix some $\alpha \in [\kappa, \kappa^+) \cap cof(\omega)$ with $C_\alpha \subseteq C$. Define $X' := X \setminus S_\alpha$, then $|X'| = \kappa$ because $|X \cap S_\alpha| < \kappa$ by assumption.

As $C_\alpha \subseteq C$ we can index the elements in C_α with an increasing sequence $(\alpha_{i_n})_{n<\omega}$ of type ω. Then we have for every $n < \omega$ that $h_{i_n+1} <^* f_{\alpha_{i_n+1}} <^* f_\alpha$, in particular $h_{i_n+1} <^* f_\alpha$ which means that $\{\beta < \kappa : h_{i_n+1}(\beta) \geq f_\alpha(\beta)\}$ is bounded by some δ_n. Choose those δ_n and define $\delta := \sup_{n<\omega} \delta_n$. $\delta < \kappa$, because κ is regular and uncountable, and we have that $\forall \beta \geq \delta, \forall n < \omega\ h_{i_n+1}(\beta) < f_\alpha(\beta)$.

Now we define
$$X'' := X' \setminus (\bigcup_{n<\omega} \bigcup \{S_j \cap X' : j \in G_{\alpha_{i_n}}{}''\delta\})$$
As $G_{\alpha_{i_n}}$ is a bijection we have that $|G_{\alpha_{i_n}}{}''\delta| = |\delta| < \kappa$ and furthermore $|\{S_j : j \in G_{\alpha_{i_n}}{}''\delta\}| < \kappa$ for any $n \in \omega$. Also we have that for any $j < \kappa^+$, $|X' \cap S_j| < \kappa$ so $|\bigcup \{S_j \cap X' : j \in G_{\alpha_{i_n}}{}''\delta\}| < \kappa$. At last as κ is regular and uncountable also $|\bigcup_{n<\omega} \bigcup \{S_j \cap X' : j \in G_{\alpha_{i_n}}{}''\delta\}| < \kappa$ and so still $|X''| = \kappa$. In particular $X'' \neq \emptyset$ so let $\gamma \in X''$.

We have $\gamma \notin S_\alpha$. If we recall how S_α was constructed we see that this means that there is some $j < \alpha$ with $\gamma > f_\alpha(\text{next}(D_\alpha, G_\alpha^{-1}(j)))$ and $\gamma \in S_j$, else we would have that $\gamma \in S_\alpha$. Fix such a $j < \alpha$.

Next we can find a $\alpha_i \in C_\alpha$ with $j < \alpha_i < \alpha$ as $\sup C_\alpha = \alpha$ and $C_\alpha \subseteq C$. Now we must have that $G_{\alpha_i}^{-1}(j) \geq \delta$ because if $G_{\alpha_i}^{-1}(j) < \delta$ then $j \in G_{\alpha_i}{}''\delta$ but then by definition of X'', $S_j \cap X'' = \emptyset$ so $\gamma \notin X''$ which is a contradiction of our choice.
We now have the following:

$$\begin{aligned} f_\alpha(G_{\alpha_i}^{-1}(j)) > h_{i+1}(G_{\alpha_i}^{-1}(j)) &= \sup(X \cap S_{G_{\alpha_i}(G_{\alpha_i}^{-1}(j))}) \\ &= \sup(X \cap S_j) \geq \gamma > f_\alpha(\text{next}(D_\alpha, G_\alpha^{-1}(j))) \end{aligned}$$

The first inequality uses the way we defined δ, the first equation is a direct application of the definition of h_{i+1}, the next inequality just holds because $\gamma \in X \cap S_j$ and the last one is because of the choice of j. This leads to:

$$f_\alpha(G_{\alpha_i}^{-1}(j)) > f_\alpha(\text{next}(D_\alpha, G_\alpha^{-1}(j)))$$

and moreover, as we chose our dominating family to contain only strictly increasing functions:
$$G_{\alpha_i}^{-1}(j) > \text{next}(D_\alpha, G_\alpha^{-1}(j))$$
Recapitulate what D_α is: $D_\alpha = \{\beta < \kappa : \forall \alpha_i \in C_\alpha (G_\alpha{}''\beta \cap \alpha_i = G_{\alpha_i}{}''\beta)\}$. Now let $\beta := \text{next}(D_\alpha, G_\alpha^{-1}(j)) \in D_\alpha$. On one side we see that $j \in G_\alpha{}''\beta \cap \alpha_i$ because $j < \alpha_i$ and $G_\alpha^{-1}(j)) < \beta$. On the other hand we have $G_{\alpha_i}^{-1}(j) > \beta$ so $j \notin G_{\alpha_i}{}''\beta$ (G_{α_i} is a bijection).

But as $\beta \in D_\alpha$ we must have that $G_\alpha{}''\beta \cap \alpha_i = G''_{\alpha_i}\beta$. We have a contradiction.

We have thus shown that no unbounded $X \subseteq \kappa$ can be almost-disjoint from all $S \in \mathcal{S}$. So we have constructed a mad family $\mathcal{S} \cap [\kappa]^\kappa$. $\mathcal{S} \cap [\kappa]^\kappa$ has at most κ^+ elements and thus must have exactly κ^+ elements ($\kappa < \mathfrak{a}(\kappa)$), so $\mathfrak{a}(\kappa) = \kappa^+$.

$\square$

A question that is still open is if the analogous statement for ω holds: Is it true that $\mathfrak{d} = \aleph_1 \rightarrow \mathfrak{a} = \aleph_1$? This question was first asked by Roitman in the 70's as: Is it consistent that $\mathfrak{d} = \aleph_1 < \mathfrak{a}$?

This last diagram sums up all inequalities that are known so far:

$$
\begin{tikzcd}
\mathfrak{s}(\kappa) \arrow[dr] & & \mathfrak{d}(\kappa) \arrow[dr] & \\
 & \mathfrak{b}(\kappa) \arrow[ur] \arrow[dr, dashed, "?=?"] & & 2^\kappa \\
\kappa^+ \arrow[ur] & & \mathfrak{a}(\kappa) \arrow[uu, dotted, "?"] \arrow[ur] &
\end{tikzcd}
$$

Diagram 3

The question marks indicate what is still open. Note that if we can prove that $\mathfrak{b}(\kappa) = \mathfrak{a}(\kappa)$, we would simultaneously prove $\mathfrak{a}(\kappa) \leq \mathfrak{d}(\kappa)$ and can refute the consistency of $\mathfrak{d}(\kappa) < \mathfrak{a}(\kappa)$. In the same way, if we prove the consistency of $\mathfrak{d}(\kappa) < \mathfrak{a}(\kappa)$, we will already have proven that of $\mathfrak{b}(\kappa) < \mathfrak{a}(\kappa)$.

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