THE CONSISTENCY OF ARBITRARILY LARGE SPREAD BETWEEN THE BOUNDING AND THE SPLITTING NUMBERS

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Abstract

In the following κ and λ are arbitrary regular uncountable cardinals.

What was known?

THEOREM 1 (Balcar-Pelant-Simon, [**2**]). *It is relatively consistent with ZFC that* $\mathfrak{s} = \omega_1 < \mathfrak{b} = \kappa$.

THEOREM 2 (Shelah, [**31**]). *It is relatively consistent with ZFC that* $\mathfrak{s} = \kappa < \mathfrak{b} = \lambda$.

THEOREM 3 (Baumgartner and Dordal, [**7**]). *Adding* κ *Hechler reals to a model of GCH gives a generic extension in which* $\mathfrak{s} = \omega_1 < \mathfrak{b} = \kappa$.

THEOREM 4 (Shelah, [**31**]). *There is a proper forcing notion of size continuum, which is almost* ${}^{\omega}\omega$*-bounding and adds a real not split by the ground model reals.*

THEOREM 5 (Shelah, [**31**]). *Assume CH. There is a proper forcing extension in which* $\mathfrak{b} = \omega_1 < \mathfrak{s} = \omega_2$.

THEOREM 6 (Brendle, [**11**]). *Assume GCH. Then there is a ccc generic extension in which* $\mathfrak{b} = \omega_1 < \mathfrak{s} = \kappa$.

THEOREM 7 (M. Canjar, [**14**]). *If* $\mathfrak{d} = \mathfrak{c}$*, then there is an ultafilter* U *such that* $\mathbb{M}_U$ *does not add a dominating real.*

Theorem 8 (Velickovic, [**36**]). *Let $\kappa > \aleph_1$ be a regular cardinal. Then there is a ccc generic extension satisfying $MA + 2^{\aleph_0} = \kappa$ together with the following statement: For every family D of $2^{\aleph_0}$ dense subsets of the partial order $\mathcal{I}$ of all perfect trees, there is a ccc perfect suborder $\mathcal{P}$ of $\mathcal{I}$ such that $D \cap \mathcal{P}$ is dense in $\mathcal{P}$, for all $D \in \mathcal{D}$.*

What is new?

Theorem 9. *If $cov(\mathcal{M}) = \kappa$ and $\mathcal{H} \subseteq {}^{\omega}\omega$ is an unbounded, $<^*$-directed family of size κ, $\forall\lambda < \kappa(2^{\lambda} \leq \kappa)$, then there is a σ-centered suborder of Shelah's proper poset from Teorem 4, which preserves $\mathcal{H}$ unbounded and adds a real not split by the ground model reals.*

Theorem 10. *Assume GCH. Then there is a ccc generic extension in which $\mathfrak{b} = \kappa < \mathfrak{s} = \kappa^+$.*

The above result is an improvement of S. Shelah's consistency of $\mathfrak{b} = \omega_1 < \mathfrak{s} = \omega_2$. In chapter V, see Definition 5.4.2, we suggest a countably closed, $\aleph_2$-c.c. forcing notion $\mathbb{P}$ which adds a σ-centered forcing notion of $\mathbb{C}(\omega_2)$-names for pure conditions $Q(C)$, such that $Q(C)$ preserves all unbounded families unbounded and adds a real not split by $V^{\mathbb{C}(\omega_2)} \cap [\omega]^{\omega}$. An appropriate iteration of the forcing notion could provide the consistency of $\mathfrak{b} = \kappa < \mathfrak{s} = \lambda$.

To my parents, Teodora and Velizar

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CHAPTER 1

Introduction

Before proceeding with a brief account of the historical development of mathematical ideas which lead to the establishment of the cardinal invariants of the continuum as a separate subject, we will introduce some basic notions and give the contemporary definitions of the bounding and the splitting numbers since they present the main object of study of this work. Following standard notation we denote by ${}^{\omega}\omega$ the set of all functions from the natural numbers to the natural numbers and by $[\omega]^{\omega}$ the set of all infinite subsets of ω. Let f and g be functions in ${}^{\omega}\omega$. The function f is said to be *dominated by the function* g if there is a natural number n such that $f \leq_n g$, i.e. $(\forall i \geq n)(f(i) \leq g(i))$. Then $<^* = \bigcup_{n\in\omega} \leq_n$ is called *the bounding relation* on ${}^{\omega}\omega$. A family of functions $\mathcal{F}$ in ${}^{\omega}\omega$ is said to be *dominated* by the function g, denoted $\mathcal{F} <^* g$ if for every $f \in \mathcal{F}$, $f <^* g$. Also $\mathcal{F}$ is said to be *unbounded* (equiv. *not dominated*) if there is no function g which dominates it. Then *the bounding number* is defined as the minimal size of an unbounded family. That is

$$\mathfrak{b} = \min\{|\mathcal{B}| : \mathcal{B} \subseteq {}^{\omega}\omega \text{ and } \mathcal{B} \text{ is unbounded}\}.$$

If $A, B \in [\omega]^{\omega}$ and both of the sets $A \cap B$ and $A \cap B^c$ are infinite, then A is said to be *split* by the set B. A family S of infinite subsets of ω is said to be *splitting* if for every $A \in [\omega]^\omega$ there is $B \in S$ which splits A. Then *the splitting number* is defined as the minimal size of a splitting family. That is

$$\mathfrak{s} = \min\{|S| : S \subseteq [\omega]^\omega \text{ and } S \text{ is splitting}\}.$$

Recall also that if A, B are subsets of ω, then the set A is said to be *almost contained* in the set B, denoted $A \subseteq^* B$ if $A \backslash B$ is finite.

1.1. The Bounding and the Splitting Numbers

With the development of analysis in the nineteenth century, emerged a necessity of better understanding of the set of irrational numbers and the properties of the real line. In 1871 answering a question of Riemann, Georg Cantor obtained uniqueness of the trigonometric series representation of a function. That is he showed that *if two trigonometric series converge to the same function, except on finitely many points, then they must be equal everywhere*. A year later, he generalized his result to infinite sets of exceptional points. Recall that if $S \subseteq \mathbb{R}$, then the derived set S' of S consists of all limit points of S. Cantor showed that *if two trigonometric series converge to the same function except on a set S such that for some $n \in \mathbb{N}$ the n-th derived set $S^{(n)}$ is finite, then the series must be equal everywhere.* Although results concerning infinite sets of exceptional points were already presented in the literature by that time, for example in 1829 Dirichlet suggested that *a function whose point of discontinuity form a nowhere dense set is integrable*, Cantor's generalized uniqueness theorem was one of the first results that took extensive use of the structure of an infinite set (see [**25**]). The result was followed in 1874 by Cantor's proof that the real numbers can not be placed in bijective correspondence with the natural numbers, the surprising fact in 1878 as Cantor himself admits, that n-dimensional Euclidean space is in bijective correspondence with the real line and the continuum hypothesis, that is the hypothesis that *every infinite subset of* $\mathbb{R}$ *is either in bijective correspondence with the natural numbers or the real line*. By 1879 the study of combinatorial structure of infinite sets of reals has already emerged as an important direction in further studies of the properties of the continuum. For example, Cantor defined and studied perfect sets of reals, i.e. sets which contain all of their accumulation points [**25**], and showed that every perfect set is in bijective correspondence with the real line. However later Bernstein constructed an uncountable set, such that neither it nor its complement contained a perfect set, thus it became clear that the study of perfect sets was insufficient to settle the continuum hypothesis.

The cardinal invariants of the continuum arise from various combinatorial structures on the real line. Of particular interest for this work is *the covering number of the meager ideal* $\mathrm{cov}(\mathcal{M})$. Let $\mathcal{M}$ denote the family of all meager subsets of $\mathbb{R}$. Then $\mathrm{cov}(\mathcal{M})$ is the minimal size of a family $\mathcal{F} \subseteq \mathcal{M}$ such that $\bigcup \mathcal{F} = \mathcal{M}$. In 1899 Rene Baire showed that countably many meager sets do not cover the real line. Under the CH the minimal size of a family of meager sets which covers the real line is $\aleph_1 = \mathfrak{c}$. However if CH fails and for example $\mathfrak{c} = \aleph_2$, then it is consistent with ZFC that $\mathrm{cov}(\mathcal{M}) = \aleph_1$ and also it is consistent that

$\mathrm{cov}(\mathcal{M}) = \aleph_2$. Another cardinal invariant which should be mentioned is the dominating number $\mathfrak{d}$. A family $D \subseteq {}^\omega\omega$ is said to be *dominating* if for every function $f \in {}^\omega\omega$ there is $d \in D$ such that $f \leq^* d$. The dominating number $\mathfrak{d}$ is the minimal size of a dominating family. Note that every dominating family is unbounded and so $\mathfrak{b} \leq \mathfrak{d}$.

In his "Calculus of Infinity" (see [**18**]) Paul du Bois-Reymond in the late 1870's studied the collection of continuous, monotone increasing positive valued functions and suggested to rank them according to their rate of divergence, or convergence to zero. That is he wanted to find a linear order $\prec$ on this set of functions, and an equivalence relation $\sim$ such that $f \prec g$ provided that the "rate of growth of f is smaller than the rate of growth of g" and $f \sim g$ provided they have the same rate of growth, and such that the equivalence $\sim$ respects $\prec$ (see [**29**]). He defines $f \prec g$ if

$$\lim_{x\to\infty} f(x)/g(x) = 0 \text{ or } \lim_{x\to\infty} g(x)/f(x) = \infty$$

and $f \sim g$ if

$$0 < \lim_{x\to\infty} f(x)/g(x) < \infty.$$

The major problem in this ranking is the existence of incomparable infinities, that is the existence of functions for which the above limit does not exist. Cantor considered this a significant drawback of du Bois-Reymond's idea, a drawback which under the axiom of choice his cardinal arithmetic did not have. However, Hausdorff further pursued the study of maximal linearly ordered subsets of $({}^{\mathbb{N}}\mathbb{R}, \leq^*)$. In 1909 Hausdorff ([**22**]) showed that these maximal linearly ordered subsets have the cardinality of the continuum and established his main result the existence of (ω_1, ω_1)-gaps. However it was not until 1936 (see [**23**]), that Hausdorff published the proof for binary sequences, i.e. established that in $(2^\omega, \leq^*)$ there is a (ω_1, ω_1)-gap, but there are no (ω, ω)-gaps and no ω-limits, which leads to the contemporary concepts of scale, unboundedness, tower, pseudo-intersection and correspondingly to the cardinal invariants $\mathfrak{d}$, $\mathfrak{b}$, $\mathfrak{t}$ and $\mathfrak{p}$.

In fact, the first to give the contemporary definition of the bounding number is Rothberger. A subset A of the n-dimensional Euclidean space $\mathbb{R}^n$ is said to have the property λ, if each of its countable subsets is relative G_δ. A subset A of $\mathbb{R}^n$ is said to have the property λ' if $A \cup B$ has the property λ for every countable subset B of $\mathbb{R}^n$. Answering a question of Sierpinski, Rothberger constructs a set which has the property λ, and at the same time does not have property λ'. In [**28**] he defines $B(\aleph_\xi)$ to be the proposition that *all sequences of natural numbers of cardinality* $\aleph_\xi$ *are bounded*, and then defines $\aleph_\eta$ to be *the minimal cardinal for which* $B(\aleph_\xi)$ *does not hold*. Thus in contemporary notation $\aleph_\eta$ is the cardinal invariant $\mathfrak{b}$ and Rothberger's result states that a subset A of $\mathbb{R}^n$ has the property λ' if and only if the cardinality of A is less than the bounding number. By the time the splitting number appeared in the literature, the dependence of the topological, measure theoretic properties of the continuum and its cardinal combinatorial characteristics was well established. In fact the splitting number appeared as an algebraic characteristic of sequential compactness. In [**10**] David Booth states that *for every regular uncountable cardinal* λ*, the space* 2^λ *is sequentially compact if and only if for every sequence* $\langle a_\alpha : \alpha \in \lambda \rangle$ *of infinite subsets of* $\mathbb{N}$ *there is* $b \subseteq \mathbb{N}$ *such that for all* $\alpha \in \lambda$*,* $b \subseteq^* a_\alpha$ *or* $b \subseteq^* \mathbb{N} - a_\alpha$. In contemporary notation that is, 2^λ is sequentially compact if and only if λ is smaller than the splitting number.

Below is a list of the known consistency relations between the bounding and the splitting numbers, as well as the main results of this work; κ and λ denote arbitrary regular uncountable cardinals.

THEOREM 1.1.1 (Balcar-Pelant-Simon, [**2**]). *It is relatively consistent with ZFC that* $\mathfrak{s} = \omega_1 < \mathfrak{b} = \kappa$.

THEOREM 1.1.2 (Shelah, [**31**]). *It is relatively consistent with ZFC that* $\mathfrak{s} = \kappa < \mathfrak{b} = \lambda$.

THEOREM 1.1.3 (Shelah, [**31**]). *It is relatively consistent with ZFC that* $\mathfrak{b} = \omega_1 < \mathfrak{s} = \omega_2$.

THEOREM 1.1.4 (Brendle, [**11**]). *It is relatively consistent with ZFC that* $\mathfrak{b} = \omega_1 < \mathfrak{s} = \kappa$.

Our main result, is the following.

THEOREM 1.1.5 (Main result). *It is relatively consistent with ZFC that* $\mathfrak{b} = \kappa < \mathfrak{s} = \kappa^+$.

In chapters IV-VI we give a first step towards the proof of the relative consistency of $\mathfrak{b} = \kappa < \mathfrak{s} = \lambda$.

1.2. Forcing

In 1963 Paul Cohen introduced the method of forcing (see [**15**]) to obtain the independence of the continuum hypothesis. Since then the method of forcing is largely used to obtain different relative consistency results, including results regarding the combinatorial cardinal characteristics of the real line. This is a general method for obtaining models of large finite fragments of ZFC, which satisfy some additional axioms. Excellent exposition of the method of forcing can be found in [**24**], [**20**]. I will give some basic notions and outline some of the fundamental properties of the method of forcing, since this is a major technique for the presented work. A forcing notion is a partially ordered set, that is a set $\mathbb{P}$ together with a reflexive and transitive relation $\leq$ on $\mathbb{P}$. We will work with strong forcing notions, i.e. partial orders which are also antisymmetric. That is for all $p, q \in \mathbb{P}$ if $p \leq q$ and $q \leq p$ then $p = q$. The elements of the forcing notion are called *conditions*. If $p \leq q$ then p is said to be an *extension* of q, also to be *stronger* than q and q is said to be *weaker* than p. The intuitive idea is that stronger conditions have more information about the intended model than weaker conditions. Conditions which do not have common extensions are said to be *incompatible* and respectively conditions which have a common extensions are said to be *compatible*. A set $D \subseteq \mathbb{P}$ is *dense* if every condition $p \in \mathbb{P}$ has an extension in D. A set $A \subseteq \mathbb{P}$ is *an antichain* if its elements are pairwise incompatible. A set $D \subseteq P$ is *pre-dense* if every element of the forcing notion is compatible with some $d \in D$. A set $C \subseteq \mathbb{P}$ is *centered* if for all $p, q \in C$ there is $r \in C$

which is their common extension. A centered $G \subseteq \mathbb{P}$ which is closed with respect to weaker conditions is *a filter*. In the following c.t.m. abbreviates "countable transitive model of sufficiently large finite portion of ZFC".

DEFINITION. Let V be a c.t.m., $\mathbb{P} \in V$ a forcing notion. A filter $G \subseteq \mathbb{P}$ is *generic over* V, if $G \cap D \neq \emptyset$ for all dense $D \subseteq \mathbb{P}$, $D \in V$.

Equivalently, in the above definition one can require that the filter G meets all pre-dense sets, or all maximal antichains, or all dense open sets which belong to the model V. Recall that a dense open set, is a dense subset which is closed with respect to stronger conditions.

THEOREM. Let V be a c.t.m., $\mathbb{P} \in V$ a forcing notion and $G \subseteq \mathbb{P}$ a filter generic over V. Then there is a countable transitive extension $V[G]$ of V which contains G, has the same ordinals as V and is minimal, in the sense that if W is a transitive extension of V such that $G \in W$, then $V[G] \subseteq W$.

The model V is called *the ground model* and $V[G]$ *the generic extension*. We will be working with forcing notions, which have the property that every element has incompatible extensions. Such posets are known to provide generic extensions which are distinct from the ground model. Indeed, it is not hard to verify that for such forcing notions the generic set does not belong to the ground model (see [**24**]). The elements of the generic extension have names in the ground model, which are recursively defined relations.

DEFINITION. Let $\mathbb{P}$ be a forcing notion. Then $\dot{X}$ is a $\mathbb{P}$-*name* if $\dot{X}$ is a relation and for all $\langle \dot{Y}, p\rangle \in \dot{X}$, $\dot{Y}$ is a $\mathbb{P}$-name and $p \in \mathbb{P}$.

The collection of all $\mathbb{P}$-names is a proper class. However if V is a c.t.m. and $\mathbb{P} \in V$, then the collection $V^{\mathbb{P}}$ of all $\mathbb{P}$-names in V is a set. The notion of a $\mathbb{P}$-name is absolute and so $V^{\mathbb{P}} = \{\dot{X} : (\dot{X} \text{ is a } \mathbb{P} \text{ name})^V\}$. The generic filter determines an evaluation of the names. More precisely if $\dot{X}$ is a $\mathbb{P}$-name and G is a $\mathbb{P}$-generic filter then the set $\dot{X}[G] = \{\dot{Y}[G] : \exists p \in G(\langle \dot{Y}, p\rangle \in \dot{X})\}$ is the evaluation of $\dot{X}$ determined by G. Furthermore $V[G] = \{\dot{X}[G] : \dot{X} \in V^{\mathbb{P}}\}$. With the forcing notion we associate a forcing language which is an extension of the language of set theory. An important characteristics of the forcing extension is the fact that there is a clear relationship between its semantic properties and the forcing notion, given by the forcing relation. The forcing relation $\Vdash$ is a relation between the elements of the forcing notion and the sentences of the forcing language. This relation is definable in the ground model and gives a description of the generic extension within the ground model. The following statement is known as the forcing theorem.

THEOREM. If V is a c.t.m., $\mathbb{P} \in V$ is a forcing notion, ϕ is a sentence in the forcing language and $G \subseteq \mathbb{P}$ is a filter generic over V then

$$V[G] \vDash \phi \text{ iff } \exists p \in G(p \Vdash \phi).$$

We would like to obtain generic extensions in which $\omega_1 < \mathfrak{b} < \mathfrak{s}$. Note that prior to this work, the existence of such models was unknown. In 1984, see [**31**], S. Shelah obtains a generic extension in which $\mathfrak{b} = \omega_1 < \mathfrak{s} = \omega_2$. About 15 years later modifying a model of A. Blass and S. Shelah which gives an arbitrarily large spread between $\mathfrak{u}$ and $\mathfrak{d}$, J. Brendle obtains the consistency of $\mathfrak{b} = \omega_1 < \mathfrak{s} = \kappa$ for κ arbitrary regular uncountable cardinal (see [**31**] and [**11**]). For our purposes, it is particularly important to obtain generic extensions in which cardinals are not collapsed. Observe that the notion of a cardinal is not absolute. For example forcing with the partial order of all finite partial functions from ω to ω_1 with extension relation reverse inclusion, over a model M of CH, produces a generic extension in which ω_1^M is a countable ordinal. In between the partial orders known to produce generic extensions in which cardinals are not collapsed, are the *ccc* forcing notions and the proper forcing notions. A forcing notion is said to be *ccc*, that is to satisfy the countable chain condition, if it does not contain uncountable antichains. A forcing notion is *proper*, if for every uncountable cardinal λ, every stationary subset of $[\lambda]^\omega$ from the ground model remains stationary in the generic extension. The class of *ccc* forcing notions is contained in the class of proper forcing notions (see [**30**]). The method of forcing can be repeated in the generic extensions, which leads to the theory of iterated forcing, an excellent exposition of which can be found in [**6**]. There are certain preservation theorems concerning finite support iterations of *ccc* forcing notions, which present key points in the construction of models of $\omega_1 < \mathfrak{b} < \mathfrak{s}$. For example, the finite support iteration of *ccc* forcing notion is *ccc* and so finite support iterations of *ccc* forcing notions do not collapse cardinals. In particular the finite support iteration of *ccc* forcing notions can be used to obtain generic extensions in which cardinals are not collapsed and the continuum is arbitrarily large. Another important fact is that if $\mathcal{H}$ is an unbounded family of functions in ${}^{\omega}\omega$, every countable subfamily of which is dominated by an element of $\mathcal{H}$, then in order to preserve $\mathcal{H}$ unbounded along an iteration with finite supports of *ccc* forcing notions, it is sufficient to preserve $\mathcal{H}$ unbounded at each successor stage of the iteration (see Theorem 3.5.2). Note that iterations of proper forcing notions of length $> \omega_2$ are known to collapse the continuum. In general, there are few available iteration techniques leading to generic extension in which $2^{\aleph_0} \geq \aleph_3$ (or even $2^{\aleph_0} \geq \aleph_2$) and on the other hand there are many longstanding open questions about the combinatorial properties of the real line, whose solution would require models with continuum $\geq \aleph_3$. In between those are (see for example [**33**])

- the consistency of $\mathfrak{p} < \mathfrak{t}$
- the consistency of no P-point and no Q-point
- the consistency of $\mathfrak{s}$ being singular

Thus to a certain degree, results about the combinatorial characteristics of $\mathbb{R}$ leading to generic extensions with large continuum, might be considered test results for developing new iteration techniques.

The bounding and the splitting numbers are independent, that is it is consistent with ZFC that $\mathfrak{b} < \mathfrak{s}$ as well as $\mathfrak{s} < \mathfrak{b}$. The consistency of $\mathfrak{s} < \mathfrak{b}$ is first mentioned by Balcar, Pelant and Simon [**2**] in 1980.

In 1984 S. Shelah obtains a different model of $\mathfrak{s} < \mathfrak{b}$ (see [**31**]) and in 1985, [**7**] J. Baumgartner and P. Dordal show that in the Hechler model (i.e. a model obtained as a finite support iteration of Hechler forcing of length κ, for κ regular uncountable cardinal over a model of GCH) the bounding number is κ while the splitting number remains ω_1. In order to obtain a generic extension in which $\mathfrak{b} < \mathfrak{s}$ one has to accomplish two major tasks: preserve a given unbounded family unbounded and increase the splitting number. By the preservation theorem mentioned above, in order to preserve an unbounded family unbounded along a finite support iteration of *ccc* forcing notions, it is sufficient to preserve the family unbounded at successor stages of the iteration. On the other hand in order to increase the splitting number along such an iteration, it is sufficient cofinally often to add reals which are not split by the ground model reals. A forcing notion which is known to add a real not split by the ground model reals is Mathias forcing [**26**].

Definition. Mathias forcing $\mathbb{M}$ consists of all pairs (s, A) where s is a finite subset of ω and $A \in [\omega]^\omega$ such that $\max s < \min A$. We say that $p_1 = (s_1, A_1) \leq p_2 = (s_2, A_2)$ if s_1 end-extends s_2, $s_1 \backslash s_2 \subseteq A_2$ and $A_1 \subseteq A_2$. If $s_1 = s_2$ then p_1 is said to be *a pure extension* of p_2.

If $p = (s, A)$ is a Mathias condition, then the infinite set A is called *the pure part* of p and the finite set s *the stem* of p. Having in mind the notion of preprocessed conditions, observe that every extension can be obtained in two steps: extension of the stem followed by a pure extension. To see that $\mathbb{M}$ adds a real not split by the ground model reals, consider any $A \in [\omega]^\omega \cap M$ and $p = (s, B) \in \mathbb{M}$. Then $B \cap A$ or

$B \cap A^c$ is infinite. That is for every $A \in [\omega]^\omega \cap M$, the set

$$D_A = \{(s, B) : B \subseteq A \text{ or } B \subseteq A^c\}$$

is dense. Let G be $\mathbb{M}$-generic and let $U_G = \bigcup\{s : \exists B(s, B) \in G\}$. Since the conditions in G are pairwise compatible, the extension relation of $\mathbb{M}$ implies that for every B which appears as the pure part of a condition in G, the set U_G is almost contained in B. Mathias forcing notion satisfies Axiom A (see 1.3.6) and so is proper ([**5**]). Thus an iteration of $\mathbb{M}$ with countable supports over a model of CH, would produce a generic extension in which $\mathfrak{s} = \mathfrak{c} = \aleph_2$. However Mathias forcing notion is also known to add a dominating real, that is a function in ${}^\omega\omega$ which dominates all ground model reals. For every $A \in [\omega]^\omega$, the enumerating function of A, which will be denoted also by A, is obtained by defining for every $j \in \omega$, $A(j)$ to be the j-th element of A. Let G be $\mathbb{M}$-generic filter and let U_G be defined as above. It will be shown that the enumerating function f_G of U_G dominates all ground model reals. To see this consider any $f \in {}^\omega\omega \cap M$. The set

$$D_f = \{(s, A) : \forall \ell \in \omega A(\ell) \geq f(|s| + \ell)\}$$

is dense in $\mathbb{M}$. Indeed, given $(s, B) \in \mathbb{M}$, one can recursively define an infinite subset A of B so that $(s, A) \in D_f$. Then $G \cap D_f$ contains some condition (s, A). Since $U_G \backslash s \subseteq A$ and s is an initial segment of U_G, for every $\ell \in \omega$

$$f_G(\ell + |s|) \geq A(\ell) \geq f(|s| + \ell).$$

That is $f \leq^* f_G$. Therefore an iteration of $\mathbb{M}$ with countable supports of length ω_2, as suggested above, would produce a generic extension in which the bounding number is also ω_2.

By adding additional combinatorial structure on the pure Mathias conditions, in [**31**] S. Shelah obtains a forcing notion Q' (see definition 1.3.5) of size $\mathfrak{c}$ which is proper, in fact Axiom A, which adds a real not split by the ground model reals and satisfies a strong combinatorial property. This property guarantees that under an iteration of the forcing notion Q' with countable supports over a model of CH, the ground model reals will remain unbounded and so a witness to $\mathfrak{b} = \omega_1$. Therefore an iteration with countable supports of length ω_2 over a model of CH of Shelah's forcing notion Q' produces a generic extension in which $\mathfrak{b} = \omega_1 < \mathfrak{s} = \omega_2$. The additional combinatorial structure on the pure Mathias conditions, is given in the form of logarithmic measure on the finite subsets of ω.

Definition. Let $x \in [\omega]^{<\omega}$. A function $h\colon \mathcal{P}(x) \to \omega$ is *a finite logarithmic measure* if whenever $x = x_0 \cup x_1$ then $h(x_0) \geq h(x) - 1$ or $h(x_1) \geq h(x) - 1$ unless $h(x) = 0$. The value $h(x) = \|x\|$ is called *the level* of the measure.

The partial order Q' consists of pairs $p = (u, T)$ where u is a finite subset of ω and $T = \langle (x_i, h_i) : i \in \omega \rangle$ is an infinite sequence of finite logarithmic measures of strictly increasing levels. The sequence T is similarly to Mathias forcing called a pure condition, also pure part of p. Note that if $\mathrm{int}(T) = \cup\{x_i : i \in \omega\}$, then $(u, \mathrm{int}(T))$ is a Mathias condition. The properties of the finite logarithmic measure imply that if $T = \langle (x_i, h_i) : i \in \omega \rangle$ is a pure condition and $A \subseteq \omega$ is infinite, then either $\langle h_i(x_i \cap A) : i \in \omega \rangle$ or $\langle h_i(x_i \cap A^c) : i \in \omega \rangle$ is unbounded. Therefore T has a pure extension R such that $\mathrm{int}(R) \subseteq A$ or $\mathrm{int}(R) \subseteq A^c$ and so for every $A \in [\omega]^\omega \cap M$ the set $D_A = \{(u, T) : \mathrm{int}(T) \subseteq A \text{ or } \mathrm{int}(T) \subseteq A^c\}$ is dense. This implies that Q' adds a real not split by the ground model reals.

However, we would like to obtain a model of $\omega_1 < \mathfrak{b} < \mathfrak{s}$ and so we would need to produce a generic extension in which cardinals are not collapsed and $\mathfrak{c} = 2^{\aleph_0} \geq \aleph_3$. A partial order $\mathbb{P}$ which can be presented in the form $\mathbb{P} = \bigcup_{n\in\omega} X_n$ where for all $n \in \omega$, X_n is a centered subset of $\mathbb{P}$ is called *σ-centered*. Note that every σ-centered forcing notion has the countable chain condition. For every ultrafilter U let $\mathbb{M}_U$ denote the suborder of Mathias forcing notion $\mathbb{M}$ consisting of all conditions whose pure part belongs to U. Conditions in $\mathbb{M}_U$ with equal stems are compatible and so $\mathbb{M}_U$ is σ-centered. Using the fact that U is an ultrafilter, one can easily show that $\mathbb{M}_U$ adds a real not split by the ground model reals. Therefore $\mathbb{M}_U$ can be used to obtain generic extensions in which the splitting number is arbitrarily large. However $\mathbb{M}_U$ might add a dominating real. In fact if U is selective, then forcing with $\mathbb{M}_U$ does add a real dominating the ground model reals. In [**14**] M. Canjar shows that if U is an ultrafilter such that $\mathbb{M}_U$ does not add a dominating real, then U is necessarily a P-point with no Q-points below it in the Rudin-Keisler order. In [**14**] it is also shown that if $\mathfrak{d} = \mathfrak{c}$ then there is an ultrafilter U such that $\mathbb{M}_U$ does not add a dominating real. One may expect that an appropriate iteration of such $\mathbb{M}_U$'s would produce a generic extension in which $\mathfrak{b} < \mathfrak{s}$. For example given regular uncountable cardinals $\kappa < \lambda$ begin with a model of GCH, add κ Hechler reals to obtain a generic extension M in which $\mathfrak{b} = \mathfrak{d} = \mathfrak{c} = \kappa$ and iterate with finite support of length λ Canjar's $\mathbb{M}_U$ which do not add dominating reals. Note that along this iteration small dominating families are not introduced and in fact in each initial stage of the iteration $\mathfrak{d} = \mathfrak{c}$. Then in the final generic extension $\mathfrak{d} = \mathfrak{s} = \mathfrak{c} = \lambda$. However preserving the ground model reals unbounded is not sufficient to preserve a given unbounded family unbounded along such an iteration and so one can not preserve a witness for $\mathfrak{b} = \kappa$.

In chapters II and III of this work we generalize Shelah's result to models of $\mathfrak{b} = \kappa < \mathfrak{s} = \kappa^+$ for κ arbitrary regular uncountable cardinal. In fact given an unbounded family $\mathcal{H}$ of functions in ${}^\omega\omega$ such that every subfamily $\mathcal{H}'$ of cardinality strictly smaller than $|\mathcal{H}|$ is dominated by an element of $\mathcal{H}$ (such families are called $<^*$-*directed*) we will obtain a centered family $C_{\mathcal{H}}$ of pure conditions in Shelah's partial order Q' and a *ccc* suborder $Q(C_{\mathcal{H}})$ of Q', which generalizes the relativization $\mathbb{M}_U$ of Mathias forcing to an ultrafilter U. The forcing $Q(C_{\mathcal{H}})$ has the advantage to Canjar's non-dominating $\mathbb{M}_U$ that it not only adds a real which is not split by the ground model reals and preserves the ground model reals unbounded, but also preserves the given unbounded family $\mathcal{H}$ unbounded. Then under an appropriate finite support iteration of *ccc* forcing notions we obtain the consistency of $\mathfrak{b} = \kappa < \mathfrak{s} = \kappa^+$. There are certain conditions on the existence of the forcing notion $Q(C_{\mathcal{H}})$, one of which is that $|\mathcal{H}| = 2^{\aleph_0}$ and so this method can not be used to obtain generic extension in which $\omega_1 < \mathfrak{b} = \kappa < \kappa^{++} \leq \mathfrak{s}$. In the second half of this work, we suggest a generic analogue of $\mathbb{M}_U$, in fact a generic analogue of $Q(C)$, which has the countable chain condition, adds a real not split by the ground model reals and preserves a chosen unbounded family $\mathcal{H}$ of cardinality strictly smaller than $2^{\aleph_0}$ unbounded (in fact we will obtain slightly more). Thus the suggested forcing notion has the potential of providing a generic extension in which the splitting number $\mathfrak{s}$ is arbitrarily larger than $\mathfrak{b} > \omega_1$.

1.3. A proper forcing argument

Having in mind certain analogies between Shelah's model of $\mathfrak{b} = \omega_1 < \mathfrak{s} = \omega_2$ and the model of $\mathfrak{b} = \kappa < \mathfrak{s} = \kappa^+$ (sections 2.1 - 3.6), in this section we give a more detailed outline of Shelah's proof. Apart from the original paper [**31**] (and [**30**]) an excellent presentation of this material is given in [**1**]. The section is self-contained and the rest of the work does not depend on it.

Recall that a forcing notion $\mathbb{P}$ is *weakly bounding* if the ground model reals remain an unbounded family in every generic extension via $\mathbb{P}$. However, the iteration of weakly bounding forcing notions is not necessarily weakly bounding (see [**1**]) and so a stronger notion of unboundedness is needed - see [**31**]:

Definition 1.3.1 (Shelah, [**31**]). The partial order $\mathbb{P}$ is *almost* ${}^\omega\omega$-*bounding* if for every $\mathbb{P}$-name $\dot{f}$ for a function in ${}^\omega\omega$ and every condition $p \in \mathbb{P}$ there is a ground model function $g \in {}^\omega\omega$ such that for every infinite subset A of ω, there is an extension q_A of p such that

$$q_A \Vdash \exists^\infty k \in A(\dot{f}(k) \leq \check{g}(k)).$$

As mentioned in [**31**], the Cohen forcing notion is almost ${}^\omega\omega$-bounding. By [**30**] the countable support iteration of proper almost ${}^\omega\omega$-bounding forcing notions is weakly bounding, and so in such iterations the ground model reals remain an unbounded family. It remains to observe the following preservation theorems (see [**30**] or [**1**]):

THEOREM 1.3.2 (Shelah, [**30**]). *Assume CH. Let $\langle\langle \mathbb{P}_i : i \leq \delta\rangle, \langle \dot{\mathbb{Q}}_i : i < \delta\rangle\rangle$ where $\delta < \omega_2$, be a countable support iteration of proper forcing notions of size $\aleph_1$. Then CH holds in $V^{\mathbb{P}_\delta}$.*

THEOREM 1.3.3 (Shelah, [**30**]). *Assume CH. Let $\langle \mathbb{P}_i : i \leq \delta\rangle$ where $\delta \leq \omega_2$, be a countable support iteration of proper forcing notions of size $\aleph_1$. Then $\mathbb{P}_\delta$ satisfies the $\aleph_2$-chain conditions.*

Therefore beginning with a model of CH and iterating with countable support (of length ω_2) proper forcing notions of size continuum, which satisfy the almost ${}^\omega\omega$-bounding property, one can obtain a generic extension in which cardinals are not collapsed and the ground model reals remain an unbounded family. Furthermore if the forcing notion adds a real which is not split by the ground model reals, such an iteration would give the consistency of $\mathfrak{b} = \omega_1 < \mathfrak{s} = \omega_2$. Thus it is sufficient to obtain the following theorem.

THEOREM 1.3.4 (Shelah, [**31**]). *Assume CH. There is a proper, almost ${}^\omega\omega$-bounding forcing notion Q' of size $\aleph_1$ such that in every*

(V, Q')-generic extension there is an infinite subset of ω which is not split by the ground model reals.

For completeness we give the definition of Shelah's partial order Q'. The partial order (defined in section 2.2) differs slightly from the forcing notion given below.

DEFINITION 1.3.5 (Shelah). Let Q' be the set of all pairs (u, T) where u is a finite subset of ω and $T = \langle (s_i, h_i) : i \in \omega \rangle$ is a sequence of finite logarithmic measures such that

(1) $\max u < \min s_0$
(2) $\max s_i < \min s_{i+1}$
(3) $h_i(s_i) < h_{i+1}(s_{i+1})$.

Also $\mathrm{int}(T) = \cup\{s_i : i \in \omega\}$ denotes the underlying subset of ω.

We say that (u_1, T_1) is *extended by* (u_2, T_2) where $T_\ell = \langle t_i^\ell : i \in \omega \rangle$ for $\ell = 1, 2$, $t_i^\ell = (s_i^\ell, h_i^\ell)$ and denote this by $(u_2, T_2) \leq (u_1, T_1)$ if the following conditions hold:

(1) u_2 is an end-extension of u_1 and $u_2 \backslash u_1 \subseteq \mathrm{int}(T_1)$
(2) $\mathrm{int}(T_2) \subseteq \mathrm{int}(T_1)$ and there is a sequence $\langle B_i : i \in \omega \rangle$ of finite subsets of ω such that $\max u_2 < \min s_j^1$ for $j = \min B_0$, $\max B_i < \min B_{i+1}$ and $s_i^2 \subseteq \cup\{s_j^1 : j \in B_i\}$
(3) for every $e \subseteq s_i^2$ such that $h_i^2(e) > 0$ there is $j \in B_i$ such that $h_j^1(e \cap s_j^1) > 0$.

Whenever $(u, T) \in Q'$ the finite set u is called the *stem* of the condition and T the *pure part*. Conditions with empty stem are called *pure* *conditions* and are often denoted by their pure part. If q extends p, and q has the same stem as p, then q is called a *pure extension* of p.

Observe that if (u, T) is a condition in Q', then $(u, \mathrm{int}(T))$ is a condition in the Mathias forcing notion. In fact the reason that Q' adds a real not split by the ground model reals is the same as for Mathias forcing. To see that Q' adds a real not split by the ground model reals, note that if $T \in Q'$ is a pure condition and A is an infinite subset of ω, then there is a condition $T' \in Q'$ extending T such that $\mathrm{int}(T') \subseteq A$ or $\mathrm{int}(T') \subseteq A^c$. But then for every ground model infinite subset A of ω the set

$$D_A = \{(u, T) \in Q' : \mathrm{int}(T) \subseteq A \text{ or } \mathrm{int}(T) \subseteq A^c\}$$

is dense in Q' and so the real

$$U_G = \bigcup \{u : \exists T (u, T) \in G\}$$

where G is Q'-generic filter is not split by the ground model reals. Note that if $\langle \mathbb{P}_\alpha : \alpha \leq \delta \rangle$ is a countable support iteration of proper forcing notions, where δ is of uncountable cofinality, then any new real is obtained at some initial stage $\delta_0 < \delta$ of the iteration. Furthermore if $\langle \mathbb{P}_\alpha : \alpha \leq \omega_2 \rangle$ is a countable support iteration of proper forcing notions, then any set of reals of cardinality ω_1 is added at some (proper) initial stage of the iteration. Therefore, assuming Q' is proper, an iteration of Q' over a model of CH of length ω_2, would result in a model of $\mathfrak{s} = \omega_2$.

DEFINITION 1.3.6 (Baumgartner, [**5**]). A forcing notion $\mathbb{P} = (\mathbb{P}, \leq)$ is said to satisfy Axiom A, if the following holds:

(1) There is a sequence of partial orders $\{\leq_n\}_{n\in\omega}$ on $\mathbb{P}$, where $\leq_0 = \leq$, such that $\leq_n \subseteq \leq_m$ for every $m \leq n$. That is, whenever $m \leq n$ and p, q are conditions in $\mathbb{P}$ such that $p \leq_n q$, then $p \leq_m q$.

(2) If $\{p_n\}_{n\in\omega}$ is a sequence of conditions in $\mathbb{P}$ such that $p_{n+1} \leq_{n+1} p_n$ for every n, then there is a condition p such that $p \leq_{n+1} p_n$ for every n. The sequence $\{p_n\}_{n\in\omega}$ is called a *fusion sequence* and p is called *the fusion* of the sequence.

(3) For every $D \subseteq \mathbb{P}$ which is dense, and every condition p, for every $n \in \omega$ there is a condition p' such that $p' \leq_n p$ and a countable subset D_0 of D which is predense below p'.

The forcing notion Q' satisfies Axiom A and so is proper (see [**5**]). To see that indeed Q' satisfies Axiom A, define a sequence of suborders $\{\leq_i\}_{i\in\omega}$ of $\leq$ as follows. Let (u_ℓ, T_ℓ) where $T_\ell = \langle t_i^\ell : i \in \omega\rangle$ for $\ell = 1, 2$ be conditions in Q'. Define

$$(u_2, T_2) \leq_1 (u_1, T_1)$$

if $u_1 = u_2$ and $(u_2, T_2) \leq_0 (u_1, T_1)$ where $\leq_0 = \leq$ is the partial order given in Definition 1.3.5. For $i \geq 1$ let

$$(u_2, T_2) \leq_{i+1} (u_1, T_1)$$

if $u_1 = u_2$ and for every $j \in i$ $t_j^1 = t_j^2$. Then $\{\leq_i\}_{i\in\omega}$ is a decreasing sequence of partial orders on Q. Furthermore if $\{p_n\}_{n\in\omega} = \{(u, T_n)\}_{n\in\omega}$, where $T_n = \langle t^n_j : j \in \omega \rangle$ is a fusion sequence, then the condition $p = (u, T)$ where $T = \langle t_j : j \in \omega \rangle$ and for every $j \in \omega$, $t_j = t^{j+1}_j$ is a fusion of the given sequence.

In order to establish part (3) of Axiom A as well as the almost ${}^\omega\omega$-bounding property, one needs the notion of preprocessed conditions (see [**6**], Section 8). Note that in Section 2.6 we work with a slight modification of this notion.

Definition 1.3.7. Suppose $D \subseteq Q'$ is a dense open set. We say that $p = (u, T)$ where $T = \langle t_i : i \in \omega \rangle$ is preprocessed for D and $k \in \omega$ if for every subset v of k which end-extends u, $(v, \langle t_j : j \geq k \rangle)$ has a pure extension in D if and only if $(v, \langle t_j : j \geq k \rangle)$ belongs to D.

The following three Lemmas show that, whenever D is a dense open set and $p \in Q'$ there is a pure extension q of p such that for every $i \in \omega$, q is preprocessed for D and i.

Lemma 1.3.8. *Let D be a dense open subset of Q' and $k \in \omega$. If (u, T) is preprocessed for D and k, then any extension of (u, T) is also preprocessed for D and k.*

Proof. Consider any extension (w, R) of (u, T) where $R = \langle r_i : i \in \omega \rangle$. Let v be a subset of k, which end-extends w and such that $(v, \langle r_j : j \geq k \rangle)$ has a pure extension in D. Since R extends T, by definition of the extension relation on Q $\langle r_j : j \geq k \rangle)$ is an extension of $\langle t_j : j \geq k \rangle$. Therefore $(v, \langle t_j : j \geq k \rangle)$ has a pure extension in D and since (u, T) is preprocessed for D and k, $(v, \langle t_j : j \geq k \rangle)$ belongs to D (note that v also end-extends u). But D is open and since $(v, \langle r_j : j \geq k\rangle)$ extends $(v, \langle t_j : j \geq k\rangle)$, $(v, \langle r_j : j \geq k\rangle)$ also belongs to D. $\square$

LEMMA 1.3.9. *Let $(u, T) \in Q'$, $k \in \omega$. Then (u, T) has a $\leq_{k+1}$ extension which is preprocessed for D and k.*

PROOF. Let $T = \langle t_i : i \in \omega\rangle$. Fix an enumeration $v_1, \ldots, v_j$ of all subsets of k which end-extend u. Consider $(v_1, \langle t_i : i \geq k\rangle)$. If $(v_1, \langle t_i : i \geq k\rangle)$ has a pure extension in D, denote it $(v_1, \langle t_i^1 : i \geq k\rangle)$. If there is no such pure extension, let $t_i^1 = t_i$ for every $i \geq k$. In the next step consider $(v_2, \langle t_i^1 : i \geq k\rangle)$. If it has a pure extension in D, denote it $(v_2, \langle t_i^2 : i \geq k\rangle$. If there is no such pure extension, then for every $i \geq k$ let $t_i^2 = t_i^1$. At the j-th step we will obtain condition $(v_j, \langle t_i^j : i \geq k\rangle)$. Then $(u, \langle t_i^j : i \in \omega\rangle)$ where for every $i < k$, $t_i^j = t_i$ is a $\leq_{k+1}$-extension of (u, T) which is preprocessed for D and k.

To see this, suppose $(v, \langle t_i^j : i \geq k\rangle)$ has a pure extension in D where $v \subseteq k$, v end-extends u. Then $v = v_m$ for some m, $1 \leq m \leq j$. Then at step m, we must have had that $(v_m, \langle t_i^{m-1} : i \geq k\rangle$ has a pure extension in D, and so we have fixed such a pure extension $(v_m, \langle t_i^m : i \geq k\rangle) \in D$. However since $m - 1 < j$, we have $\langle t_i^j : i \geq k\rangle \leq \langle t_i^m : i \geq k\rangle$. But D is open and so $(v, \langle t_i^j : i \geq k\rangle)$ is an element of D itself. $\square$

LEMMA 1.3.10. *Let D be a dense open set. Then any condition has a pure extension which is preprocessed for D and every $i \in \omega$.*

PROOF. Let $p = (u, T)$ be an arbitrary condition. By Lemma 1.3.9 there is a fusion sequence $\{p_i\}_{i\in\omega}$ such that $p_0 = p$, $p_{i+1} \leq_{i+1} p_i$ and p_{i+1} is preprocessed for D and i. Let q be the fusion of the sequence.

Then for every $i \in \omega$ we have that $q \leq_{i+1} p_{i+1}$ and so in particular $q \leq p_{i+1}$. Therefore by Lemma 1.3.8 q is preprocessed for D and i. $\square$

Observe that q is obtained as the fusion of a sequence. This fact will appear very important in obtaining the almost-${}^{\omega}\omega$ bounding property, and in particular Lemma 1.3.11. With this we are ready to show that the forcing notion Q satisfies Axiom A, part (3). Let D be a dense open set and p an arbitrary condition. By Lemma 1.3.10 there is a pure extension $q = (u, T)$ for $T = \langle t_j : j \in \omega \rangle$ which is preprocessed for D and every $i \in \omega$. Since q is obtained as a fusion of a fusion sequence below p, for every $n \in \omega$, $q \leq_n p$. Furthermore the set

$$D_0 = \{(v, \langle t_j : j \geq i \rangle) \in D : v \subseteq i, i \in \omega, v \text{ end-extends } u\}$$

is a countable subset of D which is pre-dense below q. To see this consider an arbitrary extension (v, R) of q. Since D is dense, (v, R) has an extension $(v \cup w, R')$ in D. Note that $(v \cup w, R')$ is a pure extension of $(v \cup w, \langle t_j : j \geq k \rangle)$ for some $k \in \omega$ such that $w \subseteq k$. However q is preprocessed for D and k, and so $(v \cup w, \langle t_j : j \geq k \rangle) \in D$. Thus in particular $(v \cup w, \langle t_j : j \geq k \rangle)$ belongs to D_0 and is compatible with (v, R). This establishes axiom A and so properness. The main technical tool in obtaining the almost ${}^{\omega}\omega$-bounding property is the following Lemma - compare with section 3.3.

LEMMA 1.3.11. *Let $\dot{f}$ be a Q'-name for a function in ${}^{\omega}\omega$ and let $p = (u, T)$ be an arbitrary condition in Q'. Then there is a pure extension (u, R) of p where $R = \langle r_i : i \in \omega \rangle$, $r_i = (x_i, g_i)$ such that $\forall i \in \omega$, $\forall v \subseteq i$*

which end-extend u *and* $\forall s \subseteq x_i$ *such that* $g_i(s) > 0$ *there is* $w_v \subseteq s$ *such that* $(v \cup w_v, \langle r_j : j \geq i+1 \rangle) \Vdash \dot{f}(i) = \check{k}$ *for some* $k \in \omega$.

In order to obtain the pure condition R of the above Lemma, one has to consider logarithmic measures induced by positive sets (see Definition 2.1.4) and in particular to show that the logarithmic measure induced by the family $\mathcal{P}_k(T, D)$ where $T = \langle t_\ell : \ell \in \omega \rangle$ is a pure condition preprocessed for a given dense open set D and $k \in \omega$ consisting of all finite subsets x of $\mathrm{int}(T)$ such that for some $\ell \in \omega$, $x \cap \mathrm{int}(t_\ell)$ is positive and $\forall v \subseteq k \exists w \subseteq x$ such that $(v \cup w, T) \in D$, takes arbitrarily high values - compare with section 3.1. Because of the analogy with Theorem 3.3.2 we give a proof of the almost ${}^\omega\omega$-bounding property of Q' - see [**1**].

Theorem 1.3.12. *The forcing notion* Q' *is almost* ${}^\omega\omega$*-bounding.*

Proof. Let $\dot{f}$ be arbitrary Q-name of a function and p a condition in Q. Let $q = (u, T)$, where $T = \langle t_i : i \in \omega \rangle$ and $t_i = (x_i, g_i)$, be a pure extension of p which satisfies Lemma 1.3.11. Then for every $i \in \omega$ let $g(i)$ be the maximal k such that there are $v \subseteq i$ and $w \subseteq \mathrm{int}(t_i)$ such that $(v \cup w, \langle t_j : j \geq i+1 \rangle) \Vdash \dot{f}(i) = \check{k}$. Consider any $A \in [\omega]^\omega$ and let $q_A = (u, \langle t_i : i \in A \rangle)$. We claim that $q_A \Vdash \exists^\infty k \in A(\dot{f}(k) \leq g(k))$.

Let $n \in \omega$ and let (v, R) be an arbitrary extension of q_A. Then by definition of the extension relation there is $i \in A$ such that $i_0 > n$, $v \subseteq i$ and $s = \mathrm{int}(R) \cap \mathrm{int}(t_i)$ is such that $g_i(s) > 0$. But then by Lemma 1.3.11 there is $w \subseteq s$ such that $(v \cup w, \langle t_j : j \geq i_0 + 1 \rangle) \Vdash \dot{f}(i) = \check{k}$ and so $(v \cup w, \langle t_j : j \geq i+1 \rangle) \Vdash \dot{f}(i) \leq g(i)$. However

$(v \cup w, R)$ extends $(v \cup w, \langle t_j : j \geq i+1 \rangle)$ and so $(v \cup w, R) \Vdash \dot{f}(i) \leq g(i)$. Note also that $(v \cup w, R)$ extends (v, R). Then, since (v, R) was an arbitrary extension of q_A, the set of conditions which force "$\exists k \in A(k > n \wedge \dot{f}(k) \leq g(k))$" is dense below q_A. Since n was arbitrary as well, we obtain $q_A \Vdash \exists^\infty k \in A(\dot{f}(k) \leq g(k))$. $\square$

CHAPTER 2

Centered Families of Pure Conditions

2.1. Logarithmic Measures

The notion of logarithmic measure is due to S. Shelah. In our presentation of logarithmic measures and their basic properties (Definitions 2.1.1, 2.1.4 and Lemmas 2.1.3, 2.1.7, 2.1.9, 2.1.10) we follow [**1**].

Definition 2.1.1 (Shelah). Let s be a subset of ω and $h : [s]^{<\omega} \to \omega$, where $[s]^{<\omega}$ is the family of all finite subsets of s. The function h is called *a logarithmic measure*, if for every $A \in [s]^{<\omega}$ and for every A_0, A_1 such that $A = A_0 \cup A_1$, $h(A_i) \geq h(A) - 1$ for $i = 0$ or $i = 1$ unless $h(A) = 0$. Whenever s is a finite set and h a logarithmic measure on s, the pair $x = (s, h)$ is called *a finite logarithmic measure*. The value $h(s) = \|x\|$ is called *the level of* x.

Definition 2.1.2. Whenever h is a finite logarithmic measure on x and $e \subseteq x$ is such that $h(e) > 0$, we will say that e is *h-positive*.

Lemma 2.1.3 (Shelah). *If h is a logarithmic measure and $h(A_0 \cup \cdots \cup A_{n-1}) \geq \ell + 1$ then $h(A_j) \geq \ell - j$ for some j, $0 \leq j \leq n - 1$.*

Definition 2.1.4 (Shelah). Let $P \subseteq [\omega]^{<\omega}$ be an upwards closed family. Then P induces a logarithmic measure h on $[\omega]^{<\omega}$ defined inductively on $|s|$ for $s \in [\omega]^{<\omega}$ in the following way:

(1) $h(e) \geq 0$ for every $e \in [\omega]^{<\omega}$

(2) $h(e) > 0$ iff $e \in P$

(3) for $\ell \geq 1$, $h(e) \geq \ell+1$ iff $e \in P$, $|e| > 1$ and whenever $e_0, e_1 \subseteq e$ are such that $e = e_0 \cup e_1$, then $h(e_0) \geq \ell$ or $h(e_1) \geq \ell$.

Then $h(e) = \ell$ iff ℓ is the maximal natural number for which $h(e) \geq \ell$. The elements of P are called *positive sets* and h is said to be *induced by the positive sets* P.

Definition 2.1.5. Let h be an induced logarithmic measure. Then h is said to be atomic, if there is a singleton $\{n\}$ such that $h(\{n\}) > 0$.

Remark 2.1.6. From now on we assume that all logarithmic measures are non-atomic.

Lemma 2.1.7 (Shelah). *If h is a logarithmic measure induced by positive sets and $h(e) \geq \ell$, then for every a such that $e \subseteq a$, $h(a) \geq \ell$.*

Example 2.1.8 (Shelah, [**1**] or [**32**]). *Let P be the family of all sets containing at least two points and h the logarithmic measure induced by P on $[\omega]^{<\omega}$. Then for every $x \in P$, $h(x) = i$ where i is the minimal natural number such that $|x| \leq 2^i$. This logarithmic measure is also called a standard measure.*

An easy application of König's Lemma gives the following:

Lemma 2.1.9 (Abraham, [**1**]). *Let P be an upwards closed family of finite non-empty subsets of ω and h the induced logarithmic measure. Let $\ell \geq 1$. Then for every subset A of ω, if A does not contain a set* *of measure* $\geq \ell + 1$*, then there are* A_0, A_1 *such that* $A = A_0 \cup A_1$ *and neither of* A_0*,* A_1 *contains a set of measure greater or equal* ℓ*.*

Proof. If A is a finite set, then the given statement is the contrapositive of part 3 of Definition 2.1.4. Thus assume A is infinite. For every natural number k, let $A_k = A \cap k$ and let T be the family of all functions $f : m \to \bigcup_{0 \leq k \leq m} \mathcal{P}(A_k) \times \mathcal{P}(A_k)$, where $m \in \omega$, such that for every k,

$$f(k) = (a_0^k, a_1^k) \in \mathcal{P}(A_k) \times \mathcal{P}(A_k)$$

where $a_0^k \cup a_1^k = A_k$, $h(a_0^k) \not\geq \ell$, $h(a_1^k) \not\geq \ell$ and for every $k : 1 \leq k \leq m$, $a_0^{k-1} \subseteq a_0^k$, $a_1^{k-1} \subseteq a_1^k$.

Then T together with the end-extension relation is a tree. Furthermore for every $m \in \omega$, the m-th level of T is nonempty. Really consider an arbitrary natural number m. Then $A \cap m = A_m$ is a finite set which is not of measure greater or equal $\ell + 1$. By Definition 2.1.4, part (3), there are sets a_0^m, a_1^m such that $A_m = a_0^m \cup a_1^m$ and $h(a_0^m) \not\geq \ell$, $h(a_1^m) \not\geq \ell$. Let $a_0^{m-1} = A_{m-1} \cap a_0^m$ and $a_1^{m-1} = A_{m-1} \cap a_1^m$. Then by Lemma 2.1.7 the measure of each of a_0^{m-1}, a_1^{m-1} is not greater or equal to ℓ and $A_{m-1} = A \cap (m-1) = a_0^{m-1} \cup a_1^{m-1}$. Therefore in m steps we can define finite sequences $\langle a_0^k : 0 \leq k \leq m \rangle$, $\langle a_1^k : 0 \leq k \leq m \rangle$ such that for every k, $A_k = a_0^k \cup a_1^k$, $h(a_0^k) \not\geq \ell$, $h(a_1^k) \not\geq \ell$ and $\forall k : 0 \leq k \leq m - 1$ $a_0^k \subseteq a_0^{k+1}$, $a_1^k \subseteq a_1^{k+1}$. Therefore $f : m \to \bigcup_{0 \leq k \leq m} \mathcal{P}(A_k) \times \mathcal{P}(A_k)$ defined by $f(k) = (a_0^k, a_1^k)$ is a function in the m'th level of T.

Therefore by König's Lemma there is an infinite branch through T. Let $f : \omega \to \bigcup_{k \in \omega} \mathcal{P}(A_k) \times \mathcal{P}(A_k)$ where $f(k) = (a_0^k, a_1^k)$, $a_0^k \cup a_1^k = A_k$, etc., be such an infinite branch. Then if $B_0 = \bigcup_{k \in \omega} a_0^k$, $B_1 = \bigcup_{k \in \omega} a_1^k$

we have that $A = B_0 \cup B_1$ and none of the sets B_0, B_1 contains a set of measure greater or equal ℓ. Consider an arbitrary finite subset x of B_0. Then $x \subseteq a_0^k$ for some $k \in \omega$. But $h(a_0^k) \ngeq \ell$ and so $h(x) \ngeq \ell$. The same argument applies to B_1. □

Lemma 2.1.10 (Abraham, [**1**]). *(Sufficient Condition for High Values) Let P be an upwards closed family of finite subsets of ω and h the logarithmic measure induced by P. Then if for every $n \in \omega$ and every partition of ω into n sets $\omega = A_0 \cup \cdots \cup A_{n-1}$ there is some $j \leq n-1$ such that A_j contains a positive set x (such that $|x| \geq 2$), then for every natural number k, for every $n \in \omega$ and partition of ω into n sets $\omega = A_0 \cup \cdots \cup A_{n-1}$ there is some $j \leq n-1$ such that A_j contains a set of measure greater or equal k.*

Remark. Note that if the measure of x is ≥ 2, then $|x| \geq 2$. However for non-atomic measures, $\|x\| > 0$ implies that $|x| \geq 2$.

Proof. The proof proceeds by induction on k. If $k = 1$ this is just the assumption of the Lemma. So suppose we have proved the claim for $k = \ell$ and furthermore that it is false for $k = \ell + 1$. Then there is some $n \in \omega$ and partition of ω into n sets $\omega = A_0 \cup \cdots \cup A_{n-1}$ such that none of $A_0, \ldots, A_{n-1}$ contains a set of measure greater or equal $\ell + 1$. By Lemma 2.1.9 for each $j \in n - s$ there are sets A_j^0, A_j^1 none of which contains a set of measure greater or equal ℓ and such that $A_j = A_j^0 \cup A_j^1$. Then $\omega = A_0^0 \cup A_0^1 \cup \cdots \cup A_{n-1}^0 \cup A_{n-1}^1$ is a finite partition of ω, none of the elements of which contains a set of measure $\geq \ell$. This contradicts the inductive hypothesis for $k = \ell$. □

2.2. Centered Families of Pure Conditions

DEFINITION 2.2.1 (Shelah). Let Q be the set of all pairs (u, T) where u is a finite subset of ω and $T = \langle (s_i, h_i) : i \in \omega \rangle$ is a sequence of finite logarithmic measures such that

(1) $\max u < \min s_0$
(2) $\max s_i < \min s_{i+1}$ for all $i \in \omega$
(3) $\langle h_i(s_i) : i \in \omega \rangle$ is unbounded.

The finite part u is called *the stem* of the condition $p = (u, T)$, and T *the pure part* of p. Also $\text{int}(T) = \cup\{s_i : s \in \omega\}$. In case that $u = \emptyset$ we say that $(\emptyset, T)$ is *a pure condition* and usually denote it simply by T.

We say that (u_1, T_1) is extended by (u_2, T_2), where $T_\ell = \langle t^\ell_i : i \in \omega \rangle$ and $t^\ell_i = (s^\ell_i, h^\ell_s)$ for $\ell = 1, 2$, and denote it by $(u_2, T_2) \leq (u_1, T_1)$ if the following conditions hold:

(1) u_2 is an end-extension of u_1 and $u_2 \backslash u_1 \subseteq \text{int}(T_1)$
(2) $\text{int}(T_2) \subseteq \text{int}(T_1)$ and furthermore there is an infinite sequence $\langle B_i : i \in \omega \rangle$ of finite subsets of ω such that $\max u_2 < \min s^1_j$ for $j = \min B_0$, $\max(B_i) < \min(B_{i+1})$ and $s^2_i \subseteq \cup\{s^1_j : j \in B_i\}$.
(3) for every subset e of s^2_i such that $h^2_i(e) > 0$ there is $j \in B_i$ such that $h^1_j(e \cap s^1_j) > 0$.

If $u_1 = u_2$, then (u_2, T_2) is called a *pure extension* of (u_1, T_1).

The partial order Q' in Definition 1.3.5 differs with Q, in the requirement that the sequence $\langle h_i(s_i) : i \in \omega \rangle$ is strictly increasing rather then simply unbounded. However from every unbounded sequence one can choose a strictly increasing subsequence and so the partial order Q' (see Definition 1.3.5) is dense in Q.

DEFINITION 2.2.2. Let $T = \langle t_i : i \in \omega \rangle$ be a pure condition. Then for every $k \in \omega$, let $i_T(k) = \min\{i : k < \min \operatorname{int}(t_i)\}$ and let $T \backslash k = T_{i_T(k)} = \langle t_i : i \geq i_T(k) \rangle$. Whenever u is a finite subset of ω let $T \backslash u = T_{i_T(\max u)}$ and $(u, T) = (u, T \backslash u)$.

DEFINITION 2.2.3. Let $\mathcal{F}$ be a family of pure conditions. Then $Q(\mathcal{F})$ is the suborder of Q of all $(u, T) \in Q$ such that $\exists R \in \mathcal{F}(R \leq T)$.

DEFINITION 2.2.4. A family of pure conditions C is *centered* if whenever $X, Y \in C$ there is $R \in C$ which is their common extension.

We will be interested in centered families C of pure conditions and the associated partial order $Q(C)$.

LEMMA 2.2.5. *Let C be a centered family of pure conditions. Then $Q(C)$ is σ-centered.*

PROOF. Any two conditions in $Q(C)$ with equal stems have a common extension in $Q(C)$. $\square$

From now on by *centered family* we mean a centered family of pure conditions, unless otherwise specified. Furthermore we assume that all centered families are closed with respect to final sequences, that is if C is a centered family and $T \in C$ then $T \backslash v \in C$ for every $v \in [\omega]^{<\omega}$. Note that $Q(C)$ is the upwards closure of $\{(u, T) : T \in C\}$.

LEMMA 2.2.6. *Any two conditions of $Q(C)$ are compatible as conditions in $Q(C)$ if and only if they are compatible in Q.*

PROOF. Let $p = (u, T)$ and $q = (v, R)$. Suppose that p, q are compatible as conditions in Q. Let (w, Z) be their common extension in Q. Then in particular w is a common end-extension of u and v, which implies that u is an end-extension of v or v is an end-extension of u. Say u is an end-extension of v. Then $u\backslash v \subseteq w\backslash v \subset \text{int}(R)$. Since p and q belong to $Q(C)$, by definition there are pure conditions T', R' in C such that $T' \leq T$ and $R' \leq R$. However C is centered, and so there is a pure condition $Z' \in C$ which is a common extension of T' and R' and so a common extension of T and R. But then (u, Z') is a common extension of p and q from $Q(C)$. $\square$

A pure condition, which is compatible with every element of a centered family, is said to be *compatible* with the centered family. If C' is a centered family which contains a centered family C in its downwards closure, i.e. $C \subseteq Q(C')$, then C' is said to *extend* C. In particular if $C \subseteq Q(C')$ and there is $R \in Q(C')$ such that $\forall X \in C'(X \leq R)$ we say that C' *extends* C *below* R.

2.3. Partitioning of Pure Conditions

LEMMA 2.3.1. *Let (x, h) be a finite logarithmic measure and $h(x) \leq n$. Then $x = \cup\{x_i : i \in 2^n\}$ where for every $i \in 2^n$, $h(x_i) = 0$.*

PROOF. We give a proof by induction on n. Let $n = 1$. Then by definition of logarithmic measure there are sets x_0, x_1 such that $x = x_0 \cup x_1$ and $h(x_0) \not\geq 1$ and $h(x_1) \not\geq 1$, that is x_0 and x_1 are not positive. Suppose we have proved the claim for every measure of level $\leq n$, where $n \geq 2$. Let (x, h) be a logarithmic measure of level $\leq n+1$.

Then, there are sets x_0, x_1 such that $x = x_0 \cup x_1$ and $h(x_0) \leq n$, $h(x_1) \leq n$. By inductive hypothesis for $\ell \in 2$ $x_\ell = \cup\{x_\ell^i : i \in 2^n\}$ where for all $i \in 2^n$ $h(x_\ell^i) = 0$. But then it is clear, that x can be presented as the union of 2×2^n sets, each of which is of measure 0. $\square$

LEMMA 2.3.2. *Let $T = \langle (s_i, h_i) : i \in \omega \rangle$ be a pure condition and let A be an infinite subset of ω. If the sequence $\langle h_i(s_i \cap A) : i \in \omega \rangle$ is bounded, then T has no pure extension R with int$(R) \subseteq A$.*

PROOF. Suppose to the contrary that R is a pure condition in Q extending T such that $\text{int}(R) \subseteq A$. Then there is $\langle B_i : i \in \omega \rangle \subseteq [\omega]^{<\omega}$ such that $\forall i \in \omega$ $x_i \subseteq \cup\{s_j : j \in B_i\}$ where $R = \langle (x_i, g_i) : i \in \omega \rangle$. Since $\text{int}(R) \subseteq A$ we have $x_i = x_i \cap A \subseteq \cup\{s_j \cap A : j \in B_i\}$. Let $M \in \omega$ be such that $h_i(s_i \cap A) \leq M$ for every $i \in \omega$. Since, R is a pure condition, the sequence $\langle g_i(x_i) : i \in \omega \rangle$ is unbounded, and so there is $\ell \in \omega$ for which $g_\ell(x_\ell) \geq 2^M + 1$. For simplicity denote (x_ℓ, g_ℓ) by (x, g). By definition $x \subseteq \cup\{s_j \cap A : j \in B_\ell\}$. However for each $j \in B_\ell$, $h_j(s_j \cap A) \leq M$ and so $\forall j \in B_\ell$ there is a family of sets $\{s_j^m : m \in 2^M\}$ such that $s_j \cap A = \cup\{s_j^m : m \in 2^M\}$ and for every $m \in 2^M$, $h_j(s_j^m) = 0$. Then for every $m \in 2^M$ let $a_m = x \cap (\cup\{s_j^m : j \in B_\ell\})$. Then $x = \cup\{a_m : m \in 2^M\}$ and so by Lemma 2.1.3 there is $m \in 2^M$ such that $g(a_m) \geq (2^M + 1) - m \geq 1$. But then $\exists j \in B_\ell$ such that $h_j(a_m \cap s_j) > 0$. However $s_j^m = a_m \cap s_j$ and so $h_j(s_j^m) > 0$ which is a contradiction. $\square$

REMARK 2.3.3. It is essential to work with non-atomic measures. If $P = [\omega]^{<\omega}$ and h is the induced logarithmic measure, then $T =$

$\langle(\{2n\}, h \restriction \{2n\}) : n \in \omega\rangle$ is a sequence of finite logarithmic measures of measure 1, however amalgamating successive measures of T one can obtain a pure condition, and so in particular an unbounded sequence of finite logarithmic measures.

Definition 2.3.4. Whenever $T = \langle(s_i, h_i) : i \in \omega\rangle$ be a pure condition and $A \subseteq \omega$, let $T \restriction A = \langle(s_i \cap A, h_i \restriction \mathcal{P}(s_i \cap A)) : i \in \omega\rangle$.

Lemma 2.3.5. *Let $T = \langle t_i : i \in \omega\rangle$, where $t_i = (s_i, h_i)$, be a pure condition and $\omega = A_0 \cup \cdots \cup A_{n-1}$ a finite partition of ω. Then there is $j \in n$ such that $T \restriction A_j$ is a pure condition.*

Proof. Suppose not. That is, for every $j \in n$ there is $M_j \in \omega$ such that $h_i(s_i \cap A_j) \leq M_j$ for every $i \in \omega$. Let $M = \max_{j\in n} M_j$ and let $t_i = (s_i, h_i)$ be a measure from T with $h_i(s_i) \geq M + (n + 1)$. Let $s_i^j = s_i \cap A_j$ for every $j \in n$. Then $s_i = s_i^0 \cup s_i^1 \cup \cdots \cup s_i^{n-1}$ is a partition of s_i into n sets and so there is $j \in n$ such that $h_i(s_i^j) \geq h_i(s_i) - j = M + (n + 1) - j \geq M + 1 > M_j$ which is a contradiction. $\square$

Lemma 2.3.6. *Let R be a pure extension of T and let A be an infinite subset of ω, such that $R \restriction A$ and $T \restriction A$ are pure conditions. Then $R \restriction A$ is a pure extension of $T \restriction A$.*

Proof. Let $R = \langle(x_i, g_i) : i \in \omega\rangle$, $T = \langle(s_i, h_i) : i \in \omega\rangle$. Since R is a pure extension of T, there is a sequence $\langle B_i : i \in \omega\rangle \subseteq [\omega]^{<\omega}$ such that $\forall i \in \omega$, $x_i \subseteq \cup\{s_j : j \in B_i\}$. Note that for every $i \in \omega$, $x_i \cap A \subseteq \cup\{s_j \cap A : j \in B_i\}$ and furthermore if $e \subseteq x_i \cap A$ is such that $h_i(e) > 0$, by definition of the extension relation there is $j \in B_i$ such that $g_j(e \cap s_j) > 0$. It remains to observe that $e \cap s_j = e \cap s_j \cap A$. Thus $R \restriction A$ is a pure extension of $T \restriction A$. $\square$

LEMMA 2.3.7. *Let C be a centered family, T a pure condition compatible with C and $\omega = A_0 \cup \cdots \cup A_{n-1}$ a finite partition of ω. Then there is $j \in n$ such that $T \restriction A_j$ is a pure condition compatible with C.*

PROOF. Suppose the claim is not true and let $I \subseteq n$ be the set of all indexes $j \in n$ for which $T \restriction A_j$ is a pure condition in Q. By Lemma 2.3.5, $I \neq \emptyset$. By hypothesis, $\forall j \in I$ there is $T_j \in C$ such that T_j is incompatible with $T \restriction A_j$. However I is finite, C is centered and so there is $X \in C$ which is a common extension of $\langle T_j : j \in I \rangle$. By assumption X and T have a common extension $R \in Q$. Again by Lemma 2.3.5 there is an $i \in n$ such that $R \restriction A_i$ is a pure condition. Furthermore by Lemma 2.3.6 $R \restriction A_i \leq T \restriction A_i$ and so by Lemma 2.3.2 $i \in I$. Also $R \restriction A_i \leq R \leq X \leq T_i$ and so T_i and $T \restriction A_i$ are compatible which is a contradiction. $\square$

2.4. Good Names for Reals

REMARK 2.4.1. We will use the fact that whenever $f \in V^{\mathbb{P}} \cap {}^\omega\omega$ for some forcing notion $\mathbb{P}$, then f has a $\mathbb{P}$-name of the form $\dot{f} = \cup\{\langle\langle i, j^i_p\rangle, p\rangle : p \in \mathcal{A}_i, i \in \omega, j^i_p \in \omega\}$ where for every $i \in \omega$, $\mathcal{A}_i = \mathcal{A}_i(\dot{f})$ is a maximal antichain.

DEFINITION 2.4.2. Let C be a centered family of pure conditions and let $\dot{f}$ be a $Q(C)$-name for a real. Then $\dot{f}$ is a *good name* if for every centered family C' extending C, $\dot{f}$ is a $Q(C')$-name for a real.

REMARK 2.4.3. That is $\dot{f}$ is a good $Q(C)$-name for a real if and only if for every centered family C' extending C and for every $i \in \omega$, $\mathcal{A}_i(\dot{f})$ remains a maximal antichain in $Q(C')$.

LEMMA 2.4.4. *Let C be a centered family of pure conditions and let $\dot{f}$ be a $Q(C)$-name for a real. Then the following are equivalent:*

(1) $\dot{f}$ is a good $Q(C)$-name for a real.

(2) For every centered family C' extending C such that $|C'| = |C|$, $\dot{f}$ is a $Q(C')$-name for a real.

PROOF. The implication from (1) to (2) is straightforward. To obtain that (2) implies (1) consider any centered family C' extending C such that $|C'| > |C|$ and suppose that $\dot{f}$ is not a $Q(C')$-name. Then there is condition $p = (u, T) \in Q(C')$ which is incompatible with all elements of $\mathcal{A}_i = \mathcal{A}_i(\dot{f})$ for some $i \in \omega$. Note that $C \cup \{T\} \subseteq Q(C')$. Inductively we will construct a centered family C'' contained in $Q(C')$ such that $C \cup \{T\} \subseteq C''$ and $|C''| = |C|$. Let $C_0 = C \cup \{T\}$. Since $C_0 \subseteq Q(C')$ for all $X, Y \in C_0$ there is $Z_{X,Y} \in Q(C')$ such that $Z_{X,Y} \leq X$ and $Z_{X,Y} \leq Y$. Let $C'_0 = \{Z_{X,Y} : X, Y \in C_0\}$ and let $C_1 = C_0 \cup C'_0$. Suppose we have defined $C_n = C_{n-1} \cup C'_{n-1} \subseteq Q(C')$ where $n \geq 1$, such that for every $X, Y \in C_{n-1}$ there is $Z \in C_n$ such that $Z \leq X$, $Z \leq Y$ and $|C_{n-1}| = |C_n|$. Then since $C_n \subseteq Q(C')$ for all $X, Y \in C_n$ there is $Z_{X,Y} \in Q(C')$ such that $Z_{X,Y} \leq X$ and $Z_{X,Y} \leq Y$. Then let $C'_n = \{Z_{X,Y} : X, Y \in C_n\}$ and let $C_{n+1} = C_n \cup C'_n$. With this the inductive construction is complete. Then $C'' = \cup_{n \in \omega} C_n$ is a centered family of pure conditions containing $C \cup \{T\}$ and such that $|C''| = |C|$,

$C'' \subseteq Q(C')$. Note that C is infinite, since by assumption C is closed with respect to final subsequences.

By the hypothesis of (2), $\dot{f}$ is a $Q(C'')$-name for a real and so $\mathcal{A}_i(\dot{f})$ is a maximal antichain in $Q(C'')$. Since $C \cup \{T\} \subseteq C''$, $p \in Q(C'')$ and so there is a condition $q \in Q(C'')$ which is a common extension of p and an element q of $\mathcal{A}_i$. But $Q(C'') \subseteq Q(C')$ and so $q \in Q(C')$, which is a contradiction to p being incompatible with all elements of $\mathcal{A}_i$. $\square$

COROLLARY 2.4.5. *Let C be a centered family of pure conditions and let $\dot{f}$ be a $Q(C)$-name for a real. If there is a centered family C' extending C such that $\dot{f}$ is not a $Q(C')$-name for a real, then there is a centered family C'' extending C which has the same cardinality as C' and such that $\dot{f}$ is not a $Q(C')$-name for a real.*

2.5. Generic Extensions of Centered Families

DEFINITION 2.5.1. Let Q_{fin} denote the partial order of all finite sequences of strictly increasing finite logarithmic measures with the end-extension relation. That is, Q_{fin} is the set of all sequences $\bar{r} = \langle r_0, \dots, r_n \rangle$, $n \in \omega$ such that for every $i \leq n$, $r_i = (s_i, h_i)$ is a finite logarithmic measure and for every $i \leq n-1$

$$\max(s_i) < \min(s_{i+1}) \text{ and } h_i(s_i) < h_{i+1}(s_{i+1}).$$

The level of the sequence $\bar{r} = \langle r_0, \dots, r_n \rangle$ is the level of the highest measure r_n, denoted also $\|\bar{r}\|$. Whenever $\bar{r}_1$ and $\bar{r}_2$ are sequences in Q_{fin} define $\bar{r}_1 \leq \bar{r}_2$ if $\bar{r}_2$ is an initial segment of $\bar{r}_1$.

DEFINITION 2.5.2. Let $\bar{r} = \langle r_0, \ldots, r_{n-1} \rangle$ be a sequence in Q_{fin} where for every $i \in n$ $r_i = (s_i, h_i)$. Then $\bar{r}$ *extends* the pure condition $T = \langle t_i : i \in \omega \rangle$, $t_i = (x_i, g_i)$ denoted $\bar{r} \leq T$, if

(1) $\mathrm{int}(\bar{r}) = \cup\{s_i : i \in n\} \subseteq \mathrm{int}(T)$ and there is a sequence $\langle B_0, \ldots, B_{n-1} \rangle$ of finite subsets of ω such that $\max B_i < \min B_{i+1}$ for every $i \in n-1$, and $s_i \subseteq \cup\{x_j : j \in B_i\}$ for all $i \in n$

(2) for every $i \in n$ and $e \subseteq s_i$ such that $h_i(e) > 0$ there is $j \in B_i$ such that $g_j(e \cap x_j) > 0$.

The finite logarithmic measure $r = (s, h)$ *extends* the pure condition $T = \langle t_i : i \in \omega \rangle$, denoted by $r \leq T$, if the sequence $\bar{r} = \langle r \rangle$ extends the pure condition T.

DEFINITION 2.5.3. Let T be a pure condition. Then $\mathbb{P}(T)$ is the suborder of Q_{fin} consisting of all finite sequences $\bar{r}$ extending T.

LEMMA 2.5.4. *Let T be a pure condition. Then*

(1) $\forall k \in \omega$ the set $E_k = \{\bar{r} \in \mathbb{P}(T) : |\bar{r}| \geq k\}$ is dense in $\mathbb{P}(T)$.

(2) For every pure condition X compatible with T and every $n \in \omega$, the set $D_T(X, n) = \{\bar{r} \in \mathbb{P}(T) : \exists r_j \in \bar{r}(r_j \leq X \text{ and } \|r_j\| \geq n)\}$ is dense in $\mathbb{P}(T)$.

PROOF. Let $\bar{r} \in \mathbb{P}(T)$. Since $T \backslash \mathrm{int}(\bar{r})$ and X are compatible, there is a finite logarithmic measure z of level higher than $\|\bar{r}\|$ and n, which is their common extension. Then $\bar{r}^\frown\langle z \rangle$ extends $\bar{r}$ and is in $D_T(X, n)$. $\square$

COROLLARY 2.5.5. *Let C be a centered family of pure conditions, T a pure condition compatible with C and G a $\mathbb{P}(T)$-generic filter. Then* *in $V[G]$ there is a centered family C' extending C below $R_G = \cup G = \langle r_i : i \in \omega\rangle$ (and so below T) which is of the same cardinality as C.*

Proof. By Lemma 2.5.4.1 R_G is a pure condition of strictly increasing finite logarithmic measures. For every $X \in C$, $n \in \omega$ the set $D_T(X, n)$ is dense in $\mathbb{P}(T)$ and so $G \cap D_T(X, n) \neq \emptyset$. Then $I_X = \langle i : r_i \leq X\rangle$ is infinite and so $R_G \wedge X = \langle r_i : i \in I_X\rangle$ is pure condition which is a common extension of R_G and X. Furthermore if $Y \leq X$ then $I_Y \subseteq I_X$ which implies $R_G \wedge Y \leq R_G \wedge X$. Therefore the family $\{R_G \wedge X\}_{X \in C}$ is centered. $\square$

2.6. Preprocessed Conditions

Definition 2.6.1. Let C be a centered family of pure conditions, $\dot{f}$ a good $Q(C)$-name for a real, $k, i \in \omega$ and T a pure condition in $Q(C)$ such that $k < \min \operatorname{int}(T)$. Then T is *preprocessed* for $\dot{f}(i)$, k, C if for every $v \subseteq k$ the following holds:

If there are a centered family C', a pure condition $T' \in Q(C')$ and $q \in \mathcal{A}_i(\dot{f})$ such that C' extends C, $|C'| = |C|$, $T' \leq T$ and $(v, T') \leq q$, then there is $p \in \mathcal{A}_i(\dot{f})$ such that $(v, T) \leq p$.

Lemma 2.6.2. *Let C be a centered family, $\dot{f}$ a good $Q(C)$-name for a real, $i, k \in \omega$, $T \in Q(C)$ a pure condition, preprocessed for $\dot{f}(i)$, k, C. Let C' be a centered family extending C, $|C'| = |C|$ and $T' \in Q(C')$ a pure extension of T. Then T' is preprocessed for $\dot{f}(i)$, k, C'.*

Proof. Let C'' be a centered family extending C', $|C''| = |C'|$ and $T'' \in Q(C'')$ a pure condition extending T' such that for some

$p \in \mathcal{A}_i(\dot{f})$, $(v, T) \leq p$ where $v \subseteq k$. Then C'' extends C, $|C''| = |C|$, $T'' \leq T$ and since T is preprocessed for $\dot{f}(i)$, k, C there is $q \in \mathcal{A}_i(\dot{f})$ such that $(v, T) \leq q$. However $T' \leq T$ and so $(v, T') \leq q$. $\square$

REMARK 2.6.3. In particular, if T is preprocessed for $\dot{f}(i)$, k, C and C' is a centered family extending C such that $|C'| = |C|$, then T is preprocessed for $\dot{f}(i)$, k, C'.

LEMMA 2.6.4. *Let C be a centered family, $\dot{f}$ a good $Q(C)$-name for a real, $i, k \in \omega$, T a pure condition in $Q(C)$. Then there is a centered family C' extending C, $|C'| = |C|$ and a pure condition T' extending T, $T' \in Q(C')$ such that T' is preprocessed for $\dot{f}(i)$, k, C'.*

PROOF. Let $v_1, \ldots, v_s$ enumerate the subsets of k. In finitely many steps we will obtain the family C' and pure condition T'. Consider $(v_1, T \backslash k)$. If there is a centered family C_1' extending C, $|C_1'| = |C|$ and a pure condition $T_1' \in Q(C_1')$ such that $T_1' \leq T \backslash k$ and for some $p_1 \in \mathcal{A}_1(\dot{f})$, $(v_1, T_1') \leq p_1$, let $T_1 = T_1'$ and $C_1 = C_1'$. Otherwise let $T_1 = T$, $C_1 = C$. Proceed inductively. At step $(s-1)$ consider (v_s, T_{s-1}) and C_{s-1}. If there is a centered family C_s' extending C_{s-1}, $|C_s'| = |C_{s-1}|$ such that for some pure condition $T_s' \in Q(C_s)$ extending T_{s-1}, there is $p_s \in \mathcal{A}_i(\dot{f})$ such that $(v_s, T_s') \leq p_s$ let $T_s' = T_s$ and $C_s = C_s'$. Otherwise let $T_s = T_{s-1}$, $C_s = C_{s-1}$. It will be shown that $T' = T_s$ is preprocessed for $\dot{f}(i)$, k, $C' = C_s$.

Let $v \subseteq k$, C'' a centered family extending C', $|C''| = |C|$, T'' a pure condition in $Q(C'')$ extending T' and such that for some $p \in \mathcal{A}_i(\dot{f})$, $(v, T'') \leq p$. Then $v = v_j$ for some $j \in s+1$. Since C'' extends C', C''

extends C_{j-1} and furthermore $T'' \leq T' \leq T_{j-1}$. Therefore at stage j we have chosen a centered family C_j and a pure condition $T_j \in Q(C_j)$ such that $(v_j, T_j) \leq p_j$ for $p_j \in \mathcal{A}_i(\dot{f})$. But $T' \leq T_j$ and so $(v_j, T') \leq p_j$. $\square$

COROLLARY 2.6.5. *Let C be a centered family, T a pure condition in $Q(C)$ and $k \in \omega$. Then there is a centered family C' extending C, $|C'| = |C|$ and a pure condition $T' \in Q(C')$ extending T, such that for every $i \leq k$, T' is preprocessed for $\dot{f}(i)$, k, C'.*

PROOF. By Lemma 2.6.4 there is a centered family C_0 extending C, $|C_0| = |C|$ and a pure extension $T_0 \in Q(C_0)$ of $T\backslash k$, which is preprocessed for $\dot{f}(0)$, k, C_0. Applying Lemma 2.6.4 at each step, obtain a finite sequence $\langle T_i : i \leq k\rangle$ of pure conditions such that $\forall i \in k\ T_{i+1} \leq T_i$ and a finite sequence of centered families $\langle C_i : i \leq k\rangle$ $C_i \subseteq Q(C_{i+1})$, $|C_{i+1}| = |C_i|$, $T_i \in Q(C_i)$ and T_i is preprocessed for $\dot{f}(i)$, k, C_i. Let $T' = T_k$, $C' = C_k$. Then C_k extends C, $|C'| = |C|$, $T' \in Q(C')$ is an extension of C and since for every $i \leq k\ T' \leq T_i$, by Lemma 2.6.2 for every $i \leq k$, T' is preprocessed for $\dot{f}(i)$, k, C'. $\square$

2.7. Generic Preprocessed Conditions

LEMMA 2.7.1. *Let C be a centered family of pure conditions, $\dot{f}$ a good $Q(C)$-name for a real and let T be a pure condition in $Q(C)$. Then there is a centered family C' extending C, $|C'| = |C|$ and a sequence of pure conditions $\langle T_n : n \in \omega\rangle \subseteq Q(C')$, such that*

(1) $T_0 \leq T$ and $\forall n \geq 1(T_n \leq T_{n-1})$

(2) $\forall n \in \omega \forall i \leq n$, T_n is preprocessed for $\dot{f}(i)$, n, C'.

Proof. By Lemma 2.6.4 there is a centered family C_0 extending C, $|C_0| = |C|$ and a pure extension T_0 of T in $Q(C_0)$ which is preprocessed for $\dot{f}(0)$, 0, C_0. Proceed inductively. Suppose we have defined C_n, such that $|C_n| = |C_{n-1}|$, $T_n \in Q(C_n)$ such that for all $i \leq n$, T_n is preprocessed for $\dot{f}(i)$, n, C_n. Then by Corollary 2.6.5 there is a centered family C_{n+1} extending C_n, $|C_{n+1}| = |C_n|$ and a pure condition $T_{n+1} \in Q(C_{n+1})$ extending T_n such that for all $i \leq n+1$, T_{n+1} is preprocessed for $\dot{f}(i)$, $n+1$, C_{n+1}. With this the inductive construction is complete. Then $C' = \cup_{n\in\omega} C_n$ is a centered family extending C, $|C'| = |C|$, which contains the sequence $\langle T_n : n \in \omega\rangle$. For every $n \in \omega$ by construction T_n, for all $i \leq n$, T_n is preprocessed for $\dot{f}(i)$, n, C_n. Since C' extends C_n, $|C'| = |C_n|$ by Lemma 2.6.2, for all $i \leq n$, T_n is preprocessed for $\dot{f}(i)$, n, C'. □

Remark 2.7.2. The sequence $\tau = \langle T_n : n \in \omega\rangle$ is not uniquely determined. Also τ is not formally a fusion sequence. Until the end of the section fix a centered family of pure conditions C, a good $Q(C)$-name $\dot{f}$ for a real, $T \in Q(C)$ and a sequence of pure conditions $\tau = \langle T_n : n \in \omega\rangle$ contained in $Q(C)$ which satisfies the conclusion of Lemma 2.7.1 for C, T and $\dot{f}$.

Definition 2.7.3. Let $\mathbb{P}_\tau(C, T, \dot{f})$ be the suborder of $\mathbb{P}(T)$ consisting of all finite sequences $\bar{r} = \langle r_0, \ldots, r_\ell\rangle$, $\ell \in \omega$ such that $r_0 \leq T_0$ and for all $i : 1 \leq i \leq \ell$ and $j_i = \max \operatorname{int}(r_{i-1})$, $r_i \leq T_{j_i}$.

Lemma 2.7.4. *The following sets are dense in* $\mathbb{P}_\tau(T, C, \dot{f})$*:*

(1) $\forall k \in \omega$, $E_k = \{\bar{r} \in \mathbb{P}_\tau(C, T, \dot{f}) : |\bar{r}| \geq k\}$,

(2) for all $X \in C$, $n \in \omega$, $D_\tau(X,n) = \{\bar{r} \in \mathbb{P}_\tau(C,T,\dot{f}) : \exists r_j \in \bar{r}(r_j \leq X$ *and* $\|r_j\| \geq n)\}$.

Proof. Let $\bar{r} = \langle r_i : i \in \ell\rangle$ be a given sequence in $\mathbb{P}_\tau(C,T,\dot{f})$. Let $j_\ell = \max \mathrm{int}(r_{\ell-1})$. Since $T_{j_\ell}\backslash \mathrm{int}(\bar{r})$ and X are compatible, there is a finite logarithmic measure r of level higher than the measure of $r_{\ell-1}$ and n, which is a common extension of $T_{j_\ell}\backslash \mathrm{int}(\bar{r})$ and X. Then $\bar{r}^\frown\langle r\rangle$ is an extension of $\bar{r}$ in $D_\tau(X,n)$. □

Corollary 2.7.5. *Let G be a $\mathbb{P}_\tau(C,T,\dot{f})$-generic filter. Then*

(1) $R_G = \cup G = \langle r_i : i \in \omega\rangle$ *is a pure condition of strictly increasing logarithmic measures such that for every* $n \in \omega$, $R_n = \langle r_i : i \geq n\rangle$ *is a pure extension of* T_{j_n} *where* $j_n = \max int(r_{n-1})$.

(2) In $V[G]$ *there is a centered family* C' *extending* C *below* R_G *(and so below* T*) such that* $|C'| = |C|$. *Then in particular for all* $n \in \omega$, $x \in [int(R_n)]^{<\omega}$, $R_n\backslash x$ *is preprocessed for* $\dot{f}(n)$, $\max x$, C'.

Proof. By Lemma 2.7.4 R_G is a pure condition of strictly increasing finite logarithmic measures such that $\forall n \in \omega\ R_n$ is a pure extension of T_{j_n}. To obtain the second part note for every X in C and $n \in \omega$, the generic filter G meets $D_\tau(X,n)$ and so the sequence $I_X = \langle i : r_i \leq X\rangle$ is infinite. Then $R_G \wedge X = \langle r_i : i \in I_X\rangle$ is a common extension of R_G and X. Furthermore if $X \leq Y$, then $I_X \subseteq I_Y$ and so $R_G \wedge X \leq R_G \wedge Y$. Thus $C' = \{R_G \wedge X : X \in C\}$ is centered and extends C below R_G, $|C'| = |C|$.

Let $n \in \omega$, $x \in [\mathrm{int}(R_n)]^{<\omega}$. Note that $R_n \backslash x = R_k$ where $k = i_{R_G}(\max x) = \min\{j : \max x < \min \mathrm{int}(r_j)\}$. However $x \subseteq \mathrm{int}(R_n)$ and so $\max x \leq \max \mathrm{int}(r_{k-1}) = j_k$. Since $R_k \leq T_{j_k}$ and for every $i \leq j_k$, T_{j_k} is preprocessed for $\dot{f}(i)$, j_k, C', and so by Lemma 2.6.2 for every $i \leq j_k$, R_k is preprocessed for $\dot{f}(i)$, j_k, C'. However $\max x \leq j_k$ and $n \leq j_k$. Therefore R_k is preprocessed for $\dot{f}(n)$, $\max x$, C'. $\square$

CHAPTER 3

$\mathfrak{b} = \kappa < \mathfrak{s} = \kappa^+$

3.1. Induced Logarithmic Measures

In the following κ is an uncountable regular cardinal. For completeness we state $MA_{countable}(\kappa)$ (see [**24**]).

Definition 3.1.1. $MA_{countable}(\kappa)$ is the statement: for every countable partial order $\mathbb{P}$ and every family $\mathcal{D}$, $|\mathcal{D}| < \kappa$ of dense subsets of $\mathbb{P}$ there is a filter $G \subseteq \mathbb{P}$ such that $\forall D \in \mathcal{D}(G \cap D \neq \emptyset)$.

Let $\mathcal{M}$ be the ideal of meager subsets of the real line. Recall that the covering number of $\mathcal{M}$, $\mathrm{cov}(\mathcal{M})$ is the minimal size of a family of meager sets which covers the real line. For every regular uncountable cardinal κ, $\mathrm{cov}(\mathcal{M}) \geq \kappa$ if and only if $MA_{countable}(\kappa)$ (see [**3**]).

Lemma 3.1.2. *Let C be a centered family of pure conditions, $|C| < cov(\mathcal{M})$, $\dot{f}$ a good $Q(C)$-name for a real, $n \in \omega$, $T = \langle t_i : i \in \omega \rangle$ pure condition in $Q(C)$ such that for all $x \in [int(T)]^{<\omega}$, $T \backslash x$ is preprocessed for $\dot{f}(n)$, $\max x$, C. Then the logarithmic measure induced by the family $\mathcal{P}_v(C, T, \dot{f}(n))$ where $v \in [\omega]^{<\omega}$, of all $x \in [int(T)]^{<\omega}$ such that:*

(1) $\exists i \in \omega$ such that $h_i(x \cap s_i) > 0$ where $t_i = (s_i, h_i)$

(2) $\exists w \subseteq x \exists p \in \mathcal{A}_n(\dot{f})$ such that $(v \cup w, T \backslash x) \leq p$,

takes arbitrarily high values.

PROOF. To see that the logarithmic measure induced by the family $\mathcal{P}_v(C, T, \dot{f}(n))$ takes arbitrarily high values, consider an arbitrary finite partition $\omega = A_0 \cup \cdots \cup A_{M-1}$. By Lemma 2.3.7 there is a pure extension T' of T which is compatible with C and such that $\mathrm{int}(T') \subseteq A_j$ for some $j \in M$. By $|C| < \mathrm{cov}(\mathcal{M})$ and Corollary 2.5.5 there is a centered family of pure conditions C' extending C, $|C'| = |C|$ and a pure condition $R = \langle r_i : i \in \omega \rangle \in Q(C')$ of finite logarithmic measures of strictly increasing levels, which extends T', and so T. Then $\dot{f}$ is a $Q(C')$-name for a real and so $\mathcal{A}_n(\dot{f})$ is a maximal antichain in $Q(C')$. Therefore there is a condition $(v \cup w, R') \in Q(C')$ which is a common extension of (v, R) and some $q \in \mathcal{A}_n(\dot{f})$. By definition of the extension relation there is a finite subsequence $\langle r_i : i \in [m_1, m_2] \rangle$ of R, such that $w \subseteq x = \cup_{i=m_1}^{m_2} \mathrm{int}(r_i)$. We can assume that $m_2 \geq 1$ and so there is $i \in [m_1, m_2]$ such that $\|r_i\| > 0$. However $R \leq T$ and so there is $i \in \omega$ such that $h_i(x \cap s_i) > 0$. Therefore (1) holds for x.

Since R is pure extension of T and T is preprocessed for $\dot{f}(n)$, $\max x$, C, there is $p \in \mathcal{A}_n(\dot{f})$ such that $(v \cup w, T \backslash x) \leq p$ and so part (2) holds as well. It remains to observe that $x \subseteq \mathrm{int}(R) \subseteq A_j$ and so by Lemma 2.1.10 the logarithmic measure induced by $\mathcal{P}_v(C, T, \dot{f}(n))$ takes arbitrarily high values. $\square$

COROLLARY 3.1.3. *Let C be a centered family of pure conditions, $|C| < cov(\mathcal{M})$, $\dot{f}$ a good $Q(C)$-name for a real, $n, k \in \omega$, $T = \langle t_i : i \in \omega \rangle$ a pure condition in $Q(C)$ such that for all finite subsets x of $int(T)$, $T \backslash x$ is preprocessed for $\dot{f}(n)$,* $\max x$*, C. Then the logarithmic measure induced by the family $\mathcal{P}_k(C, T, \dot{f}(n))$ of all $x \in [int(T)]^{<\omega}$ such that*

(1) $\exists i \in \omega$ *such that* $h_i(s_i \cap x) > 0$ *where* $t_i = (s_i, h_i)$*,*

(2) $\forall v \subseteq k \exists w \subseteq x \exists p \in \mathcal{A}_n(\dot{f})$ *such that* $(v \cup w, T \backslash x) \leq p$*,*

takes arbitrarily high values.

Proof. Let $v_0, \ldots, v_{L-1}$ enumerate all subsets of k. Then if for all $j \in L$, $x_j \in \mathcal{P}_{v_j}(T, \dot{f}(n))$, the set $x = x_0 \cup \cdots \cup x_{L-1}$ belongs to $\mathcal{P}_k(C, T, \dot{f}(n))$. To see that the logarithmic measure induced by $\mathcal{P}_k(C, T, \dot{f}(n))$ takes arbitrarily high values consider an arbitrary finite partition $\omega = A_0 \cup \cdots \cup A_{M-1}$. By Lemma 2.3.7 there is a pure extension T' of T which is compatible with C and such that $\mathrm{int}(T') \subseteq A_j$ for some $j \in M$. By $|C| < \mathrm{cov}(\mathcal{M})$ and Corollary 2.5.5 there is a centered family of pure conditions C' extending C, $|C'| = |C|$ and a pure condition $R = \langle r_i : i \in \omega \rangle \in Q(C')$ of finite logarithmic measures of strictly increasing levels, which extends T' and so T. Then in particular for every $x \in [\mathrm{int}(R)]^{<\omega}$, $R \backslash x \leq T \backslash x$ and so R is preprocessed for $\dot{f}(n)$, $\max x$, C. By Lemma 3.1.2 for every $i \in L$ there is $x_i \in \mathcal{P}_{v_i}(C', R, \dot{f}(n))$. It will be shown that $x = \cup\{x_i : i \in L\}$ belongs to $\mathcal{P}_k(C, T, \dot{f}(n))$. It is clear that (1) holds for x.

To obtain (2) consider any $v \subseteq k$. Then $v = v_i$ for some $i \in L$. Since $x_i (x_i \subseteq x)$ belongs to $\mathcal{P}_{v_i}(C', R, \dot{f}(n))$ there is $w_i \subseteq x_i$ and $q_i \in \mathcal{A}_n(\dot{f})$ such that $(v_i \cup w_i, R \backslash x_i) \leq q_i$, and so in particular $(v_i \cup w_i, R \backslash x) \leq q_i$. However $R \leq T$, C' extends C, $|C'| = |C|$ and T is preprocessed for $\dot{f}(n)$, $\max x$, C. But then $\forall v \subseteq k$ $\exists p \in \mathcal{A}_n(\dot{f})$ such that $(v \cup w, T \backslash x) \leq p$. Therefore $x \in \mathcal{P}_k(C, T, \dot{f}(n))$. It remains to observe that $x \subseteq \mathrm{int}(R) \subseteq A_j$ and so by Lemma 2.1.10 the logarithmic measure induced by $\mathcal{P}_k(C, T, \dot{f}(n))$ takes arbitrarily high values. $\square$

3.2. Good Extensions

Until the end of the section let C be a centered family, $|C| < \mathrm{cov}(\mathcal{M})$, $\dot{f}$ a good $Q(C)$-name for a real, $T = \langle t_i : i \in \omega \rangle$ a pure condition in $Q(C)$ such that for all $n \in \omega$ and all $x \in [\mathrm{int}(T_n)]^{<\omega}$, where $T_n = \langle t_i : i \geq n \rangle$, $T \backslash x$ is preprocessed for $\dot{f}(n)$, $\max x$, C.

Definition 3.2.1. Let $\mathbb{P}(C, T, \dot{f})$ be the suborder of $\mathbb{P}(T)$ of all sequences $\bar{r} = \langle r_i : i \in \ell \rangle$ such that $\forall i \forall v \subseteq i \forall s \subseteq \mathrm{int}(r_i)$ which is r_i-positive, there are $w \subseteq s$ and $p \in \mathcal{A}_i(\dot{f})$ such that $(v \cup w, T \backslash s) \leq p$.

Lemma 3.2.2. *For every $k \in \omega$ the set $E_k(C, T, \dot{f}) = \{\bar{r} \in \mathbb{P}(C, T, \dot{f}) : |\bar{r}| \geq k\}$ is dense in $\mathbb{P}(C, T, \dot{f})$.*

Proof. Let $\bar{r} = \langle r_0, \ldots, r_{m-1} \rangle$ be a condition in $\mathbb{P}(C, T, \dot{f})$ and let $\ell = \max \mathrm{int}(\bar{r})$. We can assume that $m \leq k$. Then $i_T(\ell) \geq m$ and so $T \backslash \mathrm{int}(\bar{r})$ is an extension of T_m. Then by Corollary 3.1.3 (and $|C| < \mathrm{cov}(\mathcal{M})$) the logarithmic measure h induced by $\mathcal{P}_m = \mathcal{P}_m(C, T \backslash \mathrm{int}(\bar{r}), \dot{f}(m))$ takes arbitrarily high values and so there is x such that $h(x) > \|r_{m-1}\|$. Let $r_m = (x, h \restriction \mathcal{P}(x))$. We claim that $\bar{r}^\frown \langle r_m \rangle$ is an extension of $\bar{r}$ which belongs to $E_m(C, T, \dot{f})$.

Let $v \subseteq m$ and $s \subseteq \mathrm{int}(r_m) = x$, $h(s) > 0$. Then by definition of h there is $w \subseteq s$ and $p \in \mathcal{A}_m(\dot{f})$ such that $(v \cup w, T \backslash s) \leq p$. In finitely many steps obtain an end-extension of $\bar{r}$ which belongs to E_k. $\square$

Lemma 3.2.3. *For every $X \in C$, $n \in \omega$ the set $D_{X,n}(C, T, \dot{f}) = \{\bar{r} \in \mathbb{P}(C, T, \dot{f}) : \exists r_j \in \bar{r}(r_j \leq X$ and $\|r_j\| \geq n)\}$ is dense in $\mathbb{P}(C, T, \dot{f})$.*

PROOF. Let $\bar{r} = \langle r_0, \dots, r_{m-1}\rangle$ be a condition in $\mathbb{P}(C, T, \dot{f})$ and let $\ell = \max \mathrm{int}(\bar{r})$. Then $i_T(\ell) \geq m$ and so $T \backslash \mathrm{int}(\bar{r})$ is an extension of T_m. Furthermore since both X and $T\backslash \mathrm{int}(\bar{r})$ are in the centered family, there is $Y \in C$ which is their common extension. Then in particular $\forall x \in [\mathrm{int}(Y)]^{<\omega}$ Y is preprocessed for $\dot{f}(m)$, $\max x$ and C. Then by Corollary 3.2.2 and $|C| < \mathrm{cov}(\mathcal{M})$ the logarithmic measure h induced by $\mathcal{P}_m(C, Y, \dot{f}(m))$ takes arbitrarily high values and so we can choose $x \subseteq \mathrm{int}(Y)$ such that $h(x) > \max\{\|r_{m-1}\|, n\}$. Let $r_m = (x, h \upharpoonright \mathcal{P}(x))$. It is sufficient to show that $\bar{r}^\frown\langle r_m\rangle$ belongs to $\mathbb{P}(C, T, \dot{f})$.

Let $v \subseteq m$, $s \subseteq x$ and $h(s) > 0$. By definition of h there is $w \subseteq s$ and a condition $q \in \mathcal{A}_m(\dot{f})$ such that $(v \cup w, Y\backslash s) \leq q$. Since T is preprocessed for $\dot{f}(m)$, $\max s$ and C, there is $p \in \mathcal{A}_m(\dot{f})$ such that $(v \cup w, T\backslash s) \leq p$. $\square$

COROLLARY 3.2.4. *Let G be a filter in $\mathbb{P}(C, T, \dot{f})$ which meets all $D_{X,n}(C, T, \dot{f})$ and $E_k(C, T, \dot{f})$ for $X \in C$, $n, k \in \omega$.*

(1) *Then $R_G = \cup G = \langle r_i : i \in \omega\rangle$ is a pure condition of finite logarithmic measures of strictly increasing levels such that $\forall i \forall v \subseteq i \forall s \subseteq \mathit{int}(r_i)$ which is r_i-positive, there is $w \subseteq s$ and $p \in \mathcal{A}_i(\dot{f})$ such that $(v \cup w, R_G\backslash s) \leq p$.*

(2) *Furthermore there is a centered family C' extending C below R_G (and so below T) such that $|C'| = |C|$.*

PROOF. By Lemmas 3.2.2 and 3.2.3 R_G is a pure condition of finite logarithmic measures of strictly increasing levels which is compatible with C. Let $i \in \omega$, $v \subseteq i$ and $s \subseteq \mathrm{int}(r_i)$ which is r_i-positive. Then by definition of the partial order, there is $w \subseteq s$ and $p \in \mathcal{A}_i(\dot{f})$ such that $(v \cup w, T\backslash s) \leq p$. However $R_G \leq T$ and so $(v \cup w, R_G\backslash s) \leq p$. To obtain part (2) repeat the proof of Corollary 2.5.5 to get a centered family $C' = \{R_G \wedge X : X \in C\}$ extending C below R_G. $\square$

3.3. Mimicking the Almost Bounding Property

Definition 3.3.1. A family $\mathcal{H} \subseteq {}^\omega\omega$ is $<^*$*-directed* if for every subfamily $\mathcal{H}'$ such that $|\mathcal{H}'| < |\mathcal{H}|$ there is $h \in \mathcal{H}$ such that $\mathcal{H}' <^* h$.

Theorem 3.3.2. *Let κ be a regular uncountable cardinal, cov$(\mathcal{M}) = \kappa$ and $\mathcal{H}$ an unbounded, $<^*$-directed family of reals of size κ. Let C be a centered family, $|C| < \kappa$, $\dot{f}$ a good $Q(C)$-name for a real and $T \in Q(C)$. Then there is a centered family C', a pure condition $R \in Q(C')$ and a real $h \in \mathcal{H}$ such that $C \subseteq Q(C')$, $|C| = |C'|$, $R \leq T$ and such that for every centered family C'' extending C', for every $a \in [\omega]^{<\omega}$,*

$$(a, R) \Vdash_{Q(C'')} \exists^\infty i \in \omega(\dot{f}(i) < h(i)).$$

Proof. By Corollary 2.7.5 and $|C| < \text{cov}(\mathcal{M})$, there is a centered family C_1 extending C below T, which is of the same cardinality as C and such that there is a pure condition $T_1 \in Q(C_1)$, $T_1 \leq T$ with the property that if $T_1 = \langle t^1_i : i \in \omega\rangle$ then for every $n \in \omega$ and every finite subset x of $\text{int}(T_1\backslash\text{int}(t^1_{n-1}))$, $T_1\backslash x$ is preprocessed for $\dot{f}(n)$, $\max x$, C_1. By $|C_1| < \text{cov}(\mathcal{M})$ there is a filter $G \subseteq \mathbb{P}(C_1, T_1, \dot{f})$ meeting $E_k(C_1, T_1, \dot{f})$ and $D_{X,n}(C_1, T_1, \dot{f})$ for all $k, n \in \omega$ and $X \in C_1$. Then by Corollary 3.2.4 the pure condition $T_2 = \cup G = \langle r_i : i \in \omega\rangle$ extends T_1, consists of finite logarithmic measures of strictly increasing levels and for all $\forall i \in \omega \forall v \subseteq i \forall s \subseteq \text{int}(r_i)$ which is r_i-positive, there is $w \subseteq s$

and $p \in \mathcal{A}_i(\dot{f})$ such that $(v \cup w, T_2 \backslash s) \leq p$. For all $i \in \omega$ let $g(i)$ be the maximal k such that there are $v \subseteq i$, $w \subseteq \mathrm{int}(r_i)$, $p \in \mathcal{A}_i(\dot{f})$ such that $(v \cup w, T_2) \leq p$ and $p \Vdash \check{k} = \dot{f}(i)$. We can assume that g is nondecreasing. Otherwise redefine $g(i) = \max\{g(j) : j \leq i\}$. For every $X \in C_1$ let $J_X = \{i : r_i \leq X\}$ and let F_X be a step function defined as follows: $F_X(\ell) = g(J_X(i+1))$ iff $\ell \in (J_X(i), J_X(i+1)]$ where $J_X(m)$ is the m-th element of J_X. Since $\mathcal{H}$ is unbounded for all X in C_1 there is $h_X \in \mathcal{H}$ such that $h_X \not\leq^* F_X$. The cardinality of $\{h_X : X \in C_1\}$ does not exceed $|C_1|$ and so is less than κ. By the hypothesis on $\mathcal{H}$ there is $h \in \mathcal{H}$ such that $h_X \leq^* h$ for every $X \in C_1$. We can assume that h is nondecreasing. Note that:

(1) $\forall X \in C_1 \forall i \in \omega (g(i) \leq F_X(i))$
(2) Since $\exists^\infty i \in \omega(F_X(i) < h_X(i))$ and $\forall^\infty i \in \omega(h_X(i) \leq h(i))$ we have $\exists^\infty i \in \omega(F_X(i) < h(i))$. That is $h \not\leq^* F_X$.
(3) By part (1) and (2) the set $J = \{i \in \omega : g(i) < h(i)\}$ is infinite.
(4) Furthermore $\exists^\infty i \in J_X(F_X(i) < h(i))$. Suppose not. Then $\forall^\infty i \in J_X(h(i) \leq F_X(i))$ and so there is $m_0 \in \omega$ such that $\forall i \in J_X$ if $i > m_0$ then $(h(i) \leq F_X(i))$. Let $m \in \omega$ be such that $J_X(m) = \min J_X - m_0$. Then $\omega - J_X(m) = \cup\{(J_X(i), J_X(i+1)] : i \geq m\}$ and so if $\ell \in \omega - J_X(m)$, then there is $i \geq m$ such that $\ell \in (J_X(i), J_X(i+1)]$. Then $h(\ell) \leq h(J_X(i+1)) \leq F_X(J_X(i+1)) = F_X(\ell)$ and so $h(\ell) \leq F_X(\ell)$. This implies that $h \leq^* F_X$ which is a contradiction to part 2.
(5) However $\forall i \in J_X(F_X(i) = g(i))$ and so by part 4 the set $I_X = J_X \cap J$ is infinite.

Let $R = \langle r_i : i \in J\rangle$. Then for every $X \in C_1$ the pure condition $R \wedge X = \langle r_i : i \in I_X\rangle$ is a common extension of R and X. Furthermore if $X \leq Y$ then $I_X \subseteq I_Y$ since $J_X \subseteq J_Y$. Therefore $C' = \{R \wedge X : X \in C_1\}$ is a centered family which extends C_1 below R which is of the same cardinality as C_1. Let C'' be an arbitrary centered family which extends C'. We will show that $\forall a \in [\omega]^{<\omega}(a, R) \Vdash_{Q(C'')} \exists^\infty i \in \omega(\dot{f}(i) < \check{h}(i))$.

Fix any $a \in [\omega]^{<\omega}$ and $k \in \omega$. Let (b, R') be an arbitrary extension of (a, R) in $Q(C'')$. There is $i \in J$, $i > k$ such that $b \subseteq i$ and $s = \mathrm{int}(R') \cap \mathrm{int}(r_i)$ is r_i-positive. But then, there is $w \subseteq s$ such that $(b \cup w, T_2 \backslash s) \leq p$ for some $p \in \mathcal{A}_i(\dot{f})$. However $R' \backslash s \leq R \backslash s \leq T_2 \backslash s$. Therefore $(b \cup w, R' \backslash s) \leq (b, R')$ and $(b \cup w, R' \backslash s) \leq p$. Let $j \in \omega$ be such that $p \Vdash \check{j} = \dot{f}(i)$. Then by definition of g we have that $j \leq g(i)$. Since $i \in J$, $g(i) < h(i)$ and so

$$(b \cup w, R' \backslash s) \Vdash_{Q(C'')} \text{“}\dot{f}(i) = \check{j} \leq \check{g}(i) < \check{h}(i)\text{”}.$$

However (b, R') was an arbitrary extension of (a, R) in $Q(C'')$. Therefore $(a, R) \Vdash_{Q(C'')}$ “$\exists i \in \omega(i > k \wedge \dot{f}(i) < \check{h}(i))$”. Since k was arbitrary as well $(a, R) \Vdash_{Q(C'')}$ “$\exists^\infty i \in \omega(\dot{f}(i) < \check{h}(i))$”. □

3.4. Adding an Ultrafilter

Lemma 3.4.1. *Let κ be a regular uncountable cardinal, cov$(\mathcal{M}) = \kappa$, $\mathcal{H} \subseteq {}^\omega\omega$ an unbounded $<^*$-directed family, $|H| = \kappa$, $\forall \lambda < \kappa(2^\lambda \leq \kappa)$. Then there is a centered family C such that $|C| = \kappa$ and*

(1) $\Vdash_{Q(C)}$ “$\mathcal{H}$ is unbounded”,

(2) $Q(C)$ adds a real not split by the ground model reals.

Proof. Let $\mathcal{N} = \{\dot{f}_\alpha\}_{\alpha<\kappa}$ be an enumeration of all names for functions in ${}^\omega\omega$ for partial orders $Q(C')$ where C' is a centered family of pure conditions of size $< \kappa$. Furthermore let $\mathcal{A} = \{A_{\alpha+1}\}_{\alpha<\kappa}$ be an enumeration of $[\omega]^\omega$. The centered family C will be obtained by transfinite induction of length κ.

Begin with arbitrary pure condition T in Q and $C_0 = \{T\backslash v : v \in [\omega]^{<\omega}\}$. If α is a successor, $\alpha = \beta + 1$ and we have defined the centered family C_β, let $\dot{g}_{\beta+1}$ be the name with least index in $\mathcal{N}\backslash\{\dot{g}_{\gamma+1}\}_{\gamma<\beta}$ which is a $Q(C_\beta)$-name for a real. Suppose $\dot{g}_{\beta+1}$ is a good $Q(C_\beta)$-name. Then let $T' \in Q(C_\beta)$ be arbitrary. By Lemma 2.3.7 there is a pure extension T'' of T' such that $\mathrm{int}(T'') \subseteq A_{\beta+1}$ or $\mathrm{int}(T'') \subseteq A^c_{\beta+1}$ and T'' is compatible with C_β. By Corollary 2.5.5 there is a centered family $C'_{\beta+1}$ extending C_β below T'' such that $|C'_{\beta+1}| = |C_{\beta+1}|$. Then by Theorem 3.3.2 there is a centered family $C_{\beta+1}$ which extends $C'_{\beta+1}$, $|C_{\beta+1}| = |C'_{\beta+1}|$ and a pure condition $T_{\beta+1} \in Q(C_{\beta+1})$ such that $T_{\beta+1} \leq T''$ (and so in particular $\mathrm{int}(T_{\beta+1}) \subseteq A_{\beta+1}$ or $\mathrm{int}(T_{\beta+1}) \subseteq A^c_{\beta+1}$) and such that for some function $h_{\beta+1}$ from the unbounded family $\mathcal{H}$, for every centered family C'' extending $C_{\beta+1}$,

$$\forall a \in [\omega]^{<\omega}(a, T_{\beta+1}) \Vdash_{Q(C'')} \exists^\infty i \in \omega(\dot{g}_{\beta+1}(i) \leq \check{h}_{\beta+1}(i)).$$

If $\dot{g}_{\beta+1}$ is not a good $Q(C_\beta)$-name, then by Corollary 2.4.5 there is a centered family $C'_{\beta+1}$ extending C_β, $|C'_{\beta+1}| = |C_\beta|$ such that $\dot{g}_{\beta+1}$ is not a $Q(C'_{\beta+1})$-name for a real. Let $T' \in Q(C'_{\beta+1})$ be arbitrary. By Lemma 2.3.7 there is a pure condition $T_{\beta+1}$ extending T' which is compatible with $C'_{\beta+1}$ and such that $\mathrm{int}(T_{\beta+1}) \subseteq A_{\beta+1}$ or $\mathrm{int}(T_{\beta+1}) \subseteq$

$A^c_{\beta+1}$. By $|C'_{\beta+1}| < \text{cov}(\mathcal{M})$ and Corollary 2.5.5 there is a centered family $C_{\beta+1}$ extending $C'_{\beta+1}$ below $T_{\beta+1}$ such that $|C_{\beta+1}| = |C'_{\beta+1}|$.

If α is a limit let $C_\alpha = \cup_{\beta<\alpha} C_\beta$. Then C_α is of cardinality less than κ and extends C_β for every $\beta < \alpha$. With this the inductive construction is complete. Let $C = \cup_{\alpha<\kappa} C_\alpha$. Then C is centered, of cardinality κ and extends C_α for every $\alpha < \kappa$.

(1) Let $\dot{f}$ be a $Q(C)$-name for a real. Then

$$\dot{f} = \cup\{\langle\langle i, j^i_p\rangle, p\rangle : p \in \mathcal{A}_i(\dot{f}), i \in \omega, j^i_p \in \omega\}$$

where for every $i \in \omega$, $\mathcal{A}_i(\dot{f})$ is a maximal antichain in $Q(C)$ and so $|\mathcal{A}_i(\dot{f})| = \aleph_0$. For every $i \in \omega$, $p \in \mathcal{A}_i(\dot{f})$ let $\alpha_i(p) = \min\{\gamma : p \in Q(C_\gamma)\}$. Since κ is regular uncountable $\sup\{\alpha_i(p) : p \in \mathcal{A}_i(\dot{f})\} = \alpha_i < \kappa$ and furthermore $\alpha = \sup_{i\in\omega} \alpha_i < \kappa$ is minimal such that $\dot{f}$ is a $Q(C_\alpha)$-name for a function in ${}^\omega\omega$. Then $\dot{f}$ is a name in the list $\mathcal{N}$. Note that for every $\beta \geq \alpha$, $\dot{f}$ is a $Q(C_\beta)$-name and so there is $\delta < \kappa$ such that $\dot{f}$ is the name with least index in $\mathcal{N}\backslash\{\dot{g}_{\gamma+1}\}_{\gamma<\delta}$ which is a $Q(C_\delta)$-name (note $\alpha \leq \delta$). That is $\dot{f} = \dot{g}_{\delta+1}$. If $\dot{f}$ is not a good $Q(C_\delta)$-name, then we would have chosen the centered family $C_{\delta+1}$ such that $\dot{f}$ is not $Q(C_{\delta+1})$-name for a real. Then in particular there is $i \in \omega$ and $p \in Q(C_{\delta+1})$ such that p is incompatible with all elements of $\mathcal{A}_i(\dot{f})$ as conditions in $Q(C_{\delta+1})$. But then by Lemma 2.2.6 $\mathcal{A}_i(\dot{f}) \cup \{p\}$ is an antichain in Q and so $\mathcal{A}_i(\dot{f}) \cup \{p\}$ remains an antichain in $Q(C)$. Then in particular $\mathcal{A}_i(\dot{f})$ is not maximal in $Q(C)$, i.e. $\dot{f}$ is not a $Q(C)$-name for a real, which is a contradiction. Therefore $\dot{f} = \dot{g}_{\delta+1}$ is a good

$Q(C_\delta)$-name. But then by the choice of $T_{\delta+1}$ in $C_{\delta+1}$

$$\forall a \in [\omega]^{<\omega}(a, T_{\delta+1}) \Vdash_{Q(C)} \exists^\infty i \in \omega(\dot{f}(i) \leq \check{h}_{\delta+1}(i)).$$

It remains to observe that $\{(a, T_{\delta+1}) : a \in [\omega]^{<\omega}\}$ is predense in $Q(C)$ which implies $\Vdash_{Q(C)} \check{h}_{\delta+1} \not\leq^* \dot{f}$.

(2) Let G be a $Q(C)$ generic filter and $\cup G = \cup\{u : \exists T(u, T) \in G\}$. For every $\gamma \in \kappa$ the set $D_{\gamma+1} = \{(u, T) \in Q(C) : T \leq T_{\gamma+1}\}$ is dense and so $\cup G \subseteq^* \text{int}(T_{\gamma+1})$, which implies that $\cup G$ is almost contained in $A_{\gamma+1}$ or in $A^c_{\gamma+1}$. $\square$

3.5. Some preservation theorems

We will use the following well known fact about *ccc* forcing notions.

REMARK 3.5.1. Note that if $\mathcal{H}$ a $<^*$-directed family, then for every *ccc* forcing notion $\mathbb{P}$, ($\mathcal{H}$ is $<^*$ -directed)$^{V^{\mathbb{P}}}$.

The preservation theorem below will be of importance for the consistency result to be presented. The proof of Theorem 3.5.2 can be found in Judah and Shelah, [**21**], Theorem 2.2 (see also [**13**]).

THEOREM 3.5.2. *Let $\mathcal{H} \subseteq {}^\omega\omega$ be unbounded such that every countable subset of $\mathcal{H}$ is dominated by an element of $\mathcal{H}$. If $\langle \mathbb{P}_\gamma, \dot{\mathbb{Q}}_\gamma : \gamma \in \alpha\rangle$ is a finite support iteration, cf$(\alpha) = \omega$, such that $\forall\gamma \in \alpha$, $\Vdash_{\mathbb{P}_\gamma}$ "$\dot{\mathbb{Q}}_\gamma$ is ccc" and $\Vdash_{\mathbb{P}_\gamma}$ "$\check{\mathcal{H}}$ is unbounded". Then $\Vdash_{\mathbb{P}_\alpha}$ "$\check{\mathcal{H}}$ is unbounded".*

PROOF. Suppose there is a $\mathbb{P}_\alpha$-generic filter G such that in $V[G]$ there is $\exists f \in {}^\omega\omega$ dominating $\mathcal{H}$. Let $\dot{f}$ a $\mathbb{P}$-name for the real f. Let $\{\alpha_n\}_{n\in\omega}$ be increasing and cofinal sequence in α and for every $n \in \omega$ let

f_n be a function in $V[G_{\alpha_n}]$, where $G_{\alpha_n} = G \cap \mathbb{P}_{\alpha_n}$ such that for every $i \in \omega$, $f_n(i) = j$ iff $\exists q \in \mathbb{P}_\alpha(q \restriction \alpha_n \in G_{\alpha_n}$ and $q \Vdash_\alpha \dot{f}(i) = \check{j})$. Then for every $n \in \omega$ there is a function $h_n \in \mathcal{H}$ such that $V[G_{\alpha_n}] \vDash (h_n \not\leq^* f_n)$. Since $\mathbb{P}_\alpha$ is *ccc*, there is $\mathcal{C} \in [\mathcal{H}]^\omega \cap V$ such that $\{h_n : n \in \omega\} \subseteq \mathcal{C}$ and a function $h \in \mathcal{H} \cap V(\mathcal{C} \leq^* h)$. Then in particular for every $n \in \omega$, there is k_n such that $\forall i \geq k_n(h_n(i) \leq h(i))$.

By assumption $V[G] \vDash \mathcal{H} \leq^* f$. Then there are $p \in G$ and $k \in \omega$ such that $\forall i \geq k$, $p \Vdash \check{h}(i) \leq \dot{f}(i)$. Fix α_n such that support$(p) \subseteq \alpha_n$. Then, since $V[G_{\alpha_n}] \vDash h_n \not\leq f_n$ we have in particular

$$V[G_{\alpha_n}] \vDash \exists i > \max(k_n, k)(f_n(i) < h_n(i))$$

and so there is $i > \max(k_n, k)$ and condition $p' \in G_{\alpha_n}$ such that $p' \Vdash \dot{f}_n(i) < \check{h}_n(i)$ where $\dot{f}_n$ is a $\mathbb{P}_{\alpha_n}$-name for f_n. By definition of f_n there is a condition $q \in \mathbb{P}_\alpha$ such that $q \restriction \alpha_n \in G_{\alpha_n}$ and $q \Vdash_\alpha \dot{f}_n(i) = \dot{f}(i)$. Since $p \restriction \alpha_n$, p' and $q \restriction \alpha_n$ belong to the generic filter G_{α_n} there is $q' \in \mathbb{P}_\alpha$ which is a common extension of p, p' and q. Then

$$q' \Vdash_\alpha \dot{f}_n(i) = \dot{f}(i) < \check{h}_n(i) \leq \check{h}(i) \leq \dot{f}(i)$$

which is a contradiction. $\square$

LEMMA 3.5.3. *Let $\mathcal{H}$ be an unbounded family of reals and let $\mathbb{C}$ be the Cohen forcing notion. Then $\Vdash_{\mathbb{C}}$ "$\mathcal{H}$ is unbounded".*

PROOF. Let $\dot{f}$ be a $\mathcal{H}$-name for a function in ${}^\omega\omega$. It will be shown that there is $h \in \mathcal{H}$ such that $\Vdash \check{h} \not\leq^* \dot{f}$. For every $p \in \mathbb{C}$ let

$$g_p(i) = \min\{j : \exists q \leq p(q \Vdash \dot{f}(i) = j)\}.$$

Then $\{g_p : p \in \mathbb{C}\}$ is countable and so there is $g \in {}^\omega\omega \cap V$ such that $\forall p \in \mathbb{C}(g_p \leq^* g)$. That is for all $p \in \mathbb{C}$ there is $m_p \in \omega$ such that $\forall i \geq m_p(g_p(i) \leq g(i))$. Since $\mathcal{H}$ is unbounded there is $h \in \mathcal{H}$ which is not dominated by g, that is the set $A = \{i \in \omega : g(i) < h(i)\}$ is infinite. It is sufficient to show that $\Vdash \exists^\infty i \in A(\dot{f}(i) \leq g(i))$, since then $\Vdash \exists^\infty i \in \omega(\dot{f}(i) < \check{h}(i))$.

Let $p \in \mathbb{C}$. Suppose there is no $q \leq p$ such that $q \Vdash \exists^\infty i \in \check{A}(\dot{f}(i) \leq \check{g}(i))$. That is $p \Vdash \neg(\exists^\infty i \in A(\dot{f}(i) \leq \check{g}(i)))$ and so $p \Vdash \forall^\infty i \in A(\check{g}(i) < \dot{f}(i))$. That is there is $m \in \omega$ and and extension q of p such that for all $i \in A$, $i > m$, $q \Vdash \check{g}(i) < \dot{f}(i)$. Let $i \in A$ be greater than m and m_q. Let q' be an extension of q and let $j \in \omega$ such that $q' \Vdash \dot{f}(i) = \check{j}$ and $j = g_q(i)$. Then $q' \Vdash$ "$\dot{f}(i) = \check{g}_q(i) \leq \check{g}(i) < \dot{f}(i)$", which is a contradiction. $\square$

COROLLARY 3.5.4. *Let $\mathcal{H} \subseteq {}^\omega\omega$ be unbounded and let $\mathbb{C}(\kappa)$ be the forcing notion for adding κ Cohen reals. Then ($\mathcal{H}$ is unbounded)$^{V^{\mathbb{C}(\kappa)}}$.*

PROOF. By Lemma 3.5.3, Theorem 3.5.2 and Lemma 3.5.6. $\square$

The proof of Lemma 3.5.5 can be found in Bartoszynski, Judah, [**4**].

LEMMA 3.5.5. *Let $\mathcal{H} \subseteq {}^\omega\omega$ be an unbounded, $<^*$-directed family, $|\mathcal{H}| = \kappa$, $\mathbb{P}$ a forcing notion, $|\mathbb{P}| < \kappa$. Then ($\mathcal{H}$ is unbounded)$^{V^{\mathbb{P}}}$.*

PROOF. Let $\dot{f}$ be a $\mathbb{P}$-name for a function in ${}^\omega\omega$. For every $p \in \mathbb{P}$ and $i \in \omega$ let $g_p(i) = \min\{j : \exists q \leq p(q \Vdash \dot{f}(i) = \check{j})\}$. Since ($\mathcal{H}$ is unbounded)$^{V^{\mathbb{P}}}$ for every $p \in \mathbb{P}$ there is a function $h_p \in \mathcal{H} \cap V$ which is not dominated by g_p. However $|\{h_p : p \in \mathbb{P}\}| < \kappa$ and so there is $h \in \mathcal{H} \cap V$ which dominates all h_p's. That is for every $p \in \mathbb{P}$ there is $n_p \in \omega$ such that $\forall i \geq n_p(h_p(i) \leq h(i))$.

Suppose $p \in \mathbb{P}$ such that $p \Vdash$ "$\mathcal{H} <^* \dot{f}$". Then there is $p_0 \leq p$ and $n_0 \in \omega$ such that $\forall i \geq n_0$, $p_0 \Vdash \check{h}(i) \leq \dot{f}(i)$. Let $i > \max\{n_0, n_p\}$ be such that $g_{p_0}(i) < h_{p_0}(i)$ and let q be an extension of p_0 such that $q \Vdash g_{p_0}(i) = \dot{f}(i)$. Then $q \Vdash$ "$\dot{f}(i) = g_{p_0}(i) < h_{p_0}(i) \leq h(i) \leq \dot{f}(i)$" which is a contradiction. $\square$

The last two Lemmas in this section summarize some well known facts of finite support iterations of *ccc* forcing notions.

LEMMA 3.5.6. *Let κ be an ordinal of uncountable cofinality and let $\langle \mathbb{P}_\alpha, \dot{\mathbb{Q}}_\alpha : \alpha < \kappa \rangle$ be a finite support iteration of ccc forcing notions. Then every real in $V^{\mathbb{P}_\kappa}$ is obtained at some initial stage of the iteration of countable cofinality. That is*

$$^\omega\omega \cap V^{\mathbb{P}_\kappa} = \cup\{^\omega\omega \cap V^{\mathbb{P}_\alpha} : \alpha < \kappa, cf(\alpha) = \omega\}.$$

PROOF. Let $\dot{f}$ be a $\mathbb{P}_\kappa$-name for a real. We can assume that $\dot{f}$ is of the form $\dot{f} = \cup\{\langle\langle i, j^i_p\rangle, p\rangle : p \in A_i, i \in \omega, j^i_p \in \omega\}$ where each A_i is a maximal antichain of conditions in $\mathbb{P}_\kappa$ deciding $\dot{f}(i)$. Since $\mathbb{P}_\kappa$ is *ccc* every A_i is countable. Furthermore $\mathbb{P}$ is a finite support iteration and so in particular for every $p \in A_i$, the support of p is finite. Therefore for all $i \in \omega$, $\alpha_i = \sup\{\alpha^p_i : p \in A_i\}$ where $\alpha^p_i = \max \mathrm{support}(p)$ is of countable cofinality and so is smaller than κ. But then $\alpha = \sup_{i\in\omega} \alpha_i$ is also of countable cofinality (thus $\alpha < \kappa$) and $\dot{f}$ is a $\mathbb{P}_\alpha$-name. $\square$

LEMMA 3.5.7. *Let κ be a regular uncountable cardinal. Let $\langle \mathbb{P}_\alpha, \dot{\mathbb{Q}}_\alpha : \alpha < \kappa \rangle$ be a finite support iteration of ccc forcing notions of length κ. Let G be a $\mathbb{P}_\kappa$-generic filter and let $\mathcal{A} \subseteq V[G] \cap {}^\omega\omega$, $|\mathcal{A}| < \kappa$. Then $\mathcal{A}$ is obtained at some proper initial stage of the iteration.*

PROOF. For every f in $\mathcal{A}$ let $\dot{f}$ be a $\mathbb{P}_\kappa$-name for f. By Lemma 3.5.6 for every $\dot{f}$ there is an ordinal α_f of countable cofinality such that $\dot{f}$ is a $\mathbb{P}_{\alpha_f}$-name for a real. Let $\alpha = \sup\{\alpha_f : f \in \mathcal{A}\}$. Then $\mathrm{cf}(\alpha) \leq |\mathcal{A}| < \kappa$ and so $\alpha < \kappa$. It remains to observe that $\mathcal{A}$ is contained in $V[G_\alpha]$ where $G_\alpha = G \cap \mathbb{P}_\alpha$. $\square$

3.6. $\mathfrak{b} = \kappa < \mathfrak{s} = \kappa^+$

DEFINITION 3.6.1 (Hechler, [**20**]). Let $\mathcal{A}$ be an infinite set of functions in ${}^\omega\omega$. Then $\mathbb{H}(\mathcal{A})$ is the forcing notion consisting of all pairs (s, F) where $s \in \cup_{n\in\omega} {}^n\omega$ and $F \in [\mathcal{A}]^{<\omega}$ with extension relation defined as follows. We say that (s_1, F_1) extends (s_2, F_2) (and denote this by $(s_1, F_1) \leq (s_2, F_2)$) if

(1) $s_2 \subseteq s_1$, $F_2 \subseteq F_1$,
(2) $\forall f \in F_2 \forall k \in \mathrm{dom}(s_1)\backslash\mathrm{dom}(s_2)$ we have $s_1(k) \geq f(k)$.

The following is a well known fact about Hechler forcing, see [**20**].

LEMMA 3.6.2. *Let $\mathcal{A}$ be an infinite set of functions in ${}^\omega\omega$. Then the partial order $\mathbb{H}(\mathcal{A})$ is σ-centered, adds a real dominating $\mathcal{A}$ and is of the same cardinality as the set $\mathcal{A}$.*

PROOF. Note that if (s_1, F_1) and (s_2, F_2) are elements of $\mathbb{H}(\mathcal{A})$ such that $s_1 = s_2 = s$, then $(s, F_1 \cup F_2)$ is their common extension. To obtain the second part of claim, it is sufficient to observe that for every $f \in \mathcal{A}$ the set $D_f = \{(s, F) : f \in F\}$ is dense. Now let G be a $\mathbb{H}(\mathcal{A})$-generic filter and let $f_G = \cup\{s : \exists F \in [\mathcal{A}]^{<\omega}(s, F) \in G\}$. If $f \in \mathcal{A}$ and $(s, F) \in G \cap D_f$ then by definition of the extension relation, for every $i \geq |s| + 1$ we have $f(i) \leq f_G(i)$. $\square$

THEOREM 3.6.3 (GCH). *Let κ be a regular uncountable cardinal. Then there is a ccc generic extension in which $\mathfrak{b} = \kappa < \mathfrak{s} = \kappa^+$.*

PROOF. Obtain a model V of $\mathfrak{b} = \mathfrak{c} = \kappa$ by adding κ-many Hechler reals (i.e. do a finite support iteration of length κ of Hechler forcing, see [**19**] and [**8**]). Let $\mathcal{H} = V \cap {}^\omega\omega$. Then $\mathcal{H}$ is unbounded and every subfamily of $\mathcal{H}$ of cardinality less than κ is dominated by an element of $\mathcal{H}$. Furthermore in V for every $\lambda < \kappa$, $2^\lambda = \kappa$. By transfinite induction of length κ^+ define a finite support iteration of *ccc* forcing notions $\langle\langle \mathbb{P}_\alpha : \alpha \leq \kappa^+\rangle, \langle \dot{\mathbb{Q}}_\alpha : \alpha < \kappa\rangle\rangle$ as follows.

Suppose α is a limit and for every $\beta < \alpha$ we have defined a *ccc* forcing notion $\mathbb{P}_\beta$ and a $\mathbb{P}_\beta$-name $\dot{\mathbb{Q}}_\beta$ such that in $V^{\mathbb{P}_\beta}$ the family $\mathcal{H}$ is unbounded and $\Vdash_{\mathbb{P}_\beta}$ "$\dot{\mathbb{Q}}_\beta$ is *ccc* ". Let $\mathbb{P}_\alpha$ be the finite support iteration of $\langle \mathbb{P}_\beta, \dot{\mathbb{Q}}_\beta : \beta < \alpha\rangle$. Then:

(1) By Theorem 3.5.2 the family $\mathcal{H}$ remains unbounded in $V^{\mathbb{P}_\alpha}$.
(2) Since $\mathbb{P}_\alpha$ is *ccc*, by Remark 3.5.1 $\mathcal{H}$ is $<^*$-directed in $V^{\mathbb{P}_\alpha}$.
(3) In $V^{\mathbb{P}_\alpha}$ for every $\lambda < \kappa$, $2^\lambda \leq \kappa$.

If α is a successor, $\alpha = \beta + 1$ and $\mathbb{P}_\beta$-has been defined, then:

(1) Let $\dot{\mathbb{Q}}_\beta$ be a $\mathbb{P}_\beta$ name for $\mathbb{C}(\kappa)$, i.e. the forcing notion for adding κ Cohen reals and let $\mathbb{P}_{\beta+1} = \mathbb{P}_\alpha = \mathbb{P}_\beta * \dot{\mathbb{Q}}_\beta$. Then

(a) By Corollary 3.5.4 the family $\mathcal{H}$ is unbounded in $V^{\mathbb{P}_\alpha}$.

(b) Since $\mathbb{P}_\alpha$ is *ccc*, by Remark 3.5.1 $\mathcal{H}$ is $<^*$-directed in $V^{\mathbb{P}_\alpha}$.

(c) Forcing notions with the countable chain condition do not collapse cardinals and so $\forall\lambda < \kappa(2^\lambda \leq \kappa)$.

(d) In $V^{\mathbb{P}_\alpha}$ the covering number of the meager ideal $\mathcal{M}$ is κ.

(2) Therefore in $V^{\mathbb{P}_\alpha}$ the hypothesis of Lemma 3.4.1 holds and so there is a centered family of pure conditions C such that $Q(C)$ preserves $\mathcal{H}$ unbounded and adds a real not spit by $[\omega]^\omega \cap V^{\mathbb{P}_\alpha}$. Let $\dot{\mathbb{Q}}_\alpha$ be a $\mathbb{P}_\alpha$-name for $Q(C)$ and $\mathbb{P}_{\alpha+1} = \mathbb{P}_\alpha * \dot{\mathbb{Q}}_\alpha$. Then:

(a) By part (2) of Lemma 3.4.1, $\mathcal{H}$ is unbounded in $V^{\mathbb{P}_{\alpha+1}}$.

(b) Since $\mathbb{P}_{\alpha+1}$ is *ccc*, $\mathcal{H}$ remains $<^*$-directed in $V^{\mathbb{P}_{\alpha+1}}$.

(c) Also by the *ccc* of $\mathbb{P}_{\alpha+1}$ $\forall\lambda < \kappa(2^\lambda \leq \kappa)$.

(3) In $V^{\mathbb{P}_{\alpha+1}}$ let $\mathcal{A} \subseteq {}^\omega\omega$ be an unbounded family, $|\mathcal{A}| < \kappa$ and let $\dot{\mathbb{Q}}_{\alpha+1}$ be a $\mathbb{P}_{\alpha+1}$-name for $\mathbb{H}(\mathcal{A})$. Let $\mathbb{P}_{\alpha+2} = \mathbb{P}_{\alpha+1} * \dot{\mathbb{Q}}_{\alpha+1}$.

(a) Then since $|\mathbb{H}(\mathcal{A})| = |\mathcal{A}| < \kappa$, by Lemma 3.5.5 the family $\mathcal{H}$ remains unbounded in $V^{\mathbb{P}_{\alpha+2}}$.

(b) Since $\mathbb{H}(\mathcal{A})$ is *ccc*, by Remark 3.5.1 every subfamily of $\mathcal{H}$ of size less than κ is dominated by an element of $\mathcal{H}$.

(c) Again by the *ccc* of $\mathbb{P}_{\alpha+2}$, in $V^{\mathbb{P}_{\alpha+2}}$ for all $\lambda < \kappa(2^\lambda \leq \kappa)$.

(d) Furthermore $\mathcal{A}$ is bounded in $V^{\mathbb{P}_{\alpha+2}}$.

With this the inductive construction is complete. Let $\mathbb{P} = \mathbb{P}_{\kappa^+}$ be the finite support iteration $\langle\langle\mathbb{P}_\alpha : \alpha \leq \kappa^+\rangle, \langle\dot{\mathbb{Q}}_\alpha : \alpha < \kappa^+\rangle\rangle$. Then $\mathbb{P}$ is a *ccc* forcing notion and in $V^{\mathbb{P}}$ we have that $2^\omega = \kappa^+$. Let $\mathcal{A}$ be a subfamily of $[\omega]^\omega \cap V^{\mathbb{P}}$ of cardinality less than κ^+. Then by Lemma 3.5.7 there is $\alpha < \kappa^+$ such that $\mathcal{A} \subseteq V[G_\alpha]$ where $G_\alpha = G \cap \mathbb{P}_\alpha$ and G is a $\mathbb{P}$-generic filter over V. Then by the inductive construction of $\mathbb{P}$, in $V[G_{\alpha+3}]$ there is a real which is not split by $\mathcal{A}$. Therefore $V^{\mathbb{P}} \vDash \mathfrak{s} = \kappa^+$. By Theorem 3.5.2 and the construction of $\mathbb{P}$ the family $\mathcal{H}$ is unbounded in $V^{\mathbb{P}}$. Since every family of reals in $V^{\mathbb{P}}$ of size less than κ is obtained at some initial stage of the iteration, using a suitable bookkeeping device (by step (3) of the successor case in the inductive construction of $\mathbb{P}$) one can guarantee that any such subfamily is bounded in $V^{\mathbb{P}}$ and so $V^{\mathbb{P}} \vDash \mathfrak{b} = \kappa$. $\square$

Remark 3.6.4. Alternatively in the model V defined in the proof of Theorem 3.6.3 one can define a finite support iterated forcing construction $\langle\langle \mathbb{P}_\alpha : \alpha \leq \kappa^+\rangle, \langle \dot{\mathbb{Q}}_\alpha : \alpha < \kappa^+\rangle\rangle$ such that for every $\alpha < \kappa^+$, $\Vdash_{\mathbb{P}_\alpha}$ "$\dot{\mathbb{Q}}_\alpha$ is *ccc* and $|\dot{\mathbb{Q}}_\alpha| = \mathfrak{c}$" as follows.

If α is a limit and $\mathbb{P}_\beta, \dot{\mathbb{Q}}_\beta$ have been defined for every $\beta < \alpha$ let $\mathbb{P}_\alpha$ be the finite support iteration of $\langle \mathbb{P}_\beta, \dot{\mathbb{Q}}_\beta : \beta < \alpha\rangle$. If $\alpha = \beta + 1$ and $\mathbb{P}_\beta$ has been defined, then let $V_\beta = V^{\mathbb{P}_\beta}$ and let $\mathbb{H}_1$ be the forcing notion for adding κ Cohen reals. Then in $V_\beta^{\mathbb{H}_1}$ by Lemma 3.5.4, $\mathcal{H}$ is unbounded and $\forall\lambda < \kappa(2^\lambda \leq \kappa)$, $\mathrm{cov}(\mathcal{M}) = \kappa$. Therefore in $V_\beta^{\mathbb{H}_1}$ the hypothesis of Lemma 3.4.1 hold and so there is a centered family of pure conditions C such that $Q(C)$ adds a real not split by $V_\beta^{\mathbb{H}_1} \cap [\omega]^\omega$ (and so not split by $V_\beta \cap [\omega]^\omega$) and preserves $\mathcal{H}$ unbounded. Then let $\dot{\mathbb{H}}_2$ be a $\mathbb{H}_1$-name for $Q(C)$ and in $V_\beta^{\mathbb{H}_1 * \dot{\mathbb{H}}_2}$ let $\mathcal{A} \subseteq V_\beta \cap {}^\omega\omega$ be an unbounded family of cardinality less than κ. Let $\dot{\mathbb{H}}_3$ be a $\mathbb{H}_1 * \dot{\mathbb{H}}_2$ name for $\mathbb{H}(\mathcal{A})$. Then in $V_\beta^{(\mathbb{H}_1 * \dot{\mathbb{H}}_2) * \dot{\mathbb{H}}_3}$ the family $\mathcal{A}$ is dominated. Since $|\mathbb{H}(\mathcal{A})| < \kappa$, by Lemma 3.5.5 the family $\mathcal{H}$ is unbounded. Let $\dot{\mathbb{Q}}_\beta$ be a $\mathbb{P}_\beta$-name for $(\mathbb{H}_1 * \dot{\mathbb{H}}_2) * \dot{\mathbb{H}}_3$, and let $\mathbb{P}_\alpha = \mathbb{P}_\beta * \dot{\mathbb{Q}}_\beta$.

CHAPTER 4

Symmetry

4.1. $Q(C)$ which preserves unboundedness

Suppose for every unbounded family of reals $\mathcal{H} \subseteq {}^{\omega}\omega$ there is a centered family of pure conditions $C = C_{\mathcal{H}}$ in the partial order Q such that $Q(C)$ adds a real not split by the ground model reals and at the same time preserves $\mathcal{H}$ unbounded. Then let V be a model of GCH and V_1 a generic extension obtained by adding κ Hechler reals $\mathcal{H}$ (for κ-regular uncountable cardinal). Proceed with a finite support iteration $\langle \mathbb{Q}_\alpha : \alpha \leq \lambda \rangle$ of length λ over V_1 where

(1) for every even α, $\mathbb{Q}_\alpha = Q(C_\alpha)$ for C_α a centered family of pure conditions such that $Q(C_\alpha)$ preserves $\mathcal{H}$ unbounded and adds a real not split by the ground model reals, and
(2) for every odd α, $\mathbb{Q}_\alpha = \mathbb{H}(\mathcal{A}_\alpha)$ is the Hechler forcing associated with a family of reals $\mathcal{A}_\alpha$ obtained at a previous stage of the iteration which is of cardinality less than κ.

Then $V_1^{\mathbb{Q}_\lambda}$ would satisfy $\mathfrak{b} = \kappa < \mathfrak{s} = \lambda$. However there are certain difficulties in obtaining such centered family of pure conditions. One may try to proceed along the lines of Theorem 3.3.2, dropping the requirement that $|C| < |\mathcal{H}|$. Then it would be sufficient to guarantee that for every $X \in C_1$ the intersection $I_X = J_X \cap J$ is infinite, where C_1, J_X, J are defined as in the proof of Theorem 3.3.2. For this it would be sufficient to provide a filter G in $\mathbb{P} = \mathbb{P}(C_1, T_1, \dot{f})$ meeting

$$D_J(X, n) = \{\bar{r} \in \mathbb{P} : \exists r_i \in \bar{r} \text{ s.t. } i \in J, r_i \leq X \text{ and } \|r_i\| \geq n\}$$

for all $X \in C_1$, $n \in \omega$. It is not difficult to show that for all $A \in [\omega]^\omega$, $X \in C_1$ the corresponding set $D_A(X, n)$ is dense in $\mathbb{P}(C_1, T_1, \dot{f})$. Therefore the existence of such a filter would require meeting continuum many dense sets which is not possible. One way to sidestep this difficulty is exactly what is done in Theorem 3.3.2, namely to require that every subfamily of $\mathcal{H}$ of cardinality smaller than $|\mathcal{H}|$ is dominated by an element of $\mathcal{H}$ and to consider only centered families of cardinality less than the cardinality of the unbounded family $\mathcal{H}$. As is established in chapter III this leads to the consistency of $\mathfrak{b} = \kappa < \mathfrak{s} = \kappa^+$ for arbitrary regular uncountable κ. Observe that the restrictions on the unbounded family $\mathcal{H}$ as well as the centered family C, prevent further iteration and so also further generalization of the same construction.

Another way to sidestep the difficulty in preserving a small unbounded family unbounded, is to consider generic centered families, that is centered families of names for pure conditions (see Theorem 5.4.1). Let Γ be a set of ordinals. Then $\mathbb{C}(\Gamma)$ denotes the forcing notion of all partial functions from $\Gamma \times \omega$ to ω with extension relation reverse inclusion. That is $p \leq q$ if $q \subseteq p$ and so in particular $\mathbb{C}(\omega_2)$ is the forcing notion for adding ω_2 Cohen reals. In the last three chapters we will examine the existence of a countably closed forcing notion which has the $\aleph_2$-chain condition and which adds a centered family C of $\mathbb{C}(\omega_2)$-names for pure conditions, such that $Q(C)$ preserves the first ω_1 Cohen reals unbounded and adds a real not split by $V^{\mathbb{C}(\omega_2)} \cap [\omega]^\omega$. Consider the following partial order.

DEFINITION 4.1.1. Let $\mathbb{P}'$ be the forcing notion of all pairs $p = (\Gamma_p, C_p)$ where Γ_p is a countable subset of ω_2 and C_p is a countable centered family of $\mathbb{C}(\Gamma_p)$-names for pure conditions with extension relation defined as follows. For every p and q in $\mathbb{P}'$ let $p \leq q$ if $\Gamma_q \subseteq \Gamma_p$ and $\Vdash_{\mathbb{C}(\Gamma_p)}$ "$C_q \subseteq Q(C_p)$".

Then in particular $\mathbb{P}'$ is countably closed. Note that if G is $\mathbb{P}'$-generic then $C_G = \cup\{C_p : p \in G\}$ is centered family of $\mathbb{C}(\omega_2)$-names for pure conditions. Let G_0 be $\mathbb{C}(\omega_2)$-generic filter. Then it would be sufficient to guarantee that

$$U_H = \cup\{u : \exists X \text{ s.t. } (u, X) \in H\}$$

where H is $Q(C_G)$-generic over $V[G_0][G]$ is not split by

$$V^{\mathbb{C}(\omega_2)} \cap [\omega]^\omega = V^{\mathbb{C}(\omega_2)\times\mathbb{P}'} \cap [\omega]^\omega.$$

This amounts to obtaining the following Lemma:

LEMMA 4.1.2. *Let Γ be a countable subset of ω_2, C a countable centered family of $\mathbb{C}(\Gamma)$-names for pure conditions. Let $\dot{A}$ be a $\mathbb{C}(\Gamma)$-name for an infinite subset of ω. Let G be a $\mathbb{C}(\Gamma)$-generic filter. Then in $V[G]$ there is a pure condition X such that int$(X) \subseteq A$ or int$(X) \subseteq A^c$ and a countable centered family C' extending C below X.*

The second task is to preserve the collection of the first ω_1 Cohen reals unbounded. Note that equivalently we might aim in preserving unbounded any subfamily of the Cohen reals of size ω_1. Let $\dot{f}$ be a $\mathbb{C}(\omega_2) * Q(C_G)$-name for a real in $V[G]$. Then there is a countable subset Γ of ω_2 and a countable centered family $C \subseteq C_G$ of $\mathbb{C}(\Gamma)$-names for pure conditions such that $\dot{f}$ is a $\mathbb{C}(\Gamma) * Q(C)$-name for a real. Then it would be sufficient to show the following.

Lemma 4.1.3. *Let $\dot{f}$ be a $\mathbb{C}(\Gamma_p) * Q(C_p)$-name for a real, let $\delta \in \omega_1 \backslash \Gamma_p$ and let $\dot{h} = \cup \dot{G}_\delta$ where $\dot{G}_\delta$ is the $\mathbb{C}(\{\delta\})$-canonical name for the generic filter. Then there is a countable centered family C' of $\mathbb{C}(\Gamma \cup \{\delta\})$-names for pure conditions extending C such that for every centered family C'' of $\mathbb{C}(\omega_2)$-names for pure condition which extends C', $\Vdash_{\mathbb{C}(\omega_2)*Q(C'')}$ "$\dot{h} \not\leq^* \dot{f}$".*

Observe that if $\dot{f}$ is a $\mathbb{C}(\Gamma) * Q(C)$-name for a real, where Γ is a countable subset of ω_2, C is a countable centered family of $\mathbb{C}(\Gamma)$-names for pure conditions, then for every $\Gamma' \in [\omega_2]^\omega$, such that $\Gamma \subseteq \Gamma'$, $\dot{f}$ is also a $\mathbb{C}(\Gamma') * Q(C)$-name for a real. However if C' is a centered family of $\mathbb{C}(\Gamma')$-names for pure conditions extending C, that is $\Vdash_{\mathbb{C}(\Gamma')} C \subseteq Q(C')$ then it is not necessarily the case that $\dot{f}$ is a $\mathbb{C}(\Gamma') * Q(C')$-name for a real. Lemma 4.1.3 holds, as it will be shown later, for names $\dot{f}$ which are *good* in the sense that whenever C' is as above, $\dot{f}$ is also a $\mathbb{C}(\Gamma') * Q(C')$-name. An important point in preservation of the first ω_1 Cohen reals is the fact that for every $\mathbb{C}(\omega_1) * Q(C_G)$-name for a real $\dot{f}$, there is $a \in G$ such that $\dot{f}$ is a good $\mathbb{C}(\Gamma_a) * Q(C_a)$-name for a real (see discussion following Definition 5.4.2).

The main difficulty in realizing this project is the $\aleph_2$-chain condition. Work in a model of CH and consider a collection $\{p_\alpha : \alpha \in I\}$

of $\aleph_2$-many elements of $\mathbb{P}'$. For every $\alpha \in I$ let $\Gamma_\alpha = \Gamma_{p_\alpha}$, $C_\alpha = C_{p_\alpha}$. By CH and passing to a subset we can assume that for all $\alpha, \beta \in I$ the order types of Γ_α and Γ_β are the same. Furthermore by the Delta System Lemma we can choose a subfamily $\{p_\alpha : \alpha \in J\}$ for some $J \subseteq I$, $|J| = \aleph_2$ such that for all $\alpha, \beta \in J$, $\Gamma_\alpha \cap \Gamma_\beta = \Delta$, $\sup \Delta < \min \Gamma_\alpha \backslash \Delta$ and for all $\alpha < \beta$ in J, $\sup \Gamma_\alpha \backslash \Delta < \min \Gamma_\beta \backslash \Delta$. Also we can assume that there is an isomorphism $i_{\alpha,\beta} : \Gamma_\alpha \cong \Gamma_\beta$ such that $i_{\alpha,\beta} \restriction \Delta = \mathrm{id}$ and $\{i_{\alpha,\beta}(X) : X \in C_\alpha\} = C_\beta$. Therefore it is sufficient to obtain:

Lemma 4.1.4. *Let p, q be conditions in $\mathbb{P}'$ such that $\Delta = \Gamma_p \cap \Gamma_q$, $\sup \Delta < \min \Gamma_p \backslash \Delta < \sup \Gamma_p \backslash \Delta < \min \Gamma_q \backslash \Delta$ and there is an isomorphism $i : \Gamma_p \cong \Gamma_q$, such that $i \restriction \Delta = id$ and $C_q = \{i(X) : X \in C_p\}$. Then there is $r \in \mathbb{P}'$ such that $r \leq p$ and $r \leq q$.*

The main argument of Lemma 4.1.4 is the claim below.

Lemma 4.1.5. *Let $X \in C_p$. Then there is a $\mathbb{C}(\Gamma_p \cup \Gamma_q)$-name for a pure condition $\tilde{X}$ such that $\Vdash_{\mathbb{C}(\Gamma_p \cup \Gamma_q)} \tilde{X} \leq \dot{X}$ and $\tilde{X} \leq i(\dot{X})$.*

Indeed if 4.1.5 holds, then $r = (\Gamma_r, C_r)$ where $\Gamma_r = \Gamma_p \cup \Gamma_q$ and $C_r = C_p \cup C_q \cup \{\tilde{X}_X : X \in C_p\}$ where for every $X \in C_p$, $\tilde{X}_X$ is $\mathbb{C}(\Gamma_p \cup \Gamma_q)$-name for a pure condition extending X and $i(X)$ would be a common extension of p and q. In order to guarantee Lemma 4.1.5, we have to impose certain combinatorial property on the names for pure conditions (see Definition 4.3.2 and Definition 6.1.3). We refer to names that have this property as *symmetric* since one of its defining characteristics is that different evaluations of the name are compatible pure conditions (see Lemma 4.5.1). The same combinatorial property can be imposed on names for infinite sets of integers (Definition 4.2.2). What might be considered of independent interest is the fact that in every Cohen generic extension the collection of subsets of ω which do not have symmetric names forms an ideal (see Corollary 4.2.10). Furthermore we have to accomplish the entire construction, in particular define a partial order analogous to 4.1.1 and obtain statements analogous to Lemmas 4.1.2, 4.1.3, 4.1.4 and 4.1.5 remaining within the class of names for pure conditions which have the given combinatorial property. In chapters IV and V we develop a particular case of this combinatorial property, which completes the $\aleph_2$-chain condition in case that the root of the Delta system is empty and establish the construction within the class of names for pure conditions which have this property - see Lemma 4.4.6, Theorem 5.4.1 and Lemma 5.4.3. In the last chapter we give a generalization of this combinatorial property (Definitions 6.1.3 and 6.1.1) and demonstrate the chain condition for non-empty root (Lemma 6.2.2).

4.2. Symmetric Names for Sets of Integers

Definition 4.2.1. Let $\dot{X}$ be a Cohen name for an infinite subset of ω. Then for every $p \in \mathbb{C}$ let $\mathrm{hull}_p \dot{X} = \{j : \exists q \leq p(q \Vdash \check{j} \in \dot{X})\}$.

Definition 4.2.2. A Cohen name $\dot{X}$ for an infinite subset of ω is said to be symmetric if for every finite number of conditions $p_1, \dots, p_k$ in $\mathbb{C}$ and every $M \in \omega$, there is $m > M$ and extensions $\bar{p}_1 \leq p_1, \dots, \bar{p}_n \leq p_n$ such that for every $i \leq k$, $\bar{p}_i \Vdash \check{m} \in \dot{X}$.

LEMMA 4.2.3. *Let $\dot{X}$ be a Cohen name for an infinite subset of ω. Then $\dot{X}$ is symmetric if and only if for every finite number of conditions $p_1, \ldots, p_n$ in $\mathbb{C}$ the set $\bigcap_{i=1}^{k} hull_{p_i}(\dot{X})$ is infinite.*

EXAMPLE 4.2.4. *Every check name for an infinite subset of ω is symmetric.*

LEMMA 4.2.5. *The Cohen generic real has a symmetric name. That is if $\dot{G}$ is the canonical name for the generic filter, then $\dot{X} = \bigcup \dot{G}$ is a symmetric name.*

PROOF. Let $p_1, \ldots, p_k$ be a finite number of conditions and $n \in \omega$. Then there is $j > n$ which does not belong to the domain of of the given conditions. Then for all $\ell = 1, \ldots, k$, $q_\ell = p_\ell \cup \{(j, 1)\}$ extends of p_ℓ and $q_\ell \Vdash j \in \dot{X}$. That is $j \in \bigcap_{i=1}^{k} \mathrm{hull}_{p_i}(\dot{X})$. $\square$

In the remainder of this section it will be shown that in the Cohen extension the family of subsets of ω which do not have symmetric names forms an ideal.

LEMMA 4.2.6. *Suppose $\dot{X}$ and $\dot{Y}$ are $\mathbb{C}$-names for subsets of ω such that $\Vdash \dot{X} \subseteq \dot{Y}$ and $\dot{Y}$ is not symmetric. Then $\dot{X}$ is not symmetric.*

PROOF. Suppose $\dot{X}$ is symmetric. Since $\dot{Y}$ is not symmetric there are conditions $p_1, \ldots, p_n$ in $\mathbb{C}$ such that $\bigcap_{i=1}^{n} \mathrm{hull}_{p_i}(\dot{Y}) \subseteq M$ for some $M \in \omega$. Since $\dot{X}$ is symmetric there are extensions $\bar{p}_i \leq p_i$ and $m > M$ such that for every $i \leq n$, $\bar{p}_i \Vdash \check{m} \in \dot{X}$. Then for every $i \leq n$ $\bar{p}_i \Vdash \check{m} \in \dot{Y}$ and so $m \in \bigcap_{i=1}^{n} \mathrm{hull}_{p_i}(\dot{X})$ which is a contradiction. $\square$

DEFINITION 4.2.7. A Cohen name $\dot{X}$ is *symmetric below* a condition p if for every finite family $p_1, \ldots, p_n$ of extensions of p and $M \in \omega$ there are extensions $\bar{p}_i \leq p_i$ for all i and $m > M$ such that $\bar{p}_i \Vdash \check{m} \in \dot{X}$.

LEMMA 4.2.8. *Let $\dot{X}$ be a symmetric name for a subset of ω and let $\dot{Y}$, $\dot{Z}$ be Cohen names such that $\Vdash \dot{X} = \dot{Y} \cup \dot{Z}$. Then for every $p \in \mathbb{C}$ either there is $q \leq p$ such that $\dot{Y}$ is symmetric below q or there is $q \leq p$ such that $\dot{Z}$ is symmetric below q.*

PROOF. Suppose not. That is there is $p \in \mathbb{C}$ such that for every $q \leq p$, $\dot{Y}$ and $\dot{Z}$ are not symmetric below q. Then in particular there are extensions $p_1, \ldots, p_k$ of p and $n_0 \in \omega$ such that $\bigcap_{i=1}^{k} \mathrm{hull}_{p_i}(\dot{Y}) \subseteq n_0$. Similarly for every i there are extensions $q_{i_j} \leq p_i$ for $j = 1, \ldots, k_i$ and $n_i \in \omega$ such that $\bigcap_{i_j=1}^{k_j} \mathrm{hull}_{q_{i_j}}(\dot{Z}) \subseteq n_i$. Let $M = \max_{i \leq k} n_i$. Since $\dot{X}$ is symmetric there is $m > M$ and extensions $t_{i_j} \leq q_{i_j}$ such that for all i_j, $t_{i_j} \Vdash \check{m} \in \dot{X}$. However $\Vdash \dot{X} = \dot{Y} \cup \dot{Z}$ and so for every i_j there is an extension a_{i_j} of t_{i_j} such that $a_{i_j} \Vdash \check{m} \in \dot{Y}$ or $a_{i_j} \Vdash \check{m} \in \dot{Z}$. If there is $i \in \{1, \ldots, k\}$ such that for every a_{i_j} $(i_j = 1, \ldots, k_j)$ $a_{i_j} \Vdash \check{m} \in \dot{Z}$ we reach a contradiction with the choice of $q_{i,j}$ and $m > n_i$, since $a_{i_j} \leq q_{i_j}$ for all i_j. Otherwise for every $i = 1, \ldots, k$ there is $i_j \in \{1, \ldots, k_j\}$ such that $a_{i_j} \Vdash \check{m} \in \dot{Y}$, which is a contradiction with the choice of $p_1, \ldots, p_k$ and $m > n_0$, since $a_{i_j} \leq p_i$ for all i. □

REMARK 4.2.9. Whenever $\dot{X}$ is a $\mathbb{P}$-name for a pure condition and $p \in \mathbb{P}$, let $\dot{X} \restriction p = \{\langle \check{x}, q \rangle : q \leq p \text{ and } q \Vdash \check{x} \in \dot{X}\}$. For poset $\mathbb{P}$ and condition $p \in \mathbb{P}$ denote by $\mathbb{P}^+(p)$ the set of all extensions of p and by $\mathbb{P}(p)$ denote the set of all conditions compatible with p.

COROLLARY 4.2.10. *Let G ba a Cohen generic filter. Then the collection $\mathcal{I}_{nsym} \in V[G]$ of subsets of ω which do not have symmetric names forms an ideal.*

PROOF. Let $\Vdash \dot{X} = \dot{Y} \cup \dot{Z}$ where $\dot{X}$ is a symmetric name for an infinite subset of ω. Then the set D of all conditions $p \in \mathbb{C}$ such that $\dot{Y}$ is symmetric below p or $\dot{Z}$ is symmetric below p is dense. Let E be a maximal antichain contained in D and for every $e \in E$ define $X^* \restriction e = \dot{Y} \restriction e$ if $\dot{Y}$ is symmetric below e and let $X^* \restriction e = \dot{Z} \restriction e$ if $\dot{Z}$ is symmetric below e. Then X^* is a a Cohen name for an infinite subset of ω such that $\Vdash (X^* \subseteq \dot{X}) \wedge (X^* = \dot{Y} \vee X^* = \dot{Z})$. With every $e \in E$ associate a symmetric name X^*_e for an infinite subset of ω as follows. Let $X^*_e \restriction e = X^* \restriction e$. Let e' be a condition in E distinct from e. There is an isomorphism $i_{ee'} : \mathbb{C}^+(e) \to \mathbb{C}^+(e')$ where for every $p \in \mathbb{C}$, $\mathbb{C}^+(p) = \{q \in \mathbb{C} : q \leq p\}$. Then for every $e' \in E \backslash \{e\}$ let $X^*_e \restriction e' = i_{ee'}(X^*_e \restriction e)$.

Let G be a Cohen generic filter, $X = \dot{X}[G]$, $Y = \dot{Y}[G]$ and $Z = \dot{Z}[G]$. Then $G \cap E = \{e\}$ and so $V[G] \vDash X^*_e[G] = Y$ or $X^*_e[G] = Z$ depending on whether $\dot{Y}$ or $\dot{Z}$ is symmetric below e. Thus either Y or Z has a symmetric name. $\square$

REMARK 4.2.11. $\mathcal{I}_{nsym}$ does not contain infinite subsets from the ground model V, since every check name for an infinite subset of ω is symmetric. Note also that a symmetric name is necessarily a name for an infinite subset of ω and so every finite subset of ω belongs to $\mathcal{I}_{nsym}$.

4.3. Symmetric Names for Pure Conditions

In the following LM denotes the family of all finite logarithmic measures. For every $n \in \omega$ let L_n be the set of all finite logarithmic measures x such that $\|x\| \geq n$ and $\min \operatorname{int}(x) \geq n$. Just as in Section 4.2 we can give the following definition:

Definition 4.3.1. Let $\dot{X}$ be a Cohen name for a pure condition. Then for every $p \in \mathbb{C}$ let

$$\operatorname{hull}_p(\dot{X}) = \{x \in LM : \exists q (q \leq p)(q \Vdash \check{x} \leq \dot{X})\}.$$

Definition 4.3.2. Let $\dot{X}$ be a $\mathbb{C}$-name for a pure condition. We say that $\dot{X}$ is *symmetric* if for every $n \in \omega$ and every finite number of conditions $p_1, \ldots, p_k$ there is $x \in L_n$ and extensions $\bar{p}_1 \leq p_1, \ldots, \bar{p}_k \leq p_k$ such that for every $\ell = 1, \ldots, k$ $(\bar{p}_\ell \Vdash \check{x} \leq \dot{X})$.

Definition 4.3.3. A name for a pure condition $\dot{X}$ is *symmetric below* a given condition p if for every $M \in \omega$ and finite number of extensions $p_1, \ldots, p_n$ of p there are extensions $\bar{p}_1 \leq p_1, \ldots, \bar{p}_n \leq p_n$ and a measure $x \in L_M$ such that for every $\ell = 1, \ldots, n$ $\bar{p}_\ell \Vdash$ "$\check{x} \leq \dot{X}$".

Proposition 4.3.4. *Let $\dot{X}$ be a Cohen name for a pure condition. The following are equivalent:*

(1) $\dot{X}$ is symmetric.
(2) For every finite number of extension $p_1, \ldots, p_k$ of p and $n \in \omega$ the intersection $(\bigcap_{i=1}^k hull_{p_i}(\dot{X})) \cap L_n$ is nonempty.
(3) For every finite number of extensions $p_1, \ldots, p_k$ of p the set $\bigcap_{i=1}^k hull_{p_i}(\dot{X})$ contains a pure condition.

PROOF. Part (2) is just a reformulation of part (1).

Assume (1) and let $\dot{X}$ be a symmetric name for a pure condition. Let $p_1, \ldots, p_k$ be some finite number of conditions and $n \in \omega$. Then there is $x_n \in L_n$ and extensions $p_{1,n} \leq p_1, \ldots, p_{k,n} \leq p_k$ such that $p_{l,n} \Vdash \check{x_n} \leq \dot{X}$ $(\forall l \leq k)$ and so $x_n \in \bigcap_{i=1}^k \operatorname{hull}_{p_i}(\dot{X}) \bigcap L_n$. Then for $k_n = \max\{\|x_n\|, \max \operatorname{int}(x_n)\}$ there is $x_{n+1} \in L_{k_n}$, and extensions $p_{1,n+1} \leq p_1, \ldots, p_{k,n+1} \leq p_k$ such that $p_{\ell,n+1} \Vdash \check{x}_{n+1} \leq \dot{X}$ $(\forall \ell \leq k)$ and so in particular x_{n+1} belongs to $\bigcap_{i=1}^k \operatorname{hull}_{p_i}(\dot{X})$. Proceeding inductively we can choose a pure condition $\langle x_n : n \in \omega \rangle$ contained in $\bigcap_{i=1}^k \operatorname{hull}_{p_i}(\dot{X})$.

To see that (3) implies (1) fix any $p_1, \ldots, p_k$ finite number of conditions and let $n \in \omega$. By assumption $\bigcap_{i=1}^k \operatorname{hull}_{p_i}(\dot{X})$ contains a pure condition $\langle x_i : i \in \omega \rangle = R$ of logarithmic measures of strictly increasing hight. However $x_n \in L_n \bigcap (\bigcap_{i=1}^k \operatorname{hull}_{p_i}(\dot{X}))$ and so for some $\bar{p}_1 \leq p_1, \ldots, \bar{p}_k \leq p_k$ we have $\bar{p}_\ell \Vdash \check{x}_n \leq \dot{X}$ $(\forall \ell \leq k)$. □

REMARK 4.3.5. Thus a $\mathbb{C}$-name for a pure condition is not symmetric iff there are conditions $p_1, \ldots, p_k$ and $M \in \omega$ such that

$$(\cap_{i=1}^k \operatorname{hull}_{p_i}(\dot{X})) \cap L_M = \emptyset.$$

REMARK 4.3.6. If a finite logarithmic measure x does not belong to $\cap_{i=1}^k \operatorname{hull}_{p_i}(\dot{X})$ then there is an index $i \leq k$ such that $p_i \Vdash \check{x} \nleq \dot{X}$.

LEMMA 4.3.7. *Let $\dot{X}$ and $\dot{Y}$ be $\mathbb{C}$-names for pure conditions such that $\Vdash \dot{X} \leq \dot{Y}$. If $\dot{Y}$ is not symmetric, then $\dot{X}$ is not symmetric.*

PROOF. Suppose that $\dot{X}$ is symmetric, but $\dot{Y}$ is not symmetric. Then there are conditions $p_1, \ldots, p_k$ such that $\bigcap_{i=1}^k \operatorname{hull}_{p_i}(\dot{Y})$ does not contain measures of level greater than M for some $M \in \omega$. Let $j > M$. Since $\dot{X}$ is symmetric there are extensions $q_1 \leq p_1, \ldots, q_k \leq p_k$ and $x \in L_M$ such that for every $\ell \leq k$, $q_\ell \Vdash \check{x} \leq \dot{X}$. But then $q_\ell \Vdash \check{x} \leq \dot{Y}$ and so x belongs to $\bigcap_{i=1}^{k} \mathrm{hull}_{p_i}(\dot{Y})$ which is a contradiction. $\square$

4.4. An ultrafilter of Symmetric Names

DEFINITION 4.4.1. The finite logarithmic measure x is said to be *stronger* than the finite logarithmic measure y if x is of measure greater than the measure of y and $\min \mathrm{int}(x) > \max \mathrm{int}(y)$. We will denote the fact that x is stronger than y with $x > y$.

REMARK 4.4.2. Whenever p and q are incompatible conditions we will denote this by $p \perp q$.

LEMMA 4.4.3. *Let $\dot{X}$ be a symmetric Cohen name for a pure condition and let $\dot{A}$ be a name for an infinite subset of ω. Then for every $p_1, \ldots, p_k$ in $\mathbb{C}$ and every $M \in \omega$ there is a finite logarithmic measure $z \in L_M$ and extensions $\bar{p}_1 \leq p_1, \ldots, \bar{p}_k \leq p_k$ such that $\forall i \leq n$,*

$$\bar{p}_i \Vdash \text{“}\check{z} \leq \dot{X} \textit{ and } \check{z} \subseteq \dot{A}\text{”} \textit{ or } \bar{p}_i \Vdash \text{“}\check{z} \leq \dot{X} \textit{ and } \check{z} \subseteq \dot{A}^c\text{”}.$$

PROOF. Let $s_0 = 2^k + M$. Since $\dot{X}$ is symmetric there are extensions $p_{1,1} \leq p_1, \ldots, p_{1,k} \leq p_k$ and $x \in L_{s_0}$ such that for every $i \leq k$, $p_{1,i} \Vdash \check{x} \leq \dot{X}$. Let $s = \max x + 1$. Extend $p_{1,i}$ to a condition $p_{2,i}$ such that for some $a_i \subseteq s$, $p_{2,i} \Vdash (\dot{A} \restriction s) = \check{a}_i$ and so in particular if $b_i = s \backslash a_i$ then $p_{2,i} \Vdash (\dot{A}^c \restriction s) = \check{b}_i$.

In the ground model we can partition x into 2^k subsets $\{z_j : j \leq 2^k\}$ such that $\forall j \leq 2^k \forall i \leq k$ z_j is contained in a_i or b_i. Furthermore by

Lemma 2.1.3 there is $j_0 \leq 2^k$ such that the measure of $z = z_{j_0}$ is at least M. Then for every $i \leq k$ we have

$$p_{2,i} \Vdash \text{“}(\check{z} \leq \dot{X} \text{ and } \check{z} \subseteq \dot{A}) \text{ or } (\check{z} \leq \dot{X} \text{ and } \check{z} \subseteq \dot{A}^c)\text{”}.$$

Then for every $i \leq k$ there is a further extension $\bar{p}_i \leq p_{2,i}$ such that $\bar{p}_i \Vdash$ “$\check{z} \leq \dot{X}$ and $\check{z} \subseteq \dot{A}$” or $\bar{p}_i \Vdash$ “$\check{z} \leq \dot{X}$ and $\check{z} \subseteq \dot{A}^c$”. □

LEMMA 4.4.4. *Let $\dot{X}$ be a $\mathbb{C}$-symmetric name for a pure condition and $\dot{A}$ a $\mathbb{C}$-name for an infinite subset of ω. Then there is a Cohen symmetric name for a pure condition $\dot{Y}$ such that $\Vdash \dot{Y} \leq \dot{X}$ and $\forall i \in \omega$*

$$\Vdash \text{“}\mathit{int}(\dot{Y}(i)) \subseteq \dot{A} \text{ or } \mathit{int}(\dot{Y}(i)) \subseteq \dot{A}^c\text{”}.$$

PROOF. Fix an enumeration $\{p_n : n \in \omega\}$ of $\mathbb{C}$. Find an extension $p_{0,0}$ of p_0 and a finite measure x_0 such that $p_{0,0} \Vdash$ “$\check{x}_0 \leq \dot{X} \wedge \text{int}(\check{x}_0) \subseteq \dot{A}$” or “$p_{0,0} \Vdash \check{x}_0 \leq \dot{X} \wedge \text{int}(\check{x}_0) \subseteq \dot{A}^c$”. Let $A_0 = \{a_{0,s} : s \in \omega\}$ be a maximal antichain in $\mathbb{C} - \mathbb{C}(p_{0,0})$ such that for every $s \in \omega$ there is a measure $x_{0,s}$ such that $a_{0,s} \Vdash$ “$\check{x}_{0,s} \leq \dot{X} \wedge \text{int}(\check{x}_{0,s}) \subseteq \dot{A}$” or $a_{0,s} \Vdash$ “$\check{x}_{0,s} \leq \dot{X} \wedge \text{int}(\check{x}_{0,s}) \subseteq \dot{A}^c$”. Let $R_0 = \{\langle p_{0,0}, \check{x}_0 \rangle\} \cup \{\langle a_{0,s}, \check{x}_{0,s} \rangle : s \in \omega\}$. Proceed inductively. Suppose we have defined conditions $\{p_{n-1,\ell}\}_{\ell \in n}$ and a finite logarithmic measure x_{n-1} such that for every $\ell \in n$, $p_{n-1,\ell} \leq p_{n-1}$ and $p_{n-1,\ell} \Vdash$ “$\check{x}_{n-1} \leq \dot{X} \wedge \text{int}(\check{x}_{n-1}) \subseteq \dot{A}$” or $p_{n-1,\ell} \Vdash$ “$\check{x}_{n-1} \leq \dot{X} \wedge \text{int}(\check{x}_{n-1}) \subseteq \dot{A}^c$”. Furthermore suppose we have defined a maximal antichain $A_{n-1} = \{a_{n-1,s} : s \in \omega\}$ in $\mathbb{C} - \mathbb{C}(\{p_{n-1,\ell}\}_{\ell \in n})$ such that for every $s \in \omega$ there is a finite logarithmic measure $x_{n-1,s}$ such that $a_{n-1,s} \Vdash$ “$\check{x}_{n-1,s} \leq \dot{X} \wedge \text{int}(\check{x}_{n-1,s}) \subseteq \dot{A}$” or

$a_{n-1,s} \Vdash$ "$\check{x}_{n-1,s} \leq \dot{X} \wedge \mathrm{int}(\check{x}_{n-1,s}) \subseteq \dot{A}$" and finally we have defined $R_{n-1} = \{\langle \check{x}_{n-1}, p_{n-1,\ell}\rangle\}_{\ell \in n} \cup \{\langle \check{x}_{n-1,s}, a_{n-1,s}\rangle : s \in \omega\}$.

If $p_n \perp \{p_{n-1,i}\}_{i \leq n-1}$ then there is $a_{n-1,s} \in A_{n-1}$ compatible with p_n and we can fix a common extension b_n and let $y_n = x_{n-1,s}$. Otherwise let b_n be a common extension of p_n and some $p_{n-1,j}$ for $j \in n$ and let $y_n = x_{n-1}$. By the Lemma 4.4.3 there are extensions $p_{n,0} \leq p_{n-1,0}, \dots, p_{n,n-1} \leq p_{n-1,n-1}, p_{n,n} \leq b_n$ and a finite measure x_n stronger than x_{n-1} and y_n such that $\forall i \leq n$

$$(p_{n,i} \Vdash \check{x}_n \leq \dot{X} \wedge \mathrm{int}(\check{x}_n) \subseteq \dot{A}) \text{ or } (p_{n,i} \Vdash \check{x}_n \leq \dot{X} \wedge \mathrm{int}(\check{x}_n) \subseteq \dot{A}^c).$$

Fix a maximal antichain $A_n = \langle a_{n,s} : s \in \omega\rangle$ in $\mathbb{C} - \mathbb{C}(\{p_{n,i}\}_{i \leq n})$ such that for all $s \in \omega$

(1) $\exists i^s \in \omega$ such that $a_{n,s} \leq a_{n-1,i^s}$

(2) $\exists x_{n,s}$ measure stronger than x_{n-1,i^s} such that $a_{n,s} \Vdash$ "$\check{x}_{n,s} \leq \dot{X} \wedge \mathrm{int}(\check{x}_{n,s}) \subseteq \dot{A}$" or $a_{n,s} \Vdash$ "$\check{x}_{n,s} \leq \dot{X} \wedge \mathrm{int}(\check{x}_{n,s}) \subseteq \dot{A}^c$".

Let $R_n = \{\langle p_{n,i}, \check{x}_n\rangle\}_{i \leq n} \cup \{\langle a_{n,s}, \check{x}_{n,s}\rangle : s \in \omega\}$. With this the inductive construction in complete and $\dot{Y} = \bigcup_{n \in \omega} R_n$ is the desired symmetric name for a pure condition. $\square$

REMARK 4.4.5. Note that R_i is a name for the i-th measure of $\dot{Y}$.

LEMMA 4.4.6. *Let G be a Cohen generic filter. In $V[G]$ let X be a pure condition with symmetric name $\dot{X}$ and let $A \in V[G]$ be an infinite subset of ω. Then in $V[G]$ there is a pure condition Z extending X, which has a symmetric name and such that int(Z) $\subseteq A$ or int(Z) $\subseteq A^c$.*

PROOF. Let $\dot{Y}$ be the name constructed in Lemma 4.4.4. Then there are $\mathbb{C}$ names $\dot{R}$ and $\dot{T}$ such that $\Vdash \dot{R} = \langle \dot{Y}(i) : \dot{Y}(i) \subseteq \dot{A} \rangle$ and $\Vdash \dot{T} = \langle \dot{Y}(i) : \dot{Y}(i) \subseteq \dot{A}^c \rangle$. Then $\Vdash \dot{R} \cup \dot{T} = \dot{Y}$. We claim that for every $p \in \mathbb{C}$ there is an extension $q \leq p$ such that $\dot{R}$ is symmetric below q or $\dot{T}$ is symmetric below q.

Suppose not and let p be a condition which does not have an extension with the desired properties. Then there are extensions $p_1, \ldots, p_k$ of p such that for some n_0, $(\bigcap_{i=1}^{k} \mathrm{hull}_{p_i}(\dot{R}) \bigcap L_{n_0} = \emptyset$ and respectively for every $i \leq k$ there are $\{q_{i,j}\}_{j=1}^{\ell_i} \subseteq \mathbb{C}^+(p_i)$ such that for some n_i, $(\bigcap_{j=1}^{\ell_i} \mathrm{hull}_{q_{i,j}}(\dot{T})) \bigcap L_{n_i} = \emptyset$. By construction of $\dot{Y}$ there are extensions $\bar{q}_{i,j} \leq q_{i,j}$ and a measure x of level higher than $\{n_0, \ldots, n_k\}$ such that $\bar{q}_{i,j} \Vdash \check{x} \in \dot{Y}$. Then for all i, j $q_{i,j} \Vdash \check{x} \in \dot{R} \cup \dot{T}$ and so there is a further extension $t_{i,j} \leq q_{i,j}$ such that $t_{i,j} \Vdash \check{x} \in \dot{R}$ or $t_{i,j} \Vdash \check{x} \in \dot{T}$.

If for every $i \leq k$ there is some $j \leq \ell_i$ such that $t_{i,j} \Vdash \check{x} \in \dot{R}$, then since $t_{i,j} \leq p_i$ we obtain that x is in $\bigcap_{i=1}^{k} \mathrm{hull}_{p_i}(\dot{R})$ which is a contradiction since x is of measure greater than n_0. Otherwise, there is some $i \leq k$ such that $\forall j = 1, \ldots, \ell_i$ $t_{i,j} \Vdash \check{x} \in \dot{T}$. But then x is in $\bigcap_{j=1}^{\ell_i} \mathrm{hull}_{q_{i,j}} \dot{T}$ which is a contradiction since the measure of x is greater than n_i.

Therefore the set D of all $p \in \mathbb{C}$ such that $\dot{R}$ or $\dot{T}$ is symmetric below p is dense in $\mathbb{C}$. Fix a maximal antichain E contained in D. Define Y^* as follows: for every $e \in E$ let $Y^* \restriction e = \dot{R} \restriction e$ if $\dot{R}$ is symmetric below e and let $Y^* \restriction e = \dot{T} \restriction e$ otherwise. Then

$$\Vdash (Y^* \leq Y) \wedge (\mathrm{int}(Y^*) \subseteq \dot{A} \vee \mathrm{int}(Y^*) \subseteq \dot{A}^c).$$

Furthermore for every $e \in E$ let Y_e be a symmetric name defined as follows. Let $Y_e \restriction e = \dot{R} \restriction e$ if $\dot{R}$ is symmetric below e and let $Y_e \restriction e' = i_{ee'}(Y_e \restriction e)$ for every $e' \in E \backslash \{e\}$, where $i_{ee'}$ is an isomorphism of $\mathbb{C}^+(e)$ and $\mathbb{C}^+(e')$. Note that for every $e \in E$, $\Vdash Y_e = \dot{R} \vee Y_e = \dot{T}$. Now, since G is Cohen generic there is $e \in G \cap E$. Then Y_e is a symmetric name for $\dot{R}[G]$ or $\dot{T}[G]$, and so in particular for an extension of X the underlying infinite set of which is contained in A or in A^c. $\square$

As a straightforward generalization of the above one obtains:

Corollary 4.4.7. *Let G be a Cohen generic filter. If, in $V[G]$ X is a pure condition with a symmetric name and $A_0 \cup \cdots \cup A_{n-1}$ is a partition of ω into finitely many sets, then there is a pure condition Y extending X which has a symmetric name and such that int$(Y) \subseteq A_j$ for some $j \in \omega$.*

In particular we obtain the following result:

Corollary 4.4.8. *Let G be a Cohen generic filter and let X be a pure condition in $V[G]$ with symmetric name $\dot{X}$. Let $A \in V[G] \cap [\omega]^\omega$ which does not have a symmetric name. Then in $V[G]$ there is a pure condition Y extending X such that int$(Y) \subseteq A^c$.*

Proof. By Lemma 4.4.6 there is a symmetric name $\dot{Y}$ for a pure condition such that in $V[G]$, $\dot{Y}[G] = Y \leq X$ and $\mathrm{int}(Y) \subseteq A$ or $\mathrm{int}(Y) \subseteq A^c$. Suppose $V[G] \models \mathrm{int}(Y) \subseteq A$. Let $\dot{A}$ be a Cohen name for A and let $p \in G$ be a condition in G such that $p \Vdash \mathrm{int}(\dot{Y}) \subseteq \dot{A}$. If $\dot{A}$ is symmetric below p then the set A does have a symmetric name, which is a contradiction to the hypothesis of the lemma. Therefore there is a finite set of extensions of p, $p_1, \ldots, p_k$ such that $\bigcap_{i=1}^k \mathrm{hull}_{p_i}(\dot{A}) \subseteq M$ for some $M \in \omega$. However $\dot{Y}$ is symmetric and so there is $x \in L_M$ and extensions $q_i \leq p_i$ such that for every i $q_i \Vdash \check{x} \leq \dot{Y}$. Then for every i, $q_i \Vdash \check{x} \subseteq \dot{A}$ which implies that $\mathrm{int}(x) \subseteq \bigcap_{i=1}^k \mathrm{hull}_{p_i}(\dot{A})$. This is a contradiction, since $x \in L_M$ implies that $\min \mathrm{int}(x) \geq M$. $\square$

In his original work from 1984 S. Shelah works with a restriction of the partial order Q to a suborder $Q[I]$, where I is an ideal on $\mathcal{P}(\omega)$ containing all finite subsets, which consists of all conditions (a, T) in Q, where $T = \langle t_i : i \in \omega \rangle$, having the property that for every A in I the sequence $T \cap A^c = \langle t_i : \mathrm{int}(t_i) \cap A = \emptyset \rangle$ is a pure condition. If G is a Cohen generic filter, in $V[G]$ the collection I_{nsym} of subsets of ω which do not have symmetric names forms an ideal (containing all finite subsets of ω) and so in $V[G]$ we can consider the analogous partial order $Q_s[I_{nsym}]$ where Q_s is the suborder of Q consisting of conditions with symmetric name for the pure part.

Corollary 4.4.9. *Let G be a Cohen generic filter. Then*

$$V[G] \vDash (Q_s = Q_s[I_{nsym}]).$$

Proof. In $V[G]$ let $X = \langle x_i : i \in \omega \rangle$ be a pure condition with symmetric name $\dot{X}$ and let A be a subset of ω which does not have a symmetric name. Let $\dot{Z}$ be a name for $Z = \langle x_i : \mathrm{int}(x_i) \cap A = \emptyset \rangle$. By Lemma 4.4.8 there is a pure condition $Y \leq X$ which has a symmetric name $\dot{Y}$ such that $\mathrm{int}(Y) \subseteq A$. Then $Y \leq Z$ and so there is a condition

$p \in G$ such that $p \Vdash \dot{Y} \leq \dot{Z}$. By Lemma 4.3.7 $\dot{Z}$ is symmetric below p, which implies that Z has a symmetric name. $\square$

Whenever $\dot{X}$ is a $\mathbb{P}$-name for an infinite subset of ω (resp. a $\mathbb{P}$-name for a pure condition) where $\mathbb{P}$ is a forcing notion, such that for every finite set of conditions $p_1, \ldots, p_n$ in $\mathbb{P}$ and integer M there are extensions $\bar{p}_1 \leq p_1, \ldots, \bar{p}_n \leq p_n$ and $m > M$ (resp. a finite logarithmic measure $x \in L_M$) such that for all $j = 1, \ldots, n$ $\bar{p}_j \Vdash \check{m} \in \dot{X}$ (resp. $\bar{p}_j \Vdash \check{x} \leq \dot{X}$) we will say that the name $\dot{X}$ is symmetric. Also if we want to emphasize that $\dot{X}$ is a $\mathbb{P}$-name, we will say that $\dot{X}$ is $\mathbb{P}$-symmetric.

4.5. Extending Different Evaluations

LEMMA 4.5.1. *Let $\dot{X}$ be a Cohen symmetric name for a pure condition. Let $\mathbb{C}_n = \mathbb{C} \times \cdots \times \mathbb{C}$ be the product of n-copies of $\mathbb{C}$. Then there is a $\mathbb{C}_n$-symmetric name for a pure condition $\tilde{X}$ such that for all $\mathbb{C}_n$-generic filters G, for all $j = 1, \ldots, n$ $V[G] \models \tilde{X}[G] \leq \dot{X}[G^j]$, where G^j is the j-th projection of G.*

PROOF. Let $\{p_m\}_{m \in \omega}$ be an enumeration of $\mathbb{C}_n$. Then for every $m \in \omega$, $p_m = (p_m^1, \ldots, p_m^n)$ where $p_m^j \in \mathbb{C}$. Consider p_1. Since $\dot{X}$ is $\mathbb{C}$-symmetric name there are extensions $p_{1,1}^1 \leq p_1^1, \ldots, p_{1,1}^n \leq p_1^n$ and a finite logarithmic measure x_1 such that for every $j = 1, \ldots, n$, $p_{1,1}^j \Vdash \check{x}_1 \leq \dot{X}$. Let $p_{1,1} = (p_{1,1}^1, \ldots, p_{1,1}^n)$ and let $R_1' = \{\langle p_{1,1}, \check{x}_1 \rangle\}$. Fix a maximal antichain of conditions $A_1 = \{a_{1,s} : s \in \omega\}$ in $\mathbb{C}_n - \mathbb{C}_n^+(p_{1,1})$ such that $\forall s \in \omega$, there is a finite logarithmic measure $x_{1,s}$ such that for every $j = 1, \ldots, n$, $a_{1,s}^j \Vdash \check{x}_{1,s} \leq \dot{X}$. Let $R_1'' = \{\langle a_{1,s}, \check{x}_{1,s} \rangle : s \in \omega\}$ and let $R_1 = R_1' \cup R_1''$.

Suppose we have defined R_{m-1}. Consider $p_{m-1,1},\dots,p_{m-1,m-1}$ and p_m. If $p_m \perp \{p_{m-1,\ell}\}_{\ell=1}^{m-1}$, then there is $a_{m-1,s}\in A_{m-1}$ such that $a_{m-1,s}\not\perp p_m$ with common extension b_m. Let $y_m = x_{m-1,s}$. Otherwise there is $j\in\{1,\dots,m-1\}$ such that $p_{m-1,j}\not\perp p_m$ with common extension which we again denote b_m. In this case let $y_m = x_{m-1}$. By symmetry of $\dot{X}$ there are extensions $p^j_{m,\ell}\le p^j_{m-1,l}$ for $1\le \ell\le m-1, 1\le j\le n$ and $p^j_{m,m}\le b^j_m$ $(\forall j: 1\le j\le n)$ and a finite logarithmic measure x_m which is stronger than $\{x_{m-1},y_m\}$ such that for all j,ℓ $p^j_{m,\ell}\Vdash \check{x}_m\le\dot{X}$. Then for every $\ell=1,\dots,m$ let $p_{m,\ell}=(p^1_{m,\ell},\dots,p^n_{m,\ell})$ and let $R'_m=\{\langle p_{m,\ell},\check{x}_m\rangle\}_{\ell=1}^m$. Just as in the base case let $A_m=\{a_{m,s}:s\in\omega\}$ be a maximal antichain in $\mathbb{C}_n-\mathbb{C}^+_n(\{p_{m,\ell}\}_{\ell=1}^m)$ such that for all $s\in\omega$

(1) $\exists i^s\in\omega$ such that $a_{m,s}\le a_{m-1,i^s}$

(2) $\exists x_{m,s}$ stronger than x_{m-1,i^s} such that for all $j=1,\dots,n$, $a^j_{m,s}\Vdash\check{x}_{m,s}\le\dot{X}$.

Let $R''_m=\{\langle a_{m,s},\check{x}_{m,s}\rangle\}_{s\in\omega}$ and let $R_m=R'_m\cup R''_m$. With this the inductive construction id complete and we can define $\tilde{X}=\cup_{m\in\omega}R_m$.

Let G be $\mathbb{C}_n$-generic. Then $G\cap A_1$ contains some $a_{1,s}$ (or $p_{1,1}$) and so $\tilde{X}[G](1)=x_{1,s}$ (resp. $\tilde{X}[G](1)=x_1$). However for every $j=1,\dots,n$, $a^j_{1,s}\in G^j$ and so since $a^j_{1,s}\Vdash\check{x}_{1,s}\le\dot{X}$ we have $x_{1,s}\le\dot{X}[G^j]$ (similarly $x_1\le\dot{X}[G^j]$). The same argument holds for every $m\in\omega$. Indeed if $a_{m,s}\in G\cap A_m$, then $\tilde{X}[G](m)=x_{m,s}$. But for all $j=1,\dots,n$, $a^j_{m,s}\in G^j$ and since $a^j_{m,s}\Vdash(\check{x}_{m,s}\le\dot{X})$ we obtain $\check{x}_{m,s}=\tilde{X}[G](m)\le\dot{X}[G^j]$. Therefore $\tilde{X}[G]$ is a pure condition which is a common extension of $\dot{X}[G^1],\dots,\dot{X}[G^n]$.

The name $\tilde{X}$ is $\mathbb{C}_n$-symmetric. Consider any finite number of conditions $p^1, \dots, p^n$ and $M \in \omega$. In the fixed enumeration of $\mathbb{C}_n$ for every $j = 1, \dots, n$ there is i_j such that $p^j = p_{i_j}$. Then there is $k \in \omega$ such that $k > i_j$ for all j and $k > M$. Then $p_{k,i_1} \leq p_1, \dots, p_{k,i_n} \leq p_{i_n}$ and for $x_k \in L_k \subseteq L_M$ for all j we have $p_{k,i_j} \Vdash$ "$\check{x}_k \in \tilde{X}$". That is given any finite number of conditions $p^1, \dots, p^n$ and $M \in \omega$ there are extensions $q_1 \leq p^1, \dots, q_n \leq p^n$ and a finite logarithmic measure $x \in L_M$ such that $\forall l : 1 \leq l \leq n$, $q_l \Vdash \check{x} \leq \tilde{X}$. Therefore the name $\tilde{X}$ is $\mathbb{C}_n$-symmetric. □

Lemma 4.5.2. *Let $\dot{X} = \langle \dot{X}(i) : i \in \omega \rangle$ be a Cohen symmetric name for a pure condition, A an infinite subset of ω and G_0 a $\mathbb{C}_n$-generic filter. Then there is a $\mathbb{C}_n$-symmetric name for a pure condition $X^* = \langle X^*(i) : i \in \omega \rangle$ such that*

(1) int$(X^*[G_0]) \subseteq A$ or int$(X^*[G_0]) \subseteq A^c$ and

(2) $\forall m \in \omega$, $j \leq n$ $X^*_m[G_0] \leq \dot{X}_m[G^j_0]$, where $\dot{X}_m = \langle \dot{X}(i) : i \geq m \rangle$ and $X^*_m = \langle X^*(i) : i \geq m \rangle$.

Proof. Let $\{p_m\}_{m \in \omega}$ be a fixed enumeration of $\mathbb{C}_n$. Consider $p_1 = (p^1_1, \dots, p^n_1)$. Since $\dot{X}_1$ is $\mathbb{C}$-symmetric there is $x_1 \in L_1$ and extensions $p^j_{1,1} \leq p^j_1$ (for $j = 1, \dots, n$) such that $\mathrm{int}(x_1) \subseteq A$ or $\mathrm{int}(x_1) \subseteq A^c$ and for all j, $p^j_{1,1} \Vdash \check{x}_1 \leq \dot{X}_1$. Let $A_1 = \{a_{1,s} : s \in \omega\}$ be a maximal antichain in $\mathbb{C}_n - \mathbb{C}^+_n(p_{1,1})$ such that for all $s \in \omega$ there is a finite logarithmic measure $x_{1,s}$ such that $\mathrm{int}(x_{1,s}) \subseteq A$ or $\mathrm{int}(x_{1,s}) \subseteq A^c$ and for all $j = 1, \dots, n$, $a^j_{1,s} \Vdash \check{x}_{1,s} \leq \dot{X}_1$ where $a_{1,s} = (a^1_{1,s}, \dots, a^n_{1,s})$. Let $R_1 = \{\langle p_{1,1}, \check{x}_1 \rangle\} \cup \{\langle a_{1,s}, \check{x}_{1,s} \rangle : s \in \omega\}$. Suppose we have defined R_{m-1}. Consider $p_{m-1,1}, \dots, p_{m-1,m-1}$ and p_m. If $p_m \perp \{p_{m-1,\ell}\}^{m-1}_{\ell=1}$ then there is $s \in \omega$ such that $p_m \not\perp a_{m-1,s}$. In this case let b_m be their common extension and let $y_m = x_{m-1,s}$. If there is $j \leq m-1$ such that $p_m \not\perp p_{m-1,j}$ let b_m be their common extension and let $y_m = x_{m-1}$. Then there are extensions $p_{m,\ell} \leq p_{m-1,\ell}$ for every $\ell = 1, \ldots, m-1$ and $p_{m,m} \leq b_m$, and a finite logarithmic measure x_m stronger than $\{x_{m-1}, y_m\}$ such that $x_m \subseteq A$ or $x_m \subseteq A^c$ and for all $\ell = 1, \ldots, m$, for all $j = 1, \ldots, n$, $p^j_{m,\ell} \Vdash \check{x}_m \leq \dot{X}_m$ where $p_{m,\ell} = (p^1_{m,\ell}, \ldots, p^n_{m,\ell})$. Let $A_m = \{a_{m,s} : s \in \omega\}$ be a maximal antichian in $\mathbb{C}_n - \mathbb{C}_n^+(\{p_{m,\ell}\}_{\ell=1}^m)$ such that for all $s \in \omega$

(1) $\exists i^s \in \omega$ s.t. $a_{m,s} \leq a_{m-1,i^s}$,

(2) $\exists x_{m,s}$ stronger than x_{m-1,i^s} such that $x_{m,s} \subseteq A$ or $x_{m,s} \subseteq A^c$ and for all $j = 1, \ldots, n$, $a^j_{m,s} \Vdash \check{x}_{m,s} \leq \dot{X}_m$ where $a_{m,s} = (a^1_{m,s}, \ldots, a^n_{m,s})$

Let $R_m = \{\langle p_{m,\ell}, \check{x}_m \rangle\}_{\ell=1}^m \cup \{\langle a_{m,s}, \check{x}_{m,s} \rangle : s \in \omega\}$. With this the inductive construction is complete and we can define $\tilde{X} = \cup_{m \in \omega} R_m$.

To see that $\tilde{X}$ is symmetric consider any finite number of conditions $p^1, \ldots, p^m$ in $\mathbb{C}_n$ and some $M \in \omega$. Then $\forall j \leq m$, $\exists i_j \in \omega$ such that $p^j = p_{i_j}$ (in the fixed enumeration of $\mathbb{C}_n$). There is $k > M$ s.t. $k > i_j$ for all $j \leq m$. Then $p_{k,i_1} \leq p_{i_1}, \ldots, p_{k,i_m} \leq p_{i_m}$ and $x_k \in L_k \subseteq L_M$ are such that $p_{k,\ell} \Vdash \check{x}_k \leq \tilde{X}$ for all $\ell \in \{1, \ldots, k\}$ and so in particular $p_{k,\ell} \Vdash \check{x}_k \leq \tilde{X}$ $(\forall \ell \in \{i_1, \ldots, i_m\})$.

Let G be $\mathbb{C}_n$-generic. We will show that for every $j \leq n$, $\tilde{X}[G] \leq \dot{X}[G^j]$. Then $G \cap (\{p_{1,1}\} \cup A_1)$ contains some condition $a_{1,s}$ (or contains $p_{1,1}$). Then for every $j \leq n$, $a^j_{1,s} \Vdash (\check{x}_{1,s} \leq \dot{X})$ (resp. $p^j_{1,1} \Vdash (\check{x}_1 \leq \dot{X})$) and since $a^j_{1,s} \in G^j$ (resp. $p^j_{1,1} \in G^j$) $x_{1,s} = \tilde{X}[G](1) \leq \dot{X}[G^j]$ for every $j \leq n$ (resp. $x_1 = \tilde{X}[G](1) \leq \dot{X}[G^j]$). The same argument holds for every $A_k \cup \{p_{k,1}, \ldots, p_{k,k}\}$) and so $\tilde{X}[G]$ is a common extension of $\dot{X}[G^1], \ldots, \dot{X}[G^n]$. Furthermore $\tilde{X}_m = \cup_{k \geq m} R_k$ is symmetric and for every $\mathbb{C}_n$-generic filter in $V[G]$ for every $j \leq n$ we have $\tilde{X}_m[G] \leq \dot{X}_m[G^j]$. Observe that $\forall i \in \omega$

$$\Vdash_{\mathbb{C}_n} \text{“int}(\tilde{X}(i)) \subseteq \check{A} \text{ or } \text{int}(\tilde{X}(i)) \subseteq \check{A}^c\text{”}.$$

Let $\tilde{H}$, $\tilde{T}$ be $\mathbb{C}_n$-names such that $\Vdash$ “$\tilde{H} = \langle \tilde{X}(i) : \text{int}(\tilde{X}(i)) \subseteq \check{A} \rangle$” and $\Vdash$ “$\tilde{T} = \langle \tilde{X}(i) : \text{int}(\tilde{X}(i)) \subseteq \check{A}^c \rangle$”. Then following the proof of Lemma 4.4.6 obtain that for every $p \in \mathbb{C}_n$ there is $q \leq p$ such that $\tilde{H}$ or $\tilde{T}$ is symmetric below q. Let E be a maximal antichian of conditions with this property. Then for every $e \in E$ let $X^*_e \restriction e = \tilde{H} \restriction e$ if $\tilde{H}$ is symmetric below e and let $X^*_e \restriction e = \tilde{T} \restriction e$ if $\tilde{T}$ is symmetric below e. For every $e' \in E \backslash \{e\}$ let $i_{ee'} : \mathbb{C}^+_n(e) \cong \mathbb{C}^+_n(e')$ be a partial order isomorphism. Then for every $e' \in E \backslash \{e\}$ let $X^*_e \restriction e' = i_{e,e'}(X^*_e \restriction e)$. Then for every $e \in E$, X^*_e is a symmetric name for a pure condition such that $\mathbb{1} \Vdash$ “$\text{int}(X^*_e) \subseteq \check{A}$ or $\text{int}(X^*_e) \subseteq \check{A}^c$”. Furthermore $e \Vdash X^*_e \leq \tilde{X}$ and so in particular for all $m \in \omega$, $e \Vdash X^*_{e,m} \leq \tilde{X}_m$, where $\Vdash X^*_{e,m} = \langle X^*_e(i) : i \geq m \rangle$. Then $X^* = X^*_{e_0}$ where $\{e_0\} = G_0 \cap E$ is the desired symmetric name for a pure condition. $\square$

Remark 4.5.3. Note that in $V[G]$ the centered family

$$C^* = \{(X^*_e)_m[G]\}_{m \in \omega}$$

extends the centered family $C_j = \{\dot{X}_m[G^j]\}_{m \in \omega}$ for every $j = 1, \ldots, n$.

CHAPTER 5

Preserving small unbounded families

5.1. Preprocessed Names for Pure Conditions

DEFINITION 5.1.1. Let $\Gamma \in [\omega_2]^\omega$ and let C be a centered family of $\mathbb{C}(\Gamma)$-symmetric names for pure conditions. We say that $\dot{f}$ is a *good* $\mathbb{C}(\Gamma) * Q(C)$*-name* for a real if for every subset Γ' of ω_2 such that $\Gamma \subseteq \Gamma'$ and centered family C' of $\mathbb{C}(\Gamma')$-symmetric names for pure conditions extending C, $\dot{f}$ is a $\mathbb{C}(\Gamma') * Q(C')$-name for a real.

DEFINITION 5.1.2. Let C be a countable centered family of $\mathbb{C}(\Gamma)$-names for pure conditions, let $\dot{f}$ be a good $\mathbb{C}(\Gamma) * Q(C)$-name for a real, $i, k \in \omega$, $p \in \mathbb{C}(\Gamma)$, $\dot{X}$ a symmetric name for a pure condition in $Q(C)$ such that $\Vdash \check{k} < \min \operatorname{int}(\dot{X})$. The name $\dot{X}$ is *preprocessed* for $\dot{f}(i)$, k, p and C where $i, k \in \omega$ if for every $v \subseteq k$ the following holds:

If there is a countable centered family C' of $\mathbb{C}(\Gamma)$-symmetric names extending C, a symmetric name for a pure condition $\dot{Y}$ in $Q(C')$ extending $\dot{X}$ and a condition $A \in \mathcal{A}_i(\dot{f})$ such that $(p, (v, \dot{Y})) \leq A$ then there is $B \in \mathcal{A}_i(\dot{f})$ such that $(p, (v, \dot{X})) \leq B$.

LEMMA 5.1.3. *Let C be a countable centered family of $\mathbb{C}(\Gamma)$-symmetric names for pure conditions, $\dot{f}$ a good $\mathbb{C}(\Gamma) * Q(C)$-name for a real, $\dot{X}$ a symmetric name for a pure condition in $Q(C)$. Let C' be a countable centered family of symmetric names for pure conditions extending C*

and $\dot{Y}$ a symmetric name for a pure condition in $Q(C')$ extending $\dot{X}$. If $\dot{X}$ is preprocessed for $\dot{f}(i)$, k, p and C then $\dot{Y}$ is preprocessed for $\dot{f}(i)$, k, p and C'.

Proof. Let C'' be a countable centered family of symmetric names for pure conditions extending C' and let $\dot{Z}$ be a symmetric name for an extension of $\dot{Y}$ such that for some $A \in \mathcal{A}_i(\dot{f})$ $(p,(v,\dot{Z})) \leq A$. However C'' extends C, $\dot{Z}$ extends $\dot{X}$, $\dot{X}$ is preprocessed for $\dot{f}(i)$, k, p and C, and so there is $B \in \mathcal{A}_i(\dot{f})$ such that $(p,(v,\dot{X})) \leq B$. But $\Vdash \dot{Y} \leq \dot{X}$ and so $(p,(v,\dot{Y})) \leq (p,(v,\dot{X})) \leq B$. Therefore $\dot{Y}$ is preprocessed for $\dot{f}(i)$, k, p and C'. $\square$

Lemma 5.1.4. *Let C be a countable centered family of $\mathbb{C}(\Gamma)$-symmetric names for pure conditions, $\dot{f}$ a good $\mathbb{C}(\Gamma) * Q(C)$-name for a real, $i, k \in \omega$, $p \in \mathbb{C}(\Gamma)$, $\dot{X}$ a symmetric name for a pure condition in $Q(C)$. Then there is a countable centered family of symmetric names for pure conditions C' extending C and a symmetric name for a pure condition T' extending $\dot{X}$, $T' \in Q(C')$ such that T' is preprocessed for $\dot{f}(i)$, k, p and C'.*

Proof. Let $v_1, \dots, v_s$ enumerate the subsets of k. The name $\dot{Y}$ and the centered family C' will be obtained at finitely many steps. Consider $(p_1,(v_1,\dot{X}))$. If there is a countable centered family C_1' of symmetric names for pure conditions extending C and a symmetric name for a pure condition $T_1' \in Q(C_1')$ extending $\dot{X}\backslash k$ such that for some $A_1 \in \mathcal{A}_i(\dot{f})$, $(p,(v_1,T_1')) \leq A_1$ let $T_1 = T_1'$, $C_1 = C_1'$. Otherwise let $T_1 = \dot{X}$, $C_1 = C$. At step $(s-1)$ consider $(p,(v_s,T_{s-1}))$ and C_{s-1}.

If there is a centered family of symmetric names for pure conditions C'_s extending C_{s-1}, $|C'_s| = |C_{s-1}|$ such that for some pure condition $T'_s \in Q(C'_s)$ extending T_{s-1}, there is $A_s \in \mathcal{A}_i(\dot{f})$ such that $(p,(v_s,T'_s)) \leq A_s$ let $T_s = T'_s$, $C_s = C'_s$. Otherwise let $T_s = T_{s-1}$, $C_s = C_{s-1}$. It will be shown that $T' = T_s$ is preprocessed for $\dot{f}(i)$, k, p and $C' = C_s$.

Let $v \subseteq k$, C'' a countable centered family of $\mathbb{C}(\Gamma)$-symmetric names for pure conditions extending C', T'' a symmetric name for a pure condition in $Q(C'')$ extending T' such that for some $A \in \mathcal{A}_i(\dot{f})$, $(p,(v,T''\backslash k)) \leq A$. Then $v = v_j$ for some $j \in s+1$. Since C'' extends C', C'' extends C_{j-1} and furthermore T'' is a name for an extension of T_{j-1}. Therefore at stage j we have chosen a centered family C_j and a symmetric name for a pure condition $T_j \in Q(C_j)$ such that $(p,(v_j,T_j)) \leq A_j \in \mathcal{A}_i(\dot{f})$. However $\Vdash T' \leq T_j$ and so $(p,(v_j,T')) \leq A_j$. □

Corollary 5.1.5. *Let $\dot{X}$ be a $\mathbb{C}(\Gamma)$-symmetric name for a pure condition in $Q(C)$, where C is a countable centered family of $\mathbb{C}(\Gamma)$-symmetric names for pure conditions and let $\dot{f}$ be a good $\mathbb{C}(\Gamma) * Q(C)$-name for a real. Let $\{p_j\}_{j\in\ell}$ be a finite number of conditions in $\mathbb{C}(\Gamma)$, $k,n \in \omega$. Then there is a countable centered family C' of $\mathbb{C}(\Gamma)$-symmetric names extending C, a symmetric name $\dot{Y}$ for a pure extension of $\dot{X}$ in $Q(C')$ such that for all $j \leq \ell$ and $i \leq n$, $\dot{Y}$ is preprocessed for $\dot{f}(i)$, k, p_j and C'.*

Proof. By Lemma 5.1.4 there is a countable centered family C_0 of $\mathbb{C}(\Gamma)$-symmetric names for pure conditions, a $\mathbb{C}(\Gamma)$-symmetric name $\dot{X}_0$ for an extension of $\dot{X}$ in $Q(C_0)$ which is preprocessed for $\dot{f}(0)$, k, p_0

and C_1. Again by Lemma 5.1.4 there is a countable centered family C_1 of $\mathbb{C}(\Gamma)$-symmetric names extending C_0 and a symmetric name for a pure condition $\dot{X}_1$ extending $\dot{X}_0$, $\Vdash \dot{X}_1 \in Q(C_1)$ which is preprocessed for $\dot{f}(0)$, k, p_1 and C_1. By Lemma 5.1.3 $\dot{X}_1$ is also preprocessed for $\dot{f}(0)$, k, p_0 and C_1. Repeating the argument ℓ-times obtain a centered family of symmetric names for pure conditions $C_{\ell-1}$ and a symmetric name $\dot{X}_{\ell-1} \in Q(C_{\ell-1})$ such that for all $j \in \ell$, $\dot{X}_{\ell-1}$ is preprocessed for $\dot{f}(0)$, k, p_j and $C_{\ell-1}$. Repeating the argument above successively for $\dot{f}(1), \ldots, \dot{f}(n-1)$ obtain a centered family C' of symmetric names for pure conditions and a symmetric name for a pure condition $\dot{Y} \in Q(C')$ extending $\dot{X}$ such that for all $j \in \ell$, $i \in n$, $\dot{Y}$ is preprocessed for $\dot{f}(i)$, k, p_j and C'. $\square$

Corollary 5.1.6. *Let $\{p_n\}_{n\in\omega}$ enumerate $\mathbb{C}(\Gamma)$. Let C be a countable centered family of $\mathbb{C}(\Gamma)$-symmetric names for pure conditions, $\dot{X}$ a $\mathbb{C}(\Gamma)$-symmetric name for a pure condition in $Q(C)$ and let $\dot{f}$ be a good $\mathbb{C}(\Gamma) * Q(C)$-name for a real. Then there is a countable centered family C' of $\mathbb{C}(\Gamma)$-symmetric names for pure conditions extending C and a sequence $\langle \dot{Y}_n : n \in \omega\rangle$ of $\mathbb{C}(\Gamma)$-symmetric names for pure conditions in $Q(C')$ such that*

(1) $\Vdash \dot{Y}_0 \leq \dot{X}$ and $\forall n \in \omega \Vdash \dot{Y}_{n+1} \leq \dot{Y}_n$

(2) $\forall n \in \omega \forall i, j \leq n$ $\dot{Y}_n$ is preprocessed for $\dot{f}(i)$, n, p_j and C'.

Proof. By Corollary 5.1.5 there is a $\mathbb{C}(\Gamma)$-symmetric name for a pure condition $\dot{Y}_0$ extending $\dot{X}$, such that $\dot{Y}_0 \in Q(C_0')$ where C_0' is a countable centered family of $\mathbb{C}(\Gamma)$-symmetric names for pure conditions extending C, which is preprocessed for $\dot{f}(0)$, p_0, 0 and C_0.

Suppose $\dot{Y}_n$, C_n have been defined. Then by Corollary 5.1.5 there is a countable centered family C_{n+1} of $\mathbb{C}(\Gamma)$-symmetric names for pure conditions extending C_n and a symmetric name for a pure condition $\dot{Y}_{n+1}$ extending $\dot{Y}_n$ such that for all $i, j \leq n+1$, $\dot{Y}_{n+1}$ is preprocessed for $\dot{f}(i)$, n, p_j and C_{n+1}. Then $C' = \cup_{n\in\omega} C_n$ is a countable centered family of $\mathbb{C}(\Gamma)$-symmetric names for pure conditions extending C, such that $\langle \dot{Y}_n : n \in \omega\rangle$ is contained in $Q(C')$ and $\forall n \in \omega$, $\forall i, j \leq n$, $\dot{Y}_n$ is preprocessed for $\dot{f}(i)$, n, p_j, and C'. $\square$

Corollary 5.1.7. *Let C be a countable centered family of $\mathbb{C}(\Gamma)$-symmetric names for pure conditions, $\dot{f}$ a good $\mathbb{C}(\Gamma) * Q(C)$-name for a real and $\dot{X}$ a $\mathbb{C}(\Gamma)$-symmetric name for a pure condition in $Q(C)$. Then there is a countable centered family C' of $\mathbb{C}(\Gamma)$-symmetric names for pure conditions extending C and a $\mathbb{C}(\Gamma)$-symmetric name $\dot{Z} = \langle \dot{Z}(i) : i \in \omega\rangle$ for a pure condition in $Q(C')$ such that $\forall n \in \omega$, $\forall i, j \leq n$ $\dot{Z}_n = \langle \dot{Z}(i) : i \geq n\rangle$ is preprocessed for $\dot{f}(i)$, p_j, n and C', where $\{p_n\}_{n\in\omega}$ is a fixed enumeration of $\mathbb{C}(\Gamma)$.*

Proof. Let C' be a countable centered family extending C, $\langle \dot{Y}_n : n \in \omega\rangle$ a sequence of $\mathbb{C}(\Gamma)$-symmetric names contained in $Q(C')$ satisfying Corollary 5.1.6. Passing to a subfamily we can assume that $C' = \{\dot{X}_n\}_{n\in\omega}$ where for all $n \in \omega$, $\Vdash \dot{X}_{n+1} \leq \dot{X}_n$. Then using the fixed enumeration of $\mathbb{C}(\Gamma)$ obtain a $\mathbb{C}(\Gamma)$-symmetric name for a pure condition $\dot{Z} = \langle \dot{Z}(i) : i \in \omega\rangle$ such that for all $n \in \omega$, $\Vdash \dot{Z}_n \leq \dot{X}_n \wedge \dot{Z}_n \leq \dot{Y}_n$ where $\Vdash \dot{Z}_n = \langle \dot{Z}(i) : i \in \omega\rangle$. Then $\dot{Z}$ and $C' = \{\dot{Z}_n\}_{n\in\omega}$ are the desired pure condition and centered family of $\mathbb{C}(\Gamma)$-symmetric names. $\square$

5.2. Induced Logarithmic Measure

LEMMA 5.2.1. *Let* $\{p_i\}_{i\in\omega}$ *be a fixed enumeration of* $\mathbb{C}(\Gamma)$ *and let* $\dot{X} = \langle \dot{X}(i) : i \in \omega\rangle$ *be a* $\mathbb{C}(\Gamma)$*-symmetric name for a pure condition such that* $\forall m \in \omega$, $\forall i, j \leq m$ *the name* $\dot{X}_m = \langle \dot{X}(i) : i \geq m\rangle$ *is preprocessed for* $\dot{f}(i)$, m, p_j *and* $C = \{\dot{X}_m\}_{m\in\omega}$, *where* $\dot{f}$ *is a good* $\mathbb{C}(\Gamma)*Q(C)$*-name for a real. Then for every* $i, k \in \omega$ *and finite number of conditions* $p^1, \dots, p^n$ *in* $\mathbb{C}(\Gamma)$ *the logarithmic measure induced by the set* $\mathcal{P}_k(\dot{f}(i), \dot{X}, \{p^j\}_{j=1}^n)$ *which consists of all* $x \in [\omega]^{<\omega}$ *such that for all* $j = 1, \dots, n$ *there is* $\bar{p}_j \leq p^j$ *such that*

(1) $\bar{p}_j \Vdash (\check{x} \subseteq int(\dot{X})) \wedge (\exists j \in \omega(x \cap int(X(j)) \in X(j)^+))$,

(2) $\forall v \subseteq k \exists w^j_v \subseteq x \exists A_{v,j} \in \mathcal{A}_i(\dot{f})$ *s.t.* $(\bar{p}_j, (v \cup w^j_v, X^*)) \leq A_{v,j}$

where X^* *is a symmetric name for some final segment of* $\dot{X}$, *takes arbitrarily high values. The conditions* $\{\bar{p}_j\}_{j=1}^n$ *are said to witness the fact that* x *is positive.*

PROOF. Let G be $\mathbb{C}^* = \prod_{i=1}^n \mathbb{C}(\Gamma_i)$-generic filter where $\forall i \neq j$, $\Gamma_i \cap \Gamma_j = \emptyset$ and $\Gamma_i \cong \Gamma_j$, such that $(p^1, \dots, p^n) \in G$. Let $\omega = A_0 \cup \dots \cup A_{M-1}$ be a partition of ω into finitely many sets. By Lemma 4.5.2 there is a $\mathbb{C}^*$-symmetric name $\tilde{X} = \langle \tilde{X}(i) : i \in \omega\rangle$ such that for some $j_0 \in M$ $\text{int}(\tilde{X}[G]) \subseteq A_{j_0}$ and for all $m \in \omega$, $j = 1, \dots, n$, $\tilde{X}_m[G] = \langle \tilde{X}(i)[G] : i \geq m\rangle \leq \dot{X}_m[G^j]$. Then in particular for all $j = 1, \dots, n$, $C_j = \{\dot{X}_m[G^j]\}_{m\in\omega} \subseteq Q(\tilde{C})$ where $\tilde{C} = \{\tilde{X}_m[G]\}_{m\in\omega}$. Since $\dot{f}$ is a good name, $\dot{f}$ is also a $\mathbb{C}^* * Q(\tilde{C})$-name for a real. Let $v_1, \dots, v_L$ enumerate the subsets of k. Fix $j \in \{1, \dots, n\}$ and $s \in \{1, \dots, L\}$. Since $f_j = \dot{f}/G^j$ is $Q(\tilde{C})$-name for a real, there is $q_{js} \in G^j$, a $\mathbb{C}(\Gamma)$-symmetric name for a pure condition $\dot{R}_{js}$ in $Q(C)$ and a finite subset u_{js} of ω, such that $A_{js} = (q_{js}, (u_{js}, \dot{R}_{js})) \in \mathcal{A}_i(\dot{f})$ and in $V[G]$ the conditions $(u_{js}, R_{js}[G^j])$ and $(v_s, \tilde{X}[G])$ are compatible with common extension $(v_s \cup w_{js}, \tilde{T}[G])$ (from $Q(\tilde{C})$). Then in particular $w_{js} \subseteq \mathrm{int}(\tilde{X}[G])$ and $v_s \cup w_{js} \backslash u_{js} \subseteq \mathrm{int}(\dot{R}_{js}[G^j])$. Since $\dot{R}_{js}$ and $\dot{X}$ are names in $Q(C)$, there is a $\mathbb{C}(\Gamma)$-symmetric name for a pure condition $\dot{Z}_{js}$ (in fact a name for a final subsequence of $\dot{X}$) in $Q(C)$ which is their common extension. Then there is $t_{js} \in G^j$ extending q_{js} and p^j such that $(t_{js}, (v_s \cup w_{js}, \dot{Z}_{js})) \leq A_{js}$ and $(t_{js}, (v_s \cup w_{js}, \dot{Z}_{js})) \leq (t_{js}, (v_s \cup w_{js}, \dot{X}))$. In finitely many steps find a finite subset x of $\mathrm{int}(\tilde{X}[G])$ such that for every $s = 1, \ldots, L$ and every $j = 1, \ldots, n$ there is $w_{js} \subseteq x$, a $\mathbb{C}(\Gamma)$-symmetric name for a pure condition $\dot{Z}_{js}$ in $Q(C)$ such that $\Vdash \dot{Z}_{js} \leq \dot{X}$, a Cohen condition $t_{js} \in G^j$ and a condition $A_{js} \in \mathcal{A}_i(\dot{f})$ such that $(t_{js}, (v_s \cup w_{js}, \dot{Z}_{js})) \leq A_{js}$ and such that for some $l \in \omega$, $x \cap \mathrm{int}(\tilde{X}(l)[G])$ is $\tilde{X}(l)$-positive. Since $\tilde{X}[G] \leq \dot{X}[G^j]$ (for all $j = 1, \ldots, n$) we have that $x \subseteq \mathrm{int}(\dot{X}[G^j])$ and furthermore for every $j = 1, \ldots, n$ there is $\ell_j \in \omega$ such that $x \cap \mathrm{int}(\dot{X}(\ell_j)[G^j])$ is a positive subset of $\dot{X}(\ell_j)[G^j]$. Then for every $j = 1, \ldots, n$ there is a condition $\bar{p}_j \in G^j$ extending p^j and $\{t_{js}\}_{s=1}^{l}$ which forces "$x \subseteq \mathrm{int}(\dot{X})$" and "$x \cap \mathrm{int}(\dot{X}(\ell_j))$ is a positive subset of $\dot{X}(\ell_j)$". Since $\bar{p}_j \leq t_{js}$, then also we have that $(\bar{p}_j, (v_s \cup w_{js}, \dot{Z}_{js})) \leq A_{js}$ for all $s = 1, \ldots, \ell$.

Let N be an integer greater than the indexes of $\bar{p}_j$ for $j = 1, \ldots, n$ in the fixed enumeration of $\mathbb{C}(\Gamma)$ and also greater than i and $\max x$. Let $X^* = \dot{X}_N$. Recall that $\dot{Z}_{js}$ is a $\mathbb{C}(\Gamma)$-symmetric name for a pure condition extending $\dot{X}$ and $\dot{Z}_{js} \in Q(C)$. Then there is a $\mathbb{C}(\Gamma)$-symmetric name for a pure condition Z^*_{js} in $Q(C)$ such that

$$\Vdash \text{“}Z^*_{js} \leq Z_{js} \text{ and } Z^*_{js} \leq X^*\text{”}$$

(in fact Z^*_{js} is a name for some final subsequence of $\dot{X}$). However X^* is preprocessed for $\dot{f}(i)$, $\max x$, $\{\bar{p}_j\}_{j\leq n}$ and C. Therefore for all j, s there is a condition $B_{js} \in \mathcal{A}_i(\dot{f})$ such that $(\bar{p}_j, (v_s \cup w^j_s, X^*)) \leq B_{js}$ and so x is a positive set. It remains to observe that $x \subseteq A_{j_0}$ and so by the sufficient condition for arbitrarily high values (Lemma 2.1.10) the logarithmic measure induced by $\mathcal{P}_k(\dot{f}(i), \dot{X}, \{p^j\}^n_{j=1})$ takes arbitrarily high values. $\square$

5.3. Good Names for Pure Conditions

Corollary 5.3.1. *Let $\{p_i\}_{i\in\omega}$ enumerate $\mathbb{C}(\Gamma)$ and let $\dot{X} = \langle \dot{X}(i) : i \in \omega\rangle$ be a $\mathbb{C}(\Gamma)$-symmetric name for a pure condition such that $\forall m \in \omega \forall i, j \leq m$ the name $\dot{X}_m = \langle \dot{X}(i) : i \geq m\rangle$ is preprocessed for $\dot{f}(i)$, m, p_j and $C = \{\dot{X}_m\}_{m\in\omega}$ where $\dot{f}$ is a good $\mathbb{C}(\Gamma) * Q(C)$-name for a real. Then there is a $\mathbb{C}(\Gamma)$-symmetric name for a pure condition $\dot{Y} = \langle \dot{Y}(i) : i \in \omega\rangle$ such that*

(1) $\forall m \in \omega$, $\dot{Y}_m = \langle \dot{Y}(i) : i \geq m\rangle$ extends $\dot{X}_m$, and

(2) for all $i \in \omega$, $v \subseteq i$, $p \in \mathbb{C}(\Gamma)$, $s \in [\omega]^{<\omega}$ such that $p \Vdash s \in \dot{Y}(i)^+$ there is $w_v \subseteq s$ and $A \in \mathcal{A}_i(\dot{f})$ such that $(p, (v \cup w_v, \dot{Y}_{i+1})) \leq A$.

Proof. For every $p \in \mathbb{C}(\Gamma)$ let $C(p) = \mathbb{C}(\Gamma)(p)$. By Lemma 5.2.1 there is $x_1 \in \mathcal{P}_1(\dot{X}_1, \dot{f}(1), p_1)$ with witness $p_{1,1}$. Fix a maximal antichain $A_1 = \{a_{1,s} : s \in \omega\}$ in $\mathbb{P} - C(p_{1,1})$ such that for every $s \in \omega$,

$a_{1,s}$ witnesses that $x_{1,s}$ is in $\mathcal{P}_1(\dot{X}_1, \dot{f}(1), a_{1s})$. Let

$$R_1 = \{\langle p_{11}, \check{x}_1 \rangle\} \cup \{\langle a_{1s}, \check{x}_{1s} : s \in \omega\}.$$

Suppose R_{m-1} is defined. If $p_m \perp \{p_{m-1,l}\}_{l \leq m-1}$ then there is some $a_{m-1,s} \in A_{m-1}$ compatible with p_m with common extension b_m and let $y_m = x_{m-1,s}$. Otherwise there is $j \leq m-1$ such that $p_m \not\perp p_{m-1,j}$ with common extension which we again denote b_n. In this case let $y_m = x_{m-1}$. Then by Lemma 5.2.1 there is a measure x_m in $\mathcal{P}_m(\dot{X}_m, \dot{f}(m), \{p_{m-1,l}\}_{l=1}^{m-1} \cup \{b_m\})$ which is stronger than x_{m-1} and y_m. Thus in particular there are extensions $p_{m,l} \leq p_{m-1,l}$ for $l \leq m-1$ and $p_{m,m} \leq b_m \leq p_m$ witnessing this fact. Just as in the base case fix a maximal antichain $A_m = \{a_{m,s} : s \in \omega\}$ in $\mathbb{P} - C(\{p_{m,l}\}_{l=1}^{m})$ such that for all $s \in \omega$

(1) $\exists i^s \in \omega (a_{m,s} \leq a_{m-1,i^s})$

(2) there is a finite logarithmic measure $x_{m,s}$ stronger than x_{m-1,i^s} such that $a_{m,s}$ witness that $x_{m,s}$ is in $\mathcal{P}_m(\dot{X}_m, \dot{f}(m), a_{ms})$.

Let $R_m = \{\langle p_{ml}, \check{x}_m \rangle\}_{l=1}^{m} \cup \{\langle a_{ms}, \check{x}_{ms} \rangle : s \in \omega\}$ and let $\dot{Y} = \cup_{m \in \omega} R_m$. Then $\forall m \in \omega$, $\dot{Y}_m = \langle \dot{Y}(i) : i \geq m \rangle$ extends $\dot{X}_m$ and has the desired properties. □

5.4. Unboundedness

THEOREM 5.4.1. *Let C be a countable centered family of $\mathbb{C}(\Gamma)$-symmetric names for pure conditions, let Γ be a countable subset of ω_2, let $\dot{f}$ be a good $\mathbb{C}(\Gamma) * Q(C)$-name for a real and $\delta \in \omega_1 \backslash \Gamma$. Let $\dot{h} = \cup \dot{G}_\delta$ where $\dot{G}_\delta$ is the canonical name for the $\mathbb{C}(\{\delta\})$-generic filter. Then* *there is a countable centered family C' of $\mathbb{C}(\Gamma \cup \{\delta\})$-symmetric names for pure conditions which extends C and such that for every centered family C'' of $\mathbb{C}(\omega_2)$-symmetric names for pure conditions which extends C', $\Vdash_{\mathbb{C}(\omega_2)*Q(C'')}$ "$\dot{h} \not\leq^* \dot{f}$".*

Proof. We can assume that $C = \{\dot{Y}_m\}_{m\in\omega}$, where $\dot{Y}_m = \langle \dot{Y}(i) : i \geq m\rangle$ and $\dot{Y} = \langle \dot{Y}(i) : i \in \omega\rangle$ is the $\mathbb{C}(\Gamma)$-symmetric name constructed in Corollary 5.3.1. Let $\dot{g}$ be a $\mathbb{C}(\Gamma)$-name for a function in ${}^{\omega}\omega$ such that $\forall p \in \mathbb{C}(\Gamma)\forall i \in \omega$, $p \Vdash \dot{g}(i) = \check{k}$ if and only if

$$\begin{aligned} k = \max\{j : v \subseteq i, w \in [\omega]^{<\omega}, p \Vdash \text{“}\check{w} \subseteq \dot{Y}(i)\text{”},\\ (p, (v \cup w, \dot{Y})) \leq A \text{ for some } A \in \mathcal{A}_i(\dot{f}) \text{ and } A \Vdash \text{“}\dot{f}(i) = \check{j}\text{”}\}. \end{aligned}$$

Let $\dot{J}$ be a $\mathbb{C}(\Gamma \cup \{\delta\})$-name for a subset of ω such that

$$\Vdash \dot{J} = \{i : \dot{g}(i) < \dot{h}(i)\}$$

and for every $m \in \omega$ let $\dot{Z}_m$ be a $\mathbb{C}(\Gamma \cup \{\delta\})$-name such that

$$\Vdash \dot{Z}_m = \langle \dot{Y}(i) : i > m \text{ and } i \in \dot{J}\rangle.$$

Claim. For all $m \in \omega$ the name $\dot{Z}_m$ is $\mathbb{C}(\Gamma \cup \{\delta\})$-symmetric.

Proof. Let $p_0, \ldots, p_{n-1}$ be a finite number of conditions in $\mathbb{C}(\Gamma \cup \{\delta\})$ and let $M \in \omega$ be given. Then for every $i \in n$ $p_i = p_i^0 \cup p_i^1$ where $p_i^0 = p_i \restriction \Gamma \times \omega$ and $p_i^1 = p_i \restriction \{\delta\} \times \omega$. By construction of $\dot{Y}$ there are extensions $q_i^0 \leq p_i^0$ and a finite logarithmic measure $x \in L_M$ such that

$\forall i \in n$, $q_i^0 \Vdash \check{x} = \dot{Y}(\ell)$ where $\ell > m$, $\ell > M$ and

$$\ell > \max\{j : (\delta, j) \in \mathrm{domain}(p_i^1), i \in n\}.$$

Furthermore for every $i \in n$ there is $t_i^0 \in \mathbb{C}(\Gamma)$ extending q_i^0 such that $t_i^0 \Vdash \dot{g}(\ell) = \check{k}_i$ for some $k_i \in \omega$. Then let $L > \max_{i \in n} k_i$ and for every $i \in n$ let

$$t_i^1 = p_i^1 \cup \{\langle(\delta, \ell), \check{L}\rangle\}.$$

Then $t_i = t_i^0 \cup t_i^1 \leq p_i$ and $t_i \Vdash$ "$\dot{Y}(\ell) = \check{x} \wedge \ell > m \wedge \ell \in \dot{J}$". That is $t_i \Vdash \check{x} \leq \dot{Z}_m$. Therefore $\dot{Z}_m$ is symmetric. $\square$

Then let $C' = \{\dot{Z}_m\}_{m \in \omega}$ and let $\dot{Z} = \dot{Z}_0$. Consider arbitrary centered family C'' of $\mathbb{C}(\omega_2)$-symmetric names such that $\Vdash C' \subseteq Q(C'')$. It is sufficient to show that $\forall a \in [\omega]^{<\omega}$, $\forall k \in \omega$

$$\Vdash_{\mathbb{C}(\omega_2)} \text{“}(a, \dot{Z}) \Vdash_{Q(C'')} \text{“}\exists i > k(\dot{f}(i) < \dot{h}(i)\text{””}$$

since

$$\Vdash_{\mathbb{C}(\omega_2)} \text{“}\{(a, \dot{Z}) : a \in [\omega]^{<\omega}\} \text{ is predense in } Q(C'')\text{”}.$$

Let $a \in [\omega]^{<\omega}$, $k \in \omega$ be arbitrary. Consider any $(p, (b, \dot{R})) \in \mathbb{C}(\omega_2) * Q(C'')$ such that $p \Vdash$ "$(b, \dot{R}) \leq (a, \dot{Z})$". Then in particular $p \Vdash b \backslash a \subseteq \mathrm{int}(\dot{Z})$ and $p \Vdash \dot{R} \leq \dot{Z}$. By definition of the extension relation there is $\ell > k$ such that $b \subseteq \ell$, a finite subset s of ω and extension $\bar{p}$ of p such that

$$\bar{p} \Vdash \text{“}\check{\ell} \in \dot{J} \text{ and } \check{s} = \mathrm{int}(\dot{R}) \cap \mathrm{int}(\dot{Z}(\ell)) \text{ is } \dot{Z}(\ell)\text{- positive”}.$$

By definition of $\dot{Z}(\ell)$ there is $w \subseteq s$ and $A \in \mathcal{A}_\ell(\dot{f})$ such that

$$(\bar{p}, (b \cup w, \dot{Y})) \leq A$$

and so $(\bar{p}, (b \cup w, \dot{Z})) \leq A$ as well as $(\bar{p}, (b \cup w, R)) \leq A$. Note that $\bar{p} \Vdash \check{w} \subseteq \mathrm{int}(\dot{R})$ and so $(\bar{p}, (b \cup w, R)) \leq (p, (b, \dot{R}))$. Furthermore

$$(\bar{p}, (b \cup w, \dot{R})) \Vdash \text{“}\dot{f}(\ell) \leq \dot{g}(\ell) < \dot{h}(\ell)\text{”}.$$

$\square$

Definition 5.4.2. Let $\mathbb{P}$ be the partial order of all pairs $p = (\Gamma_p, C_p)$ where Γ is a countable subset of ω_2, C_p is a countable centered family of $\mathbb{C}(\Gamma_p)$-symmetric names for pure conditions with extension relation defined as follows: $p \leq q$ if $\Gamma_q \subseteq \Gamma_p$ and $\Vdash_{\mathbb{C}(\Gamma_p)} C_q \subseteq Q(C_p)$.

The partial order $\mathbb{P}$ is countably closed and adds a centered family of $\mathbb{C}(\omega_2)$-symmetric names for pure conditions

$$C_H = \cup\{C_p : p \in H\}$$

where H is $\mathbb{P}$-generic. By Lemma 4.4.6, forcing with $Q(C_H)$ over $V^{\mathbb{P}\times\mathbb{C}(\omega_2)}$ adds a real not split by

$$V^{\mathbb{C}(\omega_2)} \cap [\omega]^\omega = B^{\mathbb{C}(\omega_2)\times\mathbb{P}} \cap [\omega]^\omega.$$

To see that the first ω_1 Cohen reals remain an unbounded family consider an arbitrary $\mathbb{C}(\omega_2) * Q(C_H)$-name $\dot{f}$ for a real. Then there is a condition $p \in H$ such that $\dot{f}$ is a $\mathbb{C}(\Gamma_p) * Q(C_p)$-name for a real. Then for every $q \leq p$ either there is a further extension a such that $\dot{f}$ is not a $\mathbb{C}(\Gamma_a) * Q(C_a)$-name for a real, or $\dot{f}$ is a good $\mathbb{C}(\Gamma_a) * Q(C_a)$-name. Then let $A = A^- \cup A^+$ be an antichian of conditions which is maximal below p and such that $\forall a \in A^-$ $\dot{f}$ is not a $\mathbb{C}(\Gamma_a) * Q(C_a)$-name for a real and $\forall a \in A^+$ $\dot{f}$ is a good $\mathbb{C}(\Gamma_a) * Q(C_a)$-name. Since $p \in H$ and $\dot{f}$ is a $\mathbb{C}(\omega_2) * Q(C_H)$-name for a real, there is $a \in H \cap A^+$. That is there is $a \in H$ such that $\dot{f}$ is a good $\mathbb{C}(\Gamma_a) * Q(C_a)$-name for a real. Let $\mathcal{H}$ be the collection of all names $\dot{h}$ such that $\dot{h} = \cup \dot{G}_\delta$ where $\delta \in \omega_1$ and $\dot{G}_\delta$ is the canonical $\mathbb{C}(\{\delta\})$-name for the $\mathbb{C}(\{\delta\})$-generic filter (that is $\mathcal{H}$ is the set of the first ω_1 Cohen reals). Then by Theorem 5.4.1 the set

$$D_{\dot{f}} = \{q \in \mathbb{P} : \exists \dot{h} \in \mathcal{H}(q \Vdash_{\mathbb{P}} \text{``} \Vdash_{\mathbb{C}(\omega_2) * Q(C_{\dot{H}})} \text{``}\dot{h} \not\leq^* \dot{f}\text{''''})\}$$

where $\dot{H}$ is the canonical $\mathbb{P}$-name for the $\mathbb{P}$-generic filer, is dense below a. Therefore there is $\dot{h} \in \mathcal{H}$ such that

$$V[H] \models (\Vdash_{\mathbb{C}(\omega_2) * Q(C_H)} \text{``}\dot{h} \not\leq^* \dot{f}\text{''}).$$

Lemma 5.4.3. *Let Γ_1, Γ_2 be countable subsets of ω_2 such that $\Gamma_1 \cap \Gamma_2 = \emptyset$ and let $i : \Gamma_1 \cong \Gamma_2$ be an isomorphism. Let $\dot{X}$ be $\mathbb{C}(\Gamma_1)$-symmetric name for a pure condition. Then there is $\mathbb{C}(\Gamma_1 \cup \Gamma_2)$-symmetric name for a pure condition $\tilde{X}$ such that $\Vdash_{\mathbb{C}(\Gamma_1 \cup \Gamma_2)}$ "$\tilde{X} \leq \dot{X}$ and $\tilde{X} \leq i(\dot{X})$". If $\dot{Y}$ and $\dot{Z}$ are $\mathbb{C}(\Gamma_1)$-symmetric names for pure conditions such that $\Vdash$ "$\dot{X} \leq \dot{Y}$ and $\dot{X} \leq \dot{Z}$" then*

$$\Vdash_{\mathbb{C}(\Gamma_1 \cup \Gamma_2)} \text{``}\tilde{X} \leq \dot{Y} \textit{ and } \tilde{X} \leq i(\dot{Z})\text{''}.$$

We will need the following claim.

Claim. Let $p_1, \dots, p_k$ be any finite number of conditions in $\mathbb{C}(\Gamma_1 \cup \Gamma_2)$ and let $M \in \omega$. Then there is a finite logarithmic measure $x \in L_M$ and extensions $q_1 \leq p_1, \dots, q_k \leq p_k$ such that $\forall i \leq k$,

$$q_i \Vdash \text{“}\check{x} \leq \dot{X} \text{ and } \check{x} \leq i(\dot{X})\text{”}.$$

Proof. Note that $\forall i \leq k$, $p_i = p_i^1 \cup p_i^2$ where $p_i^1 = p_i \restriction \Gamma_1$, $p_i^2 = p_i \restriction \Gamma_2$. Let $q_i^1 = p_i^1$ and $q_i^2 = i^{-1}(p_i^2)$. Then since $\dot{X}$ is $\mathbb{C}(\Gamma_1)$-symmetric there is $x \in L_M$ and extensions $q_{i,1}^1 \leq q_i^1$ and $q_{i,2}^2 \leq q_i^2$ such that $\forall i \leq k$, $q_{i,1}^1 \Vdash \check{x} \leq \dot{X}$ and $q_{i,1}^2 \Vdash \check{x} \leq \dot{X}$. But then $i(q_{i,1}^2) \Vdash_{\mathbb{C}(\Gamma_2)} \check{x} \leq i(\dot{X})$ and so $r_i = q_{i,1}^1 \cup q_{i,1}^2$ is an extension of p_i such that $r_i \Vdash_{\mathbb{C}(\Gamma_1 \cup \Gamma_2)}$ “$\check{x} \leq \dot{X}$ and $\check{x} \leq i(\dot{X})$. $\square$

With this we can proceed with the proof of Lemma 5.4.3.

Proof. Fix an enumeration $\{p_n\}_{n \in \omega}$ of $\mathbb{C}(\Gamma_1 \cup \Gamma_2)$ and inductively construct a $\mathbb{C}(\Gamma_1 \cup \Gamma_2)$-symmetric name $\tilde{X}$ such that for all $n \in \omega$,

$$\Vdash_{\mathbb{C}(\Gamma_1 \cup \Gamma_2)} \tilde{X}(n) \leq \dot{X} \wedge \tilde{X}(n) \leq i(\dot{X}).$$

Suppose $\dot{Y}$ and $\dot{Z}$ are $\mathbb{C}(\Gamma_1)$-symmetric names for pure conditions such that $\Vdash_{\mathbb{C}(\Gamma_1)} \dot{X} \leq \dot{Y} \wedge \dot{X} \leq \dot{Z}$. Then $\Vdash_{\mathbb{C}(\Gamma_2)} i(\dot{X}) \leq i(\dot{Z})$ and so $\Vdash_{\mathbb{C}(\Gamma_1 \cup \Gamma_2)} \tilde{X} \leq \dot{Y} \wedge \tilde{X} \leq i(\dot{Z})$. $\square$

Begin with a model of CH and consider a subset $\{p_i : i \in I\}$ of $\mathbb{P}$ of size $\aleph_2$. By the Delta System Lemma there is a subset J of I, $|J| = \aleph_2$ such that $\{\Gamma_i : i \in J\}$ forms a delta system with root Δ where $\forall i \in I(\Gamma_i = \Gamma_{p_i})$. Furthermore J might be chosen so that for all $i, j \in J$ there is an isomorphism $\alpha_{ij} : p_i(1) \cong p_j(1)$, such that $\alpha_{ij} \restriction \Delta$

is the identity and such that $C_j = C_{p_j} = \{\alpha_{ij}(\dot{X}) : \dot{X} \in C_{p_i}\}$. If $\Delta = \emptyset$ then by Lemma 5.4.3 for every $\dot{X} \in C_i$ where $C_i = C_{p_i}$, there is $\mathbb{C}(\Gamma_i \cup \Gamma_j)$-symmetric name for a pure condition $\tilde{X}_X$ extending $\dot{X}$ and $\alpha_{i,j}(\dot{X})$ and so $p_k = (\Gamma_k, C_k)$ where $\Gamma_k = \Gamma_i \cup \Gamma_j$ and

$$C_k = C_i \cup C_j \cup \{\tilde{X}_X : X \in C_i\}$$

is a common extension of p_i and p_j. However as mentioned in section 4.1 if the root of the delta system is non-empty the above argument does not hold and a stronger combinatorial property on the names for pure conditions is needed.

CHAPTER 6

A look ahead

6.1. General Definition of Symmetric Names

DEFINITION 6.1.1. Let $\dot{X}$ be a $\mathbb{C}(\Gamma)$-name for a subset of ω, where $\Gamma \in [\omega_2]^\omega$. Then $\dot{X}$ is *symmetric* if for all finite subsets $\Gamma' = \{\gamma_0, \ldots, \gamma_n\}$ of ω_2, where $\gamma_0 = \min \Gamma < \gamma_1 < \cdots < \gamma_n = \sup \Gamma$, for all finite families of conditions $\langle p_i^j \rangle_{i \leq k_j} \subseteq \mathbb{C}(\Gamma \cap \gamma_j \backslash \gamma_{j-1})$ for $j = 1, \ldots, n$ and every $M \in \omega$, there is $m > M$ which belongs to

$$\cap_{i_1=1}^{k_1} \mathrm{hull}_{p_{i_1}^1, \Gamma \cap \gamma_1 \backslash \gamma_0} (\cap_{i_2=1}^{k_2} \mathrm{hull}_{p_{i_2}^2, \Gamma \cap \gamma_2 \backslash \gamma_1} (\ldots \cap_{i_n=1}^{k_n} \mathrm{hull}_{p_{i_n}^n, \Gamma \cap \gamma_n \backslash \gamma_{n-1}} (\dot{X}) \ldots)).$$

REMARK 6.1.2. Note that $\dot{X}$ is a $\mathbb{C}(\Gamma)$-symmetric name for a subset of ω if and only if for all finite subsets $\Gamma' = \{\gamma_0, \ldots, \gamma_n\}$ of ω_2, where

$$\gamma_0 = \min \Gamma < \gamma_1 < \cdots < \gamma_n = \sup \Gamma,$$

for all finite families of conditions $\langle p_i^j \rangle_{i \leq k_j} \subseteq \mathbb{C}(\Gamma \cap \gamma_j \backslash \gamma_{j-1})$ for $j = 1, \ldots, n$ and every $M \in \omega$, there is a tree of extensions

$$\Phi = \{\bar{\phi}(i_1 \ldots i_j) : 1 \leq j \leq n, 1 \leq i_j \leq k_j\}$$

where $\phi(i_1 \ldots i_j)$ is an extension of $p_{i_j}^j$ in $\mathbb{C}(\Gamma \cap \gamma_j \backslash \gamma_{j-1})$, $\bar{\phi}(i_1) = \phi(i_1)$ and for $j \geq 2$ $\bar{\phi}(i_1 \ldots i_j) = (\bar{\phi}(i_1 \ldots i_{j-1}), \phi(i_1 \ldots i_j))$, and there is an integer $m > M$ such that for every maximal node $\bar{\phi}$ of Φ, $\bar{\phi} \Vdash \check{m} \in \dot{X}$. We will refer to the family of all Cohen conditions $P = \langle p_i^j \rangle_{i,j}$ as a *matrix of conditions* and to the tree $\Phi = \Phi(P)$ as *an associated tree of extensions*. Note that definition 4.2.2 coincides with the particular case of the above definition in which Γ is a singleton.

This definition generalizes to names for pure conditions.

DEFINITION 6.1.3. Let $\dot{X}$ be a $\mathbb{C}(\Gamma)$-name for a pure condition, where Γ is a countable subset of ω_2. Then $\dot{X}$ is *symmetric* if for all finite subsets $\Gamma' = \{\gamma_0, \ldots, \gamma_n\}$ of ω_2, where $\gamma_0 = \min \Gamma < \gamma_1 < \cdots < \gamma_n = \sup \Gamma$, for all finite families of conditions $\langle p_i^j \rangle_{i \leq k_j} \subseteq \mathbb{C}(\Gamma \cap \gamma_j \backslash \gamma_{j-1})$ for $j = 1, \ldots, n$ and every $M \in \omega$, there is a finite logarithmic measure $x \in L_M$ such that x belongs to

$$\cap_{i_1=1}^{k_1} \mathrm{hull}_{p^1_{i_1}, \Gamma \cap \gamma_1 \backslash \gamma_0} (\cap_{i_2=1}^{k_2} \mathrm{hull}_{p^2_{i_2}, \Gamma \cap \gamma_2 \backslash \gamma_1} (\ldots \cap_{i_n=1}^{k_n} \mathrm{hull}_{p^n_{i_n}, \Gamma \cap \gamma_n \backslash \gamma_{n-1}} (\dot{X}) \ldots)).$$

REMARK 6.1.4. Note that $\dot{X}$ is a $\mathbb{C}(\Gamma)$-symmetric name for a pure condition if and only if for all finite subsets $\Gamma' = \{\gamma_0, \ldots, \gamma_n\}$ of ω_2, where

$$\gamma_0 = \min \Gamma < \gamma_1 < \cdots < \gamma_n = \sup \Gamma,$$

for all finite families of conditions $\langle p_i^j \rangle_{i \leq k_j} \subseteq \mathbb{C}(\Gamma \cap \gamma_j \backslash \gamma_{j-1})$ for $j = 1, \ldots, n$ and every $M \in \omega$, there is a tree of extensions

$$\Phi = \{\bar{\phi}(i_1 \ldots i_j) : 1 \leq j \leq n, 1 \leq i_j \leq k_j\}$$

where $\phi(i_1 \ldots i_j)$ is an extension of $p^j_{i_j}$ in $\mathbb{C}(\Gamma \cap \gamma_j \backslash \gamma_{j-1})$, $\bar{\phi}(i_1) = \phi(i_1)$ and for $j \geq 2$, $\bar{\phi}(i_1 \ldots i_j) = (\bar{\phi}(i_1 \ldots i_{j-1}), \phi(i_1 \ldots i_j))$ and there is a finite logarithmic measure $x \in L_M$ such that for every maximal node $\bar{\phi}$ of Φ, $\bar{\phi} \Vdash \check{x} \leq \dot{X}$. We will refer to the family of all Cohen conditions

$P = \langle p_i^j \rangle_{i,j}$ as a *matrix of conditions* and to the tree $\Phi = \Phi(P)$ as *an associated tree of extensions*. Note that definition 4.3.2 coincides with the particular case of Definition 6.1.3 in which Γ is a singleton.

6.2. The $\aleph_2$-chain condition

Having in mind the construction following Definitions 5.4.2 consider the following Lemma:

LEMMA 6.2.1. *Let Γ and Θ be countable subsets of ω_2, let $\Delta = \Gamma \cap \Theta$, $\Omega = \Gamma \cup \Theta$ and let $\dot{X}$ be $\mathbb{C}(\Gamma)$-symmetric name for a pure condition. Suppose*

$$\sup \Delta < \min \Gamma \backslash \Delta < \sup \Gamma \backslash \Delta < \min \Theta \backslash \Delta$$

and let $i : \Gamma \cong \Theta$ be an isomorphism such that $i \restriction \Delta = id$. Then for every finite subset $\Gamma' = \{\gamma_0, \dots, \gamma_s\}$ of ω_2 where

$$\gamma_0 = \min \Omega < \gamma_1 < \cdots < \gamma_s = \sup \Omega$$

and all finite families of conditions $\langle p_i^j \rangle_{i \le k_j} \subseteq \mathbb{C}(\Omega \cap \gamma_j \backslash \gamma_{j-1})$ for $j = 1, \dots, s$ and every $M \in \omega$ there is $x \in L_M$ and an associated tree of extensions

$$\Phi(P) = \{\bar{\phi}(i_1 \dots i_j) : 1 \le j \le s, 1 \le i_j \le k_j\}$$

for every maximal node $\bar{\phi}$ of which

$$\bar{\phi} \Vdash \text{“}\check{x} \le \dot{X} \text{ and } \check{x} \le i(\dot{X})\text{”}$$

Here P denotes the collection $\langle p_i^j \rangle_{i,j}$ of the given Cohen conditions.

PROOF. Adding more ordinals if necessary we can assume that

$$\Gamma' \cap \Delta = \{\gamma_j\}_{j \in n+1},\ \Gamma' \cap \Gamma \backslash \Delta = \{\gamma_j\}_{j \in (n,2n]},\ \Gamma' \cap \Theta \backslash \Delta = \{\gamma_j\}_{j \in (2n,3n]}.$$

Furthermore we can assume that $i(\gamma_j) = \gamma_{j+n}$ for all $j \in (n, 2n]$ and $\gamma_n = \sup \Delta$, $\gamma_{2n} = \sup \Gamma \backslash \Delta$, $\gamma_{3n} = \sup \Theta$. Also for all $j \in (0, 3n]$ let $k_j = k$. Let $M \in \omega$ be given. We have to obtain a tree of extensions $\Phi(P)$ associated with the matrix P and a finite logarithmic measure $x \in L_M$ such that the maximal nodes of Φ force that x extends $\dot{X}$ and the isomorphic copy $i(\dot{X})$.

For every $j \in (0, 2n]$, $i \in (0, k]$ let $r_i^j = p_i^j$ and for $j \in (n, 2n]$, $i \in (k, 2k]$ let $r_i^j = i^{-1}(p_{i-k}^{j+n})$. Then $R = (r_i^j)_{i,j}$ is a matrix of conditions in $\mathbb{C}(\Gamma)$ and so by symmetry of $\dot{X}$ there is a finite logarithmic measure $x \in L_M$ and a tree of extensions $\Psi(R) = \{\bar{\psi}(i_1 \dots i_j) : 1 \leq j \leq 2n, 1 \leq i_j \leq k_j'\}$ where for $j \in (0, n]$ $k_j' = k$ and for $j \in (n, 2n]$ $k_j' = 2k$, the maximal nodes of which force that x extends $\dot{X}$. For $j \in (0, 2n]$ let

$$\phi(i_1 \dots i_j) = \psi(i_1 \dots i_j)$$

and for $j \in (2n, 3n]$, say $j = 2n + m$ let

$$\phi(i_1 \dots i_j) = i(\psi(i_1 \dots i_n; i_{n+1} + k, \dots, i_{n+m} + k)).$$

Then let

$$\Phi(P) = \{\bar{\phi}(i_1 \dots i_j) : 1 \leq j \leq 3n, 1 \leq i_j \leq k\}$$

where $\bar{\phi}(i_1) = \phi(i_1)$ and $\bar{\phi}(i_1 \dots i_j) = (\bar{\phi}(i_1 \dots i_{j-1}), \phi(i_1 \dots, i_j))$ is a tree of extensions of the given matrix P. To see that Φ has the desired properties, consider arbitrary maximal node $\bar{\phi} = \bar{\phi}(i_1 \dots i_{3n})$. Then $\bar{\phi} = (\alpha_0, \alpha_1, \alpha_2)$ where $\alpha_0 = \bar{\phi}(i_1 \dots i_n) = \bar{\psi}(i_1 \dots i_n)$, $\alpha_1 = \langle \phi(i_1 \dots i_j) : n < j \leq 2n \rangle$ and $\alpha_2 = \langle \phi(i_1 \dots i_j) : 2n < j \leq 3n \rangle$. Observe in particular that $\alpha_0 \in \mathbb{C}(\Delta)$, $\alpha_1 \in \mathbb{C}(\Gamma \backslash \Delta)$ and $\alpha_2 \in \mathbb{C}(\Theta \backslash \Delta)$. Furthermore $(\alpha_0, \alpha_1) = \bar{\psi}(i_1 \dots i_{2n})$ and

$$(\alpha_0, i^{-1}(\alpha_2)) = \bar{\psi}(i_1 \dots i_n; i_{2n+1} + k, \dots, i_{3n} + k)$$

are maximal nodes of $\Psi(R)$ and so they force that x extends $\dot{X}$. It remains to observe that $i(\alpha_0, i^{-1}(\alpha_2)) = (\alpha_0, \alpha_2)$ since $i \restriction \Delta = \mathrm{id}$ and so $(\alpha_0, \alpha_2) \Vdash \check{x} \leq i(\dot{X})$. Therefore $\bar{\phi} \Vdash$ "$\check{x} \leq \dot{X}$ and $\check{x} \leq i(\dot{X})$". □

LEMMA 6.2.2. *Let Γ and Θ be countable subsets of ω_2, let $\Delta = \Gamma \cap \Theta$, $\Omega = \Gamma \cup \Theta$,*

$$\sup \Delta < \min \Gamma \backslash \Delta < \sup \Gamma \backslash \Delta < \min \Theta \backslash \Delta$$

and let $i : \Gamma \cong \Theta$ be an isomorphism such that $i \restriction \Delta = id$. Let $\dot{X}$ be $\mathbb{C}(\Gamma)$-symmetric name for a pure condition. Then $i(\dot{X})$ is a $\mathbb{C}(\Theta)$-symmetric name for a pure condition and there is a $\mathbb{C}(\Omega)$-symmetric name for a pure condition $\tilde{X}$ such that

$$\Vdash_{\mathbb{C}(\Omega)} \tilde{X} \leq \dot{X} \text{ and } \tilde{X} \leq i(\dot{X}).$$

PROOF. Enumerate all finite subsets of ω_2 and associated matrices of conditions on Ω such that each pair is enumerated cofinally often. At stage n consider the n-th pair and let m_n be an integer grater than the measures and domains of all finite logarithmic measures that have been defined up to this stage. Apply Lemma 6.2.1 to this n-th pair and the integer m_n to obtain a corresponding tree of extensions T_n and a finite logarithmic measure $x \in L_{m_n}$ such that the maximal nodes of the tree force "$\check{x} \leq \dot{X} \wedge \check{x} \leq i(\dot{X})$. Then let $\{(t, \check{x}) : t$ max node of $T_n\} \subseteq \tilde{X}$. $\square$

6.3. Conclusion and open questions

A family $\mathcal{A}$ of infinite subsets of ω, with pairwise finite intersection is *an almost disjoint family.* An almost disjoint family which is maximal, is called a *maximal almost disjoint family*, usually abbreviated as mad family. The *almost disjointness* number $\mathfrak{a}$ is the minimal size of a maximal almost disjoint family. The *ultrafilter number* $\mathfrak{u}$ is the minimal size of an ultrafilter base. A family $\mathcal{F}$ of subsets of ω has the *strong finite intersection property* if the intersection of any finite subfamily of $\mathcal{F}$ is infinite. A *pseudo-intersection* of a family $\mathcal{F}$ is an infinite set almost contained in every element of the family. *The pseudo-intersection* number $\mathfrak{p}$ is the minimal size of a family which has the strong finite intersection property and no pseudo-intersection.

Theorem 6.3.1 (GCH). *Let κ be a regular uncountable cardinal. Then there is a ccc generic extension, in which $\mathfrak{b} = \kappa < \mathfrak{s} = \mathfrak{a} = \kappa^+$.*

Proof. In [**12**] J. Brendle shows that if V is a model of ZFC^*, κ is a regular uncountable cardinal, $\langle f_\alpha : \alpha < \kappa \rangle$ is a $<^*$-well ordered sequence of strictly increasing functions from ω to ω and in V, $\mathfrak{c} = \kappa$, $2^\kappa = \kappa^+$ and $\langle f_\alpha : \alpha < \kappa \rangle$ is unbounded and $\mathcal{A}$ is a maximal almost disjoint family, then there is a *ccc* forcing notion $\mathbb{P}(\mathcal{A})$ of size $\mathfrak{c}$ such that $\Vdash_{\mathbb{P}(\mathcal{A})}$ "$\mathcal{A}$ is not mad and $\langle f_\alpha : \alpha < \kappa\rangle$ is unbounded". Using an appropriate bookkeeping device, along the finite support iteration of Theorem 3.6.3, one can destroy all mad families of size $\leq \kappa$. $\square$

THEOREM 6.3.2 (GCH). *Let κ be a regular uncountable cardinal. Then there is a ccc generic extension in which* $\mathfrak{p} = \mathfrak{b} = \kappa < \mathfrak{s} = \mathfrak{a} = \kappa^+$.

PROOF. Along the finite support iteration from the proof of Theorem 3.6.3, one can force with all σ-centered forcing notions of size $< \kappa$, and so provide that in the final generic extension $MA_{<\kappa}(\sigma\text{-centered})$ holds. Then by Bell's theorem, $V^{\mathbb{P}_{\kappa^+}} \vDash \mathfrak{p} \geq \kappa$. However, it is a ZFC theorem that $\mathfrak{p} \leq \mathfrak{b}$ and so $V^{\mathbb{P}_{\kappa^+}} \vDash \mathfrak{p} = \kappa$. $\square$

THEOREM 6.3.3 (GCH). *Let κ be a regular uncountable cardinal. Then there is a ccc generic extension in which*

$$\mathfrak{p} = \mathfrak{b} = \kappa < \mathfrak{s} = \mathfrak{a} = \mathfrak{u} = \kappa^+.$$

PROOF. Modifying an argument from A.Blass and S. Shelah [**9**], it will be shown that in the model of Theorem 3.6.3 the ultrafilter number $\mathfrak{u} = \kappa^+$. Suppose $\mathfrak{u} < \kappa^+$ and let $\mathcal{F}$ be an ultrafilter base of size $\mathfrak{u}$. Then there is $\beta < \kappa^+$ such that $\mathcal{F} \subseteq V_\beta = V[G_\beta]$ where as usual, for every $\gamma \leq \kappa^+$ $G_\gamma = G \cap \mathbb{P}_\gamma$. We can assume that in V_β, $Q_\alpha = Q(C_\alpha)$ where $\alpha = \beta + 1$ for an appropriate centered family of pure conditions C_α. Let $s_\alpha = \cup\{u : \exists T (u,T) \in G\}$ where $G_\alpha = G_\beta * G$, i.e. G is $Q(C_\alpha)$-generic over $V_\beta = V[G_\beta]$ and let $X = \{n : |s_\alpha \cap n| \text{ is even}\}$. Then $X \in V[G_\alpha] \cap [\omega]^\omega$. Let $\dot{X}$ and $\dot{s}_\alpha$ be $Q(C_\alpha)$-names for X and s_α respectively, in $V[G_\beta]$. It will be shown that neither X nor its complement contain infinite set from V_β, which contradicts the hypothesis that $\mathcal{F}$ is an ultrafilter base. Suppose to the contrary, that there is $Y \in V_\beta \cap [\omega]^\omega$ such that $V[G_\alpha] \models Y \subseteq X$ or $V[\alpha] \models Y \subseteq X^c$. Without loss of generality suppose $V[G_\alpha] \models Y \subseteq X$. Then there is $(u,T) \in Q(C_\alpha)$ such that $(u,T) \Vdash \check{Y} \subseteq \dot{X}$. Let $m = \min \mathrm{int}(T)$ and let $y \in Y$ such that $y > m$. Then $(u, T\backslash y)$ and $(u \cup \{m\}, T\backslash y)$ extend (u,T). However $(u, T\backslash y) \Vdash \dot{s}_\alpha \cap y = u$ and $(u \cup \{m\}, T\backslash y) \Vdash \dot{s}_\alpha \cap y = u \cup \{m\}$. Then one of those extensions forces "$\check{y} \notin \dot{X}$", which is a contradiction. $\square$

COROLLARY 6.3.4 (GCH). *Let κ be regular uncountable cardinal. Then there is a ccc generic extension, in which $\mathfrak{p} = \mathfrak{t} = \mathfrak{h} = \mathfrak{b} = \kappa$ and $\mathfrak{s} = \mathfrak{d} = \mathfrak{i} = \mathfrak{a} = \mathfrak{u} = \mathfrak{c} = \kappa^+$.*

QUESTION 6.3.5. *What can be said about $\mathfrak{r}$, $\mathfrak{e}$, $\mathfrak{g}$ in this model?*

In section 4.2, we showed that in the Cohen extension, the collection of all subsets of ω which do not have symmetric names forms an ideal I_{nsym}. This ideal has a very natural definition, and on the other hand its properties seem to be distinct from the properties of known ideals.

QUESTION 6.3.6. *Find a generating set for I_{nsym}. Is there an absolute analogue of I_{nsym}? What are the properties of $\mathcal{P}(\omega)/I_{nsym}$?*

One can combine the techniques of chapter III with techniques of S. Shelah, from his original paper [**31**] on the consistency of $\mathfrak{b} = \omega_1 < \mathfrak{s} = \mathfrak{a} = \omega_2$ to obtain a forcing notion which preserves a given unbounded family unbounded, which destroys a maximal almost disjoint family and adds a real not split by the ground model reals. Furthermore, it seems reasonable to expect that the construction of the countably closed, $\aleph_2$-c.c. forcing notion from chapter V, see Definition 5.4.2, can be modified to obtain a countably closed, $\aleph_2$-c.c. forcing notion (or alternatively κ-closed, κ^+-c.c.) which adds a centered family C of $\mathbb{C}(\lambda)$-names for pure conditions, such that $Q(C)$ preserves all unbounded families unbounded, destroys $V^{\mathbb{C}(\lambda)} \cap [\omega]^\omega$ as a splitting family, and adds a real almost disjoint from the elements of a given maximal almost disjoint family in $V^{\mathbb{C}(\lambda)}$. Then it becomes imperative to find an appropriate way to iterate this forcing notion and obtain the consistency of $\mathfrak{b} = \kappa < \mathfrak{s} = \mathfrak{a} = \lambda$, where κ and λ are arbitrary regular uncountable cardinals.

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