Evidence for Set-Theoretic Truth and the Hyperuniverse Programme

Sy-David Friedman
Kurt Gödel Research Center, University of Vienna
sdf@logic.univie.ac.at

I discuss three potential sources of evidence for truth in set theory, coming from set theory's roles as a branch of mathematics and as a foundation for mathematics as well as from the intrinsic maximality feature of the set concept. I predict that new *non first-order* axioms will be discovered for which there is evidence of all three types, and that these axioms will have significant first-order consequences which will be regarded as true statements of set theory. The bulk of the paper is concerned with the *Hyperuniverse Programme*, whose aim is to discover an optimal mathematical principle for expressing the maximality of the set-theoretic universe in height and width.

1 Introduction

The truth of the axioms of ZFC is commonly accepted for at least two reasons. One reason is *foundational*, as they endow set theory with the ability to serve as a remarkably good foundation for mathematics as a whole, and another is *intrinsic*, as (with the possible exception of AC, the axiom of choice) they can be seen to be derivable from the concept of set as embodied by the maximal iterative conception.

In fact a little bit more than ZFC is justifiable on intrinsic and perhaps also foundational grounds. I refer here to *reflection principles* and their related *small large cardinals*, which are also derivable from the maximal iterative conception through *height (ordinal) maximality* and, at least in the case of inaccessible cardinals, are occasionally useful for the development of certain kinds of highly abstract mathematics (such as *Grothendieck universes*). These extensions of ZFC are *mild* in the sense that they are compatible with the *powerset-minimality* principle $V = L$.

But finding strong evidence for the truth of axioms that contradict $V = L$ has been exceedingly difficult. There are a number of reasons for this. One is the fact that mild extensions of ZFC have been in a sense *too good*, in that they alone have until recently been sufficient to serve the needs of set theory as a foundation for mathematics. Another is the difficulty of squeezing more out of the maximal iterative conception through a *width (powerset) maximality* analogue of the height maximality principles that give rise to reflection. And the development of set theory as a branch of mathematics has been so dramatic, diverse and ever-changing that it has been impossible to select those perspectives on the subject whose choices of new axioms can be regarded as "the most true".

My aim in this article is to provide evidence for the following three predictions.

The Richness of Set-Theoretic Practice. The development of set theory as a branch of mathematics is so rich that there will never be a consensus about which first-order axioms (beyond ZFC plus small large cardinals) best serve this development.

A Foundational Need. Just as AC is now accepted due to its essential role for mathematical practice, a systematic study of independence results across mathematics will uncover first-order statements contradicting CH (and hence also $V = L$) which are best for resolving such independence.

An Optimal Maximality Criterion. Through the *Hyperuniverse Programme* it will be possible to arrive at an optimal *non first-order* axiom expressing the maximality of the set-theoretic universe in height and width; this axiom will have first-order consequences contradicting CH (and hence also $V = L$).

And as a synthesis of these three predictions I propose the following optimistic scenario for making progress in the study of set-theoretic truth.

Thesis of Set-Theoretic Truth. There will be first-order statements of set theory that well serve the needs of set-theoretic practice and of resolving independence across mathematics, and which are derivable[1] from the maximality of the set-theoretic universe in height and width. Such statements will come to be regarded as *true statements of set theory.*

This *Thesis* has a converse: In order for a first-order statement contradicting $V = L$ to be regarded as *true*, in my view it must well serve the needs of set-theoretic practice and of resolving independence in mathematics, and it must at least be compatible with the maximality of the set-theoretic universe as expressed

[1]For a discussion of this notion of *derivability* see the final Subsection 4.12.

by the optimal maximality criterion. Indeed the strength of the evidence for such a statement's truth is in my view measured by the extent to which it fulfills these three requirements.

An important consequence of the *Thesis* is the failure of CH. Thus part of my prediction is that CH will be regarded as false.

Note that in the *Thesis* I do not refer to *true first-order axioms* but only to *true first-order statements*. The reason is the following additional claim.

Beyond First-Order. There will never be a consensus about the truth of proposed first-order axioms that contradict $V = L$; instead true first-order statements will arise solely as consequences of true *non first-order* axioms.

One reason for this claim is the inadequancy of first-order statements to capture the maximality of the set-theoretic universe.

The plan of this paper is as follows. First I'll review some of the popular first-order axioms that well serve the needs of set-theoretic practice and argue for the *Richness* prediction above. Second I'll discuss what little is known about independence across mathematics, discussing the role of forcing axioms as evidence for the *Foundational* prediction above. And by far the bulk and central aim of the paper is the third part, in which I present the *Hyperuniverse Programme*, including its philosophical foundation and most recent mathematical developments.

2 Set-Theoretic Practice

Set theory is a burgeoning subject, rife with new ideas and new developments, constantly leading to new perspectives. Naturally certain of these perspectives stand out among the chaotic mass of new results being proved, and it is worth focusing on a few of these to expose the difficulty of settling on particular new axioms as being "the true ones".

I have emphasized the need to find evidence for the truth of axioms that contradict $V = L$, but purely in terms of the value of an axiom for the development of good set theory, what I will refer to as *Type 1 evidence*, this is not possible. Jensen's deep work unlocking the power of this axiom reveals the power of $V = L$, indeed it appears to give us, when combined with small large cardinals, a theory that is complete for all natural set-theoretic statements! That is a remarkable achievement and speaks volumes in favour of declaring $V = L$ to be *true* based on Type 1 evidence.

A natural Type 1 objection to $V = L$ is that it doesn't take forcing into account, a fundamental method for building new models of set theory. Admittedly, even in L one has forcing extensions of countable models, but it is more natural to force over the full L and not just over some small piece of it. So now we contradict $V = L$ in favour of "V contains many generic extensions of L" or something similar.

Having lots of forcing extensions of L sounds good, but then what is our canonical universe now? Shouldn't we also have a sentence that is true only in V, and not in any of its proper inner models, while at the same time having many generic extensions of L? Indeed this is possible with class forcing (see [11]). So now we have a nice Type 1 axiom: V is a canonical universe which is class-generic over L, containing many set-generic extensions of L. This is an excellent context for doing set theory, as the forcing method is now available.

In fact we can do even better and take V to be $L[0^\#]$. Not only does this model contain many generic extensions of L, it is also a canonical universe and we recover all of the powerful methods that Jensen developed under $V = L$, relativised now to the real $0^\#$. So our Type 1 evidence leads us to the superb axiom $V = L[0^\#]$.

Objection! What about measurable cardinals? Recall the important hierarchy of consistency strengths: Natural theories are wellordered (up to bi-interpretability) by their consistency strengths and the consistency strengths of large cardinal axioms provide a nice collection of consistency strengths which is cofinal in a large initial segment of (if not all of) this hierarchy. This does not mean that large cardinals must exist but at the very least there should be inner models having them. So now based on Type 1 evidence we get some version of "There are inner models with large cardinals", an attractive environment in which to do good set theory.

Moreover, notice that if we have inner models for large cardinals we haven't lost the option of looking at L or its generic extensions, they are still available as inner models. So we seem to have reached the best Type 1 axiom yet.

But we could ask for even more. Recall that L has a nice internal structure, very powerful for deriving consequences of $V = L$. Can V not only have inner models for large cardinals but also an L-like internal structure? Of course the answer is positive, as we can adopt the axiom "There are inner models with large cardinals and $V = L[x]$ for some real x". A better answer is provided in [14], where it is shown that V can be L-like together with arbitrary large cardinals, not only in inner models but in V itself. However, as attractive as this may sound, it fails to address a key problem, and this is where we see the multiple perspectives of set theory, with no single perspective having a claim to being "the best".

Even if we produce a nice axiom[2] of the form "There are large cardinals and V is a canonical generalisation of L", doing so commits us to an L-like environment in which to do set theory. Indeed there are other compelling perspectives on set theory which lead us to non L-like environments and correspondingly to entirely different Type 1 axioms. I will mention two of them. (Further information about the notions mentioned below is available in [22]).

Forcing axioms have a long history, dating back to Martin's axiom (MA), a special case of which asserts the existence of generics for ccc partial orders (i.e. partial orders with only countable antichains) over models of size $\aleph_1$. This simple axiom can be used to establish in one blow the relative consistency of a huge range of set-theoretic statements. Naturally there has been interest in strengthenings of MA, and a popular one is the Proper Forcing Axiom (PFA), which strengthens this[3] to the wider class of *proper* partial orders.

Now with regard to Type 1 evidence the point is that PFA has even more striking consequences than MA, qualifying it as a central and important tool for solving combinatorial problems in set theory. A powerful case can be made for its truth based on Type 1 evidence. But of course PFA conflicts with any axiom which asserts that V is L-like, as it implies the negation of CH. In fact PFA implies that the size of the continuum is $\aleph_2$.

The diversity of Type 1 evidence goes beyond just L-likeness and forcing axioms; there are also cardinal characteristics. These are natural and heavily-investigated cardinal numbers that arise when studying definability-theoretic and combinatorial properties of sets of real numbers. Each of these cardinal characteristics is an uncountable cardinal number of size at most the continuum. Now given the variety of such characteristics together with the fact that they can consistently differ from each other, isn't it compelling to adopt the axiom that cardinal characteristics provide a large spectrum of distinct uncountable cardinals below the size of the continuum and therefore the continuum is indeed quite large, in contradiction to both L-likeness and forcing axioms?[4]

[2] Woodin has in fact proposed such an axiom which he calls *Ultimate L*.

[3] For the experts, to get PFA one must allow non-transitive models of size $\aleph_1$.

[4] As a specific example, let $\mathfrak{a}$ denote the least size of an infinite almost disjoint family of subsets of ω, and $\mathfrak{b}$ ($\mathfrak{d}$) the least size of an unbounded (dominating) family of functions from ω to ω ordered by eventual domination. Then $\mathfrak{b} < \mathfrak{a} < \mathfrak{d}$ is consistent; shouldn't it in fact be true?

Thus we have three distinct types of axioms with excellent Type 1 evidence: L-likeness with large cardinals, forcing axioms and cardinal characteristic axioms. They contradict each other yet each is consistent with the existence of inner models for the others. In my view, this makes a clear case that Type 1 evidence is insufficient to establish the truth of axioms of set theory; it is also insufficient to decide whether or not CH is true.

3 Set Theory as a Foundation for Mathematics

Of course axiomatic set theory can be heartily congratulated for its success in providing a foundation for mathematics. An overwhelming case can be made that when theorems are proved in mathematics they can be regarded as theorems of a mild extension of ZFC (compatible with $V = L$). In particular, we routinely expect questions in mathematics to be answerable (perhaps with great difficulty!) in a mild extension of ZFC.

A consequence is that an independence result for such mild extensions is indeed an independence result for mathematics as a whole. This is of course of minor importance if the independence result in question is a statement of set theory, as set theory is just a small part of mathematics. But this is of considerable importance when independence arises with questions of mathematics outside of set theory, as is the case with the the Borel, Kaplansky and Whitehead Conjectures of measure theory, functional analysis and group theory, respectively. Let us not forget the great mathematician David Hilbert's thesis that the questions of mathematics can be resolved using the powerful tools of the subject. An understanding of how to deal with independence is needed to restore the status of mathematics as the complete and definitive field of study that Hilbert envisaged.

The time is ripe for set-theorists to focus on this problem. The central question is:

Foundational or Type 2 Evidence: Are there particular axioms of set theory which best serve the needs of resolving independence in other areas of mathematics?

Recently there are signs that a positive answer to this question is emerging, as new applications of set theory to functional analysis, topology, abstract algebra and model theory (a field of logic, but still outside of set theory) are being found. The *Foundational Need* that I expressed earlier is precisely the prediction that a pattern will emerge from these applications to reveal that particular axioms of set theory are best for bringing set theory closer to the complete foundation that Hilbert was hoping for.

Now where are these *foundationally advantageous* axioms of set theory to be found? Consider the following list of candidates with good Type 1 evidence:

$V = L$
V is a canonical and rich class-generic extension of L
Large Cardinal Axioms (like supercompacts)
Forcing Axioms like MA, PFA
Determinacy Axioms like AD in $L(R)$
Cardinal Characteristic Axioms like $\mathfrak{b} < \mathfrak{a} < \mathfrak{d}$

As already said, each of these axioms is important for the development of set theory, providing a unique perspective on the subject. But perhaps it is surprising to discover that only two of them, $V = L$ and Forcing Axioms, have had any significant impact on mathematics outside of set theory! The impact of Large Cardinal Axioms (like supercompacts) and Cardinal Characteristic Axioms has been minimal and that of Determinacy Axioms non-existent so far.

To give a bit more detail, both $V = L$ and Forcing Axioms can be used to answer the following questions (in different ways):

Functional Analysis: Must every homomorphism from $C(X)$, X compact Hausdorff, into another Banach algebra be continuous (the Kaplansky Problem)? Is the ideal of compact operators on a separable Hilbert space in the ring of all bounded operators the sum of two smaller ideals?; Are all automorphisms of the Calkin Algebra inner?

Topology and Measure Theory: Is every normal Moore Space metrizable? Are there S-spaces (regular, hereditarily separable spaces where some open cover has no countable subcover)? Is every strong measure 0 set of reals countable (the Borel conjecture)?

Abstract Algebra: Is every Whitehead group free (the Whitehead Problem)? What is the homological dimension of $R(x, y, z)$ as an $R[x, y, z]$-module where R is the field of real numbers? Does the direct product of countably many fields have global dimension 2?

One could also mention the field of Model Theory (part of Logic, but not part of Set Theory), where new axioms of Set Theory may play an important role in the study of Morley's theorem for Abstract Elementary Classes or perhaps even in the resolution of Vaught's Conjecture.

My prediction is that $V = L$ and Forcing Axioms will be the definite winners among choices of axioms of Set Theory that resolve independence across mathematics as a whole. But as $V = L$ is in conflict with the maximality of the set-theoretic universe in width, it is not suitable as a realization of the *Thesis of Set-Theoretic Truth*, leaving Forcing Axioms as the current leading candidate for that.

4 The Maximality of the Set-Theoretic Universe and the HP

The letters HP stand for the *Hyperuniverse Programme*, which I now discuss in detail.

4.1 The Iterative Conception of Set

As Gödel put it, the *iterative conception* of set expresses the idea that a set is something obtainable from well-defined objects by iterated application of the powerset operation. In more detail (following Boolos [7]; also see [27]): Sets are formed in *stages*, where only the empty set is formed at stage 0 and at any stage greater than 0, one forms collections of sets formed at earlier stages. (Said this way, a set is re-formed at every stage past where it is first formed, but that is OK.) Any set is formed at some least stage, after its elements have been formed. This conception excludes anomalies: We can't have $x \in x$, there is no set of all sets, there are no cycles $x_0 \in x_1 \in \cdots \in x_n \in x_0$ and there are no infinite sequences $\cdots \in x_n \in x_{n-1} \in x_{n-2} \in \cdots \in x_1 \in x_0$, as there must be a least stage at which one of the x_n's is formed. We'll assume that there are infinite sets[5], so the iteration process leads to a limit stage ω, which is not 0 and is not a successor stage.

The iterative conception yields that the universe of sets is a model of the axioms of Zermelo Set Theory, i.e. ZFC without Replacement and without the Axiom of Choice. The standard model for this theory is $V_{\omega+\omega}$.

Nevertheless, Replacement and AC (the Axiom of Choice) are included as part of the standard axioms of Set Theory, for very different reasons. The case for AC is typically made on *extrinsic grounds*, citing its *fruitfulness* for the development

[5]This is derivable once we add *maximality* to the iterative conception, but is convenient to assume already as part of the iterative conception.

of mathematics and its corresponding necessity for Set Theory as a foundation for mathematics (a case of what I have called *Type 2 evidence*). It is not clear to me that Choice is derivable from the iterative conception, nor from its necessity for doing good Set Theory (*Type 1 evidence*).

Replacement, on the other hand, is derivable from the concept of set. To see this, we need to extend the iterative conception to the stronger *maximal iterative conception*, also implicit in the set-concept.

4.2 Maximality and the Iterative Conception

The term *maximal* is used in many different senses in Set Theory, what I have in mind here is a very specific use associated to the iterative conception (IC). Recall that according to the IC, sets appear inside levels indexed by the ordinal numbers, where each successor level $V_{\alpha+1}$ is the powerset of the previous. As Boolos explained, the IC alone takes no stand on how many levels there are (the *height* of the universe V) or on how fat the individual levels are (the *width* of V). However it is generally regarded as implicit in the set-concept that both of these should be *maximal*:

Height (or Ordinal) maximality: The universe V is *as tall as possible*, i.e., the sequence of ordinals is *as long as possible*.

Width (or Powerset) maximality: The universe V is *as wide (or thick) as possible*, i.e., the powerset of each set is *as large as possible*.

If we conjunct the IC with maximality we arrive at the MIC, the *maximal iterative conception*, also part of the set-concept but more of a challenge to explain than the simple IC.

It is natural to see a *comparative* aspect to maximality, as to be *as large as possible* suggests *as large as possible within the realm of possibilities.* Thus a natural way to explain height and width maximality would be to compare V to other possible universes.

But now we face a serious problem. If V is the fixed universe of *all* sets, then there are no universes other than those already included in V. In other words V is maximal by default, as no other universe can threaten its maximality, and therefore we are limited in what we can say about this concept.

I will postpone this problem for now, and instead discuss an easier one: Let M denote a countable transitive model of ZFC (ctm). What could it mean to say that M is maximal?

Now we have a different problem. The natural way to express the maximality of M is to say that M cannot be expanded to a larger universe. Let us call this *structural maximality*. But under a very mild assumption (there is a set-model of ZFC containing all of the reals) this is impossible: Any ctm M is an element (and therefore proper subset) of a larger ctm.

So instead we move to a milder form of maximality, called *syntactic maximality*, expressed as follows.

In the case of (syntactic-) height maximality, we consider *lengthenings* of M, i.e. ctm's M^* of which M is a rank-initial segment (the ordinals of M form an initial segment of the ordinals of M^* and the powerset operations of these two universes agree on the sets in M).

In the case of width maximality, we consider *thickenings* of M, i.e. ctm's M^* of which M is an inner model (M and M^* have the same ordinals and M is included in M^*).

In this way we can produce forms of *height maximality* and *width maximality* for ctm's as follows.

If M is *height maximal* then a property of M also holds of some rank-initial segment of M. This is the typical formulation of *reflection*. (However we will see that *height maximality* is stronger than *reflection*.) Of course specific realizations of height maximality must specify which properties are to be taken into account.

If M is *width maximal* then a property of a thickening of M also holds of some inner model of M. In the case of first-order properties this is called the *Inner Model Hypothesis*, or *IMH* (introduced in [12]).

The above discussion of maximality for ctm's, although brief, will suffice for establishing the strategy of the HP.

We return now to the problem of maximality for V. Can the above discussion for ctm's also be applied to V? Does it make sense to talk about *lengthenings* and *thickenings* of V in the way we talk about them for ctm's? There are differences of opinion about this, which I'll take up next.

4.3 Actualism and Potentialism

Recall that in the *IC* we describe V, the universe of sets, via a process of iteration of the powerset operation. Does this process come to an end, or is it indefinite, always extendible further to a longer iteration? The former possibility, that there is a "limit" to the iteration process is referred to as *height actualism* and the latter view is called *height potentialism*. Analogously there is a question of the definiteness of the powerset operation: For a given set, is its powerset determined or is it always possible to extend it further by adding more subsets? The former is called *width actualism* and latter *width potentialism*.

There is a vast literature on this topic ([4, 19, 20, 21, 23, 24, 25, 26, 29, 31, 32]). However as the Hyperuniverse Programme is very flexible on the choice of ontology, we will not engage here in a lengthy discussion of the actualism/potentialism debate, but only mention some points in favour of a *Zermelian view*, combining height potentialism with width actualism, the view which we choose to adopt for our analysis of maximality via the HP.

We can summarize the situation as follows. Without difficulty, height potentialism facilitates an analysis of height maximality. Surprisingly, we will show that even with width actualism, it also facilitates an analysis of width maximality, using the method of *V-logic*. A further benefit of height potentialism is that we can reduce the study of maximality for V to the study of maximality for ctm's[6]. Our arguments also show that height actualism is viable for our analysis of width maximality, provided it is enhanced with a strong enough fragment of MK (Morse-Kelley class theory; one only needs Σ^1_1 comprehension). Thus the only problematic ontology for the HP is height actualism supported by only a weak class theory; otherwise the choice of ontology is not critical for the HP (although the programme develops slightly differently with width potentialism than it does with width actualism) [7].

I will now present some arguments due to Geoffrey Hellman ([33][8]) in favour of

[6]The set of ctm's is called the *Hyperuniverse*; hence we arrive at the *Hyperuniverse Programme*.

[7]Height actualism with just GB (Gödel-Bernays) appears inadequate for a fruitful analysis of maximality. A referee has informed us about *agnostic Platonism*, the view that there is a well-determined universe V of all sets but without taking a position on whether ZFC holds in it. But as this perspective allows for the possibility of height actualism with just GB, it is problematic for the HP.

[8]These comments were made during a lively e-mail exchange among numerous set-theorists and philosophers of set theory from August until November 2014, triggered by my response to Sol Feferman's preprint *The Continuum Hypothesis is neither a definite mathematical problem nor a definite logical problem*. Some of this discussion is documented at <http://logic.harvard.edu/blog/?cat=2>, but regrettably Hellman's comments do not appear there.

height potentialism and width actualism, *the Zermelian view.* Hellman says:

“The idea that any universe of sets can be properly extended (in height, not width) is extremely natural, endorsed by many mathematicians (e.g. MacLane, seemingly by Gödel, et. al.) ... As Maddy and others say, if it’s possible that sets beyond some (putatively maximal) level exist, then they do exist ... Thus, if ‘imaginable’ (end) extensions of V are not incoherent, then they are possible, and then, on an actualist, platonist reading, they are actual, and V wasn’t really maximal after all. ... such extensions are always possible, so that the notion of a single fixed, absolutely maximal universe V of sets is really an incoherent notion.”

And again:

“I have no earthly or heavenly idea what ‘as high as possible’ could mean, since the notion of a set domain that absolutely could not in logic be extended seems to me incoherent (or at any rate empty). As Putnam put it in his controversial paper, ‘Mathematics without Foundations’ (1967), ‘Even God couldn’t make a universe for Zermelo set theory that it would be impossible to extend.’ And I agree, theology aside.”

Regarding width potentialism, Hellman says ([33]):

“I have a good idea, I think, about ‘as thick as possible’, since the notion of full power set of a given set makes perfect sense to me ... Granted that forcing extensions can be viewed as ‘thickenings’ of the cumulative hierarchy, as usually described, when we assert the standard Power Sets axiom, we implicitly build in bivalence, i.e. that either x belongs to y or it doesn’t, i.e. we are in effect ruling forcing extensions or Boolean-valued generalizations as *non-standard* [my italics], i.e. ‘full power set’ is to be understood only in the standard way.”

And further:

“Thus, to my way of thinking, there is an important disanalogy between ‘all ordinals’ ... and ‘all subsets of a given set’. The latter is ‘already relativized’; there is nothing implicit in the notion of ‘subset’ that allows for indefinite extensions, so long as we are speaking of ‘subsets of a fixed, given set’ ... In contrast, ‘all ordinals’ cries out for relativization (a point I find in Zermelo’s [1930]); without it, it does allow for indefinite extensibility, by the very operations that we use to describe ordinals”

I do appreciate Hellman's point here, and indeed will (for the most part) adopt the *Zermelian perspective*, height potentialism with width actualism, in this paper. Another strong point in favour of this view is that although we have a clear and coherent way of generating the ordinals through a process of iteration, there is currently no analogous iteration process for generating increasingly rich power sets[9].

In light of this adoption of potentialism in height, I will now use the symbol V ambiguously, not to denote the fixed universe of all sets (which does not exist) but as a variable to range over universes within the *Zermelian multiverse* in which each universe is a rank initial segment of the next.

Despite my adoption of the Zermelian view, I will for expository purposes also consider a form of potentialism in both height and width which I will call *radical potentialism*. The HP can be run with either point of view. Although it is simpler with radical potentialism, there are interesting issues (both mathematical and philosophical) which arise when employing the Zermelian view which are worth exploring.

To describe radical potentialism, let me begin with something less radical, *width potentialism*. First as motivation, consider a Platonist view, so that V is the fixed universe of all sets, and consider the method of forcing for producing generic sets. If M is a ctm we can easily build a generic extension $M[G]$ of M using the countability of M. But of course generic extensions $V[G]$ of V do not exist, as our "real V" has all the sets. Despite this we can talk definably in V about what can be true in such a *generic extension* without actually having such extensions in V, by constructing the Boolean universe V^B within V and taking *true in a generic extension of* V to just mean *of nonzero Boolean truth value in* V^B. Thus the Platonist view is in fact dualistic: It allows for the possibility of making sense of truth in universes (generic extensions) without allowing these universes to actually exist.

Width potentialism is a view in which any universe can be thickened, keeping the same ordinals, even to the extent of making ordinals countable. Thus for example it allows for the existence of the generic extensions of V (now a variable ranging over the multiverse of all possible universes) that are prohibited by the Platonist. So for any ordinal α of V we can thicken V to a universe where α is countable; i.e., any ordinal is *potentially countable*. But that does not mean that every ordinal of V *is* countable in V, it is only countable in a larger universe. So this *potential countability* does not threaten the truth of the powerset axiom in V.

[9]But I am not 100% sure that there could not be such an analogous iteration process, perhaps provided by a wildly successful theory of inner models for large cardinals.

Now *radical potentialism* is in effect a unification of width and height potentialism. It entails that any V (in the multiverse of possible universes) looks countable inside a larger universe: We allow V to be lengthened and thickened simultaneously. Note that even just width potentialism (allowing universes to be thickened) forces us also into height potentialism: If we were to keep thickening to make every ordinal of V countable then after $\mathrm{Ord}(V)$ steps we are forced to also lengthen to reach a universe that satisfies the powerset axiom. In that universe, the original V looks countable. But then we could repeat the process with this new universe until it too is seen to be countable. The height potentialist aspect is that we cannot end this process by taking the union of all of our universes, as this would not be a model of ZFC (the powerset axiom will fail) and therefore would have to be lengthened. Note that once again, the *potential countability of* V does not threaten the truth of the axioms of ZFC in V.

4.4 Maximality in Height and #-Generation

The analysis of height maximality is the first major success of the HP. The programme has produced a robust principle expressing the maximality of V in height which appears to encompass all prior height maximality principles, including reflection, and to constitute the definitive expression of the height maximality of V in mathematical terms.

For our discussion of height maximality, height potentialism will suffice (radical potentialism is not needed). Thus we allow ourselves the option of lengthening V to universes V^* which have V as a rank-initial segment. Of course we can also consider shortenings of V, replacing V by one of its own rank-initial segments. Let us now make use of lengthenings and shortenings to formulate a height maximality principle for V, expressing the idea that the sequence of ordinals is *as long as possible*.

But before embarking on our analysis of height maximality we should take note of the following: No *first-order* statement φ can be adequate to fully capture height maximality. This is simply because a first-order statement true in V will reflect to one of its rank initial segments and we are then naturally led from φ to the stronger first-order statement "φ holds both in V and in some transitive set model of ZFC". We will also see that no first-order statement is adequate to capture width maximality. This is an instance of the *Beyond First-Order* claim of the introduction: True first-order statements contradicting $V = L$ only arise as consequences of true *non first-order* axioms.

But how do we capture height maximality with a non first-order axiom? We do this via a detailed analysis of the relationship between V and its lengthenings and shortenings.

Standard Lévy reflection tells us that a single first-order property of V with parameters will hold in some V_κ which contains those parameters. It is natural to strengthen this to the simultaneous reflection of all first-order properties of V to some V_κ, allowing arbitrary parameters from V_κ. Thus we have reflected V to a V_κ which is an elementary submodel of V.

Repeating this process leads us to an increasing, continuous sequence of ordinals $(\kappa_i \mid i < \infty)$, whee ∞ denotes the ordinal height of V, such that the models $(V_{\kappa_i} \mid i < \infty)$ form a continuous chain $V_{\kappa_0} \prec V_{\kappa_1} \prec \cdots$ of elementary submodels of V whose union is all of V.

Let C be the proper class consisting of the κ_i's. We can apply reflection to V with C as an additional predicate to infer that properties of (V, C) also hold of some $(V_\kappa, C \cap \kappa)$. But the unboundedness of C is a property of (V, C) so we get some $(V_\kappa, C \cap \kappa)$ where $C \cap \kappa$ is unbounded in κ and therefore κ belongs to C. As a corollary, properties of V in fact hold in some V_κ where κ belongs to C. It is convenient to formulate this in its contrapositive form: If a property holds of V_κ for all κ in C then it also holds of V.

Now note that for all κ in C, V_κ can be *lengthened* to an elementary extension (namely V) of which it is a rank-initial segment. By the contrapositive form of reflection of the previous paragraph, V itself also has such a lengthening V^*.

But this is clearly not the end of the story. For the same reason we can also infer that there is a continuous increasing sequence of such lengthenings $V = V_{\kappa_\infty} \prec V^*_{\kappa_{\infty+1}} \prec V^*_{\kappa_{\infty+2}} \prec \cdots$ of length the ordinals. For ease of notation, let us drop the *'s and write W_{κ_i} instead of $V^*_{\kappa_i}$ for $\infty < i$ and instead of V_{κ_i} for $i \leq \infty$. Thus V equals W_∞.

But which tower $V = W_{\kappa_\infty} \prec W_{\kappa_{\infty+1}} \prec W_{\kappa_{\infty+2}} \prec \cdots$ of lengthenings of V should we consider? Can we make the choice of this tower *canonical*?

Consider the entire sequence $W_{\kappa_0} \prec W_{\kappa_1} \prec \cdots \prec V = W_{\kappa_\infty} \prec W_{\kappa_{\infty+1}} \prec W_{\kappa_{\infty+2}} \prec \cdots$. The intuition is that all of these models resemble each other in the sense that they share the same first-order properties. Indeed by virtue of the fact that they form an elementary chain, these models all satisfy the same first-order sentences. But again in the spirit of "resemblance", the following should hold:

For $i_0 < i_1$ regard $(W_{\kappa_{i_1}}, W_{\kappa_{i_0}})$ as the structure $(W_{\kappa_{i_1}}, \in)$ together with $W_{\kappa_{i_0}}$ as a unary predicate. Then it should be the case that any two such pairs $(W_{\kappa_{i_1}}, W_{\kappa_{i_0}})$, $(W_{\kappa_{j_1}}, W_{\kappa_{j_0}})$ (with $i_0 < i_1$ and $j_0 < j_1$) satisfy the same first-order sentences, even allowing parameters which belong to both $W_{\kappa_{i_0}}$ and $W_{\kappa_{j_0}}$. Generalising this to triples, quadruples and n-tuples in general we arrive at the following situation:

$(*)$ V occurs in a continuous elementary chain $W_{\kappa_0} \prec W_{\kappa_1} \prec \cdots \prec V = W_{\kappa_\infty} \prec W_{\kappa_{\infty+1}} \prec W_{\kappa_{\infty+2}} \prec \cdots$ of length $\infty + \infty$, where the models W_{κ_i} form a *strongly-indiscernible chain* in the sense that for any n and any two increasing n-tuples $\vec{i} = i_0 < i_1 < \cdots < i_{n-1}$, $\vec{j} = j_0 < j_1 < \cdots < j_{n-1}$, the structures $W_{\vec{i}} = (W_{\kappa_{i_{n-1}}}, W_{\kappa_{i_{n-2}}}, \cdots, W_{\kappa_{i_0}})$ and $W_{\vec{j}}$ (defined analagously) satisfy the same first-order sentences, allowing parameters from $W_{\kappa_{i_0}} \cap W_{\kappa_{j_0}}$.

We are getting closer to the desired axiom of #-generation. Surely we can impose higher-order indiscernibility on our chain of models. For example, consider the pair of models $W_{\kappa_0} = V_{\kappa_0}$, $W_{\kappa_1} = V_{\kappa_1}$. We can require that these models satisfy the same second-order sentences; equivalently, we require that $H(\kappa_0^+)^V$ and $H(\kappa_1^+)^V$ satisfy the same first-order sentences. But as with the pair $H(\kappa_0)^V$, $H(\kappa_1)^V$ we would want $H(\kappa_0^+)^V$, $H(\kappa_1^+)^V$ to satisfy the same first-order sentences *with parameters*. How can we formulate this? For example, consider κ_0, a parameter in $H(\kappa_0^+)^V$ that is second-order with respect to $H(\kappa_0)^V$; we cannot simply require $H(\kappa_0^+)^V \vDash \varphi(\kappa_0)$ iff $H(\kappa_1^+)^V \vDash \varphi(\kappa_0)$, as κ_0 is the largest cardinal in $H(\kappa_0^+)^V$ but not in $H(\kappa_1^+)^V$. Instead we need to replace the occurence of κ_0 on the left side with a "corresponding" parameter on the right side, namely κ_1, resulting in the natural requirement $H(\kappa_0^+)^V \vDash \varphi(\kappa_0)$ iff $H(\kappa_1^+)^V \vDash \varphi(\kappa_1)$. More generally, we should be able to replace each parameter in $H(\kappa_0^+)^V$ by a "corresponding" element of $H(\kappa_1^+)^V$. It is natural to solve this parameter problem using embeddings.

Definition 1. *(See [10])*

A structure $N = (N, U)$ is called a # with critical point κ, *or just a* #, *if the following hold:*
(a) N is a model of ZFC$^-$ (ZFC minus powerset) in which κ is both the largest cardinal and strongly inaccessible.
(b) (N, U) is amenable (i.e. $x \cap U \in N$ for any $x \in N$).
(c) U is a normal measure on κ in (N, U).
(d) N is iterable, i.e., all of the successive iterated ultrapowers starting with (N, U) are well-founded, yielding iterates (N_i, U_i) and Σ_1 elementary iteration maps $\pi_{ij} : N_i \to N_j$ where $(N, U) = (N_0, U_0)$.

We let κ_i denote the largest cardinal of the i-th iterate N_i.

If N is a # and λ is a limit ordinal then LP(N_λ) denotes the union of the $(V_{\kappa_i})^{N_i}$'s for $i < \lambda$. (LP stands for *lower part.*) LP(N_∞) is a model of ZFC.

Definition 2. *We say that a transitive model V of ZFC is #-*generated *iff there is $N = (N, U)$, a # with iteration $N = N_0 \to N_1 \to \cdots$, such that V equals LP(N_∞) where ∞ denotes the ordinal height of V.*

#-generation fulfills our requirements for vertical maximality, with powerful consequences for reflection. L is #-generated iff $0^\#$ exists, so this principle is compatible with $V = L$. If V is #-generated via (N, U) then there are elementary embeddings from V to V which are canonically-definable through iteration of (N, U): In the above notation, any order-preserving map from the κ_i's to the κ_i's extends to such an elementary embedding. If $\pi : V \to V$ is any such embedding then we obtain not only the indiscernibility of the structures $H(\kappa_i^+)$, for all i but also of the structures $H(\kappa_i^{+\alpha})$ for any $\alpha < \kappa_0$ and more. Moreover, #-generation evidently provides the maximum amount of vertical reflection: If V is generated by (N, U) as LP(N_∞) where ∞ is the ordinal height of V, and x is any parameter in a further iterate $V^* = N_{\infty^*}$ of (N, U), then any first-order property $\varphi(V, x)$ that holds in V^* reflects to $\varphi(V_{\kappa_i}, \bar{x})$ in N_j for all sufficiently large $i < j < \infty$, where $\pi_{j,\infty^*}(\bar{x}) = x$. This implies any known form of vertical reflection and summarizes the amount of reflection one has in L under the assumption that $0^\#$ exists, the maximum amount of reflection in L. This is reinforced by a Jensen's *#-generated coding theorem* (Theorem 9.1. of [6]) which states that if V is #-generated then V can be coded into a #-generated model $L[x]$ for a real x where the given # which generates V extends to the natural generator $x^\#$ for the model $L[x]$.

From this we can conclude that #-generated models have the same large cardinal and reflection properties as does L when $0^\#$ exists.

#-generation also answers our question about which *canonical* tower of lengthenings of V to look at in reflection, namely the further lower parts of iterates of any # that generates V. This tower of lengthenings is independent of the choice of generating # for V and is therefore entirely *canonical.* And #-generation fully realizes the idea that V should look exactly like closed unboundedly many of its rank initial segments as well as its *canonical* lengthenings of arbitrary ordinal height.

In summary, #-generation stands out as the correct formalization of the principle of *height maximality*, and we shall refer to #-generated models as being *maximal in height.* It is not first-order (we have argued that no optimal height maximality principle can be), however it is second-order in a very restricted way: For a countable V, the property of being a # that generates V is expressible by quantifying universally over the models $L_\alpha(V)$ as α ranges over the countable ordinals.

4.5 Maximality in Width and the IMH

Whereas in the case of *maximality in height* we can use height potentialism (i.e., the option of lengthening V to taller universes) to arrive at an optimal principle, the case of *maximality in width* is of a very different nature. Unlike in the case of height maximality, we will see that there are many distinct criteria for width maximality and will not easily arrive at an optimal criterion. Moreover, to get a fair picture of maximality in both height and width, it is necessary to *synthesise* or *unify* width maximality criteria with #-generation, the optimal height maximality criterion.

A thorough analysis of the different possible width maximality criteria and their synthesis with #-generation, with an aim towards arriving at an optimal criterion, is the principal aim of the *Hyperuniverse Programme*.

I'll begin with a discussion of width maximality in the context of radical potentialism, as this offers a simpler theory than that provided by the Zermelian view. Thus we use the symbol V to be a variable ranging not over the Zermelian multiverse (in which universes are ordered by the relation of rank-initial segment) but over elements of the rich multiverse provided by radical potentialism, in which each universe is potentially countable. We begin with the fundamental:

Inner Model Hypothesis (IMH, [12]) If a first-order sentence holds in some outer model of V then it holds in some inner model of V.

For the current presentation, we may take *outer model* to mean a transitive set V^* containing V, with the same ordinals as V, which satisfies ZFC. An *inner model* in this presentation is a V-definable subclass of V with the same ordinals as V which satisfies ZFC. By radical potentialism, any transitive model of ZFC is countable in a larger such model and from this we can infer the existence of a rich collection of outer models of V.

The consistency of #-generation follows from the existence of $0^\#$. But the consistency of the IMH, i.e. the assertion that there are universes V satisfying the IMH, requires more.

Consistency of the IMH

Theorem 3. *([18]) Assuming large cardinals there exists a countable transitive model M of ZFC such that if a first-order sentence φ holds in an outer model N of M then it also holds in an inner model of M.*

Proof. For any real R let $M(R)$ denote the least transitive model of ZFC containing R. We are assuming large cardinals so indeed such an $M(R)$ exists (the existence of just an inaccessible is sufficient for this). We will need the following consequence of large cardinals:

$(*)$ There is a real R such that for any real S in which R is recursive, the (first-order) theory of $M(R)$ is the same as the theory of $M(S)$.

One can derive $(*)$ from large cardinals as follows. Large cardinals yield Projective Determinacy (PD). A theorem of Martin is that PD implies the following *Cone Theorem*: If X is a projective set of reals closed under Turing-equivalence then for some real R, either S belongs to X for all reals S in which R is recursive or S belongs to the complement of X for all reals S in which R is recursive.

Now for each sentence φ consider the set $X(\varphi)$ consisting of those reals R such that $M(R)$ satisfies φ. This set is projective and closed under Turing-equivalence. By the cone theorem we can choose a real $R(\varphi)$ so that either φ is true in $M(S)$ for all reals S in which $R(\varphi)$ is recursive or this holds for $\sim \varphi$. Now let R be any real in which every $R(\varphi)$ is recursive; as there are only countably-many φ's this is possible. Then R witnesses the property $(*)$.

We claim that if N is an outer model of $M(R)$ satisfying ZFC and φ is a sentence true in N then φ is true in an inner model of $M(R)$. For this we need the following deep theorem of Jensen.

Coding Theorem (see [6]) Let α be the ordinal height of N. Then N has an outer model of the form $L_\alpha[S]$ for some real S which satisfies ZFC and in which N is Δ_2-definable with parameters.

As R belongs to $M(R)$ it also belongs to N and hence to $L_\alpha[S]$ where S codes N as above. Also note that since α is least so that $M(R) = L_\alpha[R]$ models ZFC, it is also least so that $L_\alpha[S]$ satisfies ZFC and therefore $L_\alpha[S]$ equals $M(S)$.

Clearly we can choose S to be Turing above R (simply replace S by its join with R). But now by the special property of R, the theories of $M(R)$ and $M(S)$ are the same. As N is a definable inner model of $M(S)$, part of the theory of $M(S)$ is the statement "There is an inner model of φ which is Δ_2-definable with parameters" and therefore there is an inner model of $M(R)$ satisfying φ, as desired. □

Note that the model that we produce above for the IMH, $M(R)$ for some real R, is the minimal model containing the real R and therefore satisfies "there are no inaccessible cardinals". This is no accident:

Theorem 4. *[12] Suppose that M satisfies the IMH. Then in M: There are no inaccessible cardinals and in fact there is a real R such that there is no transitive model of ZFC containing R.*

Proof. A theorem of Beller and David (also in [6]) extends Jensen's Coding Theorem to say that any model M has an outer model of the form $M(R)$ for some real R, where as above $M(R)$ is the minimal transitive model of ZFC containing R. Now suppose that M satisfies the IMH and consider the sentence "There is no inaccessible cardinal". This is true in an outer model $M(R)$ of M and therefore in an inner model of M. It follows that there are no inaccessibles in M. The same argument with the sentence "There is a real R such that there is no transitive model of ZFC containing R" gives an inner model M_0 of M with this property for some real R; but then also M has this property as any transitive model of ZFC containing R in M would also give such a model in the $L[R]$ of M and therefore in M_0, as M_0 contains the $L[R]$ of M. □

It follows that if M satisfies the IMH then some real in M has no # and therefore boldface Π^1_1 determinacy fails in M (although $0^\#$ does exist and lightface Π^1_1 determinacy does hold).

Width actualism

So far I have presented the IMH in the context of radical potentialism, which allows us to talk freely about *outer models (thickenings)* of the universe V. This is of course unacceptable to the width actualist, who sees a fixed meaning to V_α for each ordinal α (although possibly an unfixed, potentialist view of what the ordinals are). Is it possible to nevertheless talk about the *maximality of V in width* from a width actualist perspective (where V is now a variable ranging over the Zermelian multiverse)? Can we express the idea that V is *as thick as possible* without actually comparing V to thicker universes (which do not exist)?

A positive answer to the latter question emerges through a study of V-logic, to which I turn next. A useful reference for this material is Barwise's book [5].

V-Logic

Let's start with something simpler, V_ω-logic. In V_ω-logic we have constant symbols $\bar{a}$ for $a \in V_\omega$ as well as a constant symbol $\bar{V}_\omega$ for V_ω itself (in addition to $\in$ and the other symbols of first-order logic). Then to the usual logical axioms and the rule of *Modus Ponens* we add the rules:

For $a \in V_\omega$: From $\varphi(\bar{b})$ for each $b \in a$ infer $\forall x \in \bar{a}\varphi(x)$.

From $\varphi(\bar{a})$ for each $a \in V_\omega$ infer $\forall x \in \bar{V}_\omega\ \varphi(x)$.

Introducing the second of these rules generates new provable statements via proofs which are now infinite. The idea of V_ω-logic is to capture the idea of a model *in which V_ω is standard*. By the *ω-completeness theorem*, the logically provable sentences of V_ω-logic are exactly those which hold in every model in which $\bar{a}$ is interpreted as a for $a \in V_\omega$ and $\bar{V}_\omega$ is interpreted as the (real, standard) V_ω. Thus a theory T in V_ω-logic is consistent in V_ω-logic iff it has a model in which V_ω is the real, standard V_ω.

Now the set of logically-provable formulas (i.e. validities) in V_ω-logic, unlike in first-order logic, is not arithmetical, i.e. it is not definable over the model V_ω. Instead it is definable over a larger structure, a *lengthening of* V_ω. Let me explain.

As proofs in V_ω-logic are no longer finite, they do not naturally belong to V_ω. Instead they belong to *the least admissible set* $(V_\omega)^+$ containing V_ω as an element, this is known to higher recursion-theorists as $L_{\omega_1^{ck}}$, where ω_1^{ck} is the least non-recursive ordinal. Something very nice happens: Whereas proofs in first-order logic belong to V_ω and therefore provability is Σ_1 definable over V_ω (*there exists a proof* is Σ_1), proofs in V_ω-logic belong to $(V_\omega)^+$ and provability is Σ_1 definable over $(V_\omega)^+$.

For our present purposes the point is that $(V_\omega)^+$ is a lengthening, not a thickening of V_ω and in this lengthening we can formulate theories which describle arbitrary models in which V_ω is standard. For example the existence of a real R such that (V_ω, R) satisfies a first-order property can be formulated as the consistency of a theory in V_ω-logic. As the structure (V_ω, R) can be regarded as a "thickening" of V_ω, we have described what can happen in "thickenings" of V_ω by a theory in $(V_\omega)^+$, a lengthening of V_ω. This is even more dramatic if we start not with V_ω but with $(V_\omega)^+ = L_{\omega_1^{ck}}$ and introduce $L_{\omega_1^{ck}}$*-logic*, a logic for ensuring that the recursive ordinals are standard. Then in the lengthening $(L_{\omega_1^{ck}})^+$ of $L_{\omega_1^{ck}}$, the least admissible set containing $L_{\omega_1^{ck}}$ as an element, we can express the existence of a thickening of

$L_{\omega_1^{ck}}$ in which a first-order statement holds, and such thickenings can contain new reals and more as elements.

V-logic is analogous to the above. It has the following constant symbols:

1. A constant symbol $\bar{a}$ for each set a in V.
2. A constant symbol $\bar{V}$ to denote the universe V.

Formulas are formed in the usual way, as in any first-order logic. To the usual axioms and rules of first-order logic we add the new rules:

(∗) From $\varphi(\bar{b})$ for all $b \in a$ infer $\forall x \in \bar{a}\varphi(x)$.

(∗∗) From $\varphi(\bar{a})$ for all $a \in V$ infer $\forall x \in \bar{V}\varphi(x)$.

This is the logic to describe models in which V is standard. The proofs of this logic appear in V^+, the least admissible set containing V as an element; this structure V^+ is a special lengthening of V of the form $L_\alpha(V)$, the α-th level of Gödel's L-hierarchy built over V. We refer to such lengthenings as *Gödel lengthenings.* Recall that with our height potentialist perspective, we can lengthen V to models V^* with V as a rank-initial segment, and therefore surely lengthen V to the Gödel lengthening V^+. (This is also the case with a height actualist perspective, provided we allow our classes to satisfy MK (Morse-Kelley), as in MK we can construct a class coding V^+.)

The Inner Model Hypothesis for a Width Actualist

As width actualists we cannot talk directly about outer models or even about sets that do not belong to V. However using V-logic we can talk about them indirectly, as I'll now illustrate. Consider the theory in V-logic where we not only have constant symbols $\bar{a}$ for the elements of V and a constant symbol $\bar{V}$ for V itself, but also a constant symbol $\bar{W}$ to denote an "outer model" of V. We add the new axioms:

1. The universe is a model of ZFC (or at least the weaker KP, admissibility theory).
2. $\bar{W}$ is a transitive model of ZFC containing $\bar{V}$ as a subset and with the same ordinals as V.

So now when we take a model of our axioms which obeys the rules of V-logic, we get a universe modelling ZFC (or at least KP) in which $\bar{V}$ is interpreted correctly as V and $\bar{W}$ is interpreted as an outer model of V. Note that this theory in V-logic has been formulated without "thickening" V, indeed it is defined inside V^+, the least admissible set containing V, a Gödel lengthening of V. Again the latter makes sense thanks to our adoption of height (not width) potentialism.

So what does the *IMH* really say for a width actualist? It says the following:

IMH: Suppose that φ is a first-order sentence and the above theory, together the axiom "$\bar{W}$ satisfies φ" is consistent in V-logic. Then φ holds in an inner model of V.

In other words, instead of talking directly about "thickenings" of V (i.e. "outer models") we instead talk about the consistency of a theory formulated in V-logic and defined in V^+, a (mild) Gödel lengthening of V.

Note that this also provides a powerful extension of the Definability Lemma for set-forcing. The latter says that definably in V we can express the fact that a sentence with parameters holds in a "set-generic extension" (for sentences of bounded complexity, such as Σ_n sentences for a fixed n). The above shows that we can do the same for arbitrary "thickenings" of V, but where the definability takes place not in V but in V^+. (In the case of *omniscient* universes V, we can in fact obtain definability in V, and under mild large cardinal assumptions, V will be omniscient. See Subsection 4.11 for a discussion of this.)

So far we have worked with V, its lengthenings and its "thickenings" (via theories expressed in its lengthenings). We next come to an important step, which is to reduce this discussion to the study of certain properties of countable transitive models of ZFC, i.e., to the *Hyperuniverse* (the set of countable transitive models of ZFC). The net effect of this reduction is to show that our width actualist discussion of maximality is in fact equivalent to a radical potentialist discussion in which all models under consideration belong to the Hyperuniverse.

4.6 The Reduction to the Hyperuniverse

Of course it would be much more comfortable to remove the quotes in "thickenings" of V, as we could then dispense with the need to reformulate our intuitions about outer models via theories in V-logic. Indeed, if we were to have this discussion not about V but about a countable transitive ZFC model *little-V*, then our worries evaporate, as genuine thickenings become available. For example, if P is a forcing notion in little-V then we can surely build a P-generic extension to get a little-$V[G]$.

Of course we can't do this for V itself as in general we cannot construct generic sets for partial orders with uncountably many maximal antichains.

But the way we have analysed things with V-logic allows us to *reduce* our study of maximality criteria for V to a study of countable transitive models. As the collection of countable transitive models carries the name *Hyperuniverse*, we are then led to what is known as the *Hyperuniverse Programme*.

I'll illustrate the reduction to the Hyperuniverse with the specific example of the IMH. Suppose that we formulate the IMH as above, using V-logic, and want to know what first-order consequences it has.

Lemma 5. *Suppose that a first-order sentence φ holds in all countable models of the IMH. Then it holds in all models of the IMH.*

Proof. Suppose that φ fails in some model V of the IMH, where V may be uncountable. Now notice that the IMH is first-order expressible in V^+, a lengthening of V. But then apply the downward Löwenheim-Skolem theorem to obtain a countable little-V which satisfies the IMH, as verified in its associated little-V^+, yet fails to satisfy φ. But this is a contradiction, as by hypothesis φ must hold in all *countable* models of the IMH. □

So *without loss of generality*, when looking at first-order consequences of maximality criteria as formulated in V-logic, we can restrict ourselves to countable little-V's. The advantage of this is then we can dispense with the little-V-logic and the quotes in "thickenings" altogether, as by the Completeness Theorem for little-V-logic, consistent theories in little-V-logic do have models, thanks to the countability of little-V. Thus for a countable little-V, we can simply say:

IMH for little-V's: Suppose that a first-order sentence holds in an outer model of little-V. Then it holds in an inner model of little-V.

This is exactly the radical potentialist version of the IMH with which we began. Thus the width actualist and radical potentialist versions of the IMH coincide on countable models.

#-Generation Revisited

The reduction of maximality principles to the Hyperuniverse is however not always so obvious, as we will now see in the case of #-generation. This reveals a difference in the development of the HP form a Zermelian perspective versus a radical potentialist perspective.

First consider the following encouraging analogue for #-generation of our earlier reduction claim for the IMH.

Lemma 6. *Suppose that a first-order sentence φ holds in all countable models which are #-generated. Then it holds in all models which are #-generated.*

Proof. Suppose that φ fails in some #-generated model V, where V may be uncountable. Let (N, U) be a generating # for V and place both V and (N, U) inside some transitive model of ZFC minus powerset T. Now apply Löwenhiem-Skolem to T to produce a countable transitive $\bar{T}$ in which there is a $\bar{V}$ which $\bar{T}$ believes to be generated by $(\bar{N}, \bar{U})$ with an elementary embdding of $\bar{T}$ into T, sending $\bar{V}$ to V and $(\bar{N}, \bar{U})$ to (N, U). But the fact that (N, U) is iterable and $(\bar{N}, \bar{U})$ is embedded into (N, U) is enough to conclude that also $(\bar{N}, \bar{U})$ is iterable. So we now have a countable $\bar{V}$ which is #-generated (via $(\bar{N}, \bar{U})$) in which φ fails, contrary to hypothesis. □

However the difficulty is this: How do we express #-generation from a width actualist perspective? Recall that to produce a generating # for V we have to produce a set of rank less than $\mathrm{Ord}(V)$ which does not belong to V, in violation of width actualism.

And recall that a # is a structure (N, U) meeting certain first-order conditions which is in addition *iterable*: For any ordinal α if we iterate (N, U) for α steps then it remains wellfounded. V is #-generated if there is a # which generates it. But notice that to express the iterability of a generating # for V we are forced to consider theories T_α formulated in $L_\alpha(V)$-logic for *arbitrary* Gödel lengthenings $L_\alpha(V)$ of V: T_α asserts that V is generated by a *pre-#* (i.e. by a structure that looks like a # but may not be fully iterable) which is *α-iterable*, i.e. iterable for α-steps. Thus we have no fixed theory that captures #-generation but only a tower of theories T_α (as α ranges over ordinals past the height of V) which capture closer and closer approximations to it.

Definition 7. *V is* weakly #-generated *if for each ordinal α past the height of V, the theory T_α which expresses the existence of an α-iterable pre-# which generates V is consistent.*

Weak #-generation is meaningful for a width actualist (who accepts enough height potentialism to obtain Gödel lengthenings) as it is expressed entirely in terms of theories internal to Gödel lengthenings of V.

For a countable little-V, weak #-generation can be expressed semantically. First a useful definition:

Definition 8. *Let little-V be a countable transitive model of ZFC and* α *an ordinal. Then little-V is* α-generated *if there is an* α*-iterable pre-# which generates little-V (as the union of the lower parts of its first* γ *iterates, where* γ *is the ordinal height of little-V).*

Then a countable little-V is weakly #-generated if it is α-generated for each countable ordinal α (where the witness to this may depend on α). Little-V is #-generated iff it is α-generated when $\alpha = \omega_1$ iff it is α-generated for all ordinals α.

Just as a syntactic approach is needed for a width actualist fomulation of #-generation, the reduction of this weakened form of #-generation to the Hyperuniverse takes a syntactic form:

Lemma 9. *Suppose that a first-order sentence* φ *holds in all countable little-V which are weakly #-generated, and this is provable in ZFC. Then* φ *holds in all models which are weakly #-generated.*

Proof. Let W be a weakly #-generated model (which may be uncountable). Thus for each ordinal α above the height of W, the theory $T_\alpha + \sim \varphi$ expressing that φ fails in W and W is generated by an α-iterable pre-# is consistent. If we choose α so that $L_\alpha(W)$ is a model of ZFC (or enough of ZFC where the truth of φ in countable #-generated models provable) then $L_\alpha(W)$ is a model of (enough of) ZFC in which W is weakly #-generated. Apply Löwenheim-Skolem to obtain a countable $\bar{W}$ and $\bar{\alpha}$ such that $L_{\bar{\alpha}}(\bar{W})$ embeds elementarily into $L_\alpha(W)$ and therefore satisfies (enough of) ZFC plus "$\bar{W}$ is weakly #-generated". Now let g be generic over $L_{\bar{\alpha}}(\bar{W})$ for the Lévy collapse of (the height of) $\bar{W}$ to ω; then $L_{\bar{\alpha}}(\bar{W})[g]$ is a model of (enough of) ZFC in which $\bar{W}$ is both countable and weakly #-generated. By hypothesis $L_{\bar{\alpha}}(W)[g]$ satisfies "$\bar{W}$ satisfies φ" and therefore $\bar{W}$ really does satisfy φ. Finally, by elementarity W satisfies φ as well, as desired. □

To summarise: As radical potentialists we can comfortably work with full #-generation as our principle of height maximality. But as width actualists we instead work with weak #-generation, expressed in terms of theories inside Gödel lengthenings $L_\alpha(V)$ of V. Weak #-generation is sufficient to maximise the height of the universe. And properly formulated, the reduction to the Hyperuniverse applies to weak #-generation: To infer that a first-order statement follows from weak #-generation it suffices to show that in ZFC one can prove that it holds in all weakly #-generated countable models.

Weak #-generation is indeed strictly weaker than #-generation for countable models: Suppose that $0^\#$ exists and choose α to be least so that α is the α-th Silver indiscernible (α is countable). Now let g be generic over L for Lévy collapsing α to ω. Then by Lévy absoluteness, L_α is weakly #-generated in $L[g]$, but it cannot be #-generated in $L[g]$ as $0^\#$ does not belong to a generic extension of L.

In what follows I will primarily work with #-generation, as at present the mathematics of weak #-generation is poorly understood. Indeed, as we'll see in the next section, a synthesis of #-generation with the IMH is consistent, but this remains an open problem for weak #-generation.

4.7 Synthesis

We introduced the IMH as a criterion for width maximality and #-generation as a criterion for height maximality. It is natural to see how these can be combined into a single criterion which recognises both forms of maximality. We achieve this in this section through *synthesis*. Note that the IMH implies that there are no inaccessibles yet #-generation implies that there are. So we cannot simply take the conjunction of these two criteria.

A #-generated model M satisfies the IMH# iff whenever a sentence holds in a #-generated outer model of M it also holds in an inner model of M.

Note that IMH# differs from the IMH by demanding that both M and M^*, the outer model, are #-generated (while the outer models considered in IMH are arbitrary). The motivation behind this requirement is to impose width maximality only with respect to those models which are height maximal.

Theorem 10. *[15] Assuming that every real has a # there is a real R such that any #-generated model containing R satisfies the IMH#.*

Proof. (Woodin) Let R be a real with the following property: Whenever X is a lightface and nonempty Π^1_2 set of reals, then X has an element recursive in R. We claim that any #-generated model M containing R as an element satisfies the IMH#.

Suppose that φ holds in M^*, a #-generated outer model of M. Let (m^*, U^*) be a generating # for M^*. Then the set X of reals S such that S codes such an (m^*, U^*) (generating a model of φ) is a lightface Π^1_2 set. So there is such a real recursive in

R and therefore in M. But then M has an inner model satisfying φ, namely any model generated by a # coded by an element of X in M. □

The argument of the previous theorem is special to the weakest form of IMH#. The original argument from [15], used *#-generated Jensen coding* to prove the consistency of a stronger principle, SIMH#(ω_1); see Theorem 15.

Corollary 11. *Suppose that φ is a sentence that holds in some V_κ with κ measurable. Then there is a transitive model which satisfies both the IMH# and the sentence φ.*

Proof. Let R be as in the proof of Theorem 10 and let U be a normal measure on κ. The structure $N = (H(\kappa^+), U)$ is a #; iterate N through a large enough ordinal ∞ so that $M = LP(N_\infty)$, the lower part model generated by N, has ordinal height ∞. Then M is #-generated and contains the real R. It follows that M is a model of the IMH#. Moreover, as M is the union of an elementary chain $V_\kappa = V_\kappa^N \prec V_{\kappa_1}^{N_1} \prec \cdots$ where φ is true in V_κ, it follows that φ is also true in M. □

Note that in Corollary 11, if we take φ to be any large cardinal property which holds in some V_κ with κ measurable, then we obtain models of the IMH# which also satisfy this large cardinal property. This implies the compatibility of the IMH# with arbitrarily strong large cardinal properties.

Question 12. *Reformulate IMH# using weak #-generation, as follows: V is weakly #-generated and for each sentence φ, if the theories expressing that V has an outer model satisfying φ with an α-iterable generating pre-# are consistent for each α, then φ holds in an inner model of V. Is this consistent?*

The above formulation of IMH# for weak #-generation takes the following form for a countable V: V is α-generated for each countable α and for all φ, if φ holds in an α-generated outer model of V for each countable α then φ holds in an inner model of V. It is not known if this is consistent.

Remark. An even weaker form of #-generation asserts that V is just $\mathrm{Ord}(V) + \mathrm{Ord}(V)$-generated, a sufficient amount of iterability to obtain ordinal maximality. However a synthesis of the IMH with this very weak #-generation yields a consistent principle that contradicts large cardinals (indeed the existence of #'s for arbitrary reals). These different forms of #-generation, and of their synthesis with the IMH, are in need of further philosophical discussion.

We have now laid the foundations for the HP and discussed the two most basic maximality principles, #-generation and the IMH. Most of the mathematical work in the HP remains to be done. Therefore what I will do in the remainder of this article is simply present a range of maximality criteria which are yet to be fully analysed and which give the flavour of how the HP is intended to proceed. These criteria are also referred to as *H-axioms*, formulated as properties of elements of the Hyperuniverse H, expressible as maximality properties within H.

4.8 The Strong IMH

Our discussion of the IMH has been always with regard to sentences, without parameters. Stronger forms result if we introduce parameters.

First note the difficulties with introducing parameters into the IMH. For example the statement

"If a sentence with parameter ω_1^V holds in an outer model of V then it holds in an inner model"

is inconsistent, as the parameter ω_1^V could become countable in an outer model and therefore the above cannot hold for the sentence "ω_1^V is countable". If we however require that ω_1 is preserved then we get a consistent principle.

Theorem 13. *Let SIMH(ω_1) be the following principle: If a sentence with parameter ω_1 holds in an ω_1-preserving outer model then it holds in an inner model. Then the SIMH(ω_1) is consistent (assuming large cardinals).*

Proof. Again use PD to get a real R such that the theory of $M(S)$, the least transitive ZFC model containing S, is fixed for all S Turing above R. Now suppose that $\varphi(\omega_1)$ is a sentence true in an ω_1-preserving outer model N of $M(R)$, where ω_1 denotes the ω_1 of $M(R)$. Then as in the proof of consistency of the IMH, we can code N into $M(S)$ for some real S Turing above R, and moreover this coding is ω_1-preserving. As $\varphi(\omega_1)$ holds in a definable inner model of $M(S)$ and ω_1 is the same in $M(R)$ and $M(S)$, it follows that $M(R)$ also has an inner model satisfying $\varphi(\omega_1)$. □

The above argument uses the fact that Jensen-coding is ω_1-preserving. It is however not ω_2-preserving unless CH holds, and therefore we have the following open question:

Question 14. *Let SIMH(ω_1, ω_2) be the following principle: If a sentence with parameters ω_1, ω_2 holds in an ω_1-preserving and ω_2-preserving outer model then it holds in an inner model. Then is the SIMH(ω_1, ω_2) consistent (assuming large cardinals)?*

The SIMH(ω_1, ω_2) implies that CH fails, as any model has a cardinal-preserving outer model in which there is an injection from ω_2 into the reals. Is there an analogue $M^*(R)$ of the minimal model $M(R)$ which does not satisfy CH? Is there a coding theorem which says that any outer model of $M^*(R)$ which preserves ω_1 and ω_2 has a further outer model of the form $M^*(S)$, also with the same ω_1 and ω_2? If so, then one could establish the consistency of the SIMH(ω_1, ω_2).

The most general from of the SIMH makes use of *absolute parameters*. A parameter p is *absolute* if some formula defines it in all outer models which preserve cardinals up to and including the hereditary cardinality of p, i.e. the cardinality of the transitive closure of p. Then SIMH(p) for an absolute parameter p states that if a sentence with parameter p holds in an outer model which preserves cardinals up to the hereditary cardinality of p then it holds in an inner model. The full SIMH (Strong Inner Model Hypothesis) states that this holds for every absolute parameter p.

The SIMH is closely related to strengthenings of Lévy absoluteness. For example, define Lévy(ω_1) to be the statement that Σ_1 formulas with parameter ω_1 are absolute for ω_1-preserving outer models; this follows from the $SIMH(\omega_1)$ and is therefore consistent. But the consistency of Lévy(ω_1, ω_2), i.e. Σ_1 absoluteness with parameters ω_1, ω_2 for outer models which preserve these cardinals, is open.

The SIMH#

A synthesis of the SIMH with #-generation can be formulated as follows: V satisfies the SIMH# if V is #-generated and whenever a sentence φ with absolute parameters holds in a #-generated outer model having the same cardinals as V up to the hereditary cardinality of those parameters, φ also holds in an inner model of V. A special case is SIMH#(ω_1), where the only parameter involved is ω_1 and we are concerned only with ω_1-preserving outer models.

Theorem 15. *[15] Assuming large cardinals, the SIMH#(ω_1) is consistent.*

Proof. Assume there is a Woodin cardinal with an inaccessible above. For each real R let $M^\#(R)$ be $L_\alpha[R]$ where α is least so that $L_\alpha[R]$ is #-generated. The Woodin cardinal with an inaccessible above implies enough projective determinacy to enable us to use Martin's Lemma to find a real R such that the theory of $M^\#(S)$ is constant for S Turing-above R. We claim that $M^\#(R)$ satisfies SIMH#(ω_1): Indeed, let M be a #-generated ω_1-preserving outer model of $M^\#(R)$ satisfying some sentence $\varphi(\omega_1)$. Let α be the ordinal height of $M^\#(R)$ (= the ordinal height of M). By the result of Jensen quoted before (Theorem 9.1 of [6]), M has a *#-generated* ω_1-preserving outer model W of the form $L_\alpha[S]$ for some real S with $R \leq_T S$. Of course α is least so that $L_\alpha[S]$ is #-generated. So W equals $M^\#(S)$ and the ω_1 of W equals the ω_1 of $M^\#(R)$. By the choice of R, $M^\#(R)$ also has a definable inner model satisfying $\varphi(\omega_1)$. □

However as with the SIMH(ω_1, ω_2), the consistency of SIMH#(ω_1, ω_2) is open.

4.9 A Maximality Protocol

This protocol aims to organise the study of height and width maximality into three stages.

Stage 1. Maximise the ordinals (height maximality).

Stage 2. Having maximised the ordinals, maximise the cardinals.

Stage 3. Having maximised the ordinals and cardinals, maximise powerset (width maximality).

Stage 1 is taken care of by #-generation. So we focus now on Stage 2, cardinal-maximisation.

In light of Stage 1, we assume now that V is #-generated and when discussing outer models of V we only consider those which are also #-generated.

We would like a criterion which says that for each cardinal κ, κ^+ is *as large as possible.* To get started let's consider the case $\kappa = \omega$, so we want to maximise ω_1. The basic problem of course is the following. As set-generic extensions of #-generated models are also #-generated:

Fact. V has a #-generated outer model in which ω_1^V is countable.

But surely we would want something like: $\omega_1^{L[x]}$ is countable for each real x. The reason for this is that $\omega_1^{L[x]}$, unlike ω_1^V in general, is *absolute* between V and all of its outer models.

Definition 16. *Let p be a parameter in V and P a set of parameters in V. Then p is* strongly absolute relative to P *if there is a formula φ with parameters from P*

that defines p in V and all #-generated outer models of V which preserve cardinals up to and including the hereditary cardinality of the parameters mentioned in φ[10]*.*

Typically we will take P to consist of all subsets of some infinite cardinal κ, in which case the cardinal-preservation in the above definition refers to cardinals up to and including κ.

CardMax(κ^+) (for κ an infinite cardinal). Suppose that the ordinal α is strongly absolute relative to subsets of κ. Then α has cardinality at most κ.

It is possible to show that if κ is regular then there is a set-forcing extension in which CardMax(κ^+) holds.

Question 17. *Is CardMax consistent, where CardMax denotes CardMax(κ^+) for all infinite cardinals κ, both regular and singular?*

Internal Cardinal Maximality

Another approach to cardinal maximality is to relate the cardinals of V to those of its inner models. Two large inner models are HOD, the class of hereditarily ordinal-definable sets, and the smaller inner model $\mathbb{S}$, the *Stable Core* of [13]. V is class-generic over each of these models.

Let M denote an inner model.

M-cardinal Violation. For each infinite cardinal κ, κ^+ is greater than the κ^+ of M.

In [9] it is shown that HOD-cardinal violation is consistent. Can we strengthen this?

Question 18. *Is it consistent that for each infinite cardinal κ, κ^+ is inaccessible, measurable or even supercompact in HOD? Is this consistent with HOD replaced by the Stable Core $\mathbb{S}$?*

A result of Shelah states that all subsets of κ belong to HOD_x for some fixed subset x of κ when κ is a singular strong limit cardinal of uncountable cofinality. By [8] this need not be true at countable cofinalities.

Question 19. *Is it consistent that for each infinite cardinal κ, κ^+ is greater than κ^+ of $\mathbb{S}_x$ (the Stable Core relativised to x) for each subset x of κ?*

[10] We thank one of the referees for pointing out that an earlier version of cardinal-maximality with a weaker parameter-absoluteness assumption is inconsistent. A similar phenomenon with weakly absolute parameters occurs in Theorem 10 of [18].

A major difference between HOD and $\mathbb{S}$ is that while any set is set-generic over HOD, this is not the case for $\mathbb{S}$.

Question 20. *Is it consistent that for each infinite cardinal κ, some subset of κ^+ is not set-generic over $\mathbb{S}_x$ for any subset x of κ?*

A positive answer to any of these three questions would yield a strong internal cardinal-maximality principle for V.

Stage 3: Having maximised the ordinals and cardinals, maximise powerset.

This is where we revisit the SIMH, but only in the context of #-generation and cardinal-preservation. Again assume that V is #-generated.

A parameter p in V is *cardinal-absolute* if there is a parameter-free formula which defines p in all #-generated outer models of V which have the same cardinals as V.

SIMH#(CP) (Cardinal-preserving SIMH#). Suppose that p is a cardinal-absolute parameter, V^* is a #-generated outer model of V with the same cardinals as V and φ is a sentence with parameter p which holds in V^*. Then φ holds in an inner model of V.

Question 21. *Is the SIMH#(CP) consistent?*

Note that SIMH#(CP) implies a strong failure of CH.

4.10 Width Indiscernibility

An alternative to the Maximality Protocol (which ideally should be synthesised with it) is *Width Indiscernibility.* The motivation is to provide a description of V in width analogous to its description in height provided by #-generation.

Recall that with #-generation we arrive at the following:

$$V_0 \prec V_1 \prec \cdots \prec V = V_\infty \prec V_{\infty+1} \prec \cdots$$

where for $i < j$, V_i is a *rank-initial segment* of V_j. Moreover the models V_i form a collection of *indiscernible models* in a strong sense. This picture was the result of an analysis which began with *height reflection*, starting with the idea that V must have unboundedly many rank-initial segments V_i which are elementary in V.

Analogously, we introduce *width reflection.* We would like to say that V has proper inner models which are "elementary in V". Of course this cannot literally be true, as if V_0 is an elementary submodel of V with the same ordinals as V then it is easy to see that V_0 equals V. Instead, we use elementary embeddings.

Width Reflection. For each ordinal α, there is a proper elementary submodel H of V such that $V_\alpha \subseteq H$ and H is *amenable*, i.e. $H \cap V_\beta$ belongs to V for each ordinal β.

Equivalently:

Width Reflection. For each ordinal α, there is a nontrivial elementary embedding $j : V_0 \to V$ with critical point at least α such that j is *amenable*, i.e. $j \restriction (V_\beta)^{V_0}$ belongs to V for each ordinal β.

Let's write $V_0 < V$ if there is a nontrivial amenable $j : V_0 \to V$, as in the second formulation of width reflection. This relation is transitive.

Proposition 22. *(a) If $V_0 < V$ then V_0 is a proper inner model of V.*
(b) Width Reflection is consistent relative to the existence of a Ramsey cardinal.

Proof. (a) This follows from Kunen's Theorem that there can be no nontrivial elementary embedding from V to V.
(b) Suppose that κ is Ramsey. Then it follows that any structure of the form $\mathcal{M} = (V_\kappa, \in, \ldots)$ has an unbounded set of indiscernibles, i.e. an unbounded subset I of κ such that for each n, any two increasing n-tuples from I satisfy the same formulas in $\mathcal{M}$. Now apply this to $\mathcal{M} = (V_\kappa, \in, <)$ where $<$ is a wellorder of V_κ of length κ. Let J be any unbounded subset of I such that $I \setminus J$ is unbounded and for any $\alpha < \kappa$, let $H(J \cup \alpha)$ denote the Skolem hull of $J \cup \alpha$ in $\mathcal{M}$. Then $H(J \cup \alpha)$ is an elementary submodel of V_κ and is not equal to V_κ because no element of $I \setminus J$ greater than α belongs to it. As V_κ contains all bounded subsets of κ it follows that $H(J \cup \alpha)$ is amenable. □

A variant of the argument in (b) above yields the consistency of arbitrarily long finite chains $V_0 < V_1 < \cdots < V_n$. But obtaining infinite such chains seems more difficult, and even more ambitiously we can ask:

Question 23. *Is it consistent to have $V_0 < V_1 < \cdots < V$ of length $Ord + 1$ such that the union of the V_i's equals V?*

The latter would be a good start on the formulation of a consistent criterion of *Width Indiscernibility*, as an analogue for maximality in width to the criterion of maximality in height provided by #-generation.

4.11 Omniscience

By OMT(V), the *outer model theory of* V, we mean the class of sentences with arbitrary parameters from V which hold in all outer models of V. We have seen using V-logic that OMT(V) is definable over V^+. However for many universes V, OMT(V) is in fact first-order definable over V. These universes are said to be *omniscient*.

Recall the following version of Tarski's result on the undefinability of truth:

Proposition 24. *The set of sentences with parameters from V which hold in V is not (first-order) definable in V with parameters.*

Surprisingly, Mack Stanley showed however that OMT(V) can indeed be V-definable.

Theorem 25. *(M.Stanley [30]) Suppose that in V there is a proper class of measurable cardinals, and indeed this class is V^+-stationary, i.e. Ord(V) is regular with respect to V^+-definable functions and this class intersects every club in Ord(V) which is V^+-definable. Then OMT(V) is V-definable.*

Proof. Using V-logic we can translate the statement that a first-order sentence φ (with parameters from V) holds in all outer models of V to the validity of a sentence φ^* in V-logic, a fact expressible over V^+ by a Σ_1 sentence. Using this we show that the set of φ which hold in all outer models of V is V-definable.

As Ord(V) is regular with respect to V^+-definable functions we can form a club C in Ord(V) such that for κ in C there is a Σ_1-elementary embedding from Hyp(V_κ) into V^+ (with critical point κ, sending κ to Ord(V)). Indeed C can be chosen to be V^+-definable.

For any κ in C let φ^*_κ be the sentence of V_κ-logic such that φ holds in all outer models of V_κ iff φ^*_κ is valid (a Σ_1 property of Hyp(V_κ)). By elementarity, φ^*_κ is valid iff φ^* is valid.

Now suppose that φ holds in all outer models of V, i.e. φ^* is valid. Then φ^*_κ is valid for all κ in C and since the measurables form a V^+-stationary class, there is a measurable κ such that φ^*_κ is valid.

Conversely, suppose that φ^*_κ is valid for some measurable κ. Now choose a normal measure U on κ and iterate $(H(\kappa^+), U)$ for Ord(V) steps to obtain a wellfounded structure $(H^*, U*)$. (This structure is wellfounded, as for any admissible set A, any measure in A can be iterated without losing wellfoundedness for α steps, for any ordinal α in A.) Then H^* equals $\mathrm{Hyp}(V^*)$ for some $V^* \subseteq V$. By elementarity, the sentence $\varphi^*_{V^*}$ which asserts that φ holds in all outer models of V^* is valid. But as V^* is an inner model of V, φ also holds in all outer models of V.

Thus φ belongs to $\mathrm{OMT}(V)$ exactly if it belongs to $\mathrm{OMT}(V_\kappa)$ for some measurable κ, and this is first-order expressible. □

Are measurable cardinals needed for omniscience? Actually, Stanley was able to use just Ramsey cardinals, but as far as the consistency of omniscience we have the following:

Theorem 26. *([16]) Suppose that κ is inaccessible and GCH holds. Then there is an omniscient model of the form $V_\kappa[G]$ where G is generic over V. Moreover, $V_\kappa[G]$ carries a definable wellorder.*

Omniscience demonstrates that it is possible to treat truth in arbitrary outer models internally in a way similar to how truth in set-generic extensions can be handled using the standard definability and truth lemmas of set-forcing. In fact, the situation is even better in that the entire outer model theory is first-order definable, not just the restriction of this theory to sentences of bounded complexity, as is the case for set-forcing. (The key difference is that in the case of set-forcing, the ground model V is uniformly definable in its set-generic extensions and therefore the full $\mathrm{OMT}(V)$ cannot be first-order definable in V by Proposition 24. An omniscient V cannot be uniformaly definable in its arbitrary outer models for the same reason.)

Note also that by Theorem 25, omniscience synthesises well with #-generation: We need only work with models that have sufficiently many measurable cardinals.

4.12 The Future of the HP

We have discussed evidence of Type 1, coming from set theory's role as a branch of mathematics, and evidence of Type 2, coming from set theory's role as a foundation for mathematics. In the first case, evidence is judged by its value for the mathematical development of set theory and in the second case it is judged by its value for resolving independence in (and providing tools for) other areas of mathematics. In both cases the weight of the evidence is measured by a consensus of researchers working in the field.

Type 3 evidence is also measured by a consensus of researchers working in set theory (and its philosophy) but emanates instead from an analysis of the intrinsic maximality feature of the set concept as expressed by the maximal iterative conception. The Hyperuniverse Programme provides a strategy for *deriving* mathematical consequences from this conception.

To illustrate more clearly how the HP derives consequences of the maximality of V I'll discuss the case of #-generation and the search for an *optimal maximality criterion.*

#-generation is a major success of the HP. It provides a powerful mathematical criterion for height maximality which implies all prior known height maximality principles and provides an elegant description of how the height of V is maximised in a way analogous to the way L is maximised in height by the existence of large cardinals (or equivalently, by the existence of $0^{\#}$). There are good reasons to beleive that #-generation will be accepted by the community of set-theorists and philosophers of set theory as the definitive expression of height maximality.

Width maximality is of course much more difficult than height maximality and the formulation, analysis and synthesis of the various possible width maximality criteria is at its early stages. The basic IMH is a good start, but must be synthesised with #-generation. The biggest challenge at the moment is dealing with formulations of width maximality which make use of parameters. The *maximality protocol* is a promising approach. But it is important to emphasize that the mathematical analysis of width maximality principles is challenging and there are sure to be some false turns in the development of the programme, leading to inconsistnet principles (this has already happened several times). Such false turns are not damaging to the programme, but rather provide valuable further understanding of the nature of maximality.

The aim of the HP is to arrive after extensive mathematical work at an *optimal criterion* of maximality for the height and width of the universe of sets, providing a full mathematical analysis of the maximal iterative conception. As already said, the validation of such a criterion as optimal depends on a consensus of researchers working in set theory and its philosophy. *Derivability* from the maximal iterative conceptions refers to formal derivability form this sought-after optimal criterion. Of greatest interest are the first-order statements derivable from maximality, but it is already clear that the criteria being developed in the programme, such as the ones mentioned in this paper, are almost exclusively *non first-order.* My prediction is that the optimal criterion will include some form of the SIMH and therefore imply the (first-order) failure of CH.

I remain optimistic that when the discoveries of this programme are combined with further work in set theory and its application to resolving problems of independence in other areas of mathematics, the prediction expressed by the *Thesis of Set-Theoretic Truth* will be satisfyingly realized. But there is first a lot of work to be done.

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[33] *The Thread*, an e-mail discussion during June-November 2014 (with extensive contributions by S.Feferman, H.Friedman, S.Friedman, G.Hellman, P.Koellner, P.Maddy, R.Solovay and H.Woodin).

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