A reflecting principle compatible with the continuum large (1)

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ESI workshop on large cardinals and descriptive set theory
PFA implies $2^\aleph_0 = \aleph_2$. All known proofs of this implication use forcing notions that collapse $\omega_2$ to $\omega_1$.

Question: Does $FA(\{P : P \text{ proper}, |P| = \aleph_1\})$ imply $2^{\aleph_0} = \aleph_2$?
I will isolate a certain subclass $\Gamma$ of \{$P : P$ proper, $|P| = \aleph_1$\} and will sketch a proof that $FA(\Gamma) + 2^{\aleph_0} > \aleph_2$ is consistent.

$FA(\Gamma)$ will be strong enough to imply for example the negation of Justin Moore’s $\mathcal{U}$ and other strong forms of the negation of Club Guessing.
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$FA(\Gamma)$ will be strong enough to imply for example the negation of Justin Moore’s $\mathfrak{U}$ and other strong forms of the negation of Club Guessing.
If $N$ is a set such that $N \cap \omega_1 \in \omega_1$, set $\delta_N = N \cap \omega_1$.

If $\mathcal{W}$ is a collection of countable sets and $N$ is a set, $\mathcal{W}$ is $N$–stationary if for every ordinal $\gamma \in N$ and every function $Z : [\gamma]^{<\omega} \to \gamma$, $Z \in N$ there is some $M \in \mathcal{W} \cap N$ such that $Z^{"[M]^{<\omega}} \subseteq M$.

If $\mathbb{P}$ is a partial order, $\mathbb{P}$ is *nice* if

(a) conditions in $\mathbb{P}$ are functions with domain included in $\omega_1$, and

(b) if $p, q \in \mathbb{P}$ are compatible, then the greatest lower bound $r$ of $p$ and $q$ exists, $\text{dom}(r) = \text{dom}(p) \cup \text{dom}(q)$, and $r(\nu) = p(\nu) \cup q(\nu)$ for all $\nu \in \text{dom}(r)$ (where $f(\nu) = \emptyset$ if $\nu \notin \text{dom}(f)$).

Exercise: Every set–forcing for which $\text{glb}(p, q)$ exists whenever $p$ and $q$ are compatible conditions is isomorphic to a nice forcing.
If $N$ is a set such that $N \cap \omega_1 \in \omega_1$, set $\delta_N = N \cap \omega_1$.

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Exercise: Every set–forcing for which $\text{glb}(p, q)$ exists whenever $p$ and $q$ are compatible conditions is isomorphic to a nice forcing.
More notation

Given a nice partial order \((\mathbb{P}, \leq)\), a \(\mathbb{P}\)–condition \(p\) and a set \(M\) such that \(\delta_M\) exists, we say that \(M\) is good for \(p\) iff, letting

\[
X = \{ s \in \mathbb{P} \cap M : s \leq p \upharpoonright \delta_M, \text{ s compatible with } p \},
\]

(i) \(X \neq \emptyset\), and

(ii) for every \(s \in X\) there is some \(t \leq s, t \in M\), such that for all \(t' \leq t\), if \(t' \in M\), then \(t' \in X\).
A class of posets

Let $\mathbb{P}$ be a nice poset. $\mathbb{P}$ is $\kappa$–suitable if there are a binary relation $R$ and a club $C \subseteq \omega_1$ satisfying the following properties.

(1) If $p \mathrel{R} (N, \mathcal{W})$, then the following conditions hold.

(1.1) $N$ is a countable subset of $H(\kappa)$, $\mathcal{W}$ is an $N$–stationary subset of $[H(\kappa)]^{\aleph_0}$, and all members of $\mathcal{W} \cap N$ are good for $p$.

(1.2) If $p'$ is a $\mathbb{P}$–condition extending $p$, then there is some $\mathcal{W}' \subseteq \mathcal{W}$ such that $p' \mathrel{R} (N, \mathcal{W}')$.

(1.3) If $\mathcal{W}' \subseteq \mathcal{W}$ is $N$–stationary, then $p \mathrel{R} (N, \mathcal{W}')$. 
A class of posets

(2) For every \( p \in P \) and every finite set \( \{(N_i, \mathcal{W}_i) : i < m\} \) such that

(\circ) each \( N_i \) is a countable subset of \( H(\kappa) \) containing \( p \),
\( \omega_1^{N_i} = \omega_1 \), \( \delta_{N_i} \in C \), \( N_i \models ZFC^* \), and

(\circ) each \( \mathcal{W}_i \) is \( N_i \)-stationary

there is a condition \( q \in P \) extending \( p \) and there are \( \mathcal{W}_i' \subseteq \mathcal{W}_i \) \( (i < m) \) such that \( q R (N_i, \mathcal{W}_i') \) for all \( i < m \).

We will say that a nice partial order is absolutely \( \kappa \)-suitable if it is \( \kappa \)-suitable in every inner model \( W \) containing it and such that \( \omega_1^W = \omega_1 \).
A class of posets

(2) For every $p \in \mathcal{P}$ and every finite set $\{(N_i, \mathcal{W}_i) : i < m\}$ such that

- each $N_i$ is a countable subset of $H(\kappa)$ containing $p$, $\omega_1^{N_i} = \omega_1$, $\delta_{N_i} \in C$, $N_i \models ZFC^*$, and
- each $\mathcal{W}_i$ is $N_i$–stationary

there is a condition $q \in \mathcal{P}$ extending $p$ and there are $\mathcal{W}_i' \subseteq \mathcal{W}_i (i < m)$ such that $q R (N_i, \mathcal{W}_i')$ for all $i < m$.

We will say that a nice partial order is absolutely $\kappa$–suitable if it is $\kappa$–suitable in every inner model $W$ containing it and such that $\omega_1^W = \omega_1$. 
Let $\Gamma_\kappa$ denote the class of all absolutely $\kappa$–suitable posets consisting of finite functions included in $\omega_1 \times [\omega_1]^{<\omega}$.

Easy: For all $\kappa \geq \omega_2$, $\Gamma_\kappa \subseteq \text{Proper}$.

$FA(\Gamma_\kappa)$: For every $\mathbb{P} \in \Gamma_\kappa$ and every collection $\mathcal{D}$ of size $\aleph_1$ consisting of dense subsets of $\mathbb{P}$ there is a filter $G \subseteq \mathbb{P}$ such that $G \cap D \neq \emptyset$ for all $D \in \mathcal{D}$. 
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One application of $FA(\Gamma_\kappa)$: $\Omega$

Definition (Moore) $\mathcal{U}$: There is a sequence $\langle g_\delta : \delta < \omega_1 \rangle$ such that each $g_\delta : \delta \rightarrow \omega$ is continuous with respect to the order topology and such that for every club $C \subseteq \omega_1$ there is some $\delta \in C$ with $g_\delta " C = \omega$.

(◦) Club Guessing implies $\mathcal{U}$.

(◦) $\mathcal{U}$ preserved by ccc forcing, and in fact by $\omega$–proper forcing.

(◦) Each of $BPFA$ and $MRP$ implies $\Omega := \neg \mathcal{U}$.
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**Definition (Moore)** $\mathcal{U}$: There is a sequence $\langle g_\delta : \delta < \omega_1 \rangle$ such that each $g_\delta : \delta \to \omega$ is continuous with respect to the order topology and such that for every club $C \subseteq \omega_1$ there is some $\delta \in C$ with $g_\delta^{-1}C = \omega$.

($\circ$) Club Guessing implies $\mathcal{U}$.

($\circ$) $\mathcal{U}$ preserved by ccc forcing, and in fact by $\omega$–proper forcing.

($\circ$) Each of $\text{BPFA}$ and $\text{MRP}$ implies $\Omega := \neg \mathcal{U}$. 
Theorem (Moore) $\emptyset$ implies the existence of an Aronszajn line which does not contain any Contraryman suborder.

Question (Moore):

Does $\Omega$ imply $2^{\aleph_0} \leq \aleph_2$?
**Theorem** (Moore) $\mathcal{U}$ implies the existence of an Aronszajn line which does not contain any Contryman suborder.

**Question** (Moore):

Does $\Omega$ imply $2^{\aleph_0} \leq \aleph_2$?
Proposition: For every $\kappa \geq \omega_2$, $FA(\Gamma_\kappa)$ implies $\Omega$.

Proof sketch:

Notation: Given $X$, a set of ordinals, and $\delta$, an ordinal, set

1. $\text{rank}(X, \delta) = 0$ iff $\delta$ is not a limit point of $X$, and
2. $\text{rank}(X, \delta) > \eta$ if and only if $\delta$ is a limit of ordinals $\epsilon$ such that $\text{rank}(X, \epsilon) \geq \eta$.

Given a sequence $\mathcal{G} = \langle g_\delta : \delta < \omega_1 \rangle$ of continuous colourings, let $\mathbb{P}_\mathcal{G}$ be the following poset:
Proposition: For every $\kappa \geq \omega_2$, $FA(\Gamma_\kappa)$ implies $\Omega$.

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Proposition: For every \( \kappa \geq \omega_2 \), \( FA(\Gamma_\kappa) \) implies \( \Omega \).

Proof sketch:

Notation: Given \( X \), a set of ordinals, and \( \delta \), an ordinal, set

\[ \text{rank}(X, \delta) = 0 \iff \delta \text{ is not a limit point of } X, \text{ and} \]

\( \text{rank}(X, \delta) > \eta \)

if and only if \( \delta \) is a limit of ordinals \( \epsilon \) such that \( \text{rank}(X, \epsilon) \geq \eta \).

Given a sequence \( \mathcal{G} = \langle g_\delta : \delta < \omega_1 \rangle \) of continuous colourings, let \( \mathbb{P}_\mathcal{G} \) be the following poset:
Conditions in $\mathbb{P}_g$ are pairs $p = (f, \langle k_\xi : \xi \in D \rangle)$ satisfying the following properties:

1. $f$ is a finite function that can be extended to a normal function $F : \omega_1 \rightarrow \omega_1$.

2. For every $\xi \in \text{dom}(f)$, $\text{rank}(f(\xi), f(\xi)) \geq \xi$.

3. $D \subseteq \text{dom}(f)$ and for every $\xi \in \text{dom}(f)$,
   - $k_\xi < \omega$,
   - $g_{f(\xi)} \text{"range}(f) \subseteq \omega \setminus \{k_\xi\}$, and
   - $\text{rank}(\{\gamma < f(\xi) : g_{f(\xi)}(\gamma) \neq k_\xi\}, f(\xi)) = \text{rank}(f(\xi), f(\xi))$. 
Given conditions \( p_\epsilon = (f_\epsilon, (k_\xi^\epsilon : \xi \in D_\epsilon)) \in \mathbb{P}_G \) for \( \epsilon \in \{0, 1\} \), \( p_1 \) extends \( p_0 \) iff

(i) \( f_0 \subseteq f_1 \),

(ii) \( D_0 \subseteq D_1 \), and

(iii) \( k_\xi^1 = k_\xi^0 \) for all \( \xi \in D_0 \).

Easy: If \( G \) is \( \mathbb{P}_G \)-generic and \( C = \text{range}(\bigcup \{ f : (\exists k)(\langle f, k \rangle \in G) \}) \), then \( C \) is a club of \( \omega_1^\gamma \) and for every \( \delta \in C \) there is \( k_\delta \in \omega \) such that \( g_\delta \models \text{"}C \subseteq \omega \setminus \{k_\delta\} \).
Given conditions $p_\epsilon = (f_\epsilon, (k_\xi^\epsilon : \xi \in D_\epsilon)) \in \mathbb{P}_G$ for $\epsilon \in \{0, 1\}$, $p_1$ extends $p_0$ iff

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Easy: If $G$ is $\mathbb{P}_G$–generic and $C = \text{range}(\bigcup\{ f : (\exists \vec{k})(\langle f, \vec{k} \rangle \in G) \})$, then $C$ is a club of $\omega_1^Y$ and for every $\delta \in C$ there is $k_\delta \in \omega$ such that $g_\delta''C \subseteq \omega \setminus \{k_\delta\}$. 
\( \mathbb{P}_G \in \Gamma_\kappa \) for every \( \kappa \geq \omega_2 \):

\((\bullet)\) We may easily translate \( \mathbb{P}_G \) into a nice forcing consisting of finite functions contained in \( \omega_1 \times [\omega_1]^{<\omega} \).

\((\bullet)\) Given \( p = (f, \langle k_\xi : \xi \in D \rangle) \in \mathbb{P}_G, N \subseteq H(\kappa) \) countable such that \( N \models ZFC^* \) and \( \delta_N \) exists, and given \( \mathcal{W} \) an \( N \)-stationary set, set \( p \mathrel{R} (N, \mathcal{W}) \) if and only if

\((a)\) \( \delta_N \) is a fixed point of \( f \),

\((b)\) \( \delta_N \in D \), and

\((c)\) for every \( M \in \mathcal{W} \), \( g_{\delta_N}(\delta_M) \neq k_{\delta_N} \).
\( P_G \in \Gamma_\kappa \) for every \( \kappa \geq \omega_2 \):

\[
\begin{align*}
\bullet & \quad \text{We may easily translate } P_G \text{ into a nice forcing consisting} \\
& \quad \text{of finite functions contained in } \omega_1 \times [\omega_1]^{<\omega}.
\end{align*}
\]

\[
\begin{align*}
\bullet & \quad \text{Given } p = (f, \langle k_\xi : \xi \in D \rangle) \in P_G, N \subseteq H(\kappa) \text{ countable} \\
& \quad \text{such that } N \models ZFC^* \text{ and } \delta_N \text{ exists, and given } \mathcal{W} \text{ an} \\
& \quad N\text{-stationary set, set} \\
& \quad \quad p R (N, \mathcal{W}) \\
& \quad \text{if and only if} \\
& \quad \quad (a) \ \delta_N \text{ is a fixed point of } f, \\
& \quad \quad (b) \ \delta_N \in D, \text{ and} \\
& \quad \quad (c) \ \text{for every } M \in \mathcal{W}, g_{\delta_N}(\delta_M) \neq k_{\delta_N}.
\end{align*}
\]
Easy to verify:

(1) in the definition of $\kappa$–suitable

[that is:

If $p R (N, \mathcal{W})$, then

(1.1) $N$ is a countable subset of $H(\kappa)$, $\mathcal{W}$ is an $N$–stationary subset of $[H(\kappa)]^{\aleph_0}$, and all members of $\mathcal{W} \cap N$ are good for $p$,

(1.2) if $p'$ is a $\mathcal{P}$–condition extending $p$, then there is some $\mathcal{W}' \subseteq \mathcal{W}$ such that $p' R (N, \mathcal{W}')$, and

(1.3) if $\mathcal{W}' \subseteq \mathcal{W}$ is $N$–stationary, then $p R (N, \mathcal{W}')$.]

Let us check (2) in the definition of \( \kappa \)-suitable (with \( C = \omega_1 \))

[that is:

(2) For every \( p \in \mathcal{P} \) and every finite set \( \{(N_i, \mathcal{W}_i) : i < m\} \) such that

(a) each \( N_i \) is a countable subset of \( H(\kappa) \) containing \( p \), \( \omega_1^{N_i} = \omega_1 \), \( \delta_{N_i} \in C \), \( N_i \models ZFC^* \), and

(b) each \( \mathcal{W}_i \) is \( N_i \)-stationary

there is a condition \( q \in \mathcal{P} \) extending \( p \) and there are \( \mathcal{W}_i' \subseteq \mathcal{W}_i \) (\( i < m \)) such that \( q R (N_i, \mathcal{W}_i') \) for all \( i < m \).]
Let \( p = (f, \langle k_\xi : \xi \in D \rangle) \in \mathcal{P}_g \). Let \( \{ (N_i, \mathcal{W}_i) : i < m \} \) satisfy (a) and (b).

Let \( (\delta_j)_{j<n} \) be the increasing enumeration of \( \{ \delta_{N_i} : i < m \} \).

Suppose \( \{ N_i : \delta_{N_i} = \delta_0 \} = \{ N_0, N_1, N_2 \} \).

Let \( \{ k_0, \ldots k_{2^3-1} \} \) be 8 colours not touched by \( g_{\delta_0} \) “range(f).”

There is \( k^0 \in \{ k_0, \ldots k_7 \} \) such that, for all \( i < 3 \), \( \mathcal{W}'_i = \{ M \in \mathcal{W}_i : \delta_M \neq k^0 \} \) is \( N_i \)-stationary.

Hence we may make the promise to avoid the colour \( k^0 \) in the colouring \( g_{\delta_0} \).
Let $p = (f, \langle k_\xi : \xi \in D \rangle) \in \mathbb{P}_g$. Let $\{(N_i, \mathcal{W}_i) : i < m\}$ satisfy (a) and (b).

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Suppose $\{N_i : \delta_{N_i} = \delta_0\} = \{N_0, N_1, N_2\}$.

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Hence we may make the promise to avoid the colour $k^0$ in the colouring $g_{\delta_0}$. 
Let \( p = (f, \langle k_\xi : \xi \in D \rangle) \in \mathcal{P}_g \). Let \( \{(N_i, \mathcal{W}_i) : i < m\} \) satisfy (a) and (b).

Let \( (\delta_j)_{j<n} \) be the increasing enumeration of \( \{\delta_{N_i} : i < m\} \).

Suppose \( \{N_i : \delta_{N_i} = \delta_0\} = \{N_0, N_1, N_2\} \). Let \( \{k_0, \ldots, k_{2^3-1}\} \) be 8 colours not touched by \( g_{\delta_0} \) "range(f)."

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Hence we may make the promise to avoid the colour \( k^0 \) in the colouring \( g_{\delta_0} \).
Now we continue with $\delta_1$, and get a colour $k^1$ we may avoid in the colouring $g_{\delta_1}$. And so on.

In the end there is a condition $q = (f', \langle k'_\xi : \xi \in D' \rangle), q \leq p$, and $N_i$–stationary $\mathcal{W}'_i \subseteq \mathcal{W}_i$ ($i < m$) such that

(a) $f'$ has all $\delta_j$ ($j < n$) as limit points and makes the promise $k^j$ at each $\delta_j$, and

(b) $q R (N_i, \mathcal{W}'_i)$ for all $i < m$.

Hence, $\mathbb{P}_G$ is (isomorphic to) a forcing in $\Gamma_\kappa$.

An application of $\text{FA}(\{\mathbb{P}_G\})$ gives now a witness of $\Omega$ for $G$. □
Now we continue with $\delta_1$, and get a colour $k^1$ we may avoid in the colouring $g_{\delta_1}$. And so on.

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An application of $\text{FA}(\{\mathbb{P}_g\})$ gives now a witness of $\Omega$ for $G$. $\square$
Now we continue with $\delta_1$, and get a colour $k^1$ we may avoid in the colouring $g_{\delta_1}$. And so on.

In the end there is a condition $q = (f', \langle k'_\xi : \xi \in D' \rangle), q \leq p$, and $N_i$–stationary $\mathcal{W}'_i \subseteq \mathcal{W}_i (i < m)$ such that

(a) $f'$ has all $\delta_j (j < n)$ as limit points and makes the promise $k^j$ at each $\delta_j$, and

(b) $q \not\in R (N_i, \mathcal{W}'_i)$ for all $i < m$.

Hence, $\mathbb{P}_G$ is (isomorphic to) a forcing in $\Gamma_\kappa$.

An application of $\text{FA} (\{\mathbb{P}_G\})$ gives now a witness of $\Omega$ for $G$.

\square
Another application of \( FA(\Gamma_{\kappa}) \)

**Proposition:** For every \( \kappa \geq \omega_2 \), \( FA(\Gamma_{\kappa}) \) implies:

\[ \neg VWCG: \quad \text{For every } C, \text{ if} \]

(a) \( |C| = \aleph_1 \) and

(b) for all \( X \in C, X \subseteq \omega_1 \) and \( \text{ot}(X) = \omega \),

then there is a club \( C \subseteq \omega_1 \) such that \( |X \cap C| < \omega \) for all \( X \in C \).

\( \neg VWCG \) is equivalent to the following statement:

For every \( C \), if

(a) \( |C| = \aleph_1 \) and

(b) for all \( X \in C, X \subseteq \omega_1 \) and \( X \) is such that for all nonzero \( \gamma < \omega_1 \), \( \text{rank}(X, \gamma) < \gamma \) (equivalently, \( \text{ot}(X \cap \gamma) < \omega_\gamma \)),

then there is a club \( C \subseteq \omega_1 \) such that \( |X \cap C| < \omega \) for all \( X \in C \).
Another application of $FA(\Gamma_{\kappa})$

**Proposition:** For every $\kappa \geq \omega_2$, $FA(\Gamma_{\kappa})$ implies:

$\neg VWCG$: For every $C$, if

(a) $|C| = \aleph_1$ and

(b) for all $X \in C$, $X \subseteq \omega_1$ and $ot(X) = \omega$,

then there is a club $C \subseteq \omega_1$ such that $|X \cap C| < \omega$ for all $X \in C$.

$\neg VWCG$ is equivalent to the following statement:

For every $C$, if

(a) $|C| = \aleph_1$ and

(b) for all $X \in C$, $X \subseteq \omega_1$ and $X$ is such that for all nonzero $\gamma < \omega_1$, $\text{rank}(X, \gamma) < \gamma$ (equivalently, $ot(X \cap \gamma) < \omega^\gamma$),

then there is a club $C \subseteq \omega_1$ such that $|X \cap C| < \omega$ for all $X \in C$. 
Main Theorem \((CH)\) Let \(\kappa\) be a cardinal such that \(2^{<\kappa} = \kappa\), \(\kappa^{\aleph_1} = \kappa\) and \(\mu^{\aleph_0} < \kappa\) for all \(\mu < \kappa\). Then there is a partial order \(\mathcal{P}\) such that

1. \(\mathcal{P}\) is proper,

2. \(\mathcal{P}\) has the \(\aleph_2\)–chain condition,

3. \(\mathcal{P}\) forces
   - (\(\bullet\)) \(FA(\Gamma_{\kappa})\)
   - (\(\bullet\)) \(2^{\aleph_0} = \kappa\)
Proof sketch

Let \( \Phi : \kappa \rightarrow H(\kappa) \) be a bijection.

(\( \Phi \) exists by \( 2^{<\kappa} = \kappa \).
Note: There is an \( \omega_1 \)–club of \( \gamma < \kappa \) such that \( \Phi \upharpoonright \gamma \) enumerates \( [\gamma]^\kappa_0 \).)
Proof sketch (continued)

Coherent systems of structures

\( \{ N_i : i < m \} \) is a coherent systems of structures if

1. \( m < \omega \) and every \( N_i \) is a countable subset of \( H(\kappa) \) such that \( (N_i, \in, \Phi \cap N_i) \preceq (H(\kappa), \in, \Phi) \).

2. Given distinct \( i, i' \) in \( m \), if \( \delta_{N_i} = \delta_{N_{i'}} \), then there is an isomorphism

\[ \Psi_{N_i, N_{i'}} : (N_i, \in, \Phi \cap N_i) \longrightarrow (N_{i'}, \in, \Phi \cap N_{i'}) \]

Furthermore, \( \Psi_{N_i, N_{i'}} \) is the identity on \( \kappa \cap N_i \cap N_{i'} \).
Proof sketch (continued)

\(a3\) For all \(i, j\) in \(m\), if \(\delta_{N_j} < \delta_{N_i}\), then there is some \(i' < m\) such that \(\delta_{N_{i'}} = \delta_{N_i}\) and \(N_j \in N_{i'}\).

\(a4\) For all \(i, i', j\) in \(m\), if \(N_j \in N_i\) and \(\delta_{N_i} = \delta_{N_{i'}}\), then there is some \(j' < m\) such that \(N_{j'} = \psi_{N_i,N_{i'}}(N_j)\).
Proof sketch (continued)

Our forcing will be the direct limit $\mathcal{P}_{\omega_2}$ of a sequence $\langle \mathcal{P}_\alpha : \alpha < \omega_2 \rangle$ of posets such that

\begin{itemize}
  \item[(\circ)] $\mathcal{P}_\alpha$ is a complete suborder of $\mathcal{P}_\beta$ if $\alpha < \beta \leq \omega_2$, and
  \item[(\circ)] a condition $q$ in $\mathcal{P}_\alpha$ is an $\alpha$–sequence $p$ together with a certain system $\Delta_q$ of side conditions.
\end{itemize}

Unlike in a usual iteration, $p$ will not consist of names, but of well–determined objects (finite functions included in $\omega_1 \times [\omega_1]^{<\omega}$).
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Unlike in a usual iteration, $p$ will not consist of names, but of well–determined objects (finite functions included in $\omega_1 \times [\omega_1]^{<\omega}$).
Defining $\langle P_{\alpha} : \alpha \leq \omega_2 \rangle$ 

$P_0$: Conditions are $p = \{(N_i, 0) : i < m\}$ where $\{N_i : i < m\}$ is a coherent system of structures.

$\leq_0$ is $\supseteq$. 
Defining $\langle P_\alpha : \alpha \leq \omega_2 \rangle$ (continued)

Suppose $P_\alpha$ defined and suppose conditions in $P_\alpha$ are pairs $(p, \Delta_p)$ with $p$ an $\alpha$–sequence and $\Delta_p = \{(N, \beta_i) : i < m\}$.

Suppose $P_\alpha$ has the $\aleph_2$–chain condition and $|P_\alpha| = \kappa$.

By $\kappa^{\aleph_1} = \kappa$ we may fix an enumeration $\dot{Q}_i^\alpha$ (for $i < \kappa$) of nice $\kappa$–suitable partial orders consisting of finite functions included in $\omega_1 \times [\omega_1]^{<\omega}$ such that for every $P_\alpha$–name $\dot{Q}$ for such a poset there are $\kappa$–many $i < \kappa$ such that $\models_{P_\alpha} \dot{Q} = \dot{Q}_i^\alpha$.

We also fix $P_\alpha$–names $\dot{R}_i^\alpha$ and $\dot{C}_i^\alpha$ (for $i < \kappa$) such that $P_\alpha$ forces that $\dot{R}_i^\alpha$ and $\dot{C}_i^\alpha$ witness that $\dot{Q}_i^\alpha$ is $\kappa$–suitable.
Defining $\langle P_\alpha : \alpha \leq \omega_2 \rangle$ (continued)

Suppose $P_\alpha$ defined and suppose conditions in $P_\alpha$ are pairs $(p, \Delta p)$ with $p$ an $\alpha$–sequence and $\Delta p = \{(N, \beta_i) : i < m\}$.

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Defining \( \langle P_\alpha : \alpha \leq \omega_2 \rangle \) (continued)

\( P_{\alpha+1} \): Conditions are

\[
q = (p\langle f_i : i \in a \rangle, \{(N_i, \beta_i) : i < m\})
\]

satisfying the following conditions.

\( b_1 \) For all \( i < m \), \( \beta_i \leq (\alpha + 1) \cap \sup(N_i \cap \omega_2) \).

\( b_2 \) The restriction of \( q \) to \( \alpha \) is a condition in \( P_\alpha \). This restriction is defined as the object \( q|_\alpha := (p, \{(N_i, \beta_i^\alpha) : i < m\}) \); where \( \beta_i^\alpha = \beta_i \) if \( \beta_i < \alpha + 1 \), and \( \beta_i^\alpha = \alpha \) if \( \beta_i = \alpha + 1 \). We denote \( \{(N_i, \beta_i) : i < m\} \) by \( \Delta_q \).

\( b_3 \) \( a \) is a finite subset of \( \kappa \).
Defining $\langle P_\alpha : \alpha \leq \omega_2 \rangle$ (continued)

\textit{b4)} For each $i \in a$, $f_i$ is a finite function included in $\omega_1 \times [\omega_1]^{<\omega}$ and $q|_{\alpha}$ forces (in $P_\alpha$) that $f_i \in \dot{Q}_i^\alpha$.

\textit{b5)} For every $N$ such that $(N, \alpha + 1) \in \Delta_q$ and $\alpha \in N$, $q|_{\alpha}$ forces that there is some $\mathcal{W}_N \subseteq \mathcal{W}^\alpha$ such that

$$f_i \dot{R}_i^\alpha(N, \mathcal{W}_N)$$

for all $i \in a \cap N$.

Here, $\mathcal{W}^\alpha$ denotes the collection of all $M$ such that $(M, \alpha) \in \Delta_u$ for some $u \in \dot{G}_\alpha$. 
Defining $\langle P_\alpha : \alpha \leq \omega_2 \rangle$ (continued)

Given conditions

$$q_\epsilon = (p_\epsilon \langle f^\epsilon_i : i \in a_\epsilon \rangle, \set{(N^\epsilon_i, \beta^\epsilon_i) : i < m_\epsilon})$$

(for $\epsilon \in \{0, 1\}$), we will say that $q_1 \leq_{\alpha+1} q_0$ if and only if the following holds.

\[\begin{align*}
\text{c 1)} & \quad q_1|_\alpha \leq_\alpha q_0|_\alpha \\
\text{c 2)} & \quad a_0 \subseteq a_1 \\
\text{c 3)} & \quad \text{For all } i \in a_0, q|_\alpha \text{ forces in } P_\alpha \text{ that } f^1_i \leq_{\dot{Q}_\alpha} f^0_i. \\
\text{c 4)} & \quad \text{For all } i < m_0 \text{ there exists } \tilde{\beta}_i \geq \beta^0_i \text{ such that } (N^0_i, \tilde{\beta}_i) \in \Delta_{q_1}.
\end{align*}\]
Defining $\langle \mathcal{P}_\alpha : \alpha \leq \omega_2 \rangle$ (continued)

Suppose $\alpha \leq \omega_2$ is a nonzero limit ordinal. Conditions are $q = (p, \{(N_i, \beta_i) : i < m\})$ such that:

$d\, 1)$ $p$ is a sequence of length $\alpha$.

$d\, 2)$ For all $i < m$, $\beta_i \leq \alpha \cap \sup(X_i \cap \omega_2)$. (Note that $\beta_i$ is always less than $\omega_2$, even when $\alpha = \omega_2$.)

$d\, 3)$ For every $\varepsilon < \alpha$, the restriction $q|_\varepsilon := (p | \varepsilon, \{(X_i, \beta_i^\varepsilon) : i < m\})$ is a condition in $\mathcal{P}_\varepsilon$; where $\beta_i^\varepsilon = \beta_i$ if $\beta_i \leq \varepsilon$, and $\beta_i^\varepsilon = \varepsilon$ if $\beta_i > \varepsilon$.

$d\, 4)$ The set of $\zeta < \alpha$ such that $p(\zeta) \neq \emptyset$ is finite.
Given conditions $q_1 = (p_1, \Delta_1)$ and $q_0 = (p_0, \Delta_0)$ in $\mathcal{P}_\alpha$, $q_1 \preceq_\alpha q_0$ if and only if:

\begin{enumerate}
\item[(e1)] For every $(X_i, \beta_i) \in \Delta_0$ there exists $\tilde{\beta}_i \geq \beta_i$ such that $(X_i, \tilde{\beta}_i) \in \Delta_1$.
\item[(e2)] For every $\beta < \alpha$, $q_1|_\beta \preceq_\beta q_0|_\beta$.
\end{enumerate}

Notation: If $\alpha \leq \omega_2$ and $q = (p, \{(N_i, \beta_i) : i < m\}) \in \mathcal{P}_\alpha$, we set $x_q = \{N_i : i < m\}$. 
Main facts about $\langle \mathcal{P}_\alpha : \alpha \leq \omega_2 \rangle$

Lemma Let $\alpha \leq \beta \leq \omega_2$.

If $q = (p, \Delta_q) \in \mathcal{P}_\alpha$, $s = (r, \Delta_s) \in \mathcal{P}_\beta$ and $q \leq_\alpha s|_\alpha$, then $(p \upharpoonright (r \upharpoonright [\alpha, \beta)), \Delta_q \cup \Delta_s)$ is a condition in $\mathcal{P}_\beta$ extending $s$.

Therefore, $\mathcal{P}_\alpha$ can be seen as a complete suborder of $\mathcal{P}_\beta$.

Lemma For every $\alpha \leq \omega_2$, $\mathcal{P}_\alpha$ is $\aleph_2$–Knaster.
Main facts about $\langle \mathcal{P}_\alpha : \alpha \leq \omega_2 \rangle$

**Lemma** Let $\alpha \leq \beta \leq \omega_2$.

If $q = (p, \Delta_q) \in \mathcal{P}_\alpha$, $s = (r, \Delta_s) \in \mathcal{P}_\beta$ and $q \leq_{\alpha} s|_{\alpha}$, then

\[(p \upharpoonright (r \upharpoonright [\alpha, \beta])), \Delta_q \cup \Delta_s)\]

is a condition in $\mathcal{P}_\beta$ extending $s$.

Therefore, $\mathcal{P}_\alpha$ can be seen as a complete suborder of $\mathcal{P}_\beta$.

**Lemma** For every $\alpha \leq \omega_2$, $\mathcal{P}_\alpha$ is $\aleph_2$–Knaster.
Let $\langle \theta_\alpha : \alpha \leq \omega_2 \rangle$ be the following sequence of regular cardinals: $\theta_0 = (2^\kappa)^+$, $\theta_\gamma = (\sup_{\alpha < \gamma} \theta_\alpha)^+$ if $\gamma$ is a nonzero limit ordinal, and $\theta_{\alpha+1} = (2^{\theta_\alpha})^+$.

Also, for each $\alpha \leq \omega_2$ let $\mathcal{M}_\alpha$ be the collection of all countable elementary substructures of $H(\theta_\alpha)$ containing $\langle \theta_\beta : \beta < \alpha \rangle$, $\Phi$ and $\mathcal{P}_\alpha$.

All $\mathcal{P}_\alpha$ are proper:
Let \( \langle \theta_\alpha : \alpha \leq \omega_2 \rangle \) be the following sequence of regular cardinals: \( \theta_0 = (2^\kappa)^+ \), \( \theta_\gamma = (\sup_{\alpha < \gamma} \theta_\alpha)^+ \) if \( \gamma \) is a nonzero limit ordinal, and \( \theta_{\alpha+1} = (2^{\theta_\alpha})^+ \).

Also, for each \( \alpha \leq \omega_2 \) let \( \mathcal{M}_\alpha \) be the collection of all countable elementary substructures of \( H(\theta_\alpha) \) containing \( \langle \theta_\beta : \beta < \alpha \rangle \), \( \Phi \) and \( \mathcal{P}_\alpha \).

All \( \mathcal{P}_\alpha \) are proper:
Lemma  Suppose $\alpha \leq \omega_2$ and $N^* \in M_\alpha$. Then,

(1)$\alpha$ for every $q \in N^* \cap P_\alpha$ there is $q' \leq_\alpha q$ such that $(N^* \cap H(\kappa), \alpha \cap \sup(N^* \cap \omega_2)) \in \Delta_{q'}$, and

(2)$\alpha$ for every $q \in P_\alpha$, if there is some $N$ such that $(N, \alpha \cap \sup(N \cap \omega_2)) \in \Delta_q$ and such that either

(a) $N^* \cap H(\kappa) = N$ or

(b) $N^* \cap H(\kappa) = \Phi"(\gamma \cap N)$ for some $\gamma \in N \cap \kappa$ such that $\Phi \upharpoonright \gamma$ enumerates $[\gamma]^{\aleph_0}$,

then $q$ is $(N^*, P_\alpha)$–generic.
The proof is by induction on $\alpha$.

Proof sketch of $(2)_\alpha$ in the case $\alpha = \sigma + 1$:

Let $E$ be an open and dense subset of $P_\alpha$ in $N^*$. It suffices to show that every $q$ satisfying the hypothesis of $(2)_\alpha$ is compatible with some condition in $E \cap N^*$. By density of $E$ we may assume, without loss of generality, that $q \in E$. We may also assume that $a^q \neq \emptyset$. 
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Proof sketch of $(2)_\alpha$ in the case $\alpha = \sigma + 1$:

Let $E$ be an open and dense subset of $\mathcal{P}_\alpha$ in $N^*$. It suffices to show that every $q$ satisfying the hypothesis of $(2)_\alpha$ is compatible with some condition in $E \cap N^*$. By density of $E$ we may assume, without loss of generality, that $q \in E$. We may also assume that $a^q \neq \emptyset$. 
Claim
For every $i \in \kappa \setminus N^*$ there are ordinals $\alpha_i < \beta_i$ such that

(a) $\alpha_i \in N^*$ and $\beta_i \in (\kappa \cap N^*) \cup \{\kappa\}$,
(b) $\alpha_i < i < \beta_i$, and
(c) $[\alpha_i, \beta_i) \cap N' \cap N^* = \emptyset$ whenever $N' \in \mathcal{X}_q \setminus N^*$ is such that $\delta_{N'} < \delta_{N^*}$.

[This is proved using the fact that all $\psi_{\overline{N},N}$ fix $\kappa \cap \overline{N} \cap N$ and are continuous (for $\overline{N} \in \mathcal{X}_q$ with $\delta_{\overline{N}} = \delta_N$), meaning that $\psi_{\overline{N},N}(\xi) = sup(\psi_{\overline{N},N} ''\xi)$ whenever $\xi \in \overline{N}$ is an ordinal of countable cofinality.]
Suppose \( a^q \setminus N^* = \{ i_0, \ldots, i_{n-1} \} \), and for each \( k < n \) let \( \alpha_k < \beta_k \) be ordinals realizing the above claim for \( i_k \).

Let us work in \( V^{P_\sigma \upharpoonright (q, \sigma)} \). By condition b5) in the definition of \( P_{\sigma+1} \) we know that there is a stationary \( \mathcal{W}_N \subseteq \mathcal{W}^\sigma \) such that \( f^q_i \dot{R}^\sigma_i (N, \mathcal{W}_N) \) for all \( i \in a^q \cap N \).

By an inductive construction (using (1) in the definition of \( \kappa \)-suitable) we may find an \( N \)-stationary \( \mathcal{W} \subseteq \mathcal{W}_N \) such that \( f^q_i \dot{R}^\sigma_i (N, \mathcal{W}) \) for all \( i \in a^q \cap N \) and such that each \( M \in \mathcal{W} \) is good for \( f^q_j \) for every \( j \in a^q \cap M \).
Since \( N^* \cap H(\kappa) \) is an \( \in \)-initial segment of \( N \) and since 
\[ [N^* \cap H(\kappa)]^{\aleph_0} \subseteq N^* \text{, every } N-\text{stationary subset of } [H(\kappa)]^{\aleph_0} \]
is also \( N^* \)-stationary.

Hence, we may find \( M^* \) and \( M \) in \( N^* \) such that

(a) \( M^* \in M_\sigma \) and \( M^* \) contains \( \mathcal{P}_{\sigma+1}, E, a^q \cap N^*, f_i^q \upharpoonright \delta_{N^*} \) for 
every \( i \in a^q \cap N^*, \alpha_k \) for every \( k < n \), and \( \beta_k \) for every 
\( k < n \) with \( \beta_k < \kappa \).

(b) \( (M, \sigma) \in \Delta_u \) for some \( u \in \dot{\mathcal{G}}_\sigma \),

(c) \( M^* \cap H(\kappa) = \Phi^{\sigma}((\gamma \cap M)) \) for some ordinal \( \gamma \in M \) such that 
\( \Phi \upharpoonright \gamma \) enumerates \( [\gamma]^{\aleph_0} \), and

(d) \( M \) is good for \( f_i^q \) for every \( i \in a^q \cap N^* \).
Since $N^* \cap H(\kappa)$ is an $\in$–initial segment of $N$ and since $[N^* \cap H(\kappa)]^{\aleph_0} \cap N \subseteq N^*$, every $N$–stationary subset of $[H(\kappa)]^{\aleph_0}$ is also $N^*$–stationary.

Hence, we may find $M^*$ and $M$ in $N^*$ such that

(a) $M^* \in \mathcal{M}_\sigma$ and $M^*$ contains $\mathcal{P}_{\sigma+1}$, $E$, $a^q \cap N^*$, $f^q_i \upharpoonright \delta_{N^*}$ for every $i \in a^q \cap N^*$, $\alpha_k$ for every $k < n$, and $\beta_k$ for every $k < n$ with $\beta_k < \kappa$.

(b) $(M, \sigma) \in \Delta_u$ for some $u \in \dot{G}_\sigma$,

(c) $M^* \cap H(\kappa) = \Phi^{\langle \gamma \cap M \rangle}$ for some ordinal $\gamma \in M$ such that $\Phi \upharpoonright \gamma$ enumerates $[\gamma]^{\aleph_0}$, and

(d) $M$ is good for $f^q_i$ for every $i \in a^q \cap N^*$. 
From (d), together with $\delta_M = \delta_{M^*}$, we have that $M^*$ is good for $f_i^q$ for every $i \in a^q \cap N^*$. For every such $i$ let $f_i$ be a $\dot{Q}_i^\sigma$–condition in $M^*$ extending $f_i^q \upharpoonright \delta_{M^*} = f_i^q \upharpoonright \delta_N$ and such that every $\dot{Q}_i^\sigma$–condition in $M^*$ extending $f_i$ is compatible with $f_i^q$.

By extending $q$ below $\sigma$ we may assume that $(M, \sigma) \in \Delta_q$ and that $q_\sigma$ decides $f_i$ for every $i \in a^q$.

The result of replacing $f_i^q$ with $\text{glb}(f_i, f_i^q)$ in $q$ for every $i \in a^q \cap N^*$ is a $\mathcal{P}_{\sigma+1}$–condition.

Hence, by further extending $q$ if necessary we may assume that every $\dot{Q}_i^\sigma$–condition in $M^*$ extending $f_i^q \upharpoonright \delta_{M^*}$ is compatible with $f_i^q$. 
From (d), together with $\delta_M = \delta_{M^*}$, we have that $M^*$ is good for $f_i^q$ for every $i \in a^q \cap N^*$. For every such $i$ let $f_i$ be a $\dot{Q}_i^\sigma$–condition in $M^*$ extending $f_i^q \upharpoonright \delta_{M^*} = f_i^q \upharpoonright \delta_N$ and such that every $\dot{Q}_i^\sigma$–condition in $M^*$ extending $f_i$ is compatible with $f_i^q$.

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Hence, by further extending $q$ if necessary we may assume that every $\dot{Q}_i^\sigma$–condition in $M^*$ extending $f_i^q \upharpoonright \delta_{M^*}$ is compatible with $f_i^q$. 
Let now $G$ be a $\mathcal{P}_\sigma$–generic filter over the ground model with $q|_\sigma \in G$.

By correctness of $M^*[G]$ within $H(\theta_\sigma)[G]$ we know that in $M^*[G]$ there is a condition $q^\circ$ satisfying the following conditions.

(a) $q^\circ \in E$ and $q^\circ|_\sigma \in G$.

(b) $a^{q^\circ} = (a^q \cap N^*) \cup \{i_0^\circ, \ldots, i_{n-1}\}$ with $\alpha_k < i_k^\circ < \beta_k$ for all $k < n$.

(c) For all $i \in a^q \cap N^*$, $f_i^{q^\circ}$ extends $f_i^q \upharpoonright \delta_{M^*}$ in $Q_i^\sigma$.

(the existence of such a $q^\circ$ is witnessed, in $V[G]$, by $q$ itself).
By induction hypothesis, $q|_\sigma$ is $(M^*, P_\sigma)$–generic. Hence, $M^*[G] \cap V = M^*$. It follows that $q^\circ$ is in $M^*$.

By extending $q$ below $\sigma$ we may assume that $q$ decides $q^\circ$ and also that it extends $q^\circ|_\sigma$. The proof in this case will be finished if we show that $q$ and $q^\circ$ are compatible.

It is not difficult to find $f_i^*$ (for $i \in a^q \cup \{i_0^\circ, \ldots, i_{n_1}^*\}$) extending $f_i^q$ and/or $f_{i_k}^{q^\circ}$ (for $k < n$) for which, in $V^{P_\sigma}|(q|_\sigma)$, we can verify condition $b5$) with respect to all $N'$ such that $(N', \sigma + 1) \in \Delta_q \cup \Delta_{q^\circ}$ and $\sigma \in N'$.

If $\delta_{N'} \geq \delta_N$, we use condition (2) (and (1)) in the definition of $\kappa$–suitable.

If $\delta_{N'} < \delta_N$ and $N' \in M^*$ (that is, $(N', \sigma + 1) \in \Delta_{q^\circ}$), we use condition (1) in the definition of $\kappa$–suitable.
By induction hypothesis, $q|_\sigma$ is $(M^*, \mathcal{P}_\sigma)$–generic. Hence, $M^*[G] \cap V = M^*$. It follows that $q^\circ$ is in $M^*$.

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It is not difficult to find $f^*_i$ (for $i \in a^q \cup \{i^*_0, \ldots, i^*_{n_1}\}$) extending $f^q_i$ and/or $f^q_{{i^*_k}}$ (for $k < n$) for which, in $V^{\mathcal{P}_\sigma}|(q|_\sigma)$, we can verify condition (b5) with respect to all $N'$ such that $(N', \sigma + 1) \in \Delta_q \cup \Delta_{q^\circ}$ and $\sigma \in N'$.

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It is not difficult to find \( f_i^* \) (for \( i \in a^q \cup \{i_0^\circ, \ldots i_{n_1}^*\} \)) extending \( f_i^q \) and/or \( f_{i_k}^{q^\circ} \) (for \( k < n \)) for which, in \( V^{P_\sigma}|(q|_\sigma) \), we can verify condition \( b5 \) with respect to all \( N' \) such that \((N', \sigma + 1) \in \Delta_q \cup \Delta_{q^\circ} \) and \( \sigma \in N' \).

If \( \delta_{N'} \geq \delta_N \), we use condition (2) (and (1)) in the definition of \( \kappa \)–suitable.

If \( \delta_{N'} < \delta_N \) and \( N' \in M^* \) (that is, \((N', \sigma + 1) \in \Delta_{q^\circ} \)), we use condition (1) in the definition of \( \kappa \)–suitable.
The only potentially problematic case is when $\delta_{N'} < \delta_N$ and $N' \in \mathcal{X}_q \setminus M^*$. But we are safe also in this case since then $(a^q \cup \{i_0^*, \ldots, i_{n_1}^*\}) \cap N' = a^q \cap N'$. We apply again (1) in the definition of $\kappa$–suitable.

Finally we extend $q$ below $\sigma$ once more to a condition $q'$ deciding $f_i^*$. Now we amalgamate $q'$ and $q^\circ$ and get a legal $\mathcal{P}_\alpha$–condition (note that in extending $q$ below $\sigma$ we are not adding new pairs $(N', \sigma + 1)$ to $\Delta$).

This finishes the (very sketchy) proof of the lemma in this case. □
The only potentially problematic case is when $\delta_{N'} < \delta_N$ and $N' \in X_q \setminus M^*$. But we are safe also in this case since then $(a^q \cup \{i_0^\circ, \ldots, i_{n_1}^*\}) \cap N' = a^q \cap N'$. We apply again (1) in the definition of $\kappa$–suitable.

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This finishes the (very sketchy) proof of the lemma in this case. $\square$
Given ordinals $\alpha < \omega_2$ and $i < \kappa$, we let $\dot{G}_i^\alpha$ be a $\mathcal{P}_{\alpha+1}$ for the collection of all $f_i^q$, where $q \in \dot{G}_{\alpha+1}$, $\alpha \in \text{Psupp}(q)$, and $i \in a^q$.

**Lemma**
For every $\alpha < \omega_2$ and every $i < \kappa$, $\mathcal{P}_{\alpha+1}$ forces that $\dot{G}_i^\alpha$ is a $V^{\mathcal{P}_\alpha}$–generic filter over $\dot{Q}_i^\alpha$.

From the above lemmas it is easy to see by standard arguments that $\mathcal{P}_{\omega_2}$ forces $FA(\Gamma_\kappa)$ and $2^{\aleph_0} = \kappa$. □
Given ordinals $\alpha < \omega_2$ and $i < \kappa$, we let $\dot{G}_i^\alpha$ be a $\mathcal{P}_{\alpha+1}$ for the collection of all $f_i^q$, where $q \in \dot{G}_{\alpha+1}$, $\alpha \in \text{Psupp}(q)$, and $i \in a^q$.

**Lemma**

For every $\alpha < \omega_2$ and every $i < \kappa$, $\mathcal{P}_{\alpha+1}$ forces that $\dot{G}_i^\alpha$ is a $V^{\mathcal{P}_\alpha}$–generic filter over $\dot{Q}_i^\alpha$.

From the above lemmas it is easy to see by standard arguments that $\mathcal{P}_{\omega_2}$ forces $FA(\Gamma_\kappa)$ and $2^{\aleph_0} = \kappa$. □
An enhanced version of the Main Theorem

Given a class $\Gamma$ of partial orders and a cardinal $\lambda$, $FA(\Gamma)_{<\lambda}$ means:

For every $P \in \Gamma$ and collection $D$ of size less than $\lambda$ consisting of dense subsets of $P$ there is a filter $G \subseteq P$ such that $G \cap D \neq \emptyset$ for every $D \in D$. 
An enhanced version of the Main Theorem

Theorem \((CH)\) Let \(\kappa\) be a cardinal such that \(2^{<\kappa} = \kappa\), \(\kappa^{\aleph_1} = \kappa\) and \(\mu^{\aleph_0} < \kappa\) for all \(\mu < \kappa\). Then there is a partial order \(\mathcal{P}\) such that

1. \(\mathcal{P}\) is proper,
2. \(\mathcal{P}\) has the \(\aleph_2\)–chain condition,
3. \(\mathcal{P}\) forces
   - \((\bullet)\) \(FA(\Gamma_\kappa) < cf(\kappa)\)
   - \((\bullet)\) \(2^{\aleph_0} = \kappa\)
Another strong failure of Club Guessing

**Definition (Moore):** *Measuring:* For every sequence \((C_\delta : \delta < \omega_1)\) such that each \(C_\delta\) is a closed subset of \(\delta\) there is a club \(D \subseteq \omega_1\) such that for every limit point \(\delta \in D\) of \(D\),

(a) either a tail of \(D \cap \delta\) is contained in \(C_\delta\),

(b) or a tail of \(D \cap \delta\) is disjoint from \(C_\delta\).

(◦) *Measuring* follows from *BPFA* and also from *MRP*.

(◦) *Measuring* implies the negation of Weak Club Guessing and implies \(\Omega\).
Another strong failure of Club Guessing

**Definition (Moore):** *Measuring*: For every sequence $(C_\delta : \delta < \omega_1)$ such that each $C_\delta$ is a closed subset of $\delta$ there is a club $D \subseteq \omega_1$ such that for every limit point $\delta \in D$ of $D$,

(a) either a tail of $D \cap \delta$ is contained in $C_\delta$,

(b) or a tail of $D \cap \delta$ is disjoint from $C_\delta$.

(◦) *Measuring* follows from BPFA and also from MRP.

(◦) *Measuring* implies the negation of Weak Club Guessing and implies $\Omega$. 
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We do not know how to derive Measuring from any "natural" forcing axiom that we can force together with the continuum large.

However,
Theorem \((CH)\) Let \(\kappa\) be a cardinal such that \(2^{<\kappa} = \kappa\), \(\kappa^\aleph_1 = \kappa\) and \(\mu^\aleph_0 < \kappa\) for all \(\mu < \kappa\). Then there is a partial order \(\mathcal{P}\) such that

1. \(\mathcal{P}\) is proper,

2. \(\mathcal{P}\) has the \(\aleph_2\)–chain condition,

3. \(\mathcal{P}\) forces
   - Measuring
   - \(2^{\aleph_0} = \kappa\)