$C^{(n)}$-cardinals

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Outline

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The $C^{(n)}$-classes

Let $C^{(n)}$ denote the closed unbounded proper class of ordinals α that are Σ_n-*correct* in V. i.e., $V_\alpha \preceq_n V$.

Thus, $C^{(0)}$ is the class of all ordinals.

And $C^{(1)}$ is precisely the class of all uncountable cardinals α such that $V_\alpha = H(\alpha)$.

Thus, $C^{(1)}$ is Π_1 definable.

In general, the class $C^{(n)}$ is Π_n definable, for $n \geq 1$.

The $C^{(n)}$-classes

The classes $C^{(n)}$, $n \in \omega$, form a basis for definable club proper classes of ordinals, in the sense that every Σ_n club proper class of ordinals contains $C^{(n)}$.

More generally, every club proper class C of ordinals that is $\underset{\sim}{\Sigma}_n$ (i.e., Σ_n-definable with parameters) contains all $\alpha \in C^{(n)}$ that are greater than the rank of the parameters involved in some Σ_n definition of C.

$C^{(n)}$-embeddings

When considering non-trivial elementary embeddings $j : V \to M$, with M transitive, one would like to have some control over where the image $j(\kappa)$ of the critical point κ goes.

A especially interesting case is when one wants $V_{j(\kappa)}$ to reflect some specific property of V or, more generally, when one wants $j(\kappa)$ to belong to a particular club proper class of ordinals.

Since the $C^{(n)}$, $n \in \omega$, form a basis for such classes, the problem can be reformulated as follows:

When can we have $j(\kappa) \in C^{(n)}$, for a given $n \in \omega$?

$C^{(n)}$-cardinals

Let us call a cardinal κ $C^{(n)}$-measurable if there is an elementary embedding $j : V \to M$, some transitive class M, with critical point κ and with $j(\kappa) \in C^{(n)}$.

Proposition

Every measurable cardinal is $C^{(n)}$-measurable, for all n.

$C^{(n)}$-cardinals

Proposition

1. *Every strong cardinal is $C^{(n)}$-strong, for all n.*
2. *Every supercompact cardinal is λ-$C^{(n)}$-supercompact, for all $\lambda \in OR$ and all n.*
   Thus, every supercompact cardinal is $C^{(n)}$-supercompact, for all n.

$C^{(n)}$-superstrong cardinals

For superstrong cardinals κ, the requirement that $j(\kappa) \in C^{(n)}$, for $n > 1$, produces stronger large cardinal principles.

Definition

A cardinal κ is $C^{(n)}$-superstrong if there exists an elementary embedding $j : V \to M$, M transitive, with critical point κ, $V_{j(\kappa)} \subseteq M$, and $j(\kappa) \in C^{(n)}$.

$C^{(n)}$-superstrong cardinals

Proposition

1. *Every superstrong cardinal is $C^{(1)}$-superstrong.*
2. *For every $n \geq 1$, if κ is $C^{(n+1)}$-superstrong, then there is a κ-complete normal ultrafilter $\mathcal{U}$ over κ such that*

$$\{\alpha < \kappa : \kappa \text{ is } C^{(n)}\text{-superstrong}\} \in \mathcal{U}.$$

Hence, the first $C^{(n)}$-superstrong cardinal κ, if it exists, is not $C^{(n+1)}$-superstrong.

$C^{(n)}$-superstrong cardinals

Proposition

If κ is 2^κ-supercompact and belongs to $C^{(n)}$, then there is a κ-complete normal ultrafilter $\mathcal{U}$ over κ such that the set of $C^{(n)}$-superstrong cardinals smaller than κ belongs to $\mathcal{U}$.

Extendible cardinals

Definition (Reinhardt, ca. 1970)

κ is λ–extendible, for $\lambda > \kappa$, if there exists an elementary embedding $j : V_\lambda \to V_\mu$, some μ, such that κ is the critical point of j and $j(\kappa) > \lambda$.

κ is extendible if it is λ-extendible for all $\lambda > \kappa$.

Extendible cardinals are supercompact, and the existence of, e.g., an almost-huge cardinal κ implies the existence of many extendible cardinals in V_κ.

$C^{(n)}$-extendible cardinals

Definition

For a cardinal κ and $\lambda > \kappa$, we say that κ is λ-$C^{(n)}$-extendible if there is an elementary embedding $j : V_\lambda \to V_\mu$, some μ, with critical point κ, and such that $j(\kappa) > \lambda$ and $j(\kappa) \in C^{(n)}$.

We say that κ is $C^{(n)}$–extendible if it is λ-$C^{(n)}$-extendible for all $\lambda > \kappa$.

Proposition

Every extendible cardinal is $C^{(1)}$-extendible.

$C^{(n)}$-extendible cardinals

Proposition

For every $n \geq 1$, if κ is $C^{(n)}$-extendible and $\kappa + 1$-$C^{(n+1)}$-extendible, then the set of $C^{(n)}$-extendible cardinals is unbounded below κ.

Hence, the first $C^{(n)}$-extendible cardinal κ, if it exists, is not $\kappa + 1$-$C^{(n+1)}$-extendible. In particular, the first extendible cardinal κ is not $\kappa + 1$-$C^{(2)}$-extendible.

Proposition

If κ is $\kappa + 1$-$C^{(n)}$-extendible, then κ is $C^{(n)}$-superstrong, and there is a κ-complete normal ultrafilter $\mathcal{U}$ over κ such that the set of $C^{(n)}$-superstrong cardinals smaller than κ belongs to $\mathcal{U}$.

Vopěnka's Principle

Definition (Vopěnka's Principle (VP). P. Vopěnka, ca. 1960)

There is no rigid proper class of graphs.

Equivalently, for every proper class $\mathcal{C}$ of structures of the same type, there exist $A \neq B$ in $\mathcal{C}$ such that A is elementarily embeddable into B.

VP and extendible cardinals

Theorem (M. Magidor, 1970)

VP implies that there exists a proper class of extendible cardinals.

VP can be characterized in terms of extendibility.

Theorem (Solovay-Reinhardt-Kanamori, 1978)

VP holds iff for every proper class A there is a cardinal κ that is λ-extendible for A, for every ordinal $\lambda > \kappa$.
i.e., there is an ordinal μ and an elementary embedding

$$j : \langle V_\lambda, \in, A \cap V_\lambda \rangle \to \langle V_\mu, \in, A \cap V_\mu \rangle$$

with critical point κ and $j(\kappa) > \lambda$.

Notation

If Γ is one of $\underset{\sim}{\Sigma}_n$, $\underset{\sim}{\Pi}_n$, $\underset{\sim}{\Delta}_n$, where $n \in \omega$, and κ is an infinite cardinal, then we write $VP(\kappa, \Gamma)$ for the following assertion:

For every Γ proper class $\mathcal{C}$ of structures of the same type τ such that both τ and the parameters of some Γ-definition of $\mathcal{C}$, if any, belong to $H(\kappa)$, and for every $B \in \mathcal{C}$, there exists $A \in \mathcal{C} \cap H(\kappa)$ that is elementarily embeddable into B.

If Γ is one of $\underset{\sim}{\Sigma}_n$, $\underset{\sim}{\Pi}_n$, $\underset{\sim}{\Delta}_n$, or Σ_n, Π_n, Δ_n, where $n \in \omega$, we write $VP(\Gamma)$ for the following statement:

For every Γ proper class $\mathcal{C}$ of structures of the language of set theory with one (equivalently, finitely-many) additional 1-ary relation symbol, there exist distinct A and B in $\mathcal{C}$ with an elementary embedding of A into B.

Vopěnka's Principle

Proposition

$VP(\kappa, \underset{\sim}{\Sigma}_1)$ holds for every uncountable cardinal κ.

Proposition

If $VP(\Pi_1)$ holds, then there exists a λ-supercompact cardinal κ, for some λ in $C^{(1)}$ greater than κ.

VP and supercompact cardinals

Theorem (B-Casacuberta-Mathias-Rosičky, 2008)

The following are equivalent:

1. $VP(\kappa, \underset{\sim}{\Delta}_2)$, *for some* κ.
2. $VP(\Delta_2)$.
3. $VP(\kappa, \underset{\sim}{\Sigma}_2)$, *for some* κ.
4. $VP(\Sigma_2)$.
5. *There exists a supercompact cardinal.*

VP and supercompact cardinals

Theorem (B-C-M-R, 2008)

The following are equivalent:

1. $VP(\kappa, \underset{\sim}{\Delta}_2)$, *for a proper class of cardinals* κ.
2. $VP(\underset{\sim}{\Delta}_2)$.
3. $VP(\kappa, \underset{\sim}{\Sigma}_2)$, *for a proper class of cardinals* κ.
4. $VP(\underset{\sim}{\Sigma}_2)$.
5. *There exists a proper class of supercompact cardinals.*

What about the more complex classes, e.g., the $\underset{\sim}{\Sigma}_3$?

VP and $C^{(n)}$-extendible cardinals

Theorem

For every $n \geq 1$, if κ is a $C^{(n)}$–extendible cardinal, then $VP(\kappa, \underset{\sim}{\Sigma}_{n+2})$ holds.

Theorem (B-C-M-R, 2008)

Let $n \geq 1$, and suppose that $VP(\Sigma_{n+1} \wedge \Pi_{n+1})$ holds. Then there exists a $C^{(n)+}$-extendible cardinal.

VP and $C^{(n)}$-extendible cardinals

Corollary

The following are equivalent for $n \geq 1$:

1. $VP(\Sigma_{n+1} \wedge \Pi_{n+1})$.
2. $VP(\kappa, \underset{\sim}{\Sigma}_{n+2})$, *for some* κ.
3. *There exists a $C^{(n)}$-extendible cardinal.*

Corollary

The following are equivalent:

1. $VP(\Sigma_3)$.
2. $VP(\kappa, \underset{\sim}{\Sigma}_3)$, *for some* κ.
3. *There exists an extendible cardinal.*

VP and $C^{(n)}$-extendible cardinals

Corollary

The following are equivalent:

1. *VP.*
2. *$VP(\kappa, \underset{\sim}{\Sigma}_n)$ holds for a proper class of cardinals κ, for every n.*
3. *For every n, there exists a $C^{(n)}$-extendible cardinal.*
4. *For every n, there is a stationary proper class of $C^{(n)}$-extendible cardinals. i.e., every definable club proper class contains a $C^{(n)}$-extendible cardinal.*

Bounded categories

A full subcategory $\mathcal{D}$ of a category $\mathcal{C}$ is called dense in $\mathcal{C}$ if every object $C \in \mathcal{C}$ is a canonical colimit of objects from $\mathcal{D}$.

Examples

1. In **Set** every singleton is dense.
2. In **Gra**, the one-point graph with no edges and the two-point graph with only one edge between them form a dense subcategory.
3. Finitely-generated Abelian groups are dense in **Ab**.

$\mathcal{C}$ is bounded if it has a small (i.e., a set) dense subcategory. Thus **Set**, **Gra**, and **Ab** are bounded. But **Top** is not.

Question

When is a category bounded?

A small full subcategory $\mathcal{A}$ of a category $\mathcal{C}$ is called colimit-dense if every object of $\mathcal{C}$ is a colimit of a diagram in $\mathcal{A}$.

Question

Is every category that has a colimit-dense subcategory bounded?

Theorem (Adámek, Rosický, Trnková, 1990)

The following are equivalent:

1. *A category is bounded iff it has a colimit-dense subcategory.*
2. *VP.*

Accessible categories

Definition (Makkai - Paré, 1989)

A category $\mathcal{K}$ is accessible if it has λ-directed colimits (some regular cardinal λ), and it has a set $\mathcal{A}$ of λ-presentable objects such that every object is a λ-directed colimit of objects from $\mathcal{A}$.

Definition (Adámek - Rosický, 1994)

A formula of L_λ is called basic if it has the form $\forall x(\varphi(x) \rightarrow \psi(x))$, where φ and ψ are disjunctions of formulas of type $\exists y\, \zeta(x, y)$ in which ζ is a conjunction of atomic formulas.

A basic theory is a theory of basic sentences.

The accessible categories are the categories equivalent to categories of models of basic theories.

Accessible categories

Examples

1. For every theory T in L_λ, the category of models of T and homomorphisms is accessible.
2. **Top** is not accessible.
3. The category of complete metric spaces is accessible.

Accessible categories

Under VP, boundedness can be easily characterized.

Theorem (Fisher, 1987)

The following are equivalent:

1. *A category is bounded and has λ-directed colimits for some regular cardinal λ iff it is accessible.*
2. *VP.*

An Application of VP in Algebraic Topology

Question (D. Farjoun, ca. 1990)

Is every functor on simplicial sets that is idempotent up to homotopy equivalent to f–localization for some map?

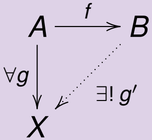
Theorem (Casacuberta – Scevenels – Smith, Adv. Math. 2005)

If there are no measurable cardinals, then there is a counterexample.
If VP holds, then the answer is yes.

Orthogonality

Definition

A morphism $f\colon A \to B$ and an object X are called orthogonal in a category $\mathcal{C}$ if for each $g\colon A \to X$ there is a unique $g'\colon B \to X$ with $g' \circ f = g$:



Small-orthogonality classes

Definition

A small-orthogonality class in a category $\mathcal{C}$ is the class of objects orthogonal to some set of morphisms $\mathcal{F} = \{f_i \colon P_i \to Q_i \mid i \in I\}$.

Boundedness and orthogonality for definable categories

Theorem (B-C-M-R, 2009)

If there is a proper class of $C^{(n)}$–extendible cardinals, then

1. *Every $\underset{\sim}{\Sigma}_{n+1} \wedge \underset{\sim}{\Pi}_{n+1}$ full subcategory of an accessible category is bounded.*
2. *Every $\underset{\sim}{\Delta}_n$ orthogonality class in an accessible category is a small-orthogonality class.*