Cardinal invariants of analytic quotients

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1 Classical cardinal invariants

2 Cardinal invariants of analytic quotients
   • Analytic ideals
   • Basic results for $F_\sigma$ quotients
   • Consistency results for $F_\sigma$ quotients
Cardinal invariants of the continuum are cardinals which typically lie between $\aleph_1$ and $c = 2^\omega$ and which describe the combinatorial structure of the real line.

Many of the classical cardinal invariants (introduced in the 70’s and 80’s) are defined in terms of $\mathcal{P}(\omega)/\text{Fin}$. 
The quotient $\mathcal{P}(\omega)/\text{Fin}$

For $A, B \subseteq \omega$:

\[ A \subseteq^* B \quad (A \text{ is almost contained in } B) \iff A \setminus B \text{ is finite} \]

$A =^* B$ iff $A \subseteq^* B$ and $B \subseteq^* A$ defines an equivalence relation. $\mathcal{P}(\omega)/\text{Fin}$ is the collection of equivalence classes, ordered by

\[ [A] \leq [B] \iff A \subseteq^* B \]

where $[A]$ is the class of $A$.

For simplicity we forget about equivalence classes and work with $([\omega]^\omega, \subseteq^*)$ while meaning $(\mathcal{P}(\omega)/\text{Fin}, \leq)$. 
Splitting and reaping: the classical case

Cardinal invariants: an example

For $A, B \in [\omega]^{\omega}$:

\[
A \text{ splits } B \iff |A \cap B| = |B \setminus A| = \aleph_0
\]

$F \subseteq [\omega]^{\omega}$ is splitting if every member of $[\omega]^{\omega}$ is split by a member of $F$.

$F \subseteq [\omega]^{\omega}$ is unsplit (or unreaped) if no member of $[\omega]^{\omega}$ splits all members of $F$, i.e.

\[
\forall A \in [\omega]^{\omega} \exists B \in F \ (|A \cap B| < \aleph_0 \text{ or } B \subseteq^* A)
\]

$s := \min\{|F| : F \text{ is splitting}\}$, the splitting number.

$t := \min\{|F| : F \text{ is unsplit}\}$, the reaping number.
Splitting and reaping 2: the classical case

\[ b := \min \{|\mathcal{F}| : \mathcal{F} \text{ is unbounded in } (\omega^\omega, \leq^*) \}, \]
the bounding number.

\[ \vartheta := \min \{|\mathcal{F}| : \mathcal{F} \text{ is cofinal in } (\omega^\omega, \leq^*) \}, \]  
the dominating number.

All these cardinal invariants are uncountable (diagonal argument!).

**Proposition**

\[ b \leq r \]  
and \[ s \leq \vartheta. \]

The order-relationship of \( b \) and \( s \) is not decidable.

The consistency of \( s < b \) is easy.

**Theorem (Shelah; Blass and Shelah)**

\[ b < s \]  
and \[ r < \vartheta \]  
are consistent.
We may also consider \((\mathcal{P}(\omega)/\text{Fin}, \leq)\) as a forcing notion. Maximal antichains correspond to maximal almost disjoint families: \(A \subseteq [\omega]^\omega\) is almost disjoint if

\[|A \cap B| < \aleph_0\]

for distinct \(A\) and \(B\) from \(A\). \(A\) is maximal almost disjoint (mad) if it is almost disjoint and for all \(C \in [\omega]^\omega\) there is \(A \in A\) with

\[|A \cap C| = \aleph_0\]

\(a := \min\{|A| : A \text{ is mad}\}\), the almost-disjointness number.

**Proposition**

\(b \leq a\).
$\mathcal{P}(\omega)/\text{Fin}$ as a forcing notion 2

$h := \min\{|D| : D \text{ is a family of open dense sets in } \mathcal{P}(\omega)/\text{Fin} \text{ and } \bigcap D \text{ is not dense}\}$, the *distributivity number*.

$h$ describes forcing-theoretic properties of $\mathcal{P}(\omega)/\text{Fin}$:

**Observation**

$h = \min\{\kappa : \mathcal{P}(\omega)/\text{Fin} \text{ adds a new function from } \kappa \text{ to } V\}$.

So $\mathcal{P}(\omega)/\text{Fin}$ preserves all cardinals $\leq h$ and $> c$.

Everything in between is collapsed:

**Theorem (Balcar, Pelant and Simon)**

$\mathcal{P}(\omega)/\text{Fin}$ forces $c = h^V$.

**Proposition**

$\aleph_1 \leq h \leq \min\{b, s\}$. 
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Quotients by ideals

Let \( I \) be an ideal on \( \omega \) with \( \text{Fin} \subseteq I \). Consider the quotient \( \mathcal{P}(\omega)/I \). Define cardinal invariants of this quotient in analogy to the classical cardinal invariants of \( \mathcal{P}(\omega)/\text{Fin} \).

We address the following questions:

- Can we prove similar inequalities for the new invariants as for their classical counterparts? Similar consistency results? (E.g., does \( s \leq \mathfrak{d} \) generalize to \( s(I) \leq \mathfrak{d} \)?)
- How do the new cardinals compare to the old ones? (E.g. what is the connection between \( s \) and \( s(I) \)?)
Which ideals should we consider?

1. $\mathcal{I}$ should be definable
   (so that it lives in any model of set theory)
2. Therefore consider only analytic ideals $\mathcal{I}$
3. Ideally, $\mathcal{I}$ should be $F_\sigma$
   (so that the quotient $\mathcal{P}(\omega)/\mathcal{I}$ is $\sigma$-closed)
4. We may also want to restrict to analytic $P$-ideals
   (because we have a nice structure theory for them)
Definable ideals 2

Via characteristic functions, identify $\mathcal{I} \subseteq \mathcal{P}(\omega)$ with a subset of $2^\omega$. Hence we may talk about $\mathcal{I}$ being Borel, analytic, $F_\sigma$, etc.

**Observation**

*If $\mathcal{I}$ is $F_\sigma$, then $\mathcal{P}(\omega)/\mathcal{I}$ is $\sigma$-closed, i.e. $t(\mathcal{I}) \geq \aleph_1$.*

$t(\mathcal{I}) := \min\{\kappa : \mathcal{P}(\omega)/\mathcal{I}$ not $\kappa$-closed\}$, the *tower number* of $\mathcal{I}$.

**Definition**

$\mathcal{I}$ is a *$P$-ideal* if for all countable $\mathcal{A} \subseteq \mathcal{I}$ there is $B \in \mathcal{I}$ with $A \subseteq^* B$ for all $A \in \mathcal{A}$. 
Structure theory for analytic ideals

Definition

$\varphi : \mathcal{P}(\omega) \to [0, \infty]$ is a lower semicontinuous submeasure if

1. $\varphi(\emptyset) = 0, \varphi(\{n\}) < \infty$
2. $\varphi(X) \leq \varphi(Y)$ for $X \subseteq Y$
3. $\varphi(X \cup Y) \leq \varphi(X) + \varphi(Y)$
4. $\varphi(X) = \lim_n \varphi(X \cap n)$
Theorem (Mazur; Solecki)

- $\mathcal{I}$ is an $F_\sigma$-ideal iff $\mathcal{I} = \text{Fin}(\varphi)$ for some $\varphi$.
- $\mathcal{I}$ is an analytic $P$-ideal iff $\mathcal{I} = \text{Exh}(\varphi)$ for some $\varphi$.
- $\mathcal{I}$ is an $F_\sigma$ $P$-ideal iff $\mathcal{I} = \text{Exh}(\varphi) = \text{Fin}(\varphi)$ for some $\varphi$. 

Exh$(\varphi) := \{X : \lim_n \varphi(X \setminus n) = 0\}$ is an $F_{\sigma\delta}$ $P$-ideal.
Fin$(\varphi) := \{X : \varphi(X) < \infty\}$ is an $F_\sigma$-ideal.
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The quotient $\mathcal{P}(\omega)/\mathcal{I}$

Instead of working with $(\mathcal{P}(\omega)/\mathcal{I}, \leq)$, consider $(\mathcal{I}^+, \subseteq)$. Here

\[ A \subseteq_\mathcal{I} B \quad (A \text{ is contained in } B \text{ modulo } \mathcal{I}) \iff A \setminus B \in \mathcal{I} \]

and

\[ \mathcal{I}^+ := \mathcal{P}(\omega) \setminus \mathcal{I} \]

denotes the $\mathcal{I}$-positive sets.
Splitting and reaping: the $F_\sigma$ case

Cardinal invariants: an example

For $A, B \in \mathcal{I}^+$:

\[
A \text{ splits } B \iff A \cap B, B \setminus A \in \mathcal{I}^+
\]

$\mathcal{F} \subseteq \mathcal{I}^+$ is I-splitting if every member of $\mathcal{I}^+$ is split by a member of $\mathcal{F}$.

$\mathcal{F} \subseteq \mathcal{I}^+$ is I-unsplit (or I-unreaped) if no member of $\mathcal{I}^+$ splits all members of $\mathcal{F}$, i.e.

\[
\forall A \in \mathcal{I}^+ \exists B \in \mathcal{F} (A \cap B \in \mathcal{I} \text{ or } B \subseteq_I A)
\]

$s(\mathcal{I}) := \min\{|\mathcal{F}| : \mathcal{F} \text{ is } \mathcal{I}\text{-splitting}\}$, the $\mathcal{I}$-splitting number.

$r(\mathcal{I}) := \min\{|\mathcal{F}| : \mathcal{F} \text{ is } \mathcal{I}\text{-unsplit}\}$, the $\mathcal{I}$-reaping number.
Basic results for $F_\sigma$ quotients

From now on assume all ideals are $F_\sigma$.

**Proposition**

$b \leq r(I)$ and $s(I) \leq \kappa$.

**Proposition**

$b < s(I)$ and $r(I) < \kappa$ are consistent.

*Proofs:* Use $\mathcal{I} = \text{Fin}(\varphi)$. □

**Proposition**

Some $A \in \mathcal{I}^+$ forces $c = h(I)^V$.

**Proposition**

$t(I) \leq h(I) \leq \min\{b, s(I)\}$. 
Basic results for $F_\sigma$ quotients 2

Proposition (Farkas and Soukup)

$b \leq a(\mathcal{I})$ for $F_\sigma$ $P$-ideals.

Proof: Use $\mathcal{I} = \text{Exh}(\varphi)$. □

Is this true for $F_\sigma$-ideals in general?

No!

Theorem

$a(\mathcal{ED}_{\text{fin}}) < b$ is consistent.

(In fact, this holds in the Hechler model.)

Here, $\mathcal{ED}_{\text{fin}}$ is one of the eventually different ideals introduced by Hernández and Hrušáč.
The ideal $\mathcal{E}D_{\text{fin}}$

Let

$$\Delta = \{(n, i) \in \omega \times \omega : i \leq n\}$$

be the triangle below the diagonal.

$\mathcal{E}D_{\text{fin}}$ is the ideal on $\Delta$ generated by graphs of functions:

for $A \subseteq \Delta$:

$$A \in \mathcal{E}D_{\text{fin}} \iff \exists m \forall n \left( |A_n| \leq m \right)$$

where $A_n = \{i : (n, i) \in A\}$ is the *vertical section of $A$ at $n$.* This is an $F_\sigma$-ideal.
Basic results for $F_\sigma$ quotients 3

Observation
$p \leq a(\mathcal{I})$.

Proposition
$a(\mathcal{I})$ can be increased by a definable $\sigma$-centered forcing $\mathbb{P}$.

Corollary
$a(\mathcal{I}) > \diamond$ is consistent.

Proof: Put $\mathbb{P}$ into Shelah’s template framework. □

... and many more.
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Summable ideals

Let \( f : \omega \to \mathbb{R}^+ \) with \( \sum_n f(n) = \infty \).

For \( A \subseteq \omega \):

\[
A \in \mathcal{I}_f \iff \sum_{n \in A} f(n) < \infty
\]

\( \mathcal{I}_f \) is an \( F_\sigma \) P-ideal.

In fact \( \mathcal{I}_f = \text{Fin}(\mu_f) = \text{Exh}(\mu_f) \) where

\[
\mu_f(A) = \sum_{n \in A} f(n)
\]
Theorem

$s(I) < s$ is consistent for any tall summable ideal $I$. 
Dually $r < r(I)$ is consistent.

Proof: $I = I_f$. Let $\varepsilon \gg \delta > 0$.
Say $g : \omega \rightarrow [\omega]^{<\omega}$ is an $\varepsilon$-function if

$$\mu_f(g(n)) \geq \varepsilon \text{ for all } n \text{ and } \limsup_n (\min g(n)) = \infty$$

$X \in [\omega]^{\omega}$ $\delta$-splits $g$ if

$$\exists \infty n \left( \mu_f(g(n) \cap X) \geq \frac{\varepsilon}{2} - \delta \text{ and } \mu_f(g(n) \setminus X) \geq \frac{\varepsilon}{2} - \delta \right)$$
Crucial Lemma

Let $M, N$ be models of ZFC. Let $\mathcal{U}$ be an ultrafilter in $M$. Assume $X \in [\omega]^\omega \cap N$ satisfies

\[ (\star^{M,N}_X) : \forall f, \varepsilon, \delta \ (X \text{ } \delta\text{-splits } f) \]

Then there is an ultrafilter $\mathcal{V} \supseteq \mathcal{U}$ in $N$ such that $(\star^{M[G],N[G]}_X)$ holds where $G$ is $\mathbb{L}_\mathcal{V}$-generic over $N$ (and thus $\mathbb{L}_\mathcal{U}$-generic over $M$).

Here $\mathbb{L}_\mathcal{U}$ denotes Laver forcing with $\mathcal{U}$, i.e. forcing with Laver trees such that successor levels of splitnodes belong to $\mathcal{U}$. 
We continue the proof sketch of the Theorem. Start with a model $V$ for $CH$. Add $\omega_1$ many Cohen reals $X_\alpha$ and obtain the model $W$. (The Cohen reals are intended as a witness for $s(\mathcal{I})$.) Use the crucial lemma to build a matrix-like iterated forcing $(\mathbb{P}_\gamma^\alpha : \alpha \leq \omega_1, \gamma \leq \omega_2)$ adding $\mathbb{L}_{U_\gamma}$-generics $\omega_2$ times with finite support and preserving $(\star_{X_\alpha}^{V,W})$ along the $\gamma$-iteration. Obtain the models $V'$ and $W'$. The $\mathbb{L}_{U_\gamma}$-generics witness $s = \aleph_2$. $(\star_{X_\alpha}^{V',W'})$ shows the Cohen reals witness $s(\mathcal{I}) = \aleph_1$. □
Distributivity for summable ideals

Conjecture

$h(I) < h$ is consistent for summable $I$.

Conjecture

$h < h(I)$ is consistent for summable $I$. 
Splitting, reaping, and distributivity for $\mathcal{ED}_{\text{fin}}$

**Theorem**

$s(\mathcal{ED}_{\text{fin}}) < s$ and $r < r(\mathcal{ED}_{\text{fin}})$ are consistent.

**Theorem**

$h(\mathcal{ED}_{\text{fin}}) < h$ is consistent.

Note that

- $h(\mathcal{ED}_{\text{fin}}) \leq h$,
- $s(\mathcal{ED}_{\text{fin}}) \leq s$, and
- $r(\mathcal{ED}_{\text{fin}}) \geq r$

in ZFC.
Next RIMS meeting in Kyoto:

Combinatorial set theory and forcing theory

November 16 - 19, 2009

at Rakuyu Kaikan, Kyoto University, Japan

organized by Teruyuki Yorioka

http://www.ipc.shizuoka.ac.jp/~styorio/rims09/