Standard universal dendrites as small Polish structures

The concept of *small Polish structures* has been introduced and studied in

K. Krupiński, *Some model theory of Polish structures*, TAMS

Goals:

- Provide a setting which allows simultaneous application of ideas and techniques from model theory and descriptive set theory.

In particular:

- prove counterparts of some results from stability theory;
- find, in this wider context, counterexamples to open problems;
- provide a (yet another) tool to measure complexity of dinamycal systems.

Polish structures

A **Polish structure** is a pair (X, G) where:

- G is a Polish group acting faithfully on a set X
  i.e $\forall g, g' \in G\ (g \neq g' \Rightarrow \exists a \in X\ ga \neq g'a)$
- the stabilisers of all singletons are closed

This generalises the notion of a profinite structure.

Notation: For $A \subseteq X$, G_A will denote the pointwise stabiliser of A.

Independence

A notion of independence in Polish structures is introduce.

Let $\vec{a} \in X^{<\omega}$ and $A \subseteq B \subseteq X$ finite.

The idea is to say that $\vec{a}$ is independent from B over A if, once A has been fixed, asking to fix B does not add too much constraint on $\vec{a}$, i.e.

$$G_B\vec{a} \text{ is big in } G_A\vec{a}.$$

Some topological notions of bigness: Open, non-meagre,...

However:

• X does not necessarily have a topology.

• Even if X has a nice topology, some orbits $G_A\vec{a}$ might behave badly, like being meagre in themselves.

Thus the relations of independence are defined via a pull back to the group G.

Let $\begin{array}{l} \pi_A : G_A \to G_A\vec{a} \\ \qquad g \mapsto g\vec{a} \end{array}$ and check whether

$$\pi_A^{-1}(G_B\vec{a}) \text{ is big in } \pi_A^{-1}(G_A\vec{a}) = G_A.$$

Definitions.
Let $\vec{a} \in X^{<\omega}$ and $A, B \subseteq_{fin} X$ (most often $A \subseteq B$).

$\vec{a} \downarrow^o_A B$: $\vec{a}$ is **o-independent** from B over A if $\pi_A^{-1}(G_{A\cup B}\vec{a})$ is open in G_A (written $\pi_A^{-1}(G_{A\cup B}\vec{a}) \subseteq_o G_A$).

$\vec{a} \downarrow^{nm}_A B$: $\vec{a}$ is **nm-independent** from B over A if $\pi_A^{-1}(G_{A\cup B}\vec{a})$ is non-meagre in G_A (written $\pi_A^{-1}(G_{A\cup B}\vec{a}) \subseteq_{nm} G_A$).

Remark. If X is separable metrisable, the action $G \times X \to X$ is continuous and $G_A\vec{a}$ is not meagre in itself, then

$$\vec{a}^*_A B \Leftrightarrow G_{A\cup B}\vec{a} \subseteq_* G_A\vec{a}$$

(for $* = nm$ it is enough X being Hausdorff)

Example: $A = \varnothing$.

$$\vec{a} \downarrow^{*}_{\varnothing} B \text{ iff } \{g \in G \mid \exists h \in G_B \; g\vec{a} = h\vec{a}\} \subseteq_{*} G$$

The opposite situation: small orbits. Let $A \subseteq_{fin} X$.

- $dcl(A) = \{a \in X \mid G_A a = \{a\}\}$: definable closure of A
- $acl(A) = \{a \in X \mid G_A a \text{ is finite}\}$: strong algebraic closure of A
- $Acl(A) = \{a \in X \mid G_A a \text{ is at most countable}\}$: algebraic closure of A

For any $A \subseteq X$, define

$$dcl(A) = \bigcup_{A_0 \subseteq_{fin} X} dcl(A_0),$$

etc.

Basic properties of independence

To develop a counterpart of basic geometric stability theory, five properties of the independence relation are needed

- **Invariance:** $\vec{a} \downarrow^*_A B \Leftrightarrow g\vec{a} \downarrow^*_{gA} gB$
- **Simmetry:** $\vec{a} \downarrow^*_A \vec{b} \Leftrightarrow \vec{b} \downarrow^*_A \vec{a}$
- **Transitivity:** $\vec{a} \downarrow^*_A B \wedge \vec{a} \downarrow^*_B C \Leftrightarrow \vec{a} \downarrow^*_A C$
- $a \in Acl(A) \Leftrightarrow \forall B\ \vec{a} \downarrow^*_A B$
- **Existence of independent extensions**

Small Polish structures

A Polish structure (X, G) is *small* if
$\forall n$, $G \times X^n \to X^n$ has at most countably many orbits

(iff $\forall a_1, \ldots, a_n \in X$, $G_{a_1,\ldots,a_n} \times X \to X$ has at most countably many orbits)

Existence of independent extensions

Theorem. Let (X, G) be a small Polish structure. Then

$$\forall \vec{a},\ \forall A \subseteq B \subseteq_{fin} X,\ \exists \vec{b} \in G_A \vec{a}$$

such that

$$\vec{b} \downarrow^{nm}_{A} B$$

Remark. The same is not true for *o*-independent extensions.

Adaptation of some concepts from stability theory

- $X^{eq} = \bigcup\{X^n/E \,|\text{E invariant eq. rel. on } X^n, \text{ s.t } Stab([a]_E) \leq_c G\}$, the *imaginary extension* of X
- Sets X^n/E are the *sorts* of X^{eq}
- $D \subseteq X^n/E$, for X^n/E a sort, is *definable* on $A \subseteq_{fin} X^{eq}$ if $G_A D = D$ and $Stab(D) \leq_c G$
- $d \in X^{eq}$ is a *name* for D if $\forall g \in G\ (gD = D \Leftrightarrow gd = d)$

Proposition. Every definable set in X^{eq} has a name in X^{eq}.

Ranks

Assume (X, G) is a small Polish structure (but in most situations it is enough to ask for the existence of *nm*-independent extensions)

Definition. $\mathcal{NM}$ is the function from the collection of orbits over finite sets (in X or X^{eq}) to $Ord \cup \{\infty\}$,

$$\mathcal{NM} : (a, A) \mapsto \mathcal{NM}(a, A) \in Ord \cup \{\infty\}$$

satisfying

$$\mathcal{NM}(a, A) \geq \alpha + 1 \Leftrightarrow \exists B \supseteq_{fin} A \ (\mathcal{NM}(a, B) \geq \alpha \wedge \neg a \downarrow_A^{nm} B)$$

Example. $\mathcal{NM}(a, A) = 0 \Leftrightarrow a \in Acl^{eq}(A)$.

Definition. (X, G) is **nm-stable** if every 1-orbit has ordinal rank, i.e. there is no infinite sequence $A_0 \subseteq A_1 \subseteq \ldots \subseteq_{fin} X$ and $a \in X$ such that a is nm-dependent from A_{i+1} over A_i.

Definition. If D is definable over A in X^{eq}, the $\mathcal{NM}$-rank of D is

$$\mathcal{NM}(D) = \sup\{\mathcal{NM}(d, A) \mid d \in D\}$$

Examples (Krupiński)

- $(S^n, Homeo(S^n))$ has rank 1
- $((S^1)^n, Homeo((S^1)^n))$ has rank 1
- $([0,1]^{\mathbb{N}}, Homeo([0,1]^{\mathbb{N}}))$ has rank 1
- if (X, G) has rank 1, then (X^n, G) has rank n

Continua

Definitions.

- A *continuum* is a compact connected metric space; it is non-degenerate if it has more than one point
- A non-degenerate continuum X is *decomposable* if $X = Y \cup Z$, for Y, Z some proper subcontinua of X. Otherwise it is *indecomposable*
- A non-degenerate continuum is *hereditarily (in)decomposable* if all its subcontinua are (in)decomposable

The pseudo-arc

Definition. The pseudo-arc is the unique continuum that is hereditarily indecomposable and arc-like:

$$\forall \varepsilon,\ \exists f : P \twoheadrightarrow [0,1] \text{ continuous },\forall y,\ diam(f^{-1}(y)) < \varepsilon$$

A construction of the pseudoarc:
Fix distinct point $p, q \in \mathbb{R}^2$.
Step 0 Draw a *simple chain* $\mathcal{U}_0 = \{U_{00}, \ldots, U_{0r_0}\}$ from p to q of connected open sets of diameter less than 1. Being a simple chain from p to q means:

- $U_i \cap U_j \Leftrightarrow |i - j| \leq 1$
- $p \in U_{0i} \Leftrightarrow i = 0$
- $q \in U_{0i} \Leftrightarrow i = r_0$

Step k+1 Draw a simple chain $\mathcal{U}_{k+1} = \{U_{k+1,0}, \ldots, U_{k+1,r_{k+1}}\}$ from p to q of connected open sets of diameter less than $\frac{1}{k+2}$ such that

- the closure of each link of $\mathcal{U}_{k+1}$ is contained in some link of $\mathcal{U}_k$
- $\mathcal{U}_{k+1}$ is *crooked* in $\mathcal{U}_k$

This last condition means that for all i, j, m, n, if

$$m + 2 < n, \;\; U_{k+1,i} \cap U_{km} \neq \varnothing, \;\; U_{k+1,j} \cap U_{kn} \neq \varnothing$$

then there are s, t with $i < s < t < j$ or $i > t > l > j$ such that

$$U_{k+1,s} \subseteq U_{k,n-1} \text{ and } U_{k+1,t} \subseteq U_{k,m+1}$$

Final step $P = \bigcap_{k \in \mathbb{N}} \bigcup \mathcal{U}_k$ is the pseudoarc.

The pseudo-arc is a quite complicated continuum. Nevertheless it is the generic continuum: the class of pseudo-arcs is dense G_δ in the space of all continua.

Theorem. (Krupiński) Let P be the pseudo-arc. Then $(P, Homeo(P))$ is a small, not nm-stable, Polish structure.

In particular, the $\mathcal{NM}$-rank of P is ∞.

Moreover P is an example of a small Polish structure not admitting o-independent extensions.

Dendrites

Among simplest continua are dendrites.

A **dendrite** is a locally connected continuum that does not contain simple closed curves.

Definition. Given a point x in a continuum X, its order $ord(x, X)$ is the smallest cardinal β such that x has a basis of open neighbourhoods whose boundaries have cardinality $\leq \beta$.

All points of a dendrite have order $\leq \aleph_0$. Points of order 1 are called end points; points of order ≥ 3 are branching points.

The following property might help to visualise a dendrite:

If X is a non-degenerate dendrite, then

$$X = \bigcup_{i \in \mathbb{N}} A_i \cup E(X)$$

where:

- each A_i is an arc, with end points p_i, q_i
- $A_{i+1} \cap \bigcup_{j=0}^{i} A_j = \{p_{i+1}\}$
- $diam(A_i) \to 0$
- $E(X)$ is the set of end points of X

The goal is to study Polish structures of the form $(D, Homeo(D))$ where D is a dendrite.

Remark. Not all dendrites are small Polish structures. Let $D \subseteq \mathbb{R}^2$ be obtained by starting with $[0,1] \times \{0\}$ as follows:

- enumerate $\{q_n\}_{n \in \mathbb{N}} = (]0,1[\cap \mathbb{Q}) \times \{0\}$
- at step n add n arcs of diameter $\leq \frac{1}{2^n}$ intersecting each other and the already achieved construction only in q_n

Then all point of $]0,1[\times \{0\}$ are in distinct orbits.

Ważewski's universal dendrites

Let $\varnothing \neq J \subseteq \{3, 4 \ldots, \omega\}$.
There is a unique dendrite D_J such that

- each branching point of D_J has order in J
- each subarc of D_J contains points of any order in J

Universality property
D_J is universal for the class of dendrites whose branching points have order in J: any such dendrite embeds in D_J.

Theorem. Each $(D_J, Homeo(D_J))$ is a small Polish structure of $\mathcal{NM}$-rank 1

Conjectures.

- Each dendrite admits *nm*-independent extensions
- If D is a dendrite and $(D, Homeo(D))$ is small, then $\mathcal{NM}(D) = 1$

Some questions.

- Characterise dendrites D such that $(D, Homeo(D))$ is small.
- Find examples of continua C with $1 < \mathcal{N}\mathcal{M}(C) < \infty$.
- The $\mathcal{N}\mathcal{M}$-gap conjecture.

An example of a small Polish structure (X, G) with $\mathcal{NM}(X) = \omega$ can be obtained as a disjoint sum of small Polish structures of increasing natural rank.
E.g., take (Y, G) of rank 1 an let $X = \bigcup_{n \geq 1} X^n$.

However, in this example the is no single orbit over finite sets with rank $\geq \omega$ (and $\neq \infty$).

The $\mathcal{NM}$-gap conjecture. Let (X, G) be a small Polish structure. Then, for any orbit o of a finite set $A \subseteq X$, one has

$$\mathcal{NM}(o) \in \omega \cup \{\infty\}.$$

This conjecture is open in the class of small profinite structures; it has been proved for small m-stable profinite groups. In this wider context it might be easier to find a counterexample.