# Non proper elementary embeddings beyond $L(V_{\lambda+1})$

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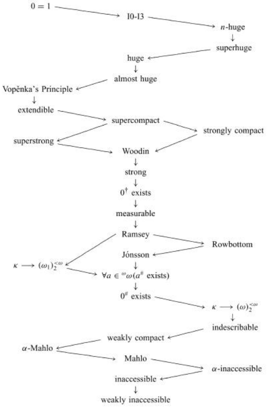
19 June 2009

**Chart of Cardinals**

The arrows indicates direct implications or relative consistency implications, often both.



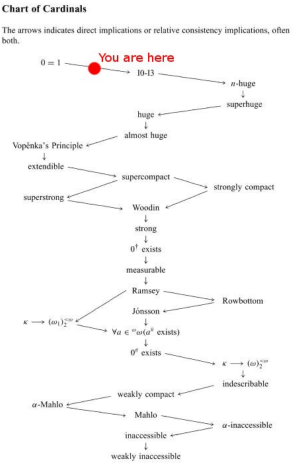


Chart of Cardinals
The arrows indicates direct implications or relative consistency implications, often both.
You are here
0 = 1
I0-I3
n-huge
superhuge
huge
almost huge
Vopěnka's Principle
extendible
supercompact
strongly compact
superstrong
Woodin
strong
0† exists
measurable
Ramsey
Rowbottom
Jónsson
κ ⟶ (ω₁)₂^{<ω}
∀a ∈ ^ωω(a# exists)
0# exists
κ ⟶ (ω)₂^{<ω}
indescribable
weakly compact
α-Mahlo
Mahlo
α-inaccessible
inaccessible
weakly inaccessible

Reinhardt Hypothesis: there exists an elementary embedding $j : V \prec V$.

It's a natural strengthening of the hypothesis with a $j : V \prec M$.

**Theorem (Kunen, 1971)**

If $j : V \prec M$, then $M \neq V$.

The *critical sequence* has an important role in the proof:

**Definition**

$\kappa_0 = crit(j)$, $\kappa_{n+1} = j(\kappa_n)$, $\lambda = \sup_{n\in\omega} \kappa_n$.

Kunen's proof uses a choice function that is in $V_{\lambda+2}$. So

**Corollary**

There is no $j : V_\eta \prec V_\eta$,with $\eta \geq \lambda + 2$.

It is natural to define the following Hypoteses-Axioms, also called rank-to-rank

**Definition**

- I3: There exists an elementary embedding $j : V_\lambda \prec V_\lambda$.
- I1: There exists an elementary embedding $j : V_{\lambda+1} \prec V_{\lambda+1}$.
- I0 (or Woodin's Axiom): There exists an elementary embedding $j : L(V_{\lambda+1}) \prec L(V_{\lambda+1})$ with critical point less than $\lambda$.

The last one was proposed by Woodin to prove the consistency of $AD_{\mathbb{R}}$, but it became obsolete for that purpose.
Nonetheless, I0 leads to interesting results.

Since the cofinality of $\lambda$ is $\omega$, $V_{\lambda+1}$ is quite similar to $V_{\omega+1}$. So $L(V_{\lambda+1})$ is quite similar to $L(\mathbb{R})$, e.g.:

- $L(V_{\lambda+1}) \vDash \mathrm{DC}_\lambda$;
- we can define $\Theta = \sup\{\alpha : \exists\pi : V_{\lambda+1} \twoheadrightarrow \alpha, \pi \in L(V_{\lambda+1})\}$ and it is regular...

Quite surprisingly, I0 is similar to $\mathrm{AD}^{L(\mathbb{R})}$.

- I0 $\rightarrow$ the Coding Lemma is true in $L(V_{\lambda+1})$;
- I0 $\rightarrow$ $\Theta$ is a limit of measurable cardinals...

So, I0 is the first example of what we can call "Higher Determinacy Axiom".

Are there other examples?
Is there a higher correspondent of $\mathrm{AD}^{L(\mathbb{R},X)}$, with $X \subseteq \mathbb{R}$?
Intuitevely, it must be "There is an elementary embedding $j : L(V_{\lambda+1}, X) \prec L(V_{\lambda+1}, X)$, with $X \subseteq V_{\lambda+1}$".
This suffices to prove the Coding Lemma, but there aren't proofs that it implies that the corresponding $\Theta$ is a limit of measurable cardinals.

However, the problem is resolved if we put another condition on the elementary embedding:

**Definition**

$j : L(X, V_{\lambda+1}) \prec L(X, V_{\lambda+1})$ is *proper* if the fixed points of $j$ are cofinal in $\Theta$.

(Actually this is not the original definition of properness, but for the purposes of the talk this is an equivalent definition)

Is there a higher correspondent of $AD_{\mathbb{R}}$?
There is no evident elementary embedding form... so the way chose by Woodin is defining an analogous of the minimum model of $AD_{\mathbb{R}}$.

**Definition**

Define a sequence of $\Gamma_\alpha \subseteq \wp(\mathbb{R})$ by induction on $\alpha$:

- $\Gamma_0 = L(\mathbb{R}) \cap \wp(\mathbb{R})$;
- If $\alpha$ is a limit ordinal then $\Gamma_\alpha = L((\bigcup_{\beta<\alpha} \Gamma_\beta)^\omega) \cap \wp(\mathbb{R})$;
- If $\mathrm{cof}(\Theta^{L(\Gamma_\alpha)}) = \omega$, then $\Gamma_{\alpha+1} = L((\Gamma_\alpha)^\omega, \mathbb{R}) \cap \wp(\mathbb{R})$, otherwise $\Gamma_{\alpha+1} = L(\Gamma_\alpha)[\mathcal{F}] \cap \wp(\mathbb{R})$, where $\mathcal{F}$ is the $\omega$-club filter in $\Theta^{L(\Gamma_\alpha)}$.

The sequence stops when $L(\Gamma_\alpha) \nvDash \mathrm{AD}$ or $\Gamma_\alpha = \Gamma_{\alpha+1}$

So, Woodin defined a sequence $\langle E^0_\alpha : \alpha < \Upsilon \rangle$ such that

- $V_{\lambda+1} \subset E^0_\alpha \subset V_{\lambda+2}$;
- if $\beta < \alpha$ then $E^0_\beta \subset E^0_\alpha$;
- $E^0_0 = L(V_{\lambda+1}) \cap V_{\lambda+2}$;
- for $\alpha$ limit, $E^0_\alpha = L(\bigcup_{\beta<\alpha} E^0_\beta) \cap V_{\lambda+2}$;
- for every $\alpha$ there exists $X \subseteq V_{\lambda+1}$ such that $L(E^0_{\alpha+1}) = L(X, V_{\lambda+1})$;
- $E^0_{\alpha+2} = L((X, V_{\lambda+1})^\sharp) \cap V_{\lambda+2}$;
- for every $\alpha < \Upsilon$ there exists an elementary embedding $j : L(E^0_\alpha) \prec L(E^0_\alpha)$;
- the sequence $E_\alpha$ has absoluteness properties.

In this definition new kinds of elementary embedding appear, i.e $j : L(E) \prec L(E)$, with $V_{\lambda+1} \subset E \subset V_{\lambda+2}$ and $L(E) \cap V_{\lambda+2} = E$.
This sequence creates a whole new playground, where the main characters are:

$$E^0_\alpha \quad \Theta^{L(E^0_\alpha)} \quad (E^0_\alpha)^\sharp$$

and their correlation, expecially at limit points. Examples:

- If $E^0_\beta = \bigcup_{\gamma<\beta} E^0_\beta$, then $\Theta^{E^0_\beta} = \sup_{\gamma<\beta} \Theta^{E^0_\gamma}$.
- If $L(E^0_\beta) = L(X, V_{\lambda+1})$, then $(E^0_\beta)^\sharp$ has no predecessor.

**Lemma (Woodin)**

Let $\eta < \Upsilon_{V_{\lambda+1}}$ be a limit ordinal. If $\Theta^{E^0_\eta} > \sup_{\beta<\eta} \Theta^{E^0_\beta}$, then there exists $Y \in E^0_\eta$ such that $L(E^0_\eta) = L(Y, V_{\lambda+1})$.

This correlations are more significant when $L(E^0_\beta) \vDash V = HOD_{V_{\lambda+1}}$, i.e in an initial segment fo $\Upsilon$.
Examples:

- $\Theta^{E^0_\beta}$ is regular.
- If $E^0_\beta = \bigcup_{\gamma<\beta} E^0_\beta$, then $\beta = \Theta^{E^0_\beta}$.
- (Woodin) If $j : L(E^0_\beta) \prec L(E^0_\beta)$ is proper, then the Coding Lemma holds and $\Theta$ is limit of measurables.

We can extend the definition of proper to this embeddings: $j$ is proper if the fixed points of $j$ are cofinal in $\Theta$.
Is this definition really relevant? Is it possible that all the elementary embeddings are proper?
Fact: if $\alpha$ is a successor ordinal or a limit ordinal with cofinality $>\omega$, every embedding is proper;
Fact 2: if we have $j: L(X, V_{\lambda+1}) \prec L(X, V_{\lambda+1})$ and $k \supset j \upharpoonright L(X, V_{\lambda+1}) \cap V_{\lambda+2}$, $k: L((X, V_{\lambda+1})^\sharp) \prec L((X, V_{\lambda+1})^\sharp)$, then $k$ is proper.

**Theorem 1**

Let $\alpha$ be the least such that $L((E^0_\alpha)^\sharp) \cap V_{\lambda+2} = E^0_\alpha$. Then there exists an elementary embedding $j: L(E^0_\alpha) \prec L(E^0_\alpha)$ that is not proper.

The fundamental property of such $\alpha$ is that $\alpha = \Theta^{L(E^0_\alpha)} = \Theta^{L((E^0_\alpha)^\sharp)}$, so this provides a model, $L((E^0_\alpha)^\sharp)$ that is big enough to "know" deeply $L(E^0_\alpha)$, but such that $\alpha$ is not too small in it.
Another important consideration is that even if $(E^0_\gamma)^\sharp \notin L(E^0_\gamma)$, its fragments are in $E^0_\gamma$, so if we have an elementary embedding from $E^0_\alpha$ to itself that conserves the fragments ($\sharp$-friendly?), it can be easily lifted to $L(E^0_\alpha)$.
In a big enough model, we can treat elementary embeddings as sets.

The proof of Theorem 1 uses this game:

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| $I$ | $k_0$ | | $k_1$ | | $k_2$ | | |
| | | | | | | | $\dots$ |
| $II$ | | $\eta_0$ | | $\eta_1$ | | | |

where the $k$s are $\sharp$-friendly elementary embeddings from $E^0_{\beta_i}$ to $E^0_{\beta_{i+1}}$, $\beta_i < \eta_1 < \beta_{i+1}$ and $k_i \subseteq k_{i+1}$.
In $L((E^0_\alpha)^\sharp)$ I has a winning strategy.

### Theorem

Let $\alpha$ be the least such that $L((E^0_\alpha)^\sharp) \cap V_{\lambda+2} = E^0_\alpha$. Then there exists an elementary embedding $j : L(E^0_\alpha) \prec L(E^0_\alpha)$ that is proper.

There are two proofs of that. One can use $j$ from $L((E^0_\alpha)^{\sharp\sharp})$ to itself or we can use again the game.

### Theorem 2

Let $\alpha$ be such that $\{\gamma < \alpha : (E^0_\gamma)^\sharp \subseteq (E^0_\alpha)^\sharp\}$ has ordertype $\lambda$. Then every $j : L(E^0_\alpha) \prec L(E^0_\alpha)$ is not proper.

We call $\alpha$ the ordinal from Theorem 1 and $\beta$ the least one between those from Theorem 2

- $\alpha > \beta$
- If $j, k : L(E^0_\beta) \prec L(E^0_\beta)$ agree upon $V_{\lambda+1}$ and the indiscernibles, than they are equal.

- Is it possible to use the game from Theorem 1 to prove other things? E.g. there are $2^\lambda$ possible elementary embeddings from $L(E^0_\alpha)$ to itself that agree on $V_{\lambda+1}$, or there are two elementary embeddings with no fixed points in common.
- Is the definition of proper relevant for the elementary embeddings between $L(X, V_{\lambda+1})$?
- Is there a value of $\Upsilon$ that is inconsistent?