Tukey Degrees of Ultrafilters

Natasha Dobrinen

University of Denver

joint work with

Stevo Todorcevic

University of Toronto
Université Paris VII

Def. $\mathcal{U} \le_T \mathcal{V}$ iff there is a *Tukey* map $g : \mathcal{U} \to \mathcal{V}$ taking unbounded subsets of $\mathcal{U}$ to unbounded subsets of $\mathcal{V}$.

Equivalently, $\mathcal{U} \le_T \mathcal{V}$ iff there is a *cofinal* map $f : \mathcal{V} \to \mathcal{U}$ taking cofinal subsets of $\mathcal{U}$ to cofinal subsets of $\mathcal{V}$.

$\mathcal{U} \equiv_T \mathcal{V}$ iff $\mathcal{U} \le_T \mathcal{V}$ and $\mathcal{V} \le_T \mathcal{U}$.

Fact. $\equiv_T$ is an equivalence relation. $\le_T$ is a partial ordering on the equivalence classes.

Motivations

1. A special class of directed systems of size $\mathfrak{c}$.

2. $\mathcal{V} \geq_{RK} \mathcal{U}$ implies $\mathcal{V} \geq_{T} \mathcal{U}$.

What is the structure of Tukey degrees of ultrafilters on ω?

[Isbell 65] There is an ultrafilter $\mathcal{U}_{top} \equiv_T [\mathfrak{c}]^{<\omega}$.

Note: $\mathcal{V} \equiv_T [\mathfrak{c}]^{<\omega}$ iff $\neg(\forall S \in [\mathcal{V}]^{\mathfrak{c}}\ \exists T \in [S]^{\omega}\ (\bigcap T \in \mathcal{V}))$.

Question. [Isbell 65] Is there always (in ZFC) an ultrafilter $\mathcal{U}$ such that $\mathcal{U} <_T \mathcal{U}_{top}$?

Note: $\mathcal{V} \equiv_T [\mathfrak{c}]^{<\omega}$ iff $\neg(\forall S \in [\mathcal{V}]^{\mathfrak{c}}\ \exists T \in [S]^{\omega}\ (\bigcap T \in \mathcal{V}))$.

Def. [Solecki/Todorcevic 04] An ultrafilter $\mathcal{V}$ is *basic* if each convergent sequence has a bounded subsequence.

Fact. Each basic ultrafilter does not have top Tukey degree.

Note: $\mathcal{V} \equiv_T [\mathfrak{c}]^{<\omega}$ iff $\neg(\forall S \in [\mathcal{V}]^{\mathfrak{c}}\ \exists T \in [S]^{\omega}\ (\bigcap T \in \mathcal{V}))$.

Def. An ultrafilter $\mathcal{V}$ is *basic* if each convergent sequence has a bounded subsequence.

Fact. A basic ultrafilter is does not have top Tukey degree.

Thm. An ultrafilter is basic iff it is a p-point.

Are there Tukey non-top ultrafilters which are not p-points?

Def. $\mathcal{U}$ is *basically generated* if there is a filter base $\mathcal{B} \subseteq \mathcal{U}$ $(\forall X \in \mathcal{U}\ \exists Y \in \mathcal{B}\ Y \subseteq X)$ such that whenever $A, A_n \in \mathcal{B}$ and $A_n \to A$, then there is a subsequence such that $\bigcap_{k<\omega} A_{n_k} \in \mathcal{U}$.

Fact. A basically generated ultrafilter is not Tukey top.

Thm. If $\mathcal{U}, \mathcal{U}_n$ are p-points, then $\lim_{n\to\mathcal{U}} \mathcal{U}_n$ is basically generated (but not a p-point).

We now focus on the structure of Tukey degrees of p-points and ultrafilters below them.

Key Theorem. If $\mathcal{U}$ is a p-point and $\mathcal{U} \geq_T \mathcal{V}$, then there is a continuous monotone map $f : \mathcal{P}(\omega) \to \mathcal{P}(\omega)$ such that $f \restriction \mathcal{U} : \mathcal{U} \to \mathcal{V}$ is a cofinal map.

Note: f is definable from its values on the Fréchet filter.

Thm. Every family of p-points of cardinality $> \mathfrak{c}^+$ contains a subfamily of equal size of pairwise Tukey incomparable p-points.

Thm. Every $\leq_T$ chain of p-points has cardinality $\leq \mathfrak{c}^+$.

Thm. If $\mathcal{U} \geq_T \mathcal{V}$ and $\mathcal{U}$ is selective, then $\mathcal{V}$ is basically generated.

Comparing with ω^ω.

Fact. If $\mathcal{U}$ is rapid, then $\mathcal{U} \geq_T \omega^\omega$.

Fact. For each ultrafilter $\mathcal{U}$, $\mathcal{U} \cdot \mathcal{U} \geq_T \omega^\omega$.

Fact. If $\mathcal{U}$ is a p-point, then $\mathcal{U}^\omega \equiv_T \mathcal{U} \times \omega^\omega$.

Thm. If $\mathcal{U}$ is a p-point, then $\mathcal{U}^\omega \geq_T \mathcal{U} \cdot \mathcal{U}$.

Thm. The following are equivalent for a p-point $\mathcal{U}$

1. $\mathcal{U} \geq_T \omega^\omega$;
2. $\mathcal{U} \equiv_T \mathcal{U} \cdot \mathcal{U}$;
3. $\mathcal{U} \equiv_T \mathcal{U}^\omega$.

Cor. If $\mathcal{U}$ is a rapid ultrafilter then $\mathcal{U} \cdot \mathcal{U} \equiv_T \mathcal{U}$.

Cor. If $\mathcal{U}$ is a p-point and $\mathcal{U} \geq_T \omega^\omega$, then $\mathcal{U} \cdot \mathcal{U} \equiv_T \mathcal{U}$.

Cor. If $\mathcal{U}$ is a p-point of cofinality $< \mathfrak{d}$, then $\mathcal{U} \not\geq_T \omega^\omega$ and therefore $\mathcal{U} \cdot \mathcal{U} >_T \mathcal{U}$.

Thm. Assuming $\mathfrak{p} = \mathfrak{c}$, there is a p-point $\mathcal{U}$ such that $\mathcal{U} \not\geq_T \omega^\omega$ and therefore $\mathcal{U} <_T \mathcal{U} \cdot \mathcal{U} <_T \mathcal{U}_{top}$.

Antichains

Thm. 1. If $\mathrm{cov}(\mathcal{M}) = \mathfrak{c}$, and $2^{<\kappa} = \mathfrak{c}$, then there are 2^{κ} pairwise incomparable selective ultrafilters.

2. If $\mathfrak{d} = \mathfrak{u} = \mathfrak{c}$ and $2^{<\kappa} = \mathfrak{c}$, then there are 2^{κ} pairwise incomparable p-points.

Chains

[Kunen 78] If $\mathcal{U}$ is κ-OK and $\kappa > \mathrm{cof}(\mathcal{U})$, then $\mathcal{U}$ is a p-point.

[Milovich 08] $\mathcal{U}$ is a p-point iff it is $\mathfrak{c}$-OK and not Tukey top.

Fact. If $\mathcal{U}$ is κ-OK but not a p-point, then $\mathcal{U} \geq_T [\kappa]^{<\omega}$. Hence, if $\mathcal{U}$ is κ-OK but not a p-point, then $\mathrm{cof}(\mathcal{U}) = \kappa$ iff $\mathcal{U} \equiv_T [\kappa]^{<\omega}$.

If there are κ-OK non p-points with cofinality κ for each uncountable $\kappa < \mathfrak{c}$, then there is a strictly increasing chain of ultrafilters of length α, where α is such that $\aleph_\alpha = \mathfrak{c}$.

Thm. (also independently by Dilip Raghavan) CH implies for each p-point $\mathcal{U}$ there is a p-point $\mathcal{V}$ such that $\mathcal{V} >_T \mathcal{U}$.

Cor. (CH) There is a Tukey strictly increasing chain of p-points of length ω_1.

Question. Is it true that there is a p-point Tukey above ANY Tukey strictly increasing chain of p-points?

Incomparable Predecessors

Thm. (MA) There is a p-point with 2 Tukey incomparable predecessors, each of which is also a p-point.

Thm. (CH) There is a block-basic ultrafilter $\mathcal{U}$ on FIN such that $\mathcal{U}_{\text{min}}$ and $\mathcal{U}_{\text{max}}$ are Tukey incomparable selective ultrafilters, and $\mathcal{U} >_T \mathcal{U}_{\text{min,max}} >_T \mathcal{U}_{\text{min}}, \mathcal{U}_{\text{max}}$.

Some Open Problems

1. Is it true in ZFC that there is an ultrafilter $\mathcal{U} <_T \mathcal{U}_{top}$ [Isbell]?

2. Is Tukey equivalent to Rudin-Keisler for selective ultrafilters?

3. Assuming the consistency of a supercompact cardinal, is there is a block-basic ultrafilter $\mathcal{U}$ on FIN such that there are exactly 5 Tukey degrees in $L(\mathbb{R})[\mathcal{U}]$?

4. Is there an ultrafilter $\mathcal{U}$ on ω such that $\mathcal{U} <_T \mathcal{U}\cdot\mathcal{U} <_T \mathcal{U}\cdot\mathcal{U}\cdot\mathcal{U}$?

5. What properties are preserved Tukey downwards?