Tukey Degrees of Ultrafilters

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joint work with

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Def. \( \mathcal{U} \leq_T \mathcal{V} \) iff there is a *Tukey map* \( g : \mathcal{U} \rightarrow \mathcal{V} \) taking unbounded subsets of \( \mathcal{U} \) to unbounded subsets of \( \mathcal{V} \).

Equivalently, \( \mathcal{U} \leq_T \mathcal{V} \) iff there is a *cofinal map* \( f : \mathcal{V} \rightarrow \mathcal{U} \) taking cofinal subsets of \( \mathcal{U} \) to cofinal subsets of \( \mathcal{V} \).

\( \mathcal{U} \equiv_T \mathcal{V} \) iff \( \mathcal{U} \leq_T \mathcal{V} \) and \( \mathcal{V} \leq_T \mathcal{U} \).

**Fact.** \( \equiv_T \) is an equivalence relation. \( \leq_T \) is a partial ordering on the equivalence classes.
Motivations

1. A special class of directed systems of size $\mathfrak{c}$.

2. $\mathcal{V} \geq_{RK} \mathcal{U}$ implies $\mathcal{V} \geq_{T} \mathcal{U}$. 
What is the structure of Tukey degrees of ultrafilters on $\omega$?
There is an ultrafilter \( U_{\text{top}} \equiv_T [\mathcal{C}]^{<\omega} \).

Note: \( \mathcal{V} \equiv_T [\mathcal{C}]^{<\omega} \) iff \( \neg (\forall S \in [\mathcal{V}]^c \exists T \in [S]^\omega (\bigcap T \in \mathcal{V})) \).

**Question.** [Isbell 65] Is there always (in ZFC) an ultrafilter \( U \) such that \( U <_T U_{\text{top}} \)?
Note: $\mathcal{V} \equiv_T [c]^\omega$ iff $\neg (\forall S \in [\mathcal{V}]^c \exists T \in [S]^\omega (\cap T \in \mathcal{V}))$.

**Def.** [Solecki/Todorcevic 04] An ultrafilter $\mathcal{V}$ is *basic* if each convergent sequence has a bounded subsequence.

**Fact.** Each basic ultrafilter does not have top Tukey degree.
Note: $\mathcal{V} \equiv_T [c]^{<\omega}$ iff $\neg(\forall S \in [\mathcal{V}]^c \exists T \in [S]^\omega (\cap T \in \mathcal{V}))$.

**Def.** An ultrafilter $\mathcal{V}$ is *basic* if each convergent sequence has a bounded subsequence.

**Fact.** A basic ultrafilter is does not have top Tukey degree.

**Thm.** An ultrafilter is basic iff it is a p-point.
Are there Tukey non-top ultrafilters which are not p-points?
**Def.** \( U \) is *basically generated* if there is a filter base \( B \subseteq U \) \((\forall X \in U \ \exists Y \in B \ Y \subseteq X)\) such that whenever \( A, A_n \in B \) and \( A_n \to A \), then there is a subsequence such that \( \bigcap_{k<\omega} A_{n_k} \in U \).

**Fact.** A basically generated ultrafilter is not Tukey top.

**Thm.** If \( U, U_n \) are p-points, then \( \lim_{n \to U} U_n \) is basically generated (but not a p-point).
We now focus on the structure of Tukey degrees of $p$-points and ultrafilters below them.
Key Theorem. If $\mathcal{U}$ is a p-point and $\mathcal{U} \geq_T \mathcal{V}$, then there is a continuous monotone map $f : \mathcal{P}(\omega) \to \mathcal{P}(\omega)$ such that $f \upharpoonright \mathcal{U} : \mathcal{U} \to \mathcal{V}$ is a cofinal map.

Note: $f$ is definable from its values on the Fréchet filter.
**Thm.** Every family of p-points of cardinality $> c^+$ contains a subfamily of equal size of pairwise Tukey incomparable p-points.

**Thm.** Every $\leq_T$ chain of p-points has cardinality $\leq c^+$.

**Thm.** If $\mathcal{U} \geq_T \mathcal{V}$ and $\mathcal{U}$ is selective, then $\mathcal{V}$ is basically generated.
Comparing with $\omega^\omega$.

**Fact.** If $\mathcal{U}$ is rapid, then $\mathcal{U} \geq_T \omega^\omega$.

**Fact.** For each ultrafilter $\mathcal{U}$, $\mathcal{U} \cdot \mathcal{U} \geq_T \omega^\omega$.

**Fact.** If $\mathcal{U}$ is a p-point, then $\mathcal{U}^\omega \equiv_T \mathcal{U} \times \omega^\omega$.

**Thm.** If $\mathcal{U}$ is a p-point, then $\mathcal{U}^\omega \geq_T \mathcal{U} \cdot \mathcal{U}$.

**Thm.** The following are equivalent for a p-point $\mathcal{U}$

1. $\mathcal{U} \geq_T \omega^\omega$;
2. $\mathcal{U} \equiv_T \mathcal{U} \cdot \mathcal{U}$;
3. $\mathcal{U} \equiv_T \mathcal{U}^\omega$. 
Cor. If $\mathcal{U}$ is a rapid ultrafilter then $\mathcal{U} \cdot \mathcal{U} \equiv_T \mathcal{U}$.

Cor. If $\mathcal{U}$ is a p-point and $\mathcal{U} \geq_T \omega^\omega$, then $\mathcal{U} \cdot \mathcal{U} \equiv_T \mathcal{U}$.

Cor. If $\mathcal{U}$ is a p-point of cofinality $< \varnothing$, then $\mathcal{U} \not\geq_T \omega^\omega$ and therefore $\mathcal{U} \cdot \mathcal{U} >_T \mathcal{U}$.

Thm. Assuming $p = c$, there is a p-point $\mathcal{U}$ such that $\mathcal{U} \not\geq_T \omega^\omega$ and therefore $\mathcal{U} <_T \mathcal{U} \cdot \mathcal{U} <_T \mathcal{U}_{top}$. 
Antichains

**Thm.** 1. If $\text{cov}(\mathcal{M}) = c$, and $2^{<\kappa} = c$, then there are $2^\kappa$ pairwise incomparable selective ultrafilters.

2. If $\mathfrak{d} = \mathfrak{u} = c$ and $2^{<\kappa} = c$, then there are $2^\kappa$ pairwise incomparable $p$-points.
Chains

[Kunen 78] If \( \mathcal{U} \) is \( \kappa \)-OK and \( \kappa > \text{cof}(\mathcal{U}) \), then \( \mathcal{U} \) is a p-point.

[Milovich 08] \( \mathcal{U} \) is a p-point iff it is \( \mathfrak{c} \)-OK and not Tukey top.

**Fact.** If \( \mathcal{U} \) is \( \kappa \)-OK but not a p-point, then \( \mathcal{U} \geq_T [\kappa]^{<\omega} \). Hence, if \( \mathcal{U} \) is \( \kappa \)-OK but not a p-point, then \( \text{cof}(\mathcal{U}) = \kappa \) iff \( \mathcal{U} \equiv_T [\kappa]^{<\omega} \).

If there are \( \kappa \)-OK non p-points with cofinality \( \kappa \) for each uncountable \( \kappa < \mathfrak{c} \), then there is a strictly increasing chain of ultrafilters of length \( \alpha \), where \( \alpha \) is such that \( \aleph_\alpha = \mathfrak{c} \).
Thm. (also independently by Dilip Raghavan) CH implies for each p-point $\mathcal{U}$ there is a p-point $\mathcal{V}$ such that $\mathcal{V} >_T \mathcal{U}$.

Cor. (CH) There is a Tukey strictly increasing chain of p-points of length $\omega_1$.

Question. Is it true that there is a p-point Tukey above ANY Tukey strictly increasing chain of p-points?
Incomparable Predecessors

**Thm. (MA)** There is a p-point with 2 Tukey incomparable predecessors, each of which is also a p-point.

**Thm. (CH)** There is a block-basic ultrafilter $\mathcal{U}$ on $\text{FIN}$ such that $\mathcal{U}_{\min}$ and $\mathcal{U}_{\max}$ are Tukey incomparable selective ultrafilters, and $\mathcal{U} >_T \mathcal{U}_{\min,\max} >_T \mathcal{U}_{\min} \cup \mathcal{U}_{\max}$. 
Some Open Problems

1. Is it true in ZFC that there is an ultrafilter $\mathcal{U} <_T \mathcal{U}_{top}$ [Isbell]?

2. Is Tukey equivalent to Rudin-Keisler for selective ultrafilters?

3. Assuming the consistency of a supercompact cardinal, is there a block-basic ultrafilter $\mathcal{U}$ on FIN such that there are exactly 5 Tukey degrees in $L(\mathbb{R})[\mathcal{U}]$?

4. Is there an ultrafilter $\mathcal{U}$ on $\omega$ such that $\mathcal{U} <_T \mathcal{U} \cdot \mathcal{U} <_T \mathcal{U} \cdot \mathcal{U} \cdot \mathcal{U}$?

5. What properties are preserved Tukey downwards?