Nonseparable UHF algebras
(or: Graphs, groups, and noncommutative tori)

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ESI, June 19, 2009
**Hilbert space**

\[ H: \text{ a complex Hilbert space} \]

\[ \mathcal{B}(H), +, \cdot, *, \| \cdot \|): \text{ the algebra of bounded linear operators on } H \]

**Definition**

A (concrete) \( C^* \)-algebra is a norm-closed subalgebra of \( \mathcal{B}(H) \).

**Theorem (Gelfand–Naimark–Segal)**

A Banach algebra with involution \( A \) is isomorphic to a concrete \( C^* \)-algebra if and only if

\[ \| aa^* \| = \| a \|^2 \]

for all \( a \in A \).
The simplest C*-algebras

\[ \mathcal{B}(H) \]

\[ M_n(\mathbb{C}), \text{ for } n \in \mathbb{N}. \]

Full matrix algebras
(Unital) embeddings

\[ M_2(\mathbb{C}) \hookrightarrow M_4(\mathbb{C}) \]

via

\[
\begin{pmatrix}
a_{11} & a_{12} \\
a_{21} & a_{22}
\end{pmatrix}
\mapsto
\begin{pmatrix}
a_{11} & a_{12} & 0 & 0 \\
a_{21} & a_{22} & 0 & 0 \\
0 & 0 & a_{11} & a_{12} \\
0 & 0 & a_{21} & a_{22}
\end{pmatrix}
\]

or, in short

\[ a \mapsto a \otimes \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \]

Fact

*All embeddings between C*-algebras are norm-preserving.*
CAR (Fermion) algebra: ‘the $E_0$ of C*-algebras’

$$M_2(\mathbb{C}) \hookrightarrow M_4(\mathbb{C}) \hookrightarrow M_8(\mathbb{C}) \hookrightarrow M_{16}(\mathbb{C}) \hookrightarrow \ldots$$

$$M_{2\infty}(\mathbb{C}) = \lim_{\longrightarrow} M_{2^n}(\mathbb{C}) = \bigotimes_{n \in \mathbb{N}} M_2(\mathbb{C}).$$

(where $\lim_{\longrightarrow}$ means ‘completion of the direct limit.’)
Uniformly Hyperfinite algebras, Approximately Matricial algebras and Locally Matricial algebras

Definition

1. $A$ is UHF if $A$ is a tensor product of full matrix algebras.
2. $A$ is AM if $A$ is a direct limit of full matrix algebras.
3. $A$ is LM if $\forall \varepsilon > 0$ and for every finite $F \subseteq A$ there is a full matrix algebra $M \subseteq A$ such that $F \subseteq_{\varepsilon} M$. 
The question

Theorem (J. Glimm)

*If A is separable and unital then*

\[ UHF \iff AM \iff LM \]

Question (J. Dixmier)

*If A is unital, does*

\[ UHF \iff AM \iff LM? \]

We have a complete answer to the problem, but I will concentrate on AM vs. UHF in this talk.
Characterizing $M_2(\mathbb{C})$

\[
\begin{align*}
    u &= \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} & v &= \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \\
    u^2 &= 1 & v^2 &= 1 \\
    uv &= \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} & vu &= \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}
\end{align*}
\]

**Lemma**

*A is isomorphic to $M_2(\mathbb{C})$ if and only if $A = C^*(\{u, v\})$, with $u^2 = v^2 = 1$ and $uv = -vu$.*

**Proof.**

($\Leftarrow$) $A$ is a linear span of $u, v, uv$, and 1.

The only noncommutative $C^*$-algebra that is 4-dimensional as a vector space is $M_2(\mathbb{C})$. $\square$
Graphs and noncommutative tori

\[ M_2(\mathbb{C}) \]

means

\[ uv = -vu \]

(and \( u^2 = v^2 = 1 \)).
More graphs and noncommutative tori

\[ uv = vu \]

(And \( u^2 = v^2 = 1 \)).
Example 2

Which $C^*$-algebra is coded by the following graph?

\[
\begin{array}{c}
\bullet_{v_1} \\
\downarrow \\
\bullet_{u_1} \\
\end{array}
\quad
\begin{array}{c}
\bullet_{v_2} \\
\downarrow \\
\bullet_{u_2} \\
\end{array}
\]

\[M_4(\mathbb{C})\]

means

\[M_2(\mathbb{C}) \otimes M_2(\mathbb{C})\]
Example # 3

Which C*-algebra is coded by the following graph?

\[ M_4(\mathbb{C}) \]
Examples \#4—\(\omega + 1\)

Let's denote this algebra by \(B(\kappa)\), where \(\kappa = |G|\).

\[M_4(\mathbb{C})M_8(\mathbb{C})M_{16}(\mathbb{C})M_{2\infty}(\mathbb{C}) — the\text{CAR algebra}\]
Analysis of $B(\aleph_0)$

So we have proved

**Lemma**

$B(\aleph_0)$ is isomorphic to $M_{2\alpha}(\mathbb{C})$.  
For every infinite $\kappa$, $B(\kappa)$ is AM.
Relative commutant

Definition
If A is a subalgebra of B, let

$$Z_B(A) = \{ b \in B : ab = ba \text{ for all } a \in A \}.$$  

Let $Z(A) = Z_A(A)$.

Fact
1. if A is LM (or AM, or UHF) then $Z(A) = \mathbb{C}I$.
2. $Z_{A \otimes B}(A) \supseteq B$. 
Complemented subalgebras

A subalgebra $A$ of $B$ is complemented in $B$ if

$$C^*(A, Z_B(A)) = B.$$ 

Note that $A$ is complemented in $A \otimes C$.

**Lemma**

*If $A$ is UHF then club many of its separable subalgebras are complemented.*

**Theorem (Farah–Katsura)**

$B(\kappa)$ is AM but not UHF if $\kappa$ is uncountable.

**Proof.**

Club many of its separable subalgebras are not complemented.  \(\square\)
Modifying $M_{2^{\aleph_1}}(\mathbb{C})$ further

$M_{2^{\aleph_1}}(\mathbb{C})$

For $S \subseteq \aleph_1$ let $B(S)$ be given by the graph with vertices

$$\{ u_\gamma, v_\gamma : \gamma < \aleph_1 \} \cup \{ w_\gamma : \gamma \in S \}.$$
Many AM algebras

Theorem (Farah–Katsura)
If $B(S) \cong B(T)$ then $S \Delta T \in NS_{\omega_1}$.

Corollary
There are $2^{\aleph_1}$ nonisomorphic AM algebras of character density $\aleph_1$ for any uncountable regular $\kappa$.

UHF algebras can be classified
\[ \bigotimes_{\kappa_2} M_2(\mathbb{C}) \otimes \bigotimes_{\kappa_3} M_3(\mathbb{C}) \otimes \bigotimes_{\kappa_5} M_5(\mathbb{C}) \otimes \bigotimes_{\kappa_7} M_7(\mathbb{C}) \otimes \ldots \]

so there are only $2^{\aleph_0}$ in character density $\aleph_{\omega_1}$.
Theorem (Farah–Katsura)

AM \nRightarrow UHF in any uncountable character density.

LM \Leftrightarrow AM in character density \leq \aleph_1.

LM \nRightarrow AM in character density \geq \aleph_2.
Question (M. Takesaki, 2008)

What about C*-algebras faithfully represented on a separable Hilbert space?

Does \( \text{LM} \Rightarrow \text{AM} \) in this case?

A nonseparable UHF algebra cannot be faithfully represented on a separable Hilbert space.
An isomorphic embedding

\[ A \xrightarrow{\pi} \mathcal{B}(H) \]

\( \pi \) is an *irreducible representation* (*irrep*) if \( H \) has no nontrivial closed subspace invariant for \( \pi[A] \).
Theorem (Kishimoto–Ozawa–Sakai, 2003)

Assume $A$ is simple and separable. Then its space of irreps is homogeneous: For all irreps $\pi_1, \pi_2$ there exist automorphisms $\alpha, \beta$ such that

\[
\begin{array}{c}
A \xrightarrow{\pi_1} B(H_1) \\
\downarrow \alpha \\
A \xrightarrow{\pi_2} B(H_1)
\end{array}
\]

\[
\begin{array}{c}
\downarrow \beta
\end{array}
\]

commutes.

There is a nonseparable simple algebra with nonhomogeneous space of irreps.
The class of nuclear C*-algebras is the most studied class of C*-algebras. All LM algebras are nuclear.

**Question (Kishimoto–Ozawa–Sakai, 2003)**

Assume $A$ is simple and nuclear. Is its space of irreps homogeneous?

A positive answer would imply that $\diamondsuit_\kappa$ implies there is a counterexample to Naimark’s problem.

(The case $\kappa = \aleph_1$ is a theorem of Akemann–Weaver.)
More graphs

For $\mathbb{A} \subseteq 2^\kappa$ define a bipartite graph $G = G(\kappa, \mathbb{A})$ and a the corresponding C*-algebra $B(\kappa, \mathbb{A})$.

$$V(G) = \kappa \cup \mathbb{A}.$$ 

For $i \in \kappa$ and $x \in \mathbb{A}$ let $u_i$ and $v_x$ be adjacent if $i \in x$.

$x = \{1, 2, 4\}.$

Lemma

If $\mathbb{A}$ is dense in $2^\kappa$ and independent, then $B(\kappa, \mathbb{A})$ is AM and it has a faithful irreducible representation on $\ell_2(\kappa)$. 
An answer to a question that Takesaki did not ask

Theorem (Farah–Katsura)

\[ AM \not\Rightarrow UHF \text{ for separably represented } C^*\text{-algebras.} \]

Proof.
Take \( M(\mathbb{N}, \mathbb{A}) \) for an uncountable dense independent family \( \mathbb{A} \subseteq 2^\mathbb{N} \). It is nonseparable, AM, and has a faithful irreducible representation on \( \ell_2(\mathbb{N}) \). So it cannot be UHF.

Proposition (Farah–Katsura)

CH implies that for separably represented algebras LM implies AM.
The dual of $\mathcal{A}$

For $\mathcal{A}$ define the dual $\hat{\mathcal{A}} = \{ y_i : i \in \kappa \} \subseteq 2^\mathcal{A}$ by

$$x \in y_i \iff i \in x$$

Lemma

$B(\kappa, \mathcal{A}) \cong B(\mathcal{A}, \hat{\mathcal{A}})$.

Lemma

1. $\hat{\hat{\mathcal{A}}} = \mathcal{A}$.
2. $\mathcal{A}$ is dense iff $\hat{\mathcal{A}}$ is independent.
3. $\mathcal{A}$ is independent iff $\hat{\mathcal{A}}$ is dense.
Theorem (Farah)

There is an AM (therefore simple nuclear) $C^*$-algebra that has nonhomogeneous space of irreps.

Proof.
Take a dense, independent $A \subseteq 2^\mathbb{N}$ of cardinality $2^{\aleph_0}$. Then $B(\mathbb{N}, A) \cong B(2^{\aleph_0}, \hat{A})$ has irreps on $\ell_2(\mathbb{N})$ and on $\ell_2(2^{\aleph_0})$. \qed
Recall that the UHF algebras can be classified

\[ M_2(\mathbb{C}) \otimes M_3(\mathbb{C}) \otimes M_5(\mathbb{C}) \otimes M_7(\mathbb{C}) \otimes \ldots \]
An embarrassing open problem.

Question

Does $\bigotimes_{\kappa_1} M_2(\mathbb{C})$ embed into $\bigotimes_{\kappa_0} M_2(\mathbb{C}) \otimes \bigotimes_{\kappa_1} M_3(\mathbb{C})$ for all $\kappa$?

(The answer is ‘no’ for smaller cardinals.)