The Descriptive Complexity of Free Bernoulli Subflows

Su Gao

Department of Mathematics
University of North Texas

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This is joint work with Steve Jackson and Brandon Seward.

A coloring property for countable groups, to appear in the Mathematical Proceedings of the Cambridge Philosophical Society.

Group colorings and Bernoulli subflows, in preparation.

Part I: Definitions and Problems

In this first part we give the basic definitions for group colorings and raise the main problems about their descriptive complexity.

Free Bernoulli Subflows

Let G be a countable group.

Bernoulli G-flow: the G-space $2^G = \{0,1\}^G$ with the shift action

$$(g \cdot x)(h) = x(g^{-1}h)$$

subflow: closed invariant subset of 2^G

free subflow: closed invariant subset of $F(G)$, the free part of 2^G

Theorem (GJS)
For every countably infinite group G there exists a free Bernoulli subflow.

Constructing free subflows

$\iff$

constructing $x \in 2^G$ so that $\overline{[x]} \subseteq F(G)$

i.e., $x \in 2^G$ such that every $y \in \overline{[x]}$ is aperiodic

2-Colorings

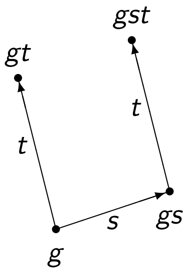
Let G be a countable group. A 2-coloring on G is a function $x : G \to \{0, 1\}$ such that

> *for any $s \in G$ with $s \neq 1_G$, there is a finite set $T \subseteq G$ such that*
>
> $$\forall g \in G \; \exists t \in T \; x(gst) \neq x(gt).$$

Lemma (GJS, Pestov)
x is a 2-coloring on G iff $\overline{[x]}$ is a free subflow.

for any $s \in G$ *with* $s \neq 1_G$*, there is a finite set* $T \subseteq G$ *such that*

$$\forall g \in G \; \exists t \in T \; x(gst) \neq x(gt).$$



Descriptive Complexity

Observation The set of all 2-colorings for any G is $\mathbf{\Pi}^0_3$.

Problem Given any countably infinite group G, is the set of all 2-colorings on G $\mathbf{\Pi}^0_3$-complete?

Minimality

Let G be a countably infinite group. $x \in 2^G$ is minimal if $\overline{[x]}$ is a minimal subflow, i.e., if $[y]$ is dense in $\overline{[x]}$ for every $y \in \overline{[x]}$.

Lemma x is minimal iff

for every finite $A \subseteq G$ there is a finite $T \subseteq G$ such that

$$\forall g \in G \ \exists t \in T \ \forall a \in A \ x(gta) = x(a)$$

Corollary The set of all minimal elements of 2^G is $\mathbf{\Pi}^0_3$.

Summary of Problems

Problem 1 Given any countably infinite group G, is the set of all 2-colorings on G $\mathbf{\Pi}^0_3$-complete?

Problem 2 Given any countably infinite group G, is the set of all minimal elements in 2^G $\mathbf{\Pi}^0_3$-complete?

Problem 3 Given any countably infinite group G, is there a simultaneous reduction for $\mathbf{\Pi}^0_3$-completeness of minimality and 2-colorings?

Simultaneous reduction for Π^0_3-completeness of minimality and 2-colorings

$$P = \{x \in 2^{\omega\times\omega} : \forall n\, \exists m\, \forall k \geq m\ x(n,k) = 0\}$$

For any countably infinite group G, is there a continuous function

$$\varphi : 2^{\omega\times\omega} \to 2^G$$

such that

$$x \in P \Longrightarrow \varphi(x) \text{ is a minimal 2-coloring on } G$$

$$x \notin P \Longrightarrow \varphi(x) \text{ is neither minimal nor a 2-coloring?}$$

Part II: Solutions and *flecc* Groups

What was thought as a routine application of descriptive set theoretic concepts and a minor generalization of previous proofs took a surprising turn...

The Canonical Construction of Colorings

Given a countably infinite group G, we first define infinitely many layers of marker sets and regions with the following properties:

F_n: a finite "basic" marker region on the n-th layer

Δ_n: the n-th layer marker set serving as the centers of the marker regions

Each marker region other than F_n itself is a translate of F_n, i.e., of the form γF_n where $\gamma \in \Delta_n$

The marker regions $\{\gamma F_n : \gamma \in \Delta_n\}$ form a maximal disjoint family in G

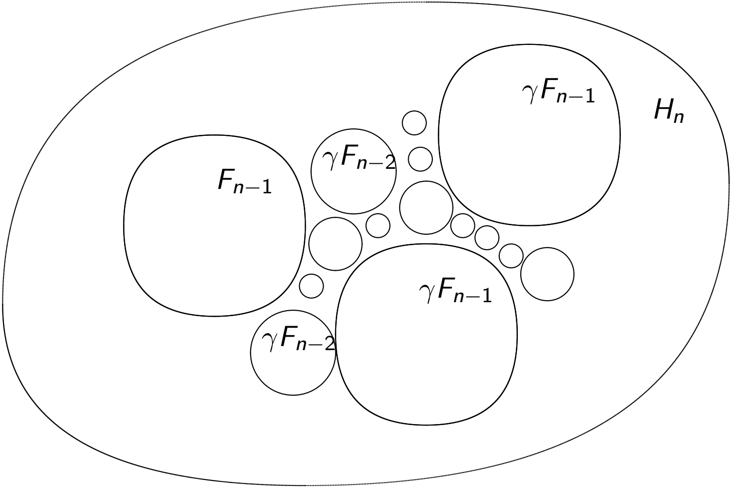
Cofinality: $\bigcup_n F_n = G$

Coherence: $F_n \subseteq F_{n+1}$, $\Delta_n \supseteq \Delta_{n+1}$; in fact, each F_{n+1} is the union of a number of disjoint translates of F_n with other disjoint successive translates of F_m, $m < n$

These are achieved by starting with preliminary finite but confinal regions

$$H_0 \subseteq H_1 \subseteq H_2 \subseteq \ldots$$

with $\bigcup_n H_n = G$, and successively "filling" H_{n+1} by disjoint translates of F_m, $m \leq n$


γF_{n-1}
H_n
γF_{n-2}
F_{n-1}
γF_{n-1}
γF_{n-2}

Next we introduce a partial coloring c of G in such a way that elements of Δ_n can be detected by a membership test:

$$g \in \Delta_1 \iff \forall f \in \lambda_1 F_0 \; c(gf) = c(f)$$

for some fixed element $\lambda_1 \in F_1$

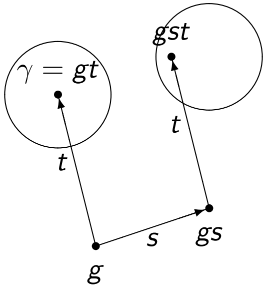
$$g \in \Delta_n \iff \forall f \in \Lambda_n \; c(gf) = c(f)$$

for some fixed finite set $\Lambda_n \subseteq F_n$

In particular, if $\gamma \in \Delta_n$ and $\eta \notin \Delta_n$, then there is some $f \in F_n$ such that

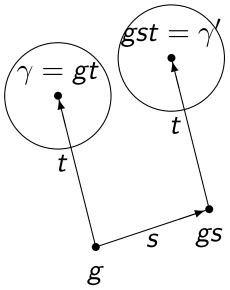
$$c(\gamma f) \neq c(\eta f)$$

If we are lucky we are done:



To be independent of luck we have to work more:

The partial coloring c also has the property that there exist at least two elements $a_n, b_n \in F_n$ such that for any $\gamma \in \Delta_n$, $\gamma a_n, \gamma b_n \notin \mathrm{dom}(c)$, i.e, each marker region γF_n contains at least two "free" elements to be colored at strategic positions

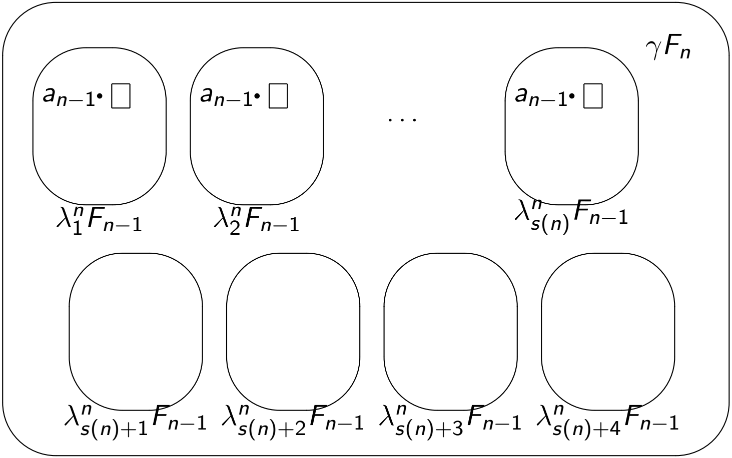


In this situation

$$\gamma' = gst = (gt)t^{-1}st = \gamma t^{-1}st$$

and so

$$\gamma^{-1}\gamma' \in F_n^{-1}F_nF_nF_nF_n^{-1}$$


γF_n
$a_{n-1}\cdot \square$
$a_{n-1}\cdot \square$
$a_{n-1}\cdot \square$
$\cdots$
$\lambda^n_1 F_{n-1}$
$\lambda^n_2 F_{n-1}$
$\lambda^n_{s(n)} F_{n-1}$
$\lambda^n_{s(n)+1} F_{n-1}$
$\lambda^n_{s(n)+2} F_{n-1}$
$\lambda^n_{s(n)+3} F_{n-1}$
$\lambda^n_{s(n)+4} F_{n-1}$

We are done if we made sure that

for any two elements $\gamma, \gamma' \in \Delta_n$ with $\gamma^{-1}\gamma \in F_n^{-1}F_n^3F_n^{-1}$, some of the "$a_{n-1}$" points in γF_n and in $\gamma' F_n$ are colored differently.

This is achieved by

- making sure there are enough copies of F_{n-1} in F_n (so that $2^{s(n)} > |F_n|^5$)
- pairs of marker points related as above are assigned different binary labels of length $s(n)$

Additional Ideas Toward Π^0_3-Completeness

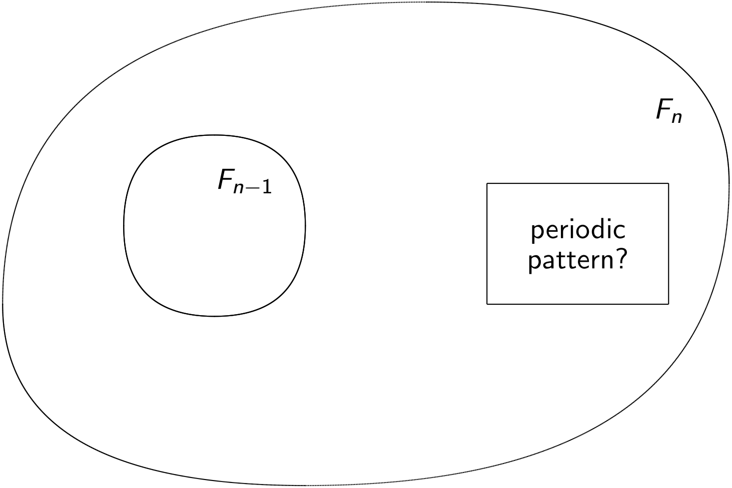
We need to uniformly create $c_x \in 2^G$ according to whether an "index" x belongs to 2^G

so that

$$x \in P \iff c_x \text{ is a coloring}$$

where

$$P = \{x \in 2^{\omega \times \omega} \,:\, \forall n\, \exists m\, \forall k \geq m\; x(n,k) = 0\}$$


F_n
F_{n-1}
periodic pattern?

At stage n we consider the digits

$$x(0,n),\ x(1,n),\ldots,\ x(n,n)$$

If $x(k,n)=1$ (where k is the least such) a periodic pattern with a specific period s_k is used

If $x(k,n)=0$ for all $k\leq n$, then the canonical construction is followed

If $x \notin P$ then by a compactness argument we can obtain $y \in \overline{[x]}$ with period s_k

Otherwise we will obtain colorings as before

In the implementation of this idea a number of things have to be fixed before the coding starts:

- the "free" coding region
- the specific periods s_k for all k

It turns out there is an obstacle when the group satisfies the following condition:

there exists a finite set $A \subseteq G - \{1_G\}$ *such that for all* $g \in G - \{1_G\}$ *there is* $i \in \mathbb{Z}$ *and* $h \in G$ *such that*

$$hg^i h^{-1} \in A.$$

We call such groups flecc.

Theorem Let G be a countable non-flecc group. Then the set of all 2-colorings on G is $\mathbf{\Pi}^0_3$-complete. In fact, there is a simultaneous reduction from P to the set of all minimal 2-colorings.

A Characterization of Flecc Groups

Given any group G and $g \in G$, the extended conjugacy class of g is defined as the set

$$C(g) = \{hg^i h^{-1} : i \in \mathbb{Z},\ h \in G\}.$$

For g of infinite order, we call the set $\bigcap_{n\in\mathbb{N}} C(g^n)$ the limit extended conjugacy class (lecc) of g.

If g is of finite order, *a* lecc of g is any $C(g^k)$ where $\mathrm{order}(g)/k$ is prime.

A group G is flecc iff

- for any $g \in G$ of infinite order, the lecc of g is not $\{1_G\}$, and
- there are only finitely many distinct lecc's in G.

Examples of Flecc Groups

$\mathbb{Z}(p^\infty)$: the additive group of all p-adic rationals mod 1

Flecc groups are closed under finite products, but not under infinite direct sums.

Every countable torsion-free group is the subgroup of a flecc group (in fact, every countable group is the subgroup of a group with only two conjugacy classes).

Theorem If G is a countably infinite flecc group, then the set of all 2-colorings on G is $\mathbf{\Sigma}^0_2$-complete.

Summary For a countably infinite group G, the set of all 2-colorings on G is $\mathbf{\Pi}^0_3$-complete iff G is not flecc.

The proof of the last theorem is surprisingly simple:

First it is easy to see that for every infinite group G the set of 2-colorings on G is $\mathbf{\Sigma}^0_2$-hard.

Now, let G be flecc. We show that the set of all 2-colorings on G is $\mathbf{\Sigma}^0_2$.

Fix a finite set $A \subseteq G$ such that for all $g \in G$ there is $i \in \mathbb{Z}$ and $h \in G$ such that $hg^i h^{-1} \in A$.

We claim that c is a 2-coloring iff for all $s \in A$ there exists a finite set F such that for all $g \in G$ there is $t \in F_s$ such that $c(gst) \neq c(gt)$. The claim gives a $\mathbf{\Sigma}^0_2$ computation for the set of 2-colorings.

To show the nontrivial direction of the claim, suppose c is not a 2-coloring. Then there is a periodic element $y \in \overline{[c]}$ with period g: $g \cdot y = y$. By flecc-ness there is $i \in \mathbb{Z}$ and $h \in G$ with $hg^i h^{-1} \in A$, and we have $(hg^i h^{-1}) \cdot (h \cdot y) = h \cdot y$. This means that there is $s = hg^i h^{-1} \in A$ and $z = h \cdot y \in \overline{[c]}$ such that $s \cdot z = z$.

Let F_s be the finite set given by the assumption. Since $z \in \overline{[c]}$ there is $g \in G$ such that $g \cdot c$ and z agree on all elements of F_s. Then in particular for any $t \in F_s$,

$$s \cdot (g \cdot c)(t) = (g \cdot c)(t).$$

This means that

$$c(g^{-1}st) = c(g^{-1}t),$$

contradicting the assumption.

Part III: The Isomorphism Relation

We consider the problem of classifying minimal free Bernoulli subflows up to (conjugacy) isomorphism.

Isomorphism Relation

Given any countably infinite group G and two subflows $S, T \subseteq 2^G$, S and T are isomorphic, denoted $S \cong T$, if there is a G-homeomorphism ϕ from S onto T, i.e., a homeomorphism $\phi : S \to T$ such that for any $g \in G$ and $x \in S$,

$$\phi(g \cdot x) = g \cdot \phi(x).$$

Problem What is the complexity of the isomorphism relation for all Bernoulli subflows?

The Space of All Bernoulli Subflows

Given a countable group G, consider the standard Borel space $F(2^G)$ of all closed subsets of 2^G, equipped with the Effros Borel structure. The space of all Bernoulli subflows

$$\mathcal{S} = \{S \in F(2^G) \,:\, S \text{ is } G\text{-invariant}\}$$

is a Borel subspace of $F(2^G)$, and hence a standard Borel space.

The space of all free Bernoulli subflows

$$\mathcal{F} = \{F \in \mathcal{S} \,:\, \forall g \in G \ \forall x \in F \ g \cdot x \neq x\}$$

is also a standard Borel space.

Likewise for the space of all minimal Bernoulli subflows

$$\mathcal{M} = \{M \in \mathcal{S} \,:\, M \text{ is minimal}\}.$$

Theorem The isomorphism relation $\cong$ on $\mathcal{S}$ is a countable Borel equivalence relation.

For $G = \mathbb{Z}$ this is a theorem of Curtis–Hedlund–Lyndon.

Theorem (Clemens)

The isomorphism relation $\cong$ for Bernoulli subshifts (i.e. $G = \mathbb{Z}$) is a universal countable Borel equivalence relation.

Questions

What about general G?
What about free Bernoulli subflows?
What about minimal free Bernoulli subflows?

Theorem

For any countably infinite group G, the isomorphism relation on the minimal free Bernoulli subflows Borel reduces E_0.

Theorem

If G is a countably infinite locally finite group (i.e., for any finite subset F of G, $\langle F \rangle$ is finite), then the isomorphism relation for the minimal free Bernoulli subflows is exactly E_0.

Questions

- What is the complexity of the isomorphism relation for general Bernoulli G-subflows for an arbitrary G?
- What about free Bernoulli subflows?
- What about minimal free Bernoulli subflows?