Ramsey-like Cardinals

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Large cardinals and elementary embeddings

- Measurable cardinals and most stronger large cardinals $\kappa$ assert the existence of elementary embeddings $j : V \rightarrow M$ with critical point $\kappa$ from the universe of sets to an inner model.
- Many smaller large cardinals $\kappa$ assert the existence of elementary embeddings $j : M \rightarrow N$ with critical point $\kappa$ from a weak $\kappa$-model (or $\kappa$-model) to a transitive set.
- A weak $\kappa$-model $M$ is a transitive set of size $\kappa$ satisfying ZFC without the Power Set Axiom and having $\kappa \in M$.
- A $\kappa$-model $M$ is a weak $\kappa$-model such that $M^{<\kappa} \subseteq M$. (important in indestructibility arguments)

Examples

- If $\kappa^{<\kappa} = \kappa$, then $\kappa$ is weakly compact if every $A \subseteq \kappa$ is contained in a weak $\kappa$-model $M$ for which there exists an elementary embedding $j : M \rightarrow N$ with critical point $\kappa$.
- If $\kappa^{<\kappa} = \kappa$, then $\kappa$ is strongly unfoldable if for every $\alpha \in \text{Ord}$, every $A \subseteq \kappa$ is contained in a weak $\kappa$-model $M$ for which there exists an elementary embedding $j : M \rightarrow N$ with critical point $\kappa$, $j(\kappa) > \alpha$, and $V_\alpha \subseteq N$. 
Iterating ultrafilters

If $j : V \rightarrow M$ is an elementary embedding with critical point $\kappa$, then $U = \{ A \subseteq \kappa \mid \kappa \in j(A) \}$ is a normal ultrafilter on $\kappa$. (closed under diagonal intersections of length $\kappa$)

We can iterate $U$ to construct an $\text{Ord}$-length directed system of elementary embeddings of inner models:

$V \xrightarrow{j_{01}} M_1 \xrightarrow{j_{12}} M_2 \xrightarrow{j_{23}} \cdots \xrightarrow{} M_n \xrightarrow{j_{nn+1}} M_{n+1} \xrightarrow{} \cdots \xrightarrow{\text{dir lim}} M_\omega \xrightarrow{j_{\omega \omega+1}} M_{\omega+1} \xrightarrow{} \cdots$

**Theorem (Kunen, 70’s)**

*If $U$ is a countably complete ultrafilter on $\kappa$, then every stage of the iteration is well-founded.*
Iterating $M$-ultrafilters

If $M$ is a weak $\kappa$-model and $j : M \to N$ is an elementary embedding with critical point $\kappa$, then $U = \{ A \subseteq \kappa \mid \kappa \in j(A) \}$ is a normal $M$-ultrafilter. That is $\langle M, \in, U \rangle \models U$ is a normal ultrafilter.

- $M$-ultrafilters need not have well-founded ultrapowers.

**Question**

Can $M$-ultrafilters that have well-founded ultrapowers be iterated?

**PROBLEM:** $U$ need not be an element of $M$! How do we obtain the next ultrafilter?

**IDEA:** Take the ultrapower of $\langle M, \in, U \rangle$ to obtain the structure $\langle N, \in, "j(U)" \rangle$.

**PROBLEM:** $[f] \in "j(U)"$ iff $\{ \xi \in \kappa \mid f(\xi) \in U \} \in U$.

But $\langle M, \in, U \rangle$ need not satisfy Comprehension!

So $\{ \xi \in \kappa \mid f(\xi) \in U \}$ need not be an element of $M$.

**Definition**

If $M$ is a weak $\kappa$-model and $U$ is an $M$-ultrafilter on $\kappa$, then $U$ is weakly amenable if for every $\langle A_\xi \mid \xi \in \kappa \rangle \in M$, the set $\{ \xi \in \kappa \mid A_\xi \in U \} \in M$. 
Weakly amenable $M$-ultrafilters

**Proposition (Folklore)**

Suppose $M$ is a weak $\kappa$-model, $U$ is a weakly amenable $M$-ultrafilter on $\kappa$ with a well-founded ultrapower, and $\langle N, \in, "j(U)" \rangle$ is the ultrapower of $\langle M, \in, U \rangle$ by $U$. Then "$j(U)$" is a weakly amenable $N$-ultrafilter on $j(\kappa)$.

**Definition**

Suppose $M$ is weak $\kappa$-model. An elementary embedding $j : M \to N$ is $\kappa$-powerset preserving if it has critical point $\kappa$ and $M$ and $N$ have the same subsets of $\kappa$.

**Proposition (Folklore)**

Suppose $M$ is a weak $\kappa$-model.

- If $j : M \to N$ is $\kappa$-powerset preserving, then the $M$-ultrafilter $U$ obtained from it is weakly amenable.
- If $U$ is a weakly amenable $M$-ultrafilter having a well-founded ultrapower, then the ultrapower embedding is $\kappa$-powerset preserving.
Hierarchy of iterability: the $\alpha$-good ultrafilters

**Question**

Can we iterate weakly amenable $M$-ultrafilters that have well-founded ultrapowers?

**Definition (G.)**

Suppose $M$ is a weak $\kappa$-model. An $M$-ultrafilter on $\kappa$ is

- **0-good** if it has a well-founded ultrapower,
- **1-good** if it is 0-good and weakly amenable,
- **$\alpha$-good** if it gives rise to the directed system of elementary embeddings of well-founded models of length $\alpha$.

**EX:** $M_0 \xrightarrow{j_{01}} M_1 \xrightarrow{j_{12}} M_2 \xrightarrow{j_{12}} M_3$ is the directed system of length 3.

**Theorem (Gaifman, 70’s)**

*If $M$ is a weak $\kappa$-model and $U$ is an $\omega_1$-good $M$-ultrafilter, then $U$ is $\alpha$-good for every $\alpha$.***
Ramsey-like cardinals: the $\alpha$-iterable Cardinals

**Definition (G., 08)**

$\kappa$ is $\alpha$-iterable if every $A \subseteq \kappa$ is contained in a weak $\kappa$-model $M$ for which there exists an $\alpha$-good $M$-ultrafilter on $\kappa$.

**Examples**

- If $\kappa^{<\kappa} = \kappa$, then $\kappa$ is weakly compact iff it is 0-iterable.
- We also call 1-iterable cardinals weakly Ramsey.
- $\kappa$ is Ramsey iff every $A \subseteq \kappa$ is contained in a weak $\kappa$-model $M$ for which there is a weakly amenable countably complete $M$-ultrafilter. Thus, Ramsey cardinals are $\omega_1$-iterable. [Dodd, Mitchell,...]
- $\omega_1$-iterable cardinals imply $0^\#$.
- $\omega_1$-iterable cardinals are strongly unfoldable in $L$. 

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The hierarchy of $\alpha$-iterable cardinals

**Theorem (G. and Welch, 08)**

*If $0^\# \text{ exists, then the Silver indiscernibles are } \alpha\text{-iterable in } L \text{ for } \alpha < \omega_1^L.*

*Sketch of proof:* Every Silver indiscernible is a critical point of an elementary embedding $j : L \rightarrow L$ that produces an $\omega_1$-iterable $L$-ultrafilter. Build in $L$, a tree of height $\omega$ of approximations to a directed system of iterated ultrapowers of countable length. Argue that the iteration that exists in $V$ is a branch through the tree. The tree is ill-founded in $V$, and hence in $L$. Thus, there is a directed system of iterated ultrapowers in $L$ as well.

**Theorem (G. and Welch, 08)**

*$\alpha$-iterable cardinals are downward absolute to $L$ for $\alpha < \omega_1^L$.*

**Theorem (G. and Welch, 08)**

*$\alpha$-iterable cardinals form a hierarchy of strength for all $\alpha \leq \omega_1$.*

**Theorem (Sharpe and Welch, 07)**

*$\omega_1$-iterable cardinals are strictly weaker than Ramsey cardinals.*
Weakly Compact Cardinals

Theorem (Folklore)

If $\kappa^\kappa = \kappa$, then TFAE:

- $\kappa$ is weakly compact.
- Every $A \subseteq \kappa$ is contained in a weak $\kappa$-model $M$ for which there exists an elementary embedding $j : M \rightarrow N$ with critical point $\kappa$.
- Every $A \subseteq \kappa$ is contained in a $\kappa$-model $M$ for which there exists an elementary embedding $j : M \rightarrow N$ with critical point $\kappa$.
- Every $A \subseteq \kappa$ is contained in a $\kappa$-model $M \prec H_{\kappa^+}$ for which there exists an elementary embedding $j : M \rightarrow N$ with critical point $\kappa$.
- For every $\kappa$-model $M$ there exists an elementary embedding $j : M \rightarrow N$ with critical point $\kappa$.

This equivalence holds for strongly unfoldable cardinals as well.

Question

What happens if we add the requirement that the embeddings have to be $\kappa$-powerset preserving?
Other Ramsey-like Cardinals

Example

\( \kappa \) is weakly Ramsey if every \( A \subseteq \kappa \) is contained in a weak \( \kappa \)-model \( M \) for which there exists a \( \kappa \)-powerset preserving elementary embedding \( j : M \rightarrow N \).

Definition (G., 07)

\( \kappa \) is strongly Ramsey if every \( A \subseteq \kappa \) is contained in a \( \kappa \)-model \( M \) for which there exists a \( \kappa \)-powerset preserving elementary embedding \( j : M \rightarrow N \).

- Strongly Ramsey cardinals are Ramsey since an \( M \)-ultrafilter for a \( \kappa \)-model \( M \) must be countably complete.

Definition (G., 07)

\( \kappa \) is super Ramsey if every \( A \subseteq \kappa \) is contained in a \( \kappa \)-model \( M \prec H_{\kappa^+} \) for which there exists a \( \kappa \)-powerset preserving elementary embedding \( j : M \rightarrow N \).

Theorem (G., 07)

It is inconsistent to assume that there is a cardinal \( \kappa \) such that every \( \kappa \)-model has a \( \kappa \)-powerset preserving elementary embedding \( j : M \rightarrow N \).
The hierarchy

Measurable cardinals

Super Ramsey cardinals

Strongly Ramsey cardinals

Ramsey cardinals

$\omega_1$-iterable cardinals

$\alpha$-iterable cardinals

Weakly Ramsey cardinals

Weakly compact cardinals