Ramsey-like Cardinals

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Large cardinals and elementary embeddings

- Measurable cardinals and most stronger large cardinals κ assert the existence of elementary embeddings $j : V \to M$ with critical point κ from the universe of sets to an inner model.
- Many smaller large cardinals κ assert the existence of elementary embeddings $j : M \to N$ with critical point κ from a weak κ-model (or κ-model) to a transitive set.
- A weak κ-model M is a transitive set of size κ satisfying ZFC without the Power Set Axiom and having $\kappa \in M$.
- A κ-model M is a weak κ-model such that $M^{<\kappa} \subseteq M$. (important in indestructibility arguments)

Examples

- If $\kappa^{<\kappa} = \kappa$, then κ is weakly compact if every $A \subseteq \kappa$ is contained in a weak κ-model M for which there exists an elementary embedding $j : M \to N$ with critical point κ.
- If $\kappa^{<\kappa} = \kappa$, then κ is strongly unfoldable if for every $\alpha \in Ord$, every $A \subseteq \kappa$ is contained in a weak κ-model M for which there exists an elementary embedding $j : M \to N$ with critical point κ, $j(\kappa) > \alpha$, and $V_\alpha \subseteq N$.

Iterating ultrafilters

If $j : V \longrightarrow M$ is an elementary embedding with critical point κ, then $U = \{A \subseteq \kappa \mid \kappa \in j(A)\}$ is a normal ultrafilter on κ. (closed under diagonal intersections of length κ)

We can iterate U to construct an *Ord*-length directed system of elementary embeddings of inner models:

$$V \xrightarrow[\text{by } U]{j_{01}} M_1 \xrightarrow[\text{by } j_{01}(U)]{j_{12}} M_2 \xrightarrow[\text{by } j_{02}(U)]{j_{23}} \cdots \longrightarrow M_n \xrightarrow[\text{by } j_{0n}(U)]{j_{nn+1}} M_{n+1} \longrightarrow \cdots \xrightarrow[\text{dir lim}]{} M_\omega \xrightarrow[\text{by } j_{0\omega}(U)]{j_{\omega\omega+1}} M_{\omega+1} \longrightarrow \cdots$$

Theorem (Kunen, 70's)

If U is a countably complete ultrafilter on κ, then every stage of the iteration is well-founded.

Iterating M-ultrafilters

If M is a weak κ-model and $j : M \to N$ is an elementary embedding with critical point κ, then $U = \{A \subseteq \kappa \mid \kappa \in j(A)\}$ is a normal M-ultrafilter.
That is $\langle M, \in, U\rangle \models U$ is a normal ultrafilter.

- M-ultrafilters need not have well-founded ultrapowers.

Question

Can M-ultrafilters that have well-founded ultrapowers be iterated?

PROBLEM: U need not be an element of M! How do we obtain the next ultrafilter?

IDEA: Take the ultrapower of $\langle M, \in, U\rangle$ to obtain the structure $\langle N, \in, \text{“}j(U)\text{”}\rangle$.

PROBLEM: $[f] \in \text{“}j(U)\text{”}$ iff $\{\xi \in \kappa \mid f(\xi) \in U\} \in U$.
But $\langle M, \in, U\rangle$ need not satisfy Comprehension!
So $\{\xi \in \kappa \mid f(\xi) \in U\}$ need not be an element of M.

Definition

If M is a weak κ-model and U is an M-ultrafilter on κ, then U is weakly amenable if for every $\langle A_\xi \mid \xi \in \kappa\rangle \in M$, the set $\{\xi \in \kappa \mid A_\xi \in U\} \in M$.

Weakly amenable M-ultrafilters

Proposition (Folklore)

Suppose M is a weak κ-model, U is a weakly amenable M-ultrafilter on κ with a well-founded ultrapower, and $\langle N, \in, \text{“}j(U)\text{”}\rangle$ is the ultrapower of $\langle M, \in, U\rangle$ by U. Then “$j(U)$” is a weakly amenable N-ultrafilter on $j(\kappa)$.

Definition

Suppose M is weak κ-model. An elementary embedding $j : M \longrightarrow N$ is κ-powerset preserving if it has critical point κ and M and N have the same subsets of κ.

Proposition (Folklore)

Suppose M is a weak κ-model.

- *If $j : M \rightarrow N$ is κ-powerset preserving, then the M-ultrafilter U obtained from it is weakly amenable.*
- *If U is a weakly amenable M-ultrafilter having a well-founded ultrapower, then the ultrapower embedding is κ-powerset preserving.*

Hierarchy of iterability: the α-good ultrafilters

Question

Can we iterate weakly amenable M-ultrafilters that have well-founded ultrapowers?

Definition (G.)

Suppose M is a weak κ-model. An M-ultrafilter on κ is

- 0-good if it has a well-founded ultrapower,
- 1-good if it is 0-good and weakly amenable,
- α-good if it gives rise to the directed system of elementary embeddings of well-founded models of length α.

EX: $M_0 \underset{\text{by } U}{\xrightarrow{j_{01}}} M_1 \underset{\text{by } ``j_{01}(U)"}{\xrightarrow{j_{12}}} M_2 \underset{\text{by } ``j_{02}(U)"}{\xrightarrow{j_{12}}} M_3$ is the directed system of length 3.

Theorem (Gaifman, 70's)

If M is a weak κ-model and U is an ω_1-good M-ultrafilter, then U is α-good for every α.

Ramsey-like cardinals: the α-iterable Cardinals

Definition (G., 08)

κ is α-iterable if every $A \subseteq \kappa$ is contained in a weak κ-model M for which there exists an α-good M-ultrafilter on κ.

Examples

- If $\kappa^{<\kappa} = \kappa$, then κ is weakly compact iff it is 0-iterable.
- We also call 1-iterable cardinals weakly Ramsey.
- κ is Ramsey iff every $A \subseteq \kappa$ is contained in a weak κ-model M for which there is a weakly amenable countably complete M-ultrafilter.
  Thus, Ramsey cardinals are ω_1-iterable. [Dodd, Mitchell,...]
- ω_1-iterable cardinals imply $0^\#$.
- ω_1-iterable cardinals are strongly unfoldable in L.

The hierarchy of α-iterable cardinals

Theorem (G. and Welch, 08)

If $0^{\#}$ exists, then the Silver indiscernibles are α-iterable in L for $\alpha < \omega_1^L$.

Sketch of proof: Every Silver indiscernible is a critical point of an elementary embedding $j : L \longrightarrow L$ that produces an ω_1-iterable L-ultrafilter. Build in L, a tree of height ω of approximations to a directed system of iterated ultrapowers of countable length. Argue that the iteration that exists in V is a branch through the tree. The tree is ill-founded in V, and hence in L. Thus, there is a directed system of iterated ultrapowers in L as well.

Theorem (G. and Welch, 08)

α-iterable cardinals are downward absolute to L for $\alpha < \omega_1^L$.

Theorem (G. and Welch, 08)

α-iterable cardinals form a hierarchy of strength for all $\alpha \leq \omega_1$.

Theorem (Sharpe and Welch, 07)

ω_1-iterable cardinals are strictly weaker than Ramsey cardinals.

Weakly Compact Cardinals

Theorem (Folklore)

If $\kappa^{<\kappa} = \kappa$, then TFAE:

- *κ is weakly compact.*
- *Every $A \subseteq \kappa$ is contained in a weak κ-model M for which there exists an elementary embedding $j : M \to N$ with critical point κ.*
- *Every $A \subseteq \kappa$ is contained in a κ-model M for which there exists an elementary embedding $j : M \to N$ with critical point κ.*
- *Every $A \subseteq \kappa$ is contained in a κ-model $M \prec H_{\kappa^+}$ for which there exists an elementary embedding $j : M \to N$ with critical point κ.*
- *For every κ-model M there exists an elementary embedding $j : M \to N$ with critical point κ.*

- This equivalence holds for strongly unfoldable cardinals as well.

Question

What happens if we add the requirement that the embeddings have to be κ-powerset preserving?

Other Ramsey-like Cardinals

Example

κ is weakly Ramsey if every $A \subseteq \kappa$ is contained in a weak κ-model M for which there exists a κ-powerset preserving elementary embedding $j : M \to N$.

Definition (G., 07)

κ is strongly Ramsey if every $A \subseteq \kappa$ is contained in a κ-model M for which there exists a κ-powerset preserving elementary embedding $j : M \to N$.

- Strongly Ramsey cardinals are Ramsey since an M-ultrafilter for a κ-model M must be countably complete.

Definition (G., 07)

κ is super Ramsey if every $A \subseteq \kappa$ is contained in a κ-model $M \prec H_{\kappa^+}$ for which there exists a κ-powerset preserving elementary embedding $j : M \to N$.

Theorem (G., 07)

It is inconsistent to assume that there is a cardinal κ such that every κ-model has a κ-powerset preserving elementary embedding $j : M \to N$.

The hierarchy

Measurable cardinals

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Super Ramsey cardinals

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Strongly Ramsey cardinals

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Ramsey cardinals

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ω_1-iterable cardinals

⋮

α-iterable cardinals

⋮

Weakly Ramsey cardinals

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Weakly compact cardinals