Yet another proof of Gaboriau-Popa

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Defn. Let E be a Borel equivalence relation on a standard Borel probability space $(\mathbb{X}, \mu)$ with all its equivalence classes countable.

E is ergodic if every E-invariant Borel set is either null or conull.

E is measure preserving $\boxed{\text{if}}$ given $A, B \subseteq \mathbb{X}$ Borel

$\Theta : A \to B$ a Borel bijection

with $\Theta \subseteq E$, (i.e. $x \, E \, \Theta(x)$ all $x \in A$)

$\boxed{\text{then}}$ we have $\mu(A) = \mu(B)$.

(This happens when $E = E_\Gamma$, the orbit equivalence relation induced by a countable group Γ acting by measure preserving automorphisms.)

For E_1, E_2 on $(\mathbb{X}_1, \mu_1), (\mathbb{X}_2, \mu_2)$ we say that E_1 is orbit equivalent to E_2,

"E_1 O.E. E_2",

if there is a measure preserving $\Theta : \mathbb{X}_1 \to \mathbb{X}_2$

s.t. for μ_1 a.e. $x \in \mathbb{X}_1$

$$[\Theta(x)]_{E_2} = \{\Theta(y) \mid y \, E_1 \, x\} \quad (=_{df.} \Theta[[x]_{E_1}])$$

Here: $[\Theta(x)]_{E_2} =_{df.} \{z \mid z \, E_2 \, \Theta(x)\}$.

<u>Theorem</u>: (Gaboriau, Popa).

$\mathbb{F}_2$, the free group on two generators, has $2^{\aleph_0}$ many free, ergodic, measure preserving actions on standard Borel probability spaces up to orbit equivalence.

I.e. $\exists (X_s, \mu_s, E_s)_{s \in [0,1]}$

each E_s induced by $\mathbb{F}_2 \curvearrowright (X_s, \mu_s)$ as above

with

E_s O.E. E_t iff $s = t$.

Afterwards, a sequence of results for other classes of groups, finally finishing in Inessa Epstein's proof of the same for any "non-amenable" group.

Their proof used deep ideas in operator algebras as well as "property (T)". Asger Törnquist gave a new proof, eliminating the use of operator algebras, but still using property (T).

Just for the sake of completeness,
here is the definition.

<u>Defn.</u> Let $\mathcal{H}$ be a Hilbert space.
An action $\Gamma \curvearrowright \mathcal{H}$ is <u>unitary</u> if
(i) it is linear
(ii) $\forall \gamma \in \Gamma \ \forall v, w \in \mathcal{H}$
$$\langle \gamma \cdot v, \gamma \cdot w \rangle = \langle v, w \rangle$$

<u>Defn.</u> For Γ a group, $\Delta \lhd \Gamma$ a normal subgroup,
Γ has <u>relative property (T) over Δ</u>
$\boxed{\text{if}}$ whenever $\Gamma \curvearrowright \mathcal{H}$ unitary with "almost invariant vectors"
$\boxed{\text{then}}$ there is a Δ-invariant vector $\neq 0$.

("Almost invariant vectors":
$\forall F \subseteq \Gamma$ finite $\forall \varepsilon > 0 \ \exists v \neq 0$
$\forall \gamma \in F \ (\|v - \gamma \cdot v\| < \varepsilon \|v\|)$).

<u>E.g.</u> $SL_2(\mathbb{Z}) \ltimes \mathbb{Z}^2$ has relative property (T) over $\mathbb{Z}^2$.

<u>Rmk:</u> Simon Thomas: "$SL_2(\mathbb{Z})$ is practically the same thing as $\mathbb{F}_2$".

<u>Defn</u>. Let (X, d) be a complete separable metric space.
Let $\Gamma \curvearrowright X$ be a Borel action.

The action is <u>expansive</u> if $\exists \varepsilon > 0$ $\exists A_1, \dots A_n \subseteq X$ Borel
$\exists F_1, \dots F_n \subseteq \Gamma$ finite

such that

(i) $\bigcup A_i \supseteq \{(x,y) \in X^2 \mid d(x,y) < \varepsilon\}$

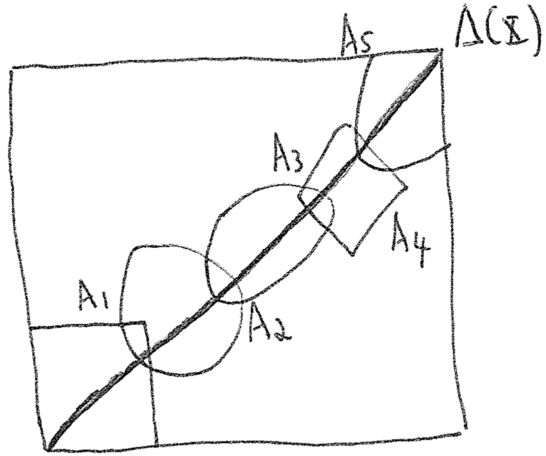
(ii) $\forall i \neq j$ $\forall \gamma \in F_i$
$(\gamma \cdot A_i \cap A_j \subseteq \Delta(X))$

"$\Delta(X)$" $= \{(x,x) \mid x \in X\}$

(iii) $\forall i$
$\bigcap_{\gamma \in F_i} \gamma \cdot A_i \subseteq \Delta(X)$

Γ acts on X^2 by
$\gamma \cdot (x,y) = (\gamma \cdot x, \gamma \cdot y)$

X^2



Lemma: Let $\Gamma \curvearrowright (X, d)$ be expansive.

Let ε, $A_1, \ldots A_n$, $F_1, \ldots F_n$ be as above, in defn.

Then: Any Borel probability measure on X^2 that "largely concentrates" on $\{(x,y) : d(x,y) < \varepsilon\}$ and is "sufficiently Γ-invariant" will "largely concentrate" on $\Delta(X)$

FORMAL TRANSLATION:

$\forall \delta > 0 \; \exists \delta' > 0$ such that

$\boxed{\text{if}}$ $\nu(\{(x,y) \mid d(x,y) < \varepsilon\}) > 1 - \delta'$ for ν a Borel prob. measure

and

$\forall i \leq n \; \forall \gamma \in F_i \; (|\nu(A_i) - \nu(\gamma \cdot A_i)| < \delta')$

then $\nu(\Delta(X)) > 1 - \delta$.

<u>Theorem:</u> Let $T \curvearrowright (X, d)$ be expansive.

Let μ be a T-invariant Borel probability measure.

Suppose we have $(E_s)_{s \in [0,1]}$ s.t.

(i) each E_s a countable, Borel, ergodic, measure preserving equivalence relation

(ii) each $E_s \supseteq E_T$

(iii) for $s \neq t$, $A \subseteq X$ with $\mu(A) > 0$

we have for a.e. $x \in A$

$$[x]_{E_s} \cap A \neq [x]_{E_t} \cap A.$$

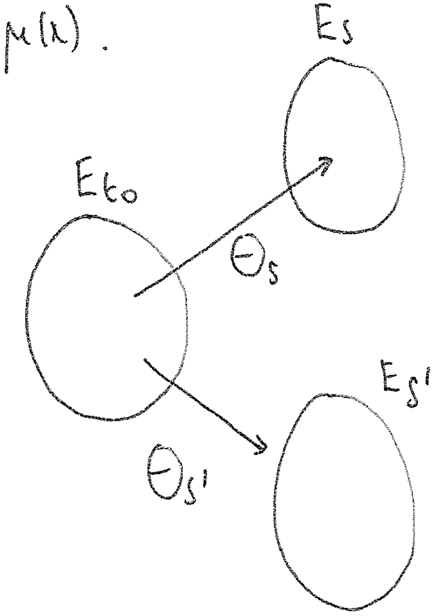
$\boxed{\text{Then:}}$ Each E_t is orbit equivalent to only countably many other E_s.

<u>Sketch of proof</u>:

Fix $\varepsilon > 0$, $F_1, \dots F_n$, $A_1, \dots A_n$ as in defn. of expansive.
Let $F = \bigcup_{i \leq n} F_i$.

Suppose instead $\exists t_0 \in [0,1]$, $W \subseteq [0,1]$, $|W| = \aleph_1$
such that at each $s \in W$
$$\Theta_s : X \to X$$
witnesses E_{t_0} O.E. E_s.

For $s, s' \in W$, let $d_0(\Theta_s, \Theta_{s'}) = \int d(\Theta_s(x), \Theta_{s'}(x))\, d\mu(x)$.



This is a separable premetric
so can easily get lots of $s \neq s'$
with
$$d_0(\Theta_s, \Theta_{s'}) \sim 0.$$

(I.e. "close to zero.")

Let Δ be a countable group acting in a measure preserving fashion with

$$E_{t_0} = E_\Delta .$$ (Feldman-Moore)

At each $s \in W$ define

$$\varphi_s : X \times F \to \Delta \cup \{\infty\}$$

$F = \bigcup_{i \leq n} F_i$: along with $A_1, \ldots A_n$ witnessing "expansive"

with $\varphi_s(x, \gamma) =$ "least" $\delta \in \Delta$ s.t. $\gamma \cdot \Theta_s(x) = \Theta_s(\delta \cdot x)$

(Here "least" refers to some enumeration of $\Delta = \{\delta_n \mid n \in \mathbb{N}\}$.)

Let $d_1(\varphi_s, \varphi_{s'})$

$$= \sum_{\gamma \in F} \mu(\{x : \varphi_s(x,\gamma) \neq \varphi_{s'}(x,\gamma)\})$$

Again a separable premetric

$\therefore$ can obtain $s \neq s'$ s.t. $d_1(\varphi_s, \varphi_{s'}), d_0(\Theta_s, \Theta_{s'}) \sim 0$

NOW:

Define $\Theta = (\Theta_s, \Theta_{s'}) : X \to X^2$

$$x \longmapsto (\Theta_s(x), \Theta_{s'}(x))$$

Let $\nu = \Theta^*[\mu]$

i.e. $\nu(A) = \mu(\Theta^{-1}[A])$

By $d_0(\Theta_s, \Theta_{s'}) \sim 0$

ν "largely concentrates" on $\{(x,y) \mid d(x,y) < \varepsilon\}$.

By $d_1(\varphi_s, \varphi_{s'}) \sim 0$

ν is "largely" $(\bigcup_{i \leq n} F_i)$ invariant.

$\boxed{\bigcup F_i = F}$

Therefore by lemma,

ν "largely" concentrates on $\Delta(X)$.

Thus we get $A \subseteq X$, $\mu(A) > 0$ s.t.

$\Theta[A] \subseteq \Delta(X)$.

I.e. $\forall x \in A$ $\Theta_s(x) = \Theta_{s'}(x)$.

Θ_s and $\Theta_{s'}$ are orbit equivalences

$\therefore E_{s_1}|_A = E_{s_2}|_A$

$\perp$. $\square$

Back to our present context (Gaboriau-Popa).

Take the usual linear action of $SL_2(\mathbb{Z})$ on $\mathbb{R}^2$.

$\mathbb{Z}^2$ is $SL_2(\mathbb{Z})$-invariant.

Let $p: \mathbb{R}^2 \to \mathbb{R}^2/\mathbb{Z}^2$ be the projection.

The $SL_2(\mathbb{Z}) \curvearrowright \mathbb{R}^2$ action pushes down to $SL_2(\mathbb{Z}) \curvearrowright \mathbb{R}^2/\mathbb{Z}^2$

$$M \cdot p(\vec{x}) = p(M \cdot \vec{x}).$$

There is an invariant Borel probability measure (basically, Lebesgue measure).

It <u>is</u> expansive, since for ε small

$$SL_2(\mathbb{Z}) \curvearrowright \{(p(\vec{x}), p(\vec{y})) : d(\vec{x}, \vec{y}) < \varepsilon\}$$

"resembles" $SL_2(\mathbb{Z}) \curvearrowright \{\vec{x} \in \mathbb{R}^2 : |\vec{x}| < \varepsilon\}$.

Now take a suitable copy of $\mathbb{F}_2 < SL_2(\mathbb{Z})$ (fact from combinatorial group theory).

Can be done so $\mathbb{F}_2 \curvearrowright \mathbb{R}^2/\mathbb{Z}^2$ expansive.

More details

Let $A = \begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 0 \\ 3 & 1 \end{bmatrix}$.

$\langle A, B \rangle \cong \mathbb{F}_2$ (Combinatorial group theory)

$\mathbb{F}_2 \curvearrowright \mathbb{R}^2/\mathbb{Z}^2$ is expansive

(linear algebra).

Then (Törnquist, see his JSL paper):

Can find $(\psi_s)_{s \in [0,1]}$ s.t. at each $s \in [0,1]$

(i) $\psi_s \curvearrowright \mathbb{R}^2/\mathbb{Z}^2$ measure preserving

(ii) $\langle A, B, \psi_s \rangle \cong \mathbb{F}_3$ (in $M_\infty(\mathbb{R}^2/\mathbb{Z}^2)$)

(iii) gives rise to a free action of $\mathbb{F}_3$

and such that for $s \neq t$

$$E_{\langle A,B,\psi_s\rangle} \neq E_{\langle A,B,\psi_t\rangle} \text{ a.e.}$$

Applying last theorem get each $E_{\langle A,B,\psi_t\rangle}$ O.E. to only countably many $E_{\langle A,B,\psi_s\rangle}$

This gives Gaborian-Popa for $\mathbb{F}_3$ rather than $\mathbb{F}_2$
– but "standard" technical tricks make possible the transition back to $\mathbb{F}_2$.

Some fanciful questions:

(1) Is there a connection between expansive actions and relative property (T)?

(2) Suppose Γ a countable group

$$\Delta \lhd \Gamma$$

an abelian normal subgroup with Γ having relative property (T).

Must Γ/Δ admit an expansive action by automorphisms on a compact abelian group?

(3) Let Γ be non-amenable countable.

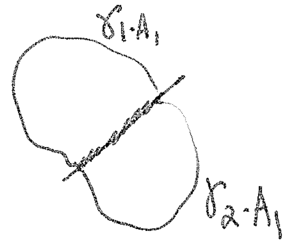
Does there exist Δ and $\Gamma \ltimes \Delta$, Δ abelian, infinite, $\Gamma \ltimes \Delta$ having relative property (T)?

Pf. of lemma on expansiveness:

Assume $n=1$ for convenience.

Say $F_1 = \{\gamma_1, \gamma_2\}$

$$\nu(A_1) \sim \nu(\gamma_1\cdot A_1) \sim \nu(\gamma\cdot A_1) \sim 1$$



$$\nu(\gamma_1\cdot A_1) = \nu(\gamma_1\cdot A_1 \cap \gamma_2\cdot A_1) + \nu(\gamma_1\cdot A_1 \setminus \gamma_2\cdot A_1) \sim 1 \quad \text{—— (1)}$$

$$\nu(\gamma_2\cdot A_1) = \nu(\gamma_2\cdot A_1 \cap \gamma_1\cdot A_1) + \nu(\gamma_2\cdot A_1 \setminus \gamma_1\cdot A_1) \sim 1 \quad \text{—— (2)}$$

$$\nu(\gamma_1\cdot A_1 \cap \gamma_2\cdot A_1) + \nu(\gamma_1\cdot A_1 \setminus \gamma_2\cdot A_1) + \nu(\gamma_2\cdot A_1 \setminus \gamma_1\cdot A_1) \sim 1 \quad \text{—— (A)}$$

(A) − (1) gives $\nu(\gamma_2\cdot A_1 \setminus \gamma_1\cdot A_1) \sim 0$

similarly $\nu(\gamma_1\cdot A_1 \setminus \gamma_2\cdot A_1) \sim 0$

The general case:

For any pair i,j $i \neq j$ can get

$$\nu(\gamma_i\cdot A_1) = \nu(\gamma_i\cdot A_1 \cap \gamma_j\cdot A_1) + \nu(\gamma_i\cdot A_1 \setminus \gamma_j\cdot A_1) \sim 1$$
$$\nu(\gamma_j\cdot A_1) = \nu(\gamma_i\cdot A_1 \cap \gamma_j\cdot A_1) + \nu(\gamma_j\cdot A_1 \setminus A_i\cdot A_1) \sim 1$$

Then similarly, $\nu(\gamma_i\cdot A_1 \setminus \gamma_j\cdot A_1) \sim 0$

Let $B = \{x \mid \forall \gamma \in F \left(\varphi_s(x,\gamma) = \varphi_{s'}(x,\gamma) \right)\}$

let $A = \Theta[B]$. $\nu(A) = 1$ by assumption. $(\nu(X^2 \setminus A) = 0)$

Given $A'_i \subseteq A$ (note: $\nu(A) = 1$), and $\gamma \in F$:

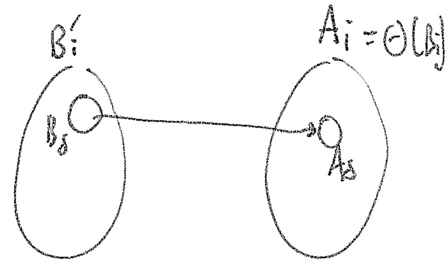
let $B'_i = \Theta^{-1}[A'_i]$.

$(A'_i = \Theta[B'_i])$.

For $\delta \in \Delta$ let $B_\delta = \{x \in B'_i \mid \varphi_s(x,\gamma) = \varphi_{s'}(x,\gamma) = \delta\}$.

$= \{x \in B'_i \mid \gamma \cdot \Theta(x) = \Theta(\delta \cdot x)\}$

$B'_i = \bigcup B_\delta$. $A_\delta = \Theta[B_\delta]$ $A'_i = \dot{\bigcup_{\delta \in \Delta}} A_\delta$



$$\nu(A'_i) = \mu(\Theta^{-1}[A'_i]) = \mu(B'_i)$$

$$= \sum_{\delta \in \Delta} \mu(B_\delta)$$

$$= \sum_{\delta \in \Delta} \mu(\delta \cdot B_\delta)$$

$$= \sum_{\delta \in \Delta} \mu(\{\delta \cdot x \mid x \in B_\delta\})$$

$$= \sum_{\delta \in \Delta} \nu(\Theta[\{\delta \cdot x \mid x \in B_\delta\}]) \quad \text{since } \Theta \text{ 1-1}$$

since by defn of $x \in B_\delta$ $\Theta(\delta \cdot x) = \gamma \cdot \Theta(x)$

$$= \sum_{\delta \in \Delta} \nu(\gamma \cdot \Theta[\{x \mid x \in B_\delta\}]) \overset{\text{def. of } A_\delta}{=} \sum_{\delta \in \Delta} \nu(\gamma \cdot A_\delta) = \nu\left(\bigcup_{\delta \in \Delta} \gamma \cdot A_\delta\right) \text{ by disjointness}$$

$$= \nu\left(\gamma \cdot \bigcup_{\delta \in \Delta} A_\delta\right) = \nu(\gamma \cdot A'_i).$$