Easton function and large cardinals

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Definition 1  \( F \) is an Easton function if for all regular cardinals \( \kappa, \mu \):

(i) If \( \kappa < \mu \), then \( F(\kappa) \leq F(\mu) \);
(ii) \( \kappa < \text{cf}(F(\kappa)) \).

By results of W.B. Easton, if we assume GCH then every Easton function \( F \) is a continuum function \( (\kappa \mapsto 2^\kappa) \) on regular cardinals in some cofinality-preserving generic extension.

Note however that the (product-style) Easton’s forcing will typically destroy large cardinals which existed in \( V \). And also, some \( F \)'s are incompatible with large cardinals.

Hence this is a result concerning what is provable about the continuum function in \textit{ZFC}, not extensions of the type \textit{ZFC} + \varphi, where \( \varphi \) is a large cardinal axiom.
**Question:** How can we generalize Easton’s result to large cardinals?

Ideally, we can formulate the task as follows:

**Given:** $V$ satisfying GCH, a property $\varphi(x)$ defining a large cardinal (such as “$x$ is measurable”), a class $E$ of cardinals satisfying $\varphi(x)$, and an Easton function $F$.

**Aim:** Find a cofinality preserving extension $V^*$ realising $F$ and preserving $\varphi(x)$ for all elements in $E$.

We look for the most general properties of $F$ which $F$ needs to satisfy to allow the above construction.
The question of optimality.

(1) Large cardinal strength. Typically, to show that $\kappa \in E$ still satisfies the large cardinal property $\varphi(x)$ in $V^*$, we will need to assume that cardinals in $E$ satisfy a stronger large cardinal property $\varphi_0(x)$ back in $V$. The result will be optimal if consistency strength of $\varphi_0(x)$ will be optimal for $\varphi(x)$.

For instance if $\varphi(x)$ is “to be measurable” and $2^\kappa = \kappa^{++}$ in $V^*$, then $\kappa$ needs to be more than measurable back in $V$ ($o(\kappa) = \kappa^{++}$).
(2) \textit{Restrictions on $F$.} Due to reflection properties at large cardinals, not all $F$’s which worked for Easton in the context of ZFC can work in the large cardinal context. The result will be optimal if every $F$ which does not contradict the consequences of the existence of large cardinals in $E$ can be realised.

For instance $2^\kappa > \kappa^+$ and $\kappa$ is measurable in $V^*$ imply that on a large set below $\kappa$, GCH must fail. Thus if back in $V$, $F$ prescribes that $F(\alpha) > \alpha^+$ on a large set below $\kappa$, then an optimal construction should realise $F$ (if this restriction is the only one governing continuum function for measurable cardinals).
For Theorem 2 below, let us assume that $F$ satisfies the property that every measurable cardinal $\kappa$ in $V$ is a closure point of $F$: $(\forall \alpha < \kappa) F(\alpha) < \kappa$ (so that $\kappa$ remains strongly inaccessible if $F$ is realised).

Note we say that $\kappa$ is $F(\kappa)$-strong (or $F(\kappa)$-hypermeasurable) when $H(F(\kappa))$ is included in $M$ for some $j : V \rightarrow M$ with $j(\kappa) > F(\kappa)$.

**Theorem 2 (Friedman,H.)** Let $F$ be an Easton function and $E = \{\kappa \mid \kappa \text{ is } F(\kappa) \text{-strong}\}$. Then if for every $\kappa \in E$ there is an embedding $j$ witnessing $F(\kappa)$-strength of $\kappa$ such that

$$F(\kappa) \leq j(F)(\kappa),$$

then there is a cofinality-preserving extension $V^*$ realising $F$ where every cardinal in $E$ remains measurable.
This theorem is “almost optimal” in both senses mentioned above.

- The consistency strength of $2^\kappa = F(\kappa)$ if $F(\kappa) > \kappa^+$ is by work of M.Gitik and W.Mitchell only slightly weaker than $F(\kappa)$-strength.
- If $\kappa$ is measurable and $j : V \rightarrow M$ is a measure ultrapower and embedding, then $2^\kappa \leq (2^\kappa)^M$; we can write this as
  
  $$C(\kappa) \leq j(C)(\kappa),$$

  where $C$ denotes the continuum function in $V$. It follows that “morally” our assumption that there exists $j$ such that $F(\kappa) \leq j(F')(\kappa)$ is necessary.
**Sketch of proof:** Iterate reverse-Easton style products composed of Cohens and Sacks with iteration points given by the closure points of $F$. Show that in the generic extension, one can lift the embedding $j$ ensured by the assumption on elements in $E$. Key points:

- Use Sacks($\alpha, F(\alpha)$) at every iteration point (and Cohens elsewhere). The inclusion of the Sacks forcing at this point allows for uniform lifting (using $\kappa^+$-fusion of Sacks at $\kappa$, and the “tuning-fork argument” of S.Friedman and K.Thompson).

- To fill in generics **in the middle interval** $[\kappa, F(\kappa)]$ on the $M$-side, the construction essentially needs that $\kappa$ is $F(\kappa)$-strong back in $V$.

The case when $F(\kappa)$ is singular is particularly tricky: A two-dimensional matrix of partial master conditions must be constructed in order to show that the intersection of a generic on the $V$ side with $M$ will give a generic.
More optimality in a special case (work in progress).

**Theorem 3 (H.)** Assume that $V$ satisfies GCH and $F$ is an Easton function. Let $X$ be the class

$$X = \{ \kappa \mid F(\kappa) = \kappa^{++}, \exists j : V \to M \text{ s.t. } (\kappa^{++})^M = \kappa^{++}$$

and $F(\kappa) \leq j(F')(\kappa) \}.$

Assume further that all elements in $X$ are closure points of $F$.

Then there is a cofinality-preserving forcing $P$ such that $V^P$ realises $F$ and satisfies for every $\kappa \in X$:

$$2^\kappa = \kappa^{++} \text{ and } \kappa \text{ is measurable.}$$
Theorem is optimal in the following sense:

- It achieves the realisation of $F$ and preservation measurability of $\kappa$ from the optimal consistency strength of $o(\kappa) = \kappa^{++}$, while realising $F$. 
Sketch of proof: Modify the above forcing to include Sacks not only at closure points of $F$, but also at double successors of closure points.

The use of Sacks at $\kappa^{++}$ of $M$ enables us to use a fusion construction meeting $\kappa^{++}$-many dense-open sets in the forcing $\text{Sacks}(\kappa^{++}, j(F)(\kappa))$, and to derive a master-condition. Using homogeneity of Sacks forcing, this ensure the existence of a generic for this stage of construction.
Generalization to models where cofinality changes. A natural generalization of above constructions considers cardinal-preserving forcings which change cofinalities to obtain a model where $F$ is realised and instead of keeping measurability of $\kappa$’s in $E$, all these $\kappa$’s are turned into singular strong limit cardinals of cof $\omega$. This can be used to derive some results concerning interactions between cardinals failing SCH and the continuum function.

**Definition 4** We say that an Easton function $F$ is toggle-like if $F(\alpha) \in \{\alpha^+, \alpha^{++}\}$ for every regular $\alpha$, and $F(\kappa^+) = \kappa^{++}$ for every $\kappa$ a Mahlo cardinal.
Theorem 5 (H.) Let $F$ be a toggle-like Easton function and let $X$ be any subclass of $\{\kappa \mid \kappa$ is $\kappa^{++}$-strong and $F(\kappa) = \kappa^{++}\}$. Then there is a cardinal-preserving extension $V^P$ which realises $F$ and where all elements in $X$ are strong limit singular cardinals of $\text{cof} \omega$. In particular, SCH fails exactly at elements of $X$.

Corollary 6 Unlike in the case of singular strong limit cardinals of uncountable cofinalities, no reflection is provable for singular strong limit cardinals of $\text{cof} \omega$ (where the reflection is formulated in terms of preservation/failure of GCH below $\kappa$).
Sketch of proof. The forcing is a two-stage forcing. Let $\theta_E$ denote all $\kappa \in X$ such that there is a witnessing embedding $j$ with $j(F)(\kappa) = \kappa^+$. Let $\theta_P = X \setminus \theta_E$.

- First force reverse-Easton style to realise $F$ everywhere except at cardinals in $\theta_E$. All cardinal in $\theta_P$ will satisfy $2^\kappa = \kappa^{++}$ and remain measurable (using the fact that every embedding witnessing $\kappa^{++}$-strength of $\kappa \in \theta_P$ satisfies $F(\kappa) \leq j(F)(\kappa)$).
- Iterate Prikry-type forcings with Easton support to simultaneously singularize all cardinals in $\theta_P$ and blow up the powerset and singularize cardinals in $\theta_E$ (using extender-based Prikry forcing for elements in $\theta_E$ and the simple Prikry forcing for elements in $\theta_P$).
Open questions.

(1) Is possible to extend the construction for relevant $\kappa$'s from the optimal cardinal-strength to include the cases where $F(\kappa) > \kappa^{++}$?

(2) Is it possible to extend the construction for relevant $\kappa$’s concerning the failure of SCH to include the cases where $F(\kappa) > \kappa^{++}$?