# Easton function and large cardinals

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**Definition 1** *$F$ is an* Easton function *if for all regular cardinals $\kappa, \mu$:*

*(i) If $\kappa < \mu$, then $F(\kappa) \leq F(\mu)$;*
*(ii) $\kappa < \mathrm{cf}(F(\kappa))$.*

By results of W.B.Easton, if we assume GCH then every Easton function F is a *continuum function* ($\kappa \mapsto 2^\kappa$) on regular cardinals in some cofinality-preserving generic extension.

Note however that the (product-style) Easton's forcing will typically destroy large cardinals which existed in $V$. And also, some $F$'s are incompatible with large cardinals.

Hence this is a result concerning what is provable about the continuum function in **ZFC**, not extensions of the type **ZFC** $+$ $\varphi$, where $\varphi$ is a large cardinal axiom.

**Question:** How can we generalize Easton's result to large cardinals?

Ideally, we can formulate the task as follows:

**Given:** $V$ satisfying GCH, a property $\varphi(x)$ defining a large cardinal (such as "$x$ is measurable"), a class $E$ of cardinals satisfying $\varphi(x)$, and an Easton function $F$.

**Aim:** Find a cofinality preserving extension $V^*$ realising $F$ and preserving $\varphi(x)$ for all elements in $E$.

We look for the most general properties of $F$ which $F$ needs to satisfy to allow the above construction.

**The question of optimality.**

(1) *Large cardinal strength.* Typically, to show that $\kappa \in E$ still satisfies the large cardinal property $\varphi(x)$ in $V^*$, we will need to assume that cardinals in $E$ satisfy a stronger large cardinal property $\varphi_0(x)$ back in $V$. The result will be optimal if consistency strength of $\varphi_0(x)$ will be optimal for $\varphi(x)$.

For instance if $\varphi(x)$ is "to be measurable" and $2^\kappa = \kappa^{++}$ in $V^*$, then $\kappa$ needs to be more than measurable back in $V$ ($o(\kappa) = \kappa^{++}$).

(2) *Restrictions on $F$.* Due to reflection properties at large cardinals, not all $F$'s which worked for Easton in the context of ZFC can work in the large cardinal context.

The result will be optimal if every $F$ which does not contradict the consequences of the existence of large cardinals in $E$ can be realised.

For instance $2^\kappa > \kappa^+$ and $\kappa$ is measurable in $V^*$ imply that on a large set below $\kappa$, GCH must fail. Thus if back in $V$, $F$ prescribes that $F(\alpha) > \alpha^+$ on a large set below $\kappa$, then an optimal construction should realise $F$ (if this restriction is the only one governing continuum function for measurable cardinals).

For Theorem 2 below, let us assume that $F$ satisfies the property that every measurable cardinal $\kappa$ in $V$ is a closure point of $F$: $(\forall\alpha < \kappa)F(\alpha) < \kappa$ (so that $\kappa$ remains strongly inaccessible if $F$ is realised).

Note we say that $\kappa$ is $F(\kappa)$-strong (or $F(\kappa)$-hypermeasurable) when $H(F(\kappa))$ is included in $M$ for some $j : V \to M$ with $j(\kappa) > F(\kappa)$.

**Theorem 2 (Friedman,H.)** *Let $F$ be an Easton function and $E = \{\kappa \,|\, \kappa$ is $F(\kappa)$-strong$\}$. Then if for every $\kappa \in E$ there is an embedding $j$ witnessing $F(\kappa)$-strength of $\kappa$ such that*

$$F(\kappa) \leq j(F)(\kappa),$$

*then there is a cofinality-preserving extension $V^*$ realising $F$ where every cardinal in $E$ remains measurable.*

This theorem is "almost optimal" in both senses mentioned above.

- The consistency strength of $2^\kappa = F(\kappa)$ if $F(\kappa) > \kappa^+$ is by work of M.Gitik and W.Mitchell only slightly weaker than $F(\kappa)$-strength.
- If $\kappa$ is measurable and $j : V \to M$ is a measure ultrapower and embedding, then $2^\kappa \leq (2^\kappa)^M$; we can write this as

$$\mathfrak{C}(\kappa) \leq j(\mathfrak{C})(\kappa),$$

  where $\mathfrak{C}$ denotes the continuum function in $V$. It follows that "morally" our assumption that there exists $j$ such that $F(\kappa) \leq j(F)(\kappa)$ is necessary.

*Sketch of proof:* Iterate reverse-Easton style products composed of Cohens and Sacks with iteration points given by the closure points of $F$. Show that in the generic extension, one can lift the embedding $j$ ensured by the assumption on elements in $E$. Key points:

- Use Sacks$(\alpha, F(\alpha))$ at every iteration point (and Cohens elsewhere). The inclusion of the Sacks forcing at this point allows for uniform lifting (using $\kappa^+$-fusion of Sacks at $\kappa$, and the "tuning-fork argument" of S.Friedman and K.Thompson).
- To fill in generics **in the middle interval** $[\kappa, F(\kappa)]$ on the $M$-side, the construction essentially needs that $\kappa$ is $F(\kappa)$-strong back in $V$.

The case when $F(\kappa)$ is singular is particularly tricky: A two-dimensional matrix of partial master conditions must be constructed in order to show that the intersection of a generic on the $V$ side with $M$ will give a generic.

More optimality in a special case (work in progress).

**Theorem 3 (H.)** *Assume that $V$ satisfies GCH and $F$ is an Easton function. Let $X$ be the class*

$$X = \{\kappa \mid F(\kappa) = \kappa^{++}, \exists j : V \to M \text{ s.t. } (\kappa^{++})^M = \kappa^{++} \text{ and } F(\kappa) \leq j(F)(\kappa)\}.$$

*Assume further that all elements in $X$ are closure points of $F$.*

*Then there is a cofinality-preserving forcing $P$ such that $V^P$ realises $F$ and satisfies for every $\kappa \in X$:*

$$2^\kappa = \kappa^{++} \text{ and } \kappa \text{ is measurable.}$$

Theorem is optimal in the following sense:

- It achieves the realisation of $F$ and preservation measurability of $\kappa$ from the optimal consistency strength of $o(\kappa) = \kappa^{++}$, while realising $F$.

*Sketch of proof:* Modify the above forcing to include Sacks not only at closure points of $F$, but also at double successors of closure points.

The use of Sacks at $\kappa^{++}$ of $M$ enables us to use a fusion construction meeting $\kappa^{++}$-many dense-open sets in the forcing Sacks$(\kappa^{++}, j(F)(\kappa))$, and to derive a master-condition. Using homogeneity of Sacks forcing, this ensure the existence of a generic for this stage of construction.

**Generalization to models where cofinality changes.** A natural generalization of above constructions considers cardinal-preserving forcings which change cofinalities to obtain a model where $F$ is realised and instead of keeping measurability of $\kappa$'s in $E$, all these $\kappa$'s are turned into singular strong limit cardinals of cof $\omega$. This can be used to derive some results concerning interactions between cardinals failing SCH and the continuum function.

**Definition 4** *We say that an Easton function $F$ is* toggle-like *if $F(\alpha) \in \{\alpha^+, \alpha^{++}\}$ for every regular $\alpha$, and $F(\kappa^+) = \kappa^{++}$ for every $\kappa$ a Mahlo cardinal.*

**Theorem 5 (H.)** *Let $F$ be a toggle-like Easton function and let $X$ be any subclass of $\{\kappa \,|\, \kappa$ is $\kappa^{++}$-strong and $F(\kappa) = \kappa^{++}\}$. Then there is a cardinal-preserving extension $V^P$ which realises $F$ and where all elements in $X$ are strong limit singular cardinals of cof $\omega$. In particular, SCH fails exactly at elements of $X$.*

**Corollary 6** *Unlike in the case of singular strong limit cardinals of uncountable cofinalities, no reflection is provable for singular strong limit cardinals of cof $\omega$ (where the reflection is formulated in terms of preservation/failure of GCH below $\kappa$).*

*Sketch of proof.* The forcing is a two-stage forcing. Let $\theta_E$ denote all $\kappa \in X$ such that there is a witnessing embedding $j$ with $j(F)(\kappa) = \kappa^+$. Let $\theta_P = X \setminus \theta_E$.

- First force reverse-Easton style to realise $F$ everywhere except at cardinals in $\theta_E$. All cardinal in $\theta_P$ will satisfy $2^\kappa = \kappa^{++}$ and remain measurable (using the fact that every embedding witnessing $\kappa^{++}$-strength of $\kappa \in \theta_P$ satisfies $F(\kappa) \leq j(F)(\kappa)$).
- Iterate Prikry-type forcings with Easton support to simultaneously singularize all cardinals in $\theta_P$ and blow up the powerset and singularize cardinals in $\theta_E$ (using extender-based Prikry forcing for elements in $\theta_E$ and the simple Prikry forcing for elements in $\theta_P$).

**Open questions.**

(1) Is possible to extend the construction for relevant $\kappa$'s from the optimal cardinal-strength to include the cases where $F(\kappa) > \kappa^{++}$?

(2) Is it possible to extend the construction for relevant $\kappa$'s concerning the failure of SCH to include the cases where $F(\kappa) > \kappa^{++}$?