Katětov order on Borel ideals

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Borel ideals

An *ideal* is a subset of $\mathcal{P}(\omega)$

- containing all finite sets
- closed under finite unions
- closed under subsets.

If $\mathcal{I}$ is an ideal, we denote by

- $\mathcal{I}^*$ the dual filter, and
- $\mathcal{I}^+$ the family of $\mathcal{I}$-positive sets.
- Given $X \in \mathcal{I}^+$, $\mathcal{I} \restriction X = \{I \cap X : I \in \mathcal{I}\}$.

An ideal is considered as a subspace of $\mathcal{P}(\omega)$, with the Cantor set (product) topology. So Borel, analytic, ... refer to this topology.

Examples of Borel ideals

- *nwd* . . . the ideal of nowhere dense subsets of the rationals
- $\mathcal{R}$ . . . the ideal generated by the cliques and free sets in the Random graph
- $\mathcal{Z}$ . . . the ideal of sets of asymptotic density zero.
- $\mathcal{ED}$ . . . the ideal on $\omega \times \omega$ generated by vertical sections and graphs of functions.
- $\mathcal{ED}_{fin}$ . . . the ideal $\mathcal{ED}$ restricted to the set below the diagonal.
- $fin \times fin$ . . . the Fubini product of *fin* with itself.
- *conv* is the ideal on the rationals in the unit interval generated by convergent sequences.

Katětov order

Definition

Given two ideals $\mathcal{I}$, $\mathcal{J}$, we say that *$\mathcal{I}$ is Katětov below $\mathcal{J}$* ($\mathcal{I} \leq_K \mathcal{J}$) if there is a function $f : \omega \longrightarrow \omega$ such that $f^{-1}[I] \in \mathcal{J}$ for every $I \in \mathcal{I}$.
We will say that $\mathcal{I}$ *is Katětov-Blass below* $\mathcal{J}$ ($\mathcal{I} \leq_{KB} \mathcal{J}$) if the function f above is finite-to-one.

The Katětov order is both downward and upward directed with the minimal element *fin*, the ideal of finite sets.

Moreover, an ideal $\mathcal{I}$ is *tall (dense)* if and only if $\mathcal{I} \not\leq_K$ *fin* ($\mathcal{I} \not\geq_K$ *fin*).

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Destructibility of ideals by forcing

Definition

Given an ideal $\mathcal{I}$ and a forcing notion $\mathbb{P}$, we say that $\mathbb{P}$ *destroys* $\mathcal{I}$ if there is a $\mathbb{P}$-name $\dot{X}$ for an infinite subset of ω such that

$$\Vdash_{\mathbb{P}} \text{"} I \cap \dot{X} \text{ is finite for every } I \in \mathcal{I}\text{"}.$$

Question: When does a given forcing destroy a given ideal?

Let Γ be the class of forcings $\mathbb{P}$ of the form $\mathit{Borel}(\omega^\omega)/I$, where I is a σ-ideal of Borel sets, such that $\mathbb{P}$ is proper, continuously homogeneous with the continuous reading of names.

Cohen, Random, Sacks, Miller, Laver, Mathias,... are all in Γ.

Trace ideals

Theorem (H.-Zapletal)

For every $\mathbb{P} \in \Gamma$ there is an ideal $\mathcal{J}_{\mathbb{P}}$ such that for any ideal $\mathcal{I}$,

$$\mathbb{P} \text{ destroys } \mathcal{I} \text{ if and only if } \mathcal{I} \leq_K \mathcal{J}_{\mathbb{P}}.$$

Theorem (H.-Zapletal)

For every $\mathbb{P} \in \Gamma$, the forcing $\mathcal{P}(\omega)/\mathcal{J}_{\mathbb{P}}$ is proper and

$$\mathcal{P}(\omega)/\mathcal{J}_{\mathbb{P}} \simeq \mathbb{P} * \dot{\mathbb{Q}},$$

where $\mathbb{Q}$ does not add reals.

Example: *nwd* is $\mathcal{J}_{Cohen}$.

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Ultrafilters

Let $\mathcal{U}$ be an ultrafilter and $\mathcal{U}^*$ the dual ideal. Then

- $\mathcal{U}$ is selective iff $\mathcal{ED} \not\leq_K \mathcal{U}^*$ iff $\mathcal{R} \not\leq_K \mathcal{U}^*$,
- $\mathcal{U}$ is a P-point iff $\mathit{fin} \times \mathit{fin} \not\leq_K \mathcal{U}^*$ iff $\mathit{conv} \not\leq_K \mathcal{U}^*$,
- $\mathcal{U}$ is a nowhere dense ultrafilter iff $\mathit{nwd} \not\leq_K \mathcal{U}^*$.
- $\mathcal{U}$ is a Q-point iff $\mathcal{ED}_{\mathit{fin}} \not\leq_{KB} \mathcal{U}^*$,
- $\mathcal{U}$ is rapid iff $\mathcal{I} \not\leq_{KB} \mathcal{U}^*$ for any analytic P-ideal $\mathcal{I}$,

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Embedding of $\mathcal{P}(\omega)/\mathit{fin}$ into Borel ideals with the Katětov order

Theorem (H.-Meza)

There is an embedding of $\mathcal{P}(\omega)/\mathit{fin}$ into Borel ideals with the Katětov order.

- There are antichains (families of pairwise incomparable ideals) of size $\mathfrak{c}$.
- There are increasing (and decreasing) chains of length $\mathfrak{b}$.

Search for minimal ideals in the Katětov order

Question: Is there a Katětov-minimal tall Borel ideal? Is there one which is locally minimal?

An Borel ideal $\mathcal{J}$ is *locally minimal* if for every Borel ideal $\mathcal{I}$ there is an $X \in \mathcal{I}^+$ such that $\mathcal{J} \leq_K \mathcal{I} \restriction X$.

Observation: Let $\mathcal{I}$ be an ideal.

$$\omega \longrightarrow (\mathcal{I}^+)^2_2 \text{ iff } \mathcal{R} \not\leq_K \mathcal{I}, \text{ i.e.}$$

$\mathcal{R}$ is locally minimal if the following has a positive answer:

Question: Is $\mathcal{I}^+ \not\longrightarrow (\mathcal{I}^+)^2_2$ true for every tall Borel ideal?

Theorem (H.-Meza)

$\mathcal{I}^+ \not\longrightarrow (\mathcal{I}^+)^2_2$ holds for all tall Borel ideals such that $\mathcal{P}(\omega)/\mathcal{I}$ is proper.

Category dichotomy

Theorem (H.)

Let $\mathcal{I}$ be a Borel ideal. Then either $\mathcal{I} \leq_K$ *nwd* or there is an $X \in \mathcal{I}^+$ such that $\mathcal{ED} \leq_K \mathcal{I} \restriction X$.

The dichotomy can really be understood as a trichotomy: Let $\mathcal{I}$ be a Borel ideal. Then

1. $\mathcal{I} \leq_K$ *nwd* or
2. there is an $X \in \mathcal{I}^+$ such that $\mathit{fin} \times \mathit{fin} \leq_K \mathcal{I} \restriction X$ or
3. there is an $X \in \mathcal{I}^+$ such that $\mathcal{ED}_{\mathit{fin}} \leq_{KB} \mathcal{I} \restriction X$.

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Submeasures

Definition

A *submesure* on a set X is a function $\varphi : \mathcal{P}(X) \longrightarrow [0, \infty]$ such that:

- $\varphi(\emptyset) = 0$
- $\varphi(A \cup B) \leq \varphi(A) + \varphi(B)$
- If $A \subseteq B$ then $\varphi(A) \leq \varphi(B)$
  If, φ satisfies
- $\varphi(A) = \sup\{\varphi(F) : F \in [A]^{<\omega}$ we say that φ is *lower semicontinuous*.

Ideals and submeasures

Definition

Given a lower semicontinuous submeasure φ on ω, let:

- $Fin(\varphi) = \{A \subseteq \omega : \varphi(A) < \infty\}$
- $Exh(\varphi) = \{A \subseteq \omega : \text{lím}_{n\to\infty} \varphi(A \cap [n, \infty)) = 0\}$

An ideal $\mathcal{I}$ is a *P-ideal* if for every sequence $\{I_n : n \in \omega\}$ of elements of $\mathcal{I}$ there is an $I \in \mathcal{I}$ such that $I_n \setminus I$ is finite for all $n \in \omega$.

Theorem

1. (Mazur) For every F_σ ideal $\mathcal{I}$ there is a l.s.c.s.m. φ such that $\mathcal{I} = Fin(\varphi)$.
2. (Solecki) For every analytic P-ideal $\mathcal{I}$ there is a l.s.c.s.m. φ such that $\mathcal{I} = Exh(\varphi)$.

Measure Dichotomy

$\mathcal{Z} = \{I \subseteq \omega : \text{lím}_{n\to\infty} \frac{|I\cap n|}{n} = 0\}$.

Let $\Omega = \{U \in Clop(2^\omega) : \mu(U) = 1/2\}$.

For $x \in 2^\omega$ let $I_x = \{U \in \Omega : x \in U\}$.

$\mathcal{S}$ is the ideal on ω generated by $\{I_x : x \in 2^\omega\}$.

Theorem (H.)

Let $\mathcal{I}$ be an analytic P-ideal. Then either $\mathcal{I} \leq_K \mathcal{Z}$ or there is an $\mathcal{I}$-positive set X such that $\mathcal{I} \restriction X \geq_K \mathcal{S}$.

Pathology of submeasures and ideals

Definition

The *degree of pathology* of a submeasure φ on a set X is defined as follows:

$$P(\varphi) = \frac{\varphi(X)}{\sup\{\mu(X) : \mu \text{ a measure on } X \text{ dominated by } \varphi\}}$$

Definition

An analytic P-ideal $\mathcal{I}$ is *non-pathological* if $\mathcal{I} = \mathit{Exh}(\varphi)$, where φ is non-pathological, i.e
$\varphi = \sup\{\mu : \mu \text{ a measure on } \omega \text{ dominated by } \varphi\}$.

Kelley's covering number

Definition

Let F be a set and $\mathcal{B} \subseteq \mathcal{P}(F)$. For any finite sequence $S = \langle S_0, \dots S_n \rangle$ of (not necessarily distinct) elements of $\mathcal{B}$ let

$$m(S) = \text{mín}\,\{|\{i \in n+1 : x \in S_i\}| : x \in F\}\,.$$

The *Kelley's covering number* $C(\mathcal{B})$ is defined by:

$$C(\mathcal{B}) = \sup\left\{\frac{m(S)}{|S|} : S \in [\mathcal{B}]^{<\omega}\right\}.$$

Crucial lemma

The following proposition, crucial for our proof of the Measure dichotomy, is a *quantitative version* of a result of Christensen:

Proposition

Let φ be a normalized submeasure on a (finite) set F and $\varepsilon > 0$. Let

$$\mathcal{A}_\varepsilon = \{A \subseteq F : \varphi(A) \leq \varepsilon\}.$$

Then

$$C(\mathcal{A}_\varepsilon) \geq 1 - \frac{1}{\varepsilon P(\varphi)}.$$

Fubini property

Definition

A Borel ideal $\mathcal{I}$ satisfies the *Fubini property* if for every Borel $A \subseteq \omega \times 2^\omega$ and $\varepsilon > 0$, if $\{n < \omega : \mu(\{x \in 2^\omega : (n,x) \in A\}) > \varepsilon\} \in \mathcal{I}^+$ then $\mu(\{x \in 2^\omega : \{n \in \omega : (n,x) \in A\} \in \mathcal{I}^+\}) \geq \varepsilon$.

Corollary

The following are equivalent for an analytic P-ideal $\mathcal{I}$

- For all $X \in \mathcal{I}^+$, $\mathcal{I} \restriction X \leq_K \mathcal{Z}$,
- For all $X \in \mathcal{I}^+$, $\mathcal{S} \not\leq_K \mathcal{I} \restriction X$,
- $\mathcal{I}$ has the Fubini property,
- $\mathcal{I}$ is non-pathological.

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Open problems

Problems

- Is there a (locally) Katětov-minimal tall Borel ideal?
- Is $\mathcal{I}^+ \not\longrightarrow (\mathcal{I}^+)^2_2$ true for every tall Borel ideal?
- Is it true that a Borel ideal $\mathcal{I}$ can be extended to an F_σ-ideal if and only if $conv \not\leq_K I$?
- Is it true that a Borel ideal $\mathcal{I}$ can be extended to an $F_{\sigma\delta}$-ideal if and only if $fin \times fin \not\leq_K \mathcal{I}$?
- Can every $F_{\sigma\delta}$-ideal be destroyed by a forcing not adding a dominating real? What about the density zero ideal $\mathcal{Z}$?