The Axiom of Real Blackwell Determinacy

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June 16th, 2009
Work in ZF+$\text{AC}_\omega(\mathbb{R})$.

$\text{AC}_\omega(\mathbb{R})$: for any countable family $(A_n \mid n \in \omega)$ of non-empty sets of reals, there is a function $f : \omega \rightarrow \mathbb{R}$ such that $f(n) \in A_n$ for every $n$. 
Perfect & imperfect information for games

Perfect information: Players know about the history of plays by both players.

E.g. Gale-Stewart games.

Imperfect information: Players do not know about what the other player does.

E.g. Blackwell games.
I’s turn.
I has played.
Il’s turn.
Il has played.
I’s turn again.
Gale-Stewart games

After infinitely many times... 

Player I wins if $x$ is in the payoff set and otherwise Player II wins.
I’s turn.
I has played.
II’s turn.
Il has played.
Blackwell games

I'm turn again.

\[
\begin{array}{c}
\emptyset \\
\frac{1}{2} & \frac{1}{2} \\
\langle 0 \rangle & \langle 1 \rangle \\
? & ? \\
\langle 00 \rangle & \langle 01 \rangle & \langle 10 \rangle & \langle 11 \rangle
\end{array}
\]
Blackwell games

After infinitely many times...
Blackwell games

Calculate the probability as below.

\[
\begin{align*}
\langle 0 \rangle & \quad \langle 01 \rangle \\
\langle 00 \rangle & \quad 2/3 & 1/3 & 1/2 \\
\langle 01 \rangle & \quad 1/2 \\
\langle 10 \rangle & \quad 1/4 & 3/4 \\
\langle 11 \rangle & \quad 3/4
\end{align*}
\]

\[
\begin{align*}
1/2 \times 2/3 &= 1/3 \\
1/2 \times 1/3 &= 1/6 \\
1/2 \times 1/4 &= 1/8 \\
1/2 \times 3/4 &= 3/8
\end{align*}
\]
Blackwell games

Player I wins if the probability of the payoff set is 1.
Player II wins if the probability of the payoff set is 0.
Formal definitions; Gale-Stewart games

Given $A \subseteq \omega 2$.

- $\sigma$ is a **strategy for I** if $\sigma : 2^{\text{Even}} \to 2$.
- $\tau$ is a **strategy for II** if $\tau : 2^{\text{Odd}} \to 2$.

For a strategy $\sigma$ for I and a strategy $\tau$ for II, define $\sigma * \tau : \omega \to 2$ as follows:

$$\begin{align*}
\sigma * \tau(n) &= \begin{cases}
\sigma(\sigma * \tau \upharpoonright n) & \text{if } n \text{ is even,} \\
\tau(\sigma * \tau \upharpoonright n) & \text{if } n \text{ is odd.}
\end{cases}
\end{align*}$$

- A strategy $\sigma$ for I is **winning in $A$** if for any strategy $\tau$ for II, $\sigma * \tau \in A$.
- A strategy $\tau$ for II is **winning in $A$** if for any strategy $\sigma$ for I, $\sigma * \tau \notin A$.
- $A$ is **determined** if either I or II has a winning strategy in $A$.
- AD: every $A \subseteq \omega 2$ is determined.

We can define AD$_X$ for any set $X$ in the same way.
Formal definitions; Blackwell games

- $\sigma$ is a \textit{mixed strategy for I} if $\sigma : 2^{\text{Even}} \rightarrow \text{Prob}(2)$.
- $\tau$ is a \textit{mixed strategy for II} if $\tau : 2^{\text{Odd}} \rightarrow \text{Prob}(2)$.
- For a mixed strategy $\sigma$ for I and a mixed strategy $\tau$ for II, define $\sigma * \tau : <\omega 2 \rightarrow \text{Prob}(2)$ as follows:

$$
\sigma * \tau(s) = \begin{cases} 
\sigma(s) & \text{if } \text{lh}(s) \text{ is even,} \\
\tau(s) & \text{if } \text{lh}(s) \text{ is odd.}
\end{cases}
$$

Then define $\mu_{\sigma,\tau} : <\omega 2 \rightarrow [0, 1]$ as follows:

$$
\mu_{\sigma,\tau}(s) = \prod_{i<\text{lh}(s)} \sigma * \tau(s \upharpoonright i)(s(i)).
$$

We can uniquely extend $\mu_{\sigma,\tau}$ to a Borel probability measure.
Given $A \subseteq \omega^2$.

- A mixed strategy $\sigma$ for $I$ is optimal in $A$ if for any mixed strategy $\tau$ for $II$, $\mu_{\sigma,\tau}(A) = 1$.
- A mixed strategy $\tau$ for $II$ is optimal in $A$ if for any mixed strategy $\sigma$ for $I$, $\mu_{\sigma,\tau}(A) = 0$.
- $A$ is Blackwell determined if either $I$ or $II$ has an optimal strategy in $A$.
- Bl-AD: every $A \subseteq \omega^2$ is Blackwell determined.

We can define Bl-AD$_X$ for a set $X$ if we have AC$_\omega(\mathbb{R} \times \omega X)$, especially we can define Bl-AD$_\mathbb{R}$ using AC$_\omega(\mathbb{R})$.

Note: there is another formulation of Blackwell games coming from game theory.
AD vs Bl-AD

1. AD implies Bl-AD.
2. $L(\mathbb{R}) \models \text{“AD } \iff \text{ Bl-AD”}$, especially they are equiconsistent. (Martin-Neeman-Vervoort)
3. Under AC, there is a set which is Blackwell-determined but not determined. (Hjorth)
4. Bl-AD implies that every set of reals is Lebesgue measurable. (Vervoort)
5. Bl-AD implies that $\omega_1$ is measurable. (Löwe)

Conjecture (Martin)

Bl-AD implies AD.
AD_\mathbb{R} vs BI-AD_\mathbb{R}

1. AD_\mathbb{R} implies BI-AD_\mathbb{R}.

2. BI-AD_\mathbb{R} implies that every relation on the reals can be uniformized, (equivalently, finite games on the reals are determined). (Löwe)

Theorem (de Kloet, Löwe and I.)

BI-AD_\mathbb{R} implies that \( \mathbb{R}^{\#} \) exists.
Especially, BI-AD_\mathbb{R} implies the consistency of AD.
What’s $\mathbb{R}^#$?

$\mathbb{R}^#$ is the complete theory of $L(\mathbb{R})$ in the language of set theory with constants for reals, $\mathbb{R}$ and $\omega$-many indiscernibles over $L(\mathbb{R})$ with appropriate properties.

Compared to $0^#$,

- We use weak Skolem terms instead of Skolem terms.
- Every existential sentence in $\mathbb{R}^#$ can be witnessed by a weak Skolem term with constants for indiscernibles only appearing in that sentence.

The latter condition is called “witness condition”.

**Theorem (Solovay)**

$\text{AD}_{\mathbb{R}}$ implies that $\mathbb{R}^#$ exists.
From Bl-AD$_\mathbb{R}$ to Con(AD)

1. By the result of Martin-Neeman-Vervoort, AD$_{L(\mathbb{R})}$ is true regarding that Bl-AD$_{L(\mathbb{R})}$ follows from Bl-AD$_\mathbb{R}$.
2. By our theorem, we get $\mathbb{R}^\#$.
3. From $\mathbb{R}^\#$, we get a set-sized elementary submodel of L(\mathbb{R}), which witnesses Con(AD).
Proof of Solovay’s theorem

**Theorem (Solovay)**

$\text{AD}_\mathbb{R}$ implies that $\mathbb{R}^\#$ exists.

**Step 1:** There is a $\sigma$-complete normal fine ultrafilter $U$ on $\mathcal{P}_{\omega_1}(\mathbb{R})$, where

1. **(Fineness)**

   $$(\forall x \in \mathbb{R}) \{ S \mid x \in S \} \in U,$$

2. **(Normality)**

   $$(\forall \langle A_x \in U \mid x \in \mathbb{R} \rangle) \triangle_{x \in \mathbb{R}} A_x = \{ S \mid (\forall x \in S) S \in A_x \} \in U.$$
Proof of Solovay’s theorem

**Theorem (Solovay)**

\( \text{AD}_{\mathbb{R}} \) implies that \( \mathbb{R}^\# \) exists.

**Step 1:** There is a \( \sigma \)-complete normal fine ultrafilter \( U \) on \( \mathcal{P}_{\omega_1}(\mathbb{R}) \).

**Proof.**

Given a subset \( A \) of \( \mathcal{P}_{\omega_1}(\mathbb{R}) \). Play the game

\[
\begin{array}{cccc}
& a_0 & a_2 & a_4 & \cdots \\
\hline
I & a_1 & a_3 & \cdots \\
\end{array}
\]

where each \( a_i \) is a finite subset of \( \mathbb{R} \).

Then

- Player I wins if \( \bigcup_{i \in \omega} a_i \in A \).
- Player II wins if \( \bigcup_{i \in \omega} a_i \notin A \).

Set \( A \in U \) if Player I has a winning strategy in the game above.
Proof of Solovay’s theorem, ctd.

Theorem (Solovay)

$AD_\mathbb{R}$ implies that $\mathbb{R}^\#$ exists.

Step 1: There is a $\sigma$-complete normal fine ultrafilter $U$ on $\mathcal{P}_{\omega_1}(\mathbb{R})$.

Step 2: Let $A$ be as follows: for a sentence $\phi$ in the language for $L(\mathbb{R})$,\[
\phi \in A \iff \{ S \in \mathcal{P}_{\omega_1}(\mathbb{R}) \mid L(S) \cap \mathbb{R} = S, \phi \in S^\# \} \in U.
\]

Then $A = \mathbb{R}^\#$. 
Proof of Theorem

**Theorem (de Kloet, Löwe and I.)**

$\text{BI-AD}_R$ implies that $\mathbb{R}^\#$ exists.

Especially, $\text{BI-AD}_R$ implies the consistency of $\text{AD}$.

**Idea:** Mimic the proof of Solovay’s theorem.

**Step 1:** There is a $\sigma$-complete normal fine ultrafilter $U$ on $\mathcal{P}_{\omega_1}(\mathbb{R})$.

Do the same argument in a Blackwell way.

**Step 2:** Exactly the same as before.
What can we do more with BI-AD$_R$?

Assume BI-AD$_R$.

1. Every set of reals has the perfect set property. (I.)
2. If every set of reals has the Baire property, then every set of reals is $\infty$-Borel. (I.)
3. If every set of reals has the Baire property, then every set of reals is Ramsey. (I.)

Question

Assume BI-AD$_R$.

1. Does every set of reals have the Baire property?
2. Is Blackwell-Wadge order well-founded?
3. Does AD$_R$ hold?
Vielen Dank für Ihre Aufmerksamkeit!!