The Axiom of Real Blackwell Determinacy

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June 16th, 2009

From now on···

Work in ZF+$\mathrm{AC}_\omega(\mathbb{R})$.

$\mathrm{AC}_\omega(\mathbb{R})$: for any countable family $(A_n \mid n \in \omega)$ of non-empty sets of reals, there is a function $f : \omega \to \mathbb{R}$ such that $f(n) \in A_n$ for every n.

Perfect & imperfect information for games

Perfect information: Players know about the history of plays by both players.

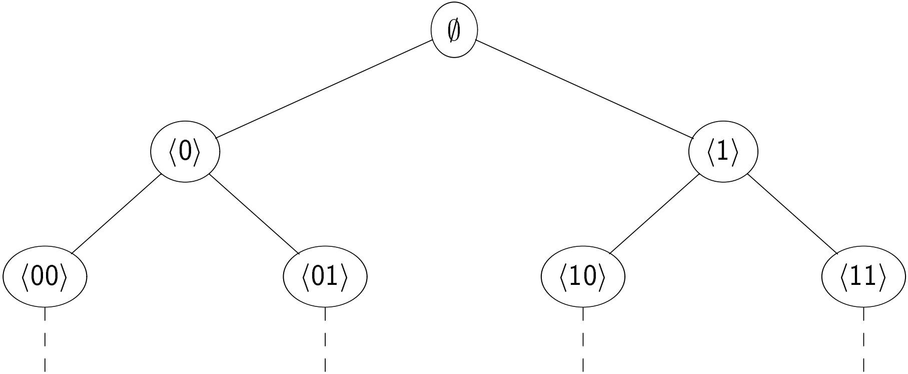
E.g. Gale-Stewart games.

Imperfect information: Players do not know about what the other player does.

E.g. Blackwell games.

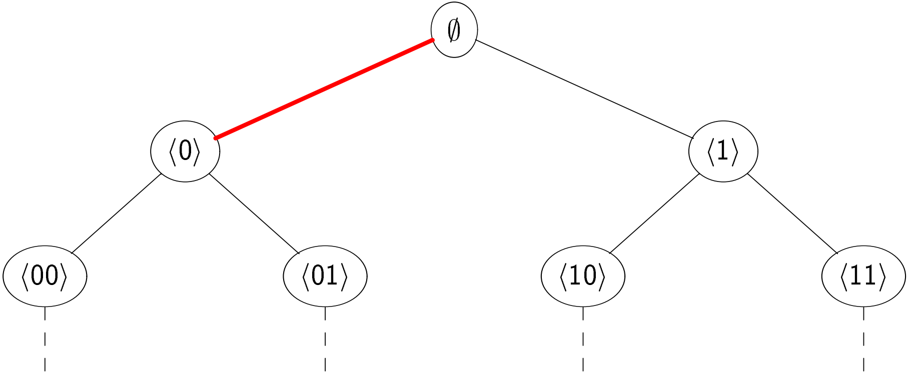
Gale-Stewart games

I's turn.



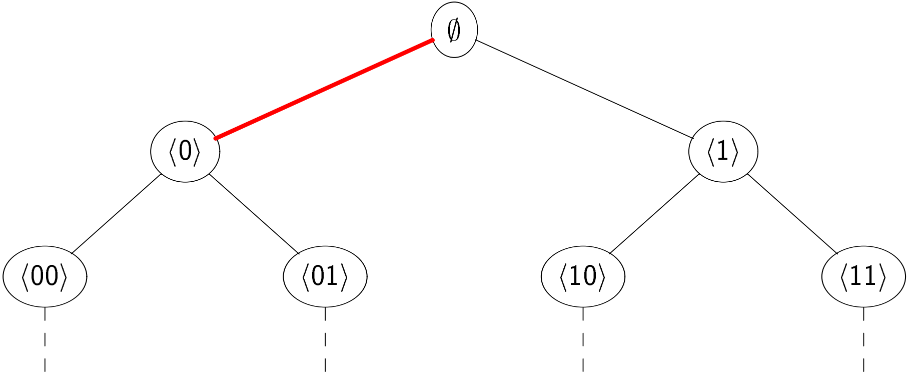
Gale-Stewart games

I has played.



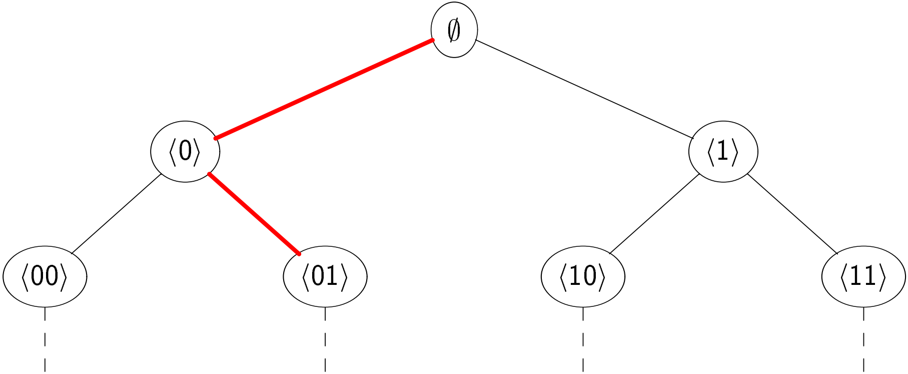
Gale-Stewart games

II's turn.



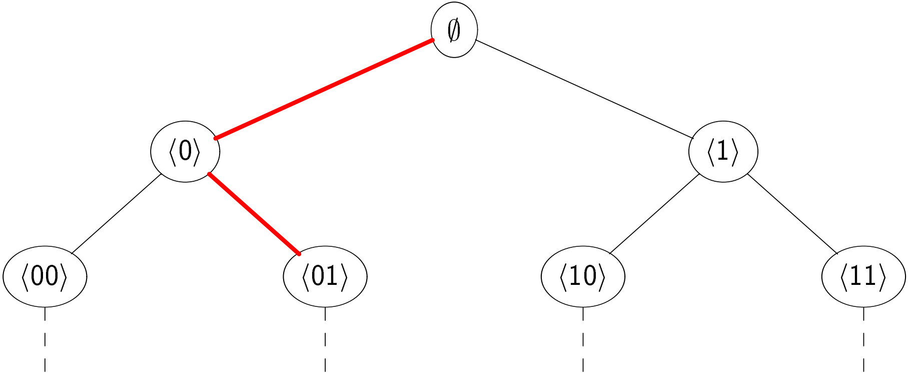
Gale-Stewart games

II has played.

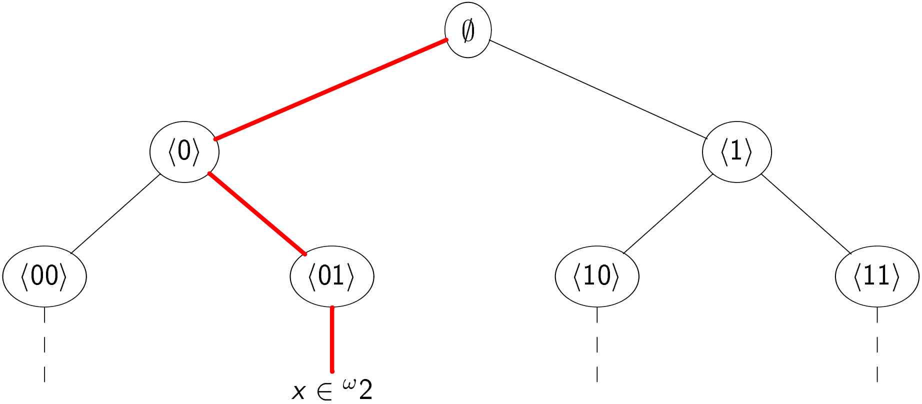


Gale-Stewart games

I's turn again.

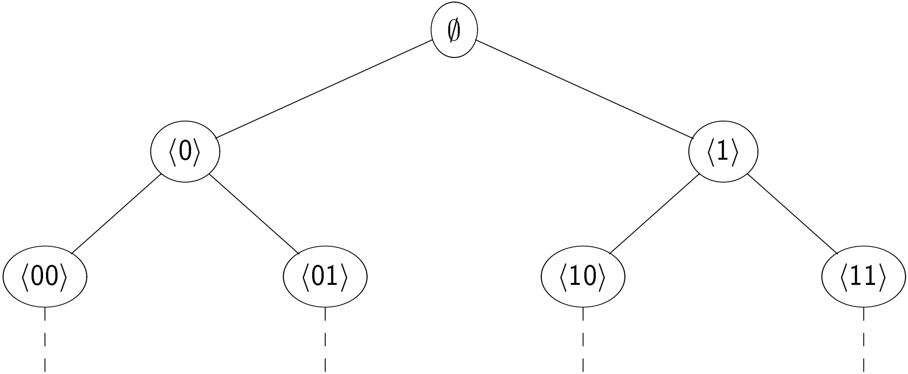


After infinitely many times· · ·

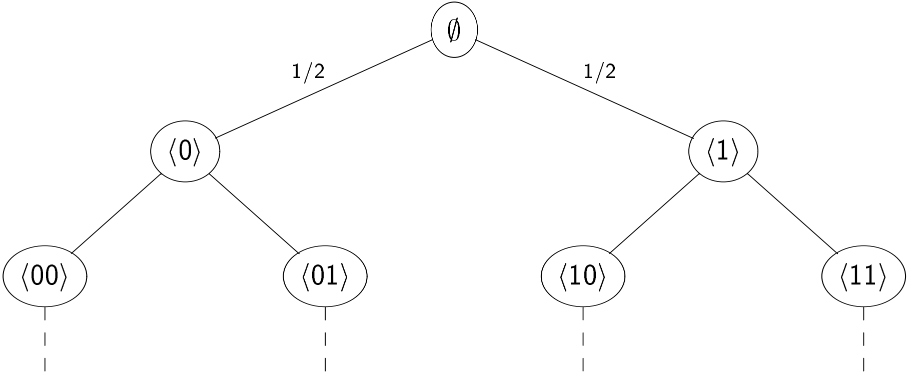


Player I wins if x is in the payoff set and otherwise Player II wins.

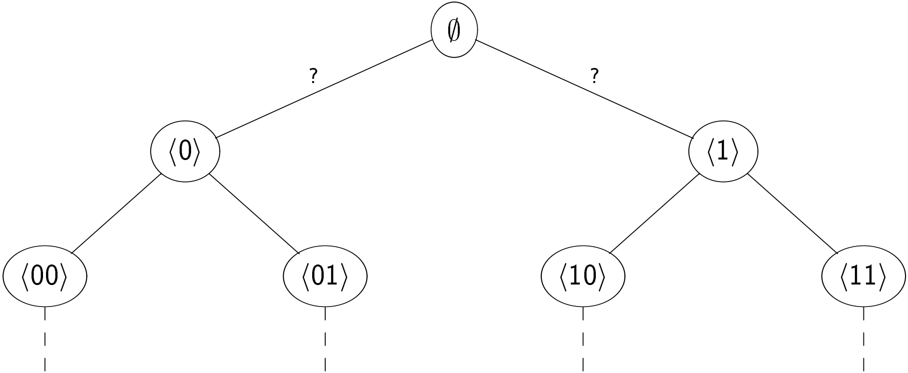
I's turn.



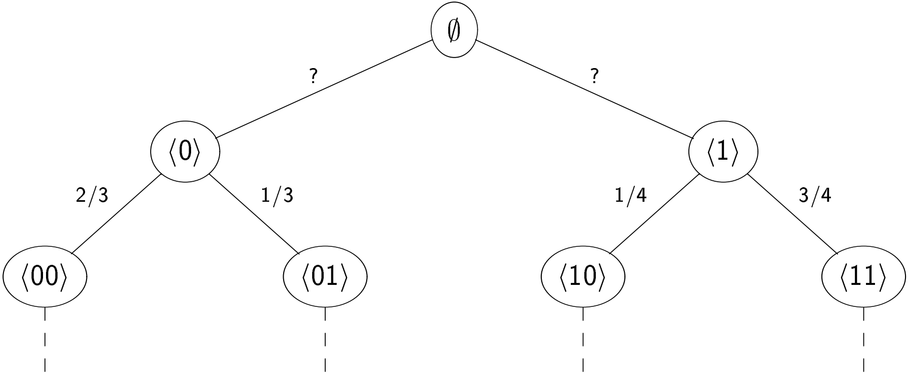
I has played.



II's turn.

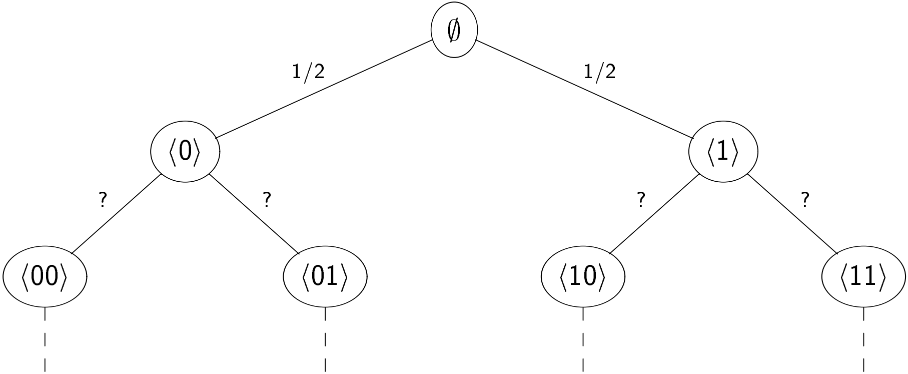


II has played.

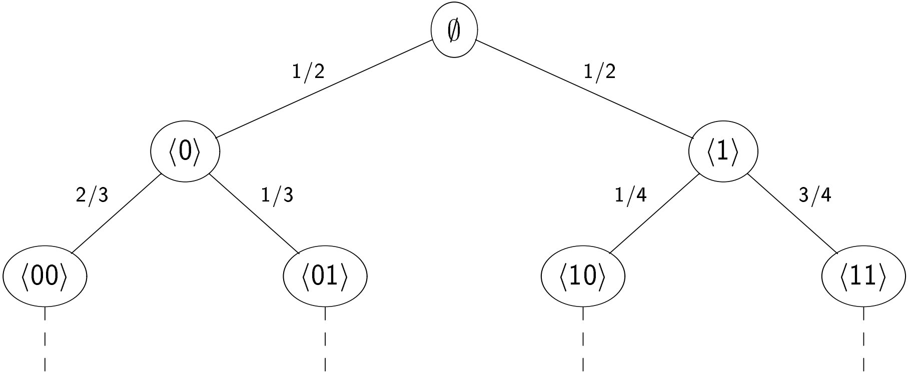


Blackwell games

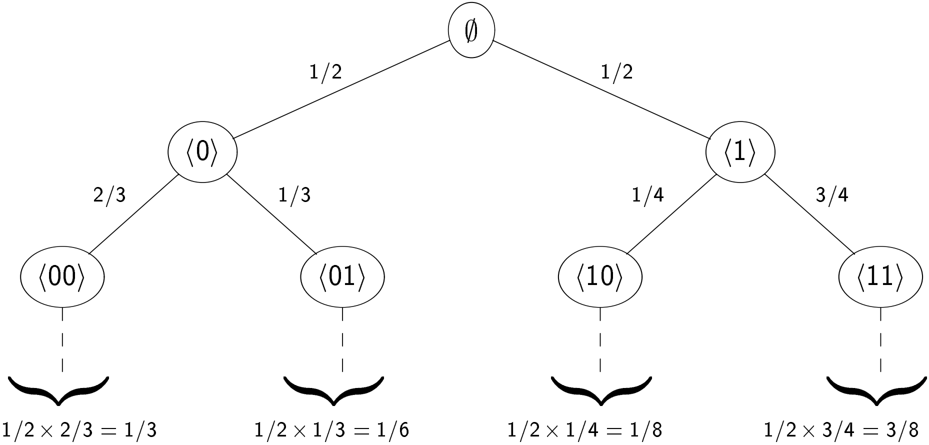
I's turn again.



After infinitely many times· · ·



Calculate the probability as below.



Blackwell games

Player I wins if the probability of the payoff set is 1.
Player II wins if the probability of the payoff set is 0.

Formal definitions; Gale-Stewart games

Given $A \subseteq {}^{\omega}2$.

- σ is a *strategy for I* if $\sigma\colon 2^{\text{Even}} \to 2$.
- τ is a *strategy for II* if $\tau\colon 2^{\text{Odd}} \to 2$.
- For a strategy σ for I and a strategy τ for II, define $\sigma * \tau\colon \omega \to 2$ as follows:

$$\sigma * \tau(n) = \begin{cases} \sigma(\sigma * \tau \restriction n) & \text{if } n \text{ is even,} \\ \tau(\sigma * \tau \restriction n) & \text{if } n \text{ is odd.} \end{cases}$$

- A strategy σ for I is *winning in A* if for any strategy τ for II, $\sigma * \tau \in A$.
- A strategy τ for II is *winning in A* if for any strategy σ for I, $\sigma * \tau \notin A$.
- A is *determined* if either I or II has a winning strategy in A.
- AD: every $A \subseteq {}^{\omega}2$ is determined.

We can define AD_X for any set X in the same way.

Formal definitions; Blackwell games

- σ is a *mixed strategy for I* if $\sigma\colon 2^{\mathrm{Even}} \to \mathrm{Prob}(2)$.
- τ is a *mixed strategy for II* if $\tau\colon 2^{\mathrm{Odd}} \to \mathrm{Prob}(2)$.
- For a mixed strategy σ for I and a mixed strategy τ for II, define $\sigma * \tau\colon {}^{<\omega}2 \to \mathrm{Prob}(2)$ as follows:

$$\sigma * \tau(s) = \begin{cases} \sigma(s) & \text{if } \mathrm{lh}(s) \text{ is even,} \\ \tau(s) & \text{if } \mathrm{lh}(s) \text{ is odd.} \end{cases}$$

Then define $\mu_{\sigma,\tau}\colon {}^{<\omega}2 \to [0,1]$ as follows:

$$\mu_{\sigma,\tau}(s) = \prod_{i<\mathrm{lh}(s)} \sigma * \tau(s \restriction i)(s(i)).$$

We can uniquely extend $\mu_{\sigma,\tau}$ to a Borel probability measure.

Formal definitions; Blackwell games ctd.

Given $A \subseteq {}^{\omega}2$.

- A mixed strategy σ for I is *optimal in A* if for any mixed strategy τ for II, $\mu_{\sigma,\tau}(A) = 1$.
- A mixed strategy τ for II is *optimal in A* if for any mixed strategy σ for I, $\mu_{\sigma,\tau}(A) = 0$.
- A is *Blackwell determined* if either I or II has an optimal strategy in A.
- Bl-AD: every $A \subseteq {}^{\omega}2$ is Blackwell determined.

We can define Bl-AD_X for a set X if we have $\text{AC}_{\omega}(\mathbb{R} \times {}^{\omega}X)$, especially we can define $\text{Bl-AD}_{\mathbb{R}}$ using $\text{AC}_{\omega}(\mathbb{R})$.

Note: there is another formulation of Blackwell games coming from game theory.

AD vs Bl-AD

1. AD implies Bl-AD.
2. $L(\mathbb{R}) \models$ "AD $\iff$ Bl-AD", especially they are equiconsistent. (Martin-Neeman-Vervoort)
3. Under AC, there is a set which is Blackwell-determined but not determined. (Hjorth)
4. Bl-AD implies that every set of reals is Lebesgue measurable. (Vervoort)
5. Bl-AD implies that ω_1 is measurable. (Löwe)

Conjecture (Martin)

Bl-AD implies AD.

$AD_{\mathbb{R}}$ vs Bl-$AD_{\mathbb{R}}$

1. $AD_{\mathbb{R}}$ implies Bl-$AD_{\mathbb{R}}$.
2. Bl-$AD_{\mathbb{R}}$ implies that every relation on the reals can be uniformized, (equivalently, finite games on the reals are determined). (Löwe)

Theorem (de Kloet, Löwe and I.)

Bl-$AD_{\mathbb{R}}$ implies that $\mathbb{R}^{\#}$ exists.
Especially, Bl-$AD_{\mathbb{R}}$ implies the consistency of AD.

What's $\mathbb{R}^\#$?

$\mathbb{R}^\#$ is the complete theory of $\mathrm{L}(\mathbb{R})$ in the language of set theory with constants for reals, $\mathbb{R}$ and ω-many indicernibles over $\mathrm{L}(\mathbb{R})$ with appropriate properties.
Compared to $0^\#$,

- We use weak Skolem terms instead of Skolem terms.
- Every existential sentence in $\mathbb{R}^\#$ can be witnessed by a weak Skolem term with constants for indicernibles only appearing in that sentence.

The latter condition is called "witness condition".

Theorem (Solovay)

$\mathrm{AD}_{\mathbb{R}}$ implies that $\mathbb{R}^\#$ exists.

From Bl-AD$_{\mathbb{R}}$ to Con(AD)

1. By the result of Martin-Neeman-Vervoort, $\mathrm{AD}^{\mathrm{L}(\mathbb{R})}$ is true regarding that $\text{Bl-AD}^{\mathrm{L}(\mathbb{R})}$ follows from Bl-AD$_{\mathbb{R}}$.
2. By our theorem, we get $\mathbb{R}^{\#}$.
3. From $\mathbb{R}^{\#}$, we get a set-sized elementary submodel of $\mathrm{L}(\mathbb{R})$, which witnesses Con(AD).

Theorem (Solovay)

$AD_{\mathbb{R}}$ implies that $\mathbb{R}^{\#}$ exists.

Step 1: There is a σ-complete normal fine ultrafilter U on $\mathcal{P}_{\omega_1}(\mathbb{R})$, where

1. (Fineness)

$$(\forall x \in \mathbb{R})\ \{S \mid x \in S\} \in U,$$

2. (Normality)

$$(\forall \langle A_x \in U \mid x \in \mathbb{R}\rangle)\ \triangle_{x\in\mathbb{R}} A_x = \{S \mid (\forall x \in S)\ S \in A_x\} \in U.$$

Proof of Solovay's theorem

Theorem (Solovay)

$AD_{\mathbb{R}}$ implies that $\mathbb{R}^{\#}$ exists.

Step 1: There is a σ-complete normal fine ultrafilter U on $\mathcal{P}_{\omega_1}(\mathbb{R})$.

Proof.

Given a subset A of $\mathcal{P}_{\omega_1}(\mathbb{R})$. Play the game

I	a_0		a_2		a_4	$\cdots$
II		a_1		a_3		$\cdots$

where each a_i is a finite subset of $\mathbb{R}$.

Then

- Player I wins if $\bigcup_{i \in \omega} a_i \in A$.
- Player II wins if $\bigcup_{i \in \omega} a_i \notin A$.

Set $A \in U$ if Player I has a winning strategy in the game above.

□

Proof of Solovay's theorem, ctd.

Theorem (Solovay)

$AD_{\mathbb{R}}$ implies that $\mathbb{R}^{\#}$ exists.

Step 1: There is a σ-complete normal fine ultrafilter U on $\mathcal{P}_{\omega_1}(\mathbb{R})$.
Step 2: Let A be as follows: for a sentence ϕ in the language for $\mathrm{L}(\mathbb{R})$,

$$\phi \in A \iff \{S \in \mathcal{P}_{\omega_1}(\mathbb{R}) \mid \mathrm{L}(S) \cap \mathbb{R} = S, \phi \in S^{\#}\} \in U.$$

Then $A = \mathbb{R}^{\#}$.

Proof of Theorem

Theorem (de Kloet, Löwe and I.)

Bl-$\mathrm{AD}_{\mathbb{R}}$ implies that $\mathbb{R}^{\#}$ exists.
Especially, Bl-$\mathrm{AD}_{\mathbb{R}}$ implies the consistency of AD.

Idea: Mimic the proof of Solovay's theorem.

Step 1: There is a σ-complete normal fine ultrafilter U on $\mathcal{P}_{\omega_1}(\mathbb{R})$.

Do the same argument in a Blackwell way.

Step 2: Exactly the same as before.

What can we do more with Bl-AD$_{\mathbb{R}}$?

Assume Bl-AD$_{\mathbb{R}}$.

1. Every set of reals has the perfect set property. (I.)
2. If every set of reals has the Baire property, then every set of reals is ∞-Borel. (I.)
3. If every set of reals has the Baire property, then every set of reals is Ramsey. (I.)

Question

Assume Bl-AD$_{\mathbb{R}}$.

1. Does every set of reals have the Baire property?
2. Is Blackwell-Wadge order well-founded?
3. Does AD$_{\mathbb{R}}$ hold?

Vielen Dank für Ihre Aufmerksamkeit!!