A precipitous club guessing ideal on ω_1

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Convention

1. Lim denotes the class of all limit ordinals.
2. Cof(θ) denotes the class of all limit ordinals of cofinality θ.
3. For sets X and Y of ordinals, we say that X is almost contained in Y and denote $X \subseteq^* Y$ iff $X = \varnothing$ or there exists a $\zeta < \sup(X)$ such that $X \setminus \zeta \subseteq Y$.
4. Throughout this talk, we assume that κ is an uncountable regular cardinal and S a stationary subset of κ consisting of limit ordinals.

Definition of club guessing sequences

Definition

We say that a sequence $\langle C_\delta : \delta \in S \rangle$ is a tail club guessing (TCG) sequence on S iff

1. for each $\delta \in S$, C_δ is an unbounded subset of δ, and
2. for every club subset D of κ, there exists a $\delta \in S$ such that $C_\delta \subseteq^* D$.

If $\vec{C}$ satisfies the same condition with $C_\delta \subseteq^* D$ replaced by $C_\delta \subseteq D$, we say that $\vec{C}$ is a fully club guessing (FCG) sequence on S. When $S = \kappa \cap \mathrm{Lim}$, we simply say that $\vec{C}$ is a TCG(FCG)-sequence on κ.

Relation between FCG and TCG-sequences

Trivially, every FCG-sequence is a TCG-sequence.

The converse fails. If $\langle C_\delta : \delta \in S \rangle$ is an FCG-sequence, then $\langle C_\delta \cup \{0\} : \delta \in S \rangle$ is a TCG-sequence which is not FCG-sequence. However, every TCG-sequence can be made FCG very easily.

Relation between fully and TCG-sequences (Cont.)

Theorem (T. Ishiu)

Let $\langle C_\delta : \delta \in S \rangle$ *be a* TCG-*sequence on* S*. Then, there exists a* $\zeta < \kappa$ *such that* $\langle C_\delta \setminus \zeta : \delta \in S \setminus (\zeta + 1) \rangle$ *is an* FCG-*sequence on* $S \setminus (\zeta + 1)$*.*

So, for every stationary subset S, there is an FCG-sequence on S iff there is a TCG-sequence on S.

Order type of a TCG-sequence

Definition

An ordinal ε is indecomposable iff for every ordinal $\alpha, \beta < \varepsilon$, $\alpha + \beta < \varepsilon$.

Definition

We say that a club guessing sequence $\langle C_\delta : \delta \in S \rangle$ has order type ε iff for club many $\delta \in S$, $\mathrm{otp}(C_\delta) = \varepsilon$.

Remark

Not every club guessing sequence has an order type.

Order type of a TCG-sequence (Cont.)

Fact

Let $\vec{C} = \langle C_\delta : \delta \in S \rangle$ *be a* TCG*-sequence on* S *such that* $\mathrm{otp}(C_\delta) < \delta$ *for club many* $\delta \in S$*. Then, there exist a* TCG*-sequence* $\vec{C}' = \langle C'_\delta : \delta \in S' \rangle$ *of some indecomposable order type* ε *on* $S' \subseteq S$ *such that* C'_δ *is a tail of* C_δ*. So,* $\vec{C} \restriction S'$ *and* $\vec{C}'$ *are essentially the same as* TCG*-sequences.*

Existence of a club guessing sequence

Trivially, $\clubsuit_\kappa$ implies the existence of an FCG-sequence on κ.
$\longrightarrow$ For every uncountable regular cardinal κ, it is consistent that there is an FCG-sequence on κ. However, S. Shelah proved the following surprising theorem.

Theorem (S. Shelah)

Let θ and κ be regular cardinals with $\theta^+ < \kappa$. Let S be a stationary subset of κ with $S \subseteq \mathrm{Cof}(\theta)$. Then, there is an FCG-sequence on S.

In particular, if $\kappa \geq \aleph_2$, then there exists an FCG-sequence on κ.

Strong club guessing sequences

Definition

We say that a sequence $\langle C_\delta : \delta \in S \rangle$ is a strong club guessing sequence on S iff

1. for each $\delta \in S$, C_δ is an unbounded subset of δ, and
2. for every club subset D of κ, there exists a club subset E of κ such that for every $\delta \in E \cap S$, $C_\delta \subseteq^* D$.

Remark

Its existence is consistent. For example, $V = L$ implies that κ carries a strong club guessing sequence iff κ is not ineffable.
It can be also added by forcing.

Definition of club guessing ideals

Definition

Let $\vec{C} = \langle C_\delta : \delta \in S \rangle$ be a TCG-sequence. We define the tail club guessing (TCG) filter TCG($\vec{C}$) associated with $\vec{C}$ as the filter on κ generated by the sets of the form $\{\delta \in S : C_\delta \subseteq^* D\}$ for club subset D of κ. The tail club guessing (TCG) ideal means the dual ideal $\check{\text{TCG}}(\vec{C})$ of the TCG-filter.

This is a typical ‘natural ideal’, which is definable possibly from simple parameters. Every TCG-ideal is κ-complete and normal.

Definition of precipitous ideals

Definition

Let I be an ideal on κ. We define an equivalence class $\sim_I$ on $\mathcal{P}(\kappa)$ by $X \sim_I Y$ iff $(X \setminus Y) \cup (Y \setminus X) \in I$.
Let $\mathcal{P}(\kappa)/I = \{[X]_I : X \subseteq \kappa \text{ and } X \notin I\}$, which is ordered by $[X]_I \leq [Y]_I$ iff $X \setminus Y \in I$.

Definition

An ideal I on κ is precipitous iff for every generic filter $G \subseteq \mathcal{P}(\kappa)/I$, the ultrapower $(V^\kappa \cap V)/G$ is well-founded.

Consistency of precipitous ideals on a small cardinal

Theorem (T. Jech, M. Magidor, W. Mitchell, and K. Prikry)

The following are equiconsistent.

1. *There is a measurable cardinal.*
2. *There is a precipitous ideal on* ω_1*.*
3. NS_{ω_1} *is precipitous.*

Saturatedness

Definition

An ideal I on κ is saturated iff $\mathcal{P}(\kappa)/I$ is κ^+-cc.

Fact

(R. Solovay) Every saturated ideal on κ is precipitous. In fact, saturated ideals behave nicely as precipitous ideals.

Motivation

There are (at least) two reasons to consider the precipitousness of natural ideals.

1. We can often apply special arguments to prove the precipitousness (e.g. the ones of M. Foreman, M. Magidor, and S. Shelah).
2. The definition of the ideal often gives us some information about the generic elementary embedding (Details on the next slide).

Generic elementary embedding from a TCG-ideal

Definition

Let W be some model and κ an uncountable regular cardinal in W. We say that a club subset C of κ is a fast club over W iff for every club subset $D \in W$ of κ, $C \subseteq^* D$.

Let $\vec{C}$ be a TCG-sequence on κ such that $\mathrm{T\check{C}G}(\vec{C})$ is precipitous. Let $j : V \to M$ be a generic elementary embedding built from $\mathrm{T\check{C}G}(\vec{C})$. Then there is a fast club subset $C \in M$ of κ over V. Moreover, if $\vec{C}$ has order type ε, then we can have $\mathrm{otp}(C) = \varepsilon$.

Generic elementary embedding from $\mathrm{NS}_\kappa \restriction S$

Suppose that $\mathrm{NS}_\kappa \restriction S$ is precipitous and there is no strong club guessing sequence on S. Let $j : V \to M$ be a generic elementary embedding built from $\mathrm{NS}_\kappa \restriction S$.
Then there is no fast club subset $C \in M$ of κ over V.

Thus, there is a clearly distinguishable difference between these two generic elementary embeddings.

Precipitous TCG-ideals that are equal to $\mathrm{NS}_\kappa \restriction S$

Theorem (H. Woodin)

It is consistent relative to $\mathrm{ZF} + AD$ *that there exists a strong club guessing sequence* $\vec{C}$ *on* ω_1 *such that* $\mathrm{T\check{C}G}(\vec{C})$ *is saturated.*

Theorem (P. Komjáth and M. Foreman)

It is consistent relative to a huge cardinal above an uncountable regular cardinal κ *that there exists a strong club guessing sequence* $\vec{C}$ *on a stationary subset* S *of* κ *such that* $\mathrm{T\check{C}G}(\vec{C})$ *is saturated.*

Precipitous TCG-ideals not of the form $\mathrm{NS}_\kappa \restriction S$

Both theorems in the previous slide produce precipitous tail club guessing ideals, but they are of the form $\mathrm{NS}_\kappa \restriction S$.
Is it necessary? $\longrightarrow$ NO!

Theorem (T. Ishiu)

It is consistent relative to a Woodin cardinal above an uncountable regular cardinal κ that every TCG*-ideal on κ is precipitous and there is no strong club guessing sequence on κ.*

The witnessing model is obtained by Levy-collapse of the Woodin cardinal to κ^+. So, there are many different TCG-sequences on κ.

Questions

Question

What is the consistency strength of the existence of a precipitous TCG-ideal?

Question

For an uncountable regular cardinal κ, if NS_κ is precipitous, then is there a precipitous TCG-ideal on κ? Vice versa?

I will present the solutions to these questions on the rest of my talk.

Strategy of Jech, Magidor, Mitchell, and Prikry

Recall how they built a model from a measurable cardinal in which NS_{ω_1} is precipitous.

1. Let κ be a measurable cardinal.
2. Levy-collapse κ to ω_1. Then, there is a precipitous ideal on ω_1.
3. Keep shooting clubs by countable conditions so that the ideal remains precipitous but it becomes NS_{ω_1}.

Remark

The fact that this indeed works is far from trivial.

Shooting a TCG-measure one set

Definition

Let $\vec{C} = \langle C_\delta : \delta \in S \rangle$ be a TCG-sequence. For a TCG($\vec{C}$)-positive subset X of κ, the poset $R(\vec{C}, X)$ to shoot a TCG($\vec{C}$)-measure one set through X is defined as the set of all closed bounded subsets p of ω_1 such that for every $\delta \in p \cap (S \setminus X)$, $C_\delta \not\subseteq^* p$. Ordered by end-extension.

$R(\vec{C}, X)$ is proper and forces that $\vec{C}$ remains a TCG-sequence on S and $X \in \mathrm{TCG}(\vec{C})$.

The forcing

Let $P = \mathrm{Coll}(\omega, {<}\kappa)$. In V^P, there is a precipitous ideal I on ${\omega_1}^{V^P} = \kappa$.

In V^P, let Q be the countable support iteration that

1. generically adds a TCG-sequence $\vec{C}$ at the zero-th stage, and
2. shoots $\mathrm{TCG}(\vec{C})$-measure one sets through all elements of $\check{I}$ at the remaining stages. (In fact, I changes as we extend the model, but it remains precipitous).

Then, in $V^{P*\dot{Q}}$, I is an $\aleph_1$-complete normal precipitous ideal and $I \subseteq \mathrm{T\check{C}G}(\vec{C})$.

$I = \text{T\v{C}G}(\vec{C})$

NS_{ω_1} is the least $\aleph_1$-complete normal ideal on ω_1, so the precipitous ideal in the model of JMMP is equal to NS_{ω_1}.

But this argument does not work for $\text{T\v{C}G}(\vec{C})$. Then does $I = \text{T\v{C}G}(\vec{C})$ really hold?

Well, the forcing is smart enough to guarantee it (though it took me long time to realize the fact).

A precipitous TCG-ideal from a measurable cardinal

So, we obtained the following result.

Theorem (T. Ishiu)

It is consistent relative to a measurable cardinal that there is a precipitous TCG*-ideal on* ω_1 *that is not a restriction of* NS_{ω_1}.

It solves our first question: the existence of a precipitous TCG-ideal (on ω_1) is equiconsistent to that of a measurable cardinal.

Toward the second question

Now, what about the second question? Particularly,

Question

Is NS_{ω_1} precipitous in the obtained model?

In fact, NS_{ω_1} is nowhere precipitous if we begin with the model of the form $L[U]$ where U is a measure on κ.

Some facts about $L[U]$

Recall some facts about the model $L[U]$.

1. If κ is a measurable cardinal, then there is a unique U such that $U \in L[U]$ and $L[U] \models$'U is a κ-complete normal filter on κ'.
2. Let I be a normal precipitous ideal on κ. Then, $\breve{I} \cap L[U] = U$.

NS_{ω_1} is nowhere precipitous

Suppose $V = L[U]$ is the ground model. Let $P = \mathrm{Coll}(\omega, {<}\kappa)$ and Q the iteration we defined. Let $G * H \subseteq P * \dot{Q}$ be generic.
Suppose that in $V[G][H] = L[U][G][H]$, $\mathrm{NS}_{\omega_1} \restriction S$ is precipitous for some stationary subset S of ω_1.
Let $j' : L[U][G][H] \to L[\hat{U}'][\hat{G}'][\hat{H}']$ be the generic elementary embedding obtained from $\mathrm{NS}_{\omega_1} \restriction S$ where

- $L[\hat{U}'] \models$ '$\hat{U}'$ is a unique $j'(\kappa)$-complete normal filter on $j'(\kappa)$',
- $\hat{G}'$ is a $j'(P)$-generic filter over $L[\hat{U}']$, and
- $\hat{H}'$ is a $j'(Q)$-generic filter over $L[\hat{U}'][\hat{G}']$.

The codomain must have this form.

NS_{ω_1} is nowhere precipitous (Cont.)

It is easy to see that in $L[U][G][H]$, there is no strong club guessing sequence on any stationary subset S of ω_1.
So, there is no fast club subset $C' \in L[\hat{U}'][\hat{G}'][\hat{H}']$ of ${\omega_1}^{L[U][G][H]}$ over $L[U][G][H]$.

NS_{ω_1} is nowhere precipitous (Cont.)

We can build a generic elementary embedding $j : L[U][G][H] \to L[\hat{U}][\hat{G}][\hat{H}]$ from $\mathrm{T\check{C}G}(\vec{C})$ so that $\hat{U} = \hat{U}'$ and $\hat{G} = \hat{G}'$.
$\hat{U} = \hat{U}'$ is automatic by the uniqueness.
$j(P)/G$ is regularly embedded into both $\mathcal{P}(\omega_1)/\mathrm{NS}_{\omega_1} \restriction S$ and $\mathcal{P}(\omega_1)/\mathrm{T\check{C}G}(\vec{C})$. So, (if we are careful enough), we can pick a common filter for that part.

NS_{ω_1} is nowhere precipitous (Cont.)

Since $j : L[U][G][H] \to L[\hat{U}][\hat{G}][\hat{H}]$ is built from $T\check{C}G(\vec{C})$, there is a fast club subset $C \in L[\hat{U}][\hat{G}][\hat{H}]$ of $\omega_1{}^{L[U][G][H]}$ over $L[U][G][H]$.

But we can show that in $L[U][G]$, Q adds no new countable sequence of ordinals. Hence, $j(Q)$ adds no new countable sequence of ordinals in $L[\hat{U}][\hat{G}]$. Thus, $C \in L[\hat{U}][\hat{G}]$.

However, there shouldn't be such a thing in $L[\hat{U}'][\hat{G}'][\hat{H}'] = L[\hat{U}][\hat{G}][\hat{H}']$

Contradiction!

TCG-sequences with (essentially) different order types

With a little modification, for every indecomposable ordinal $\varepsilon < \kappa$, we can arrange that $\vec{C}$ has order type ε.

Let $G * H \subseteq P * \dot{Q}$ be generic. By the same argument as we presented, we can show that in $V[G][H] = L[U][G][H]$, if ε' is an indecomposable ordinal $\neq \varepsilon$ and $\vec{C}'$ is a TCG-sequence on ω_1 of order type ε', then $\check{\mathrm{TCG}}(\vec{C}')$ is not precipitous.

This is not vacuous because in $L[U][G][H]$, for every indecomposable ordinal $\varepsilon' < \omega_1$, there is a TCG-sequence $\vec{C}'$ on ω_1 of order type ε'.

As a result...

Theorem (T. Ishiu)

Let κ be a measurable cardinal, and $\varepsilon < \kappa$ an indecomposable ordinal. Let $P = \mathrm{Coll}(\omega, {<}\kappa)$ and $G \subseteq P$ be generic. Then, in $V[G]$, there exists a forcing notion Q that forces

1. *there exists a* TCG*-sequence $\vec{C}$ on ω_1 of order type ε such that $\mathrm{T\check{C}G}(\vec{C})$ is precipitous,*
2. NS_{ω_1} *is nowhere precipitous, and*
3. *for every indecomposable ordinal $\varepsilon' \neq \varepsilon$, for every* TCG*-sequence $\vec{C}'$ on ω_1 of order type ε', $\mathrm{T\check{C}G}(\vec{C}')$ is not precipitous.*

What happens in the model of JMMP?

By applying the same argument to the model of JMMP, we can show that

- NS_{ω_1} is precipitous, and
- for every TCG-sequence $\vec{C}$, $\mathrm{T\check{C}G}(\vec{C})$ is not precipitous.

So, we answered the presented questions!

Conclusion

1. Con($\exists$measurable) $\Leftrightarrow$ Con($\exists$precipitous TCG-ideal on ω_1).
2. NS$_{\omega_1}$ is precipitous $\not\Rightarrow$ $\exists$ precipitous TCG-ideal on ω_1.
3. $\exists$ precipitous TCG-ideal on ω_1 $\not\Rightarrow$ NS$_{\omega_1}$ is precipitous.
4. $\exists$ precipitous TCG-ideal on ω_1 $\not\Rightarrow$ every TCG-ideal on ω_1 is precipitous.

Open questions

Question

Can we distinguish TCG-ideals of the same order type?

Question

What about regular cardinals $\geq \aleph_2$?

Question

What about other natural ideals?

Question

What is the consistency strength of the existence of two ‘essentially’ different natural precipitous ideals on the same cardinal?