Substituting Supercompactness by Strong Unfoldability

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Vienna, June 15, 2009

This talk presents joint work with **Joel D. Hamkins**.

The two main results can be viewed as analogues of the following two theorems, but in the context of strong unfoldability:

Theorem (Laver '78)

If κ is supercompact, then after suitable preparatory forcing, the supercompactness of κ becomes indestructible by all $<\kappa$-directed closed forcing.

Theorem (Baumgartner '79)

If there exists a supercompact cardinal in V, then there is a forcing extension of V in which PFA holds.

Strongly Unfoldable Cardinals
are defined via embeddings whose domain is a set, not the whole universe V

Definition

For an inaccessible cardinal κ, a κ-model of set theory is a transitive set M of size κ such that $M \models \mathrm{ZFC}^-$, $\kappa \in M$, and $M^{<\kappa} \subseteq M$.

Definition (Villaveces '98)

An inaccessible cardinal κ is strongly unfoldable
if for every ordinal θ and every κ-model M there is an elementary embedding
$j : M \to N$ with $\mathrm{cp}(j) = \kappa$, $j(\kappa) > \theta$ and $V_\theta \subseteq N$.

- view them as "miniature strong" cardinals
- Strong cardinals are strongly unfoldable

Theorem (Villaveces '98)

Strongly unfoldable cardinals

- *are weakly compact*
- *are totally indescribable*
- *are downwards absolute to L*

Moreover

- *measurable cardinals are strongly unfoldable in L, but not necessarily in V*
- *same for Ramsey cardinals*

In consistency strength, strongly unfoldable cardinals are

- *bounded below by the indescribable cardinals*
- *bounded above by the subtle cardinals*
- *relatively low in the hierarchy of large cardinals*

Strongly unfoldable cardinals can be viewed as "miniature supercompact" also!

Theorem (Miyamoto '98, indep. Dzamonja/Hamkins '06)

The following are equivalent:

- *For every ordinal* θ *and every* κ*-model* M *there is* $j : M \to N$ *with* $cp(j) = \kappa$, $j(\kappa) > \theta$ *and* $V_\theta \subseteq N$
- *For every ordinal* θ *and every* κ*-model* M *there is* $j : M \to N$ *with* $cp(j) = \kappa$, $j(\kappa) > \theta$ *and* $N^\theta \subseteq N$

This equivalence was discovered independently by Miyamoto '98 in the context of his H_{κ^+}reflecting cardinals, an equivalent large cardinal notion.

Indestructibility

Question (Villaveces '98)

Can we make a strongly unfoldable cardinal κ indestructible by Add$(\kappa, 1)$? How about Add(κ, θ)? What's the strength of a strongly unfoldable κ where GCH fails?

Idea: Borrow lifting techniques from other large cardinals.

- Hamkins '01 used strongness methods to lift through fast function forcing, through Add$(\kappa, 1)$ and Easton support iterations that control GCH
- Dzamonja and Hamkins '06 used supercompactness methods to show that $\diamondsuit_\kappa(\mathrm{REG})$ can fail at a strongly unfoldable cardinal κ

This hinted at a general indestructibility phenomenon.

The κ-proper posets

- recall that proper forcing is defined by considering whether the generic filter is generic over countable elementary submodels $X \prec H_\lambda$.
- κ-proper forcing generalizes this situation to those elementary submodels $X \prec H_\lambda$ of size κ.
- κ^+-c.c. forcing is κ-proper; so is $\leq\kappa$-closed forcing.
- κ-proper forcing preserves κ^+.

Idea:

- Take a large κ-proper poset $\mathbb{P}$
- Put $\mathbb{P}$ into $X \prec H_\lambda$ of size κ
- If $\pi : X \to M$ is Mostowski collapse, then M is a κ-model
- $\mathbb{P}$ would never fit into M, but we work with $\pi(\mathbb{P})$
- Key point: The pointwise image $\pi" G$ is an M-generic filter for $\pi(\mathbb{P})$, by κ-properness!
- Lift the embedding $j : M \to N$ to $j^* : M[\pi" G] \to N^*$

Theorem (J.,'06)

If κ is strongly unfoldable, then after suitable preparatory forcing, the strong unfoldability of κ becomes indestructible by all $<\kappa$-closed κ-proper forcing. This includes all $<\kappa$-closed κ^+-c.c forcing and all $\leq\kappa$-closed forcing.

- proof uses supercompactness methods (as in [Laver78])
- the preparatory forcing is the lottery preparation of κ (as in [Hamkins00])
- indestructibility by all $<\kappa$-closed forcing, not merely $<\kappa$-directed closed
- indestructibility by $\text{Add}(\kappa, 1)$, $\text{Add}(\kappa, \theta)$, and $\text{Coll}(\theta, \kappa^+)$ for $\theta \geq \kappa^+$
- finite iterations of $<\kappa$-closed κ-proper posets are $<\kappa$-closed κ-proper

Question (J.'06)

Can we make κ indestructible by all $<\kappa$-closed κ^+-preserving forcing?

Answer: Yes!

Main Theorem (Hamkins and J.,'07)

If κ is strongly unfoldable, then after suitable preparatory forcing, the strong unfoldability of κ becomes indestructible by all $<\kappa$-closed κ^+-preserving forcing.

- a key technical step allows us to reduce the case of a κ^+-preserving poset to the main idea that worked with κ-proper posets
- this result is optimal within the class of $<\kappa$-closed posets!
  (If κ is weakly compact in a $<\kappa$-closed forcing extension $V[G]$ collapsing κ^{+V}, then $\square_\kappa$ fails in V. But this is a very strong hypothesis, already infinitely many Woodin cardinals.)
- it is impossible to relax $<\kappa$-closure to $<\kappa$-strategic closure
  (the standard forcing to add a κ-Souslin tree is $<\kappa$-strategically closed, but destroys the weak compactness of κ)

Corollary

If there is a model of ZFC *with a strongly unfoldable cardinal, then there is a model of* ZFC *with a weakly compact cardinal* κ *that is indestructible by all* $<\kappa$*-closed* κ^+ *preserving forcing.*

Open Question

What is the exact consistency strength of a weakly compact cardinal κ that is indestructible by all $<\kappa$-closed κ^+ preserving forcing?

The question is also open for a weakly compact cardinal κ indestructible by all $<\kappa$-closed κ-proper forcing, or even only $<\kappa$-closed κ^+-c.c. forcing.

The forcing axioms PFA and $\mathrm{PFA}(\Gamma)$ and PFA_δ

Definition

PFA is the principle asserting that for every proper poset $\mathbb{Q}$ and for every collection $\mathcal{D}$ of $\aleph_1$ many maximal antichains of $\mathbb{Q}$, there exists a $\mathcal{D}$-generic filter $G \subseteq \mathbb{Q}$.

- If Γ is any class of posets, then $\mathrm{PFA}(\Gamma)$ is the corresponding assertion restricted to proper posets $\mathbb{Q} \in \Gamma$.
- If δ is a cardinal, then PFA_δ is the corresponding assertion where the antichains in $\mathcal{D}$ must have size at most δ.

Definition

The PFA lottery preparation of a cardinal κ, relative to a function $f : \kappa \to \kappa$, is the countable support κ-iteration, which forces at stages $\gamma \in \mathrm{dom}(f)$ with the lottery sum of all proper forcing $\mathbb{Q}$ in $V[G_\gamma]$ having hereditary size at most $f(\gamma)$.

The PFA lottery preparation

- modifies Hamkins' lottery preparation [Hamkins00] in a similar way as Baumgartner's iteration modifies Laver's preparation [Laver78]
- works best when f exhibits a certain fast-growing behavior
- is flexible tool for various large cardinal notions–no need for Laver functions
- forces $\mathfrak{c} = 2^\omega = \kappa = \aleph_2$
- of a supercompact cardinal forces PFA
- of a strongly unfoldable cardinal forces what?...

Answer:

Theorem (Hamkins & J. '06)

The PFA *lottery preparation of a strongly unfoldable cardinal* κ *forces* $\mathrm{PFA}(\aleph_2\text{-proper})$, *with* $\mathfrak{c} = \aleph_2 = \kappa$.

- recall: $\aleph_2$-proper posets include all $\aleph_3$-c.c posets and all $\leq\aleph_2$-closed posets.

Theorem (Hamkins & J. '06)

The PFA *lottery preparation of a strongly unfoldable cardinal* κ *forces* $\mathrm{PFA}_{\aleph_2}$, *with* $\mathfrak{c} = \aleph_2 = \kappa$.

- If the given antichains have size at most $\aleph_2 = \kappa$, then they are small enough to be subsets of the elementary submodel $X \prec H_\lambda$ of size κ. The generic filter G need not be X-generic, but it does meet all antichains inside of X.

Question

Can we improve $\mathrm{PFA}(\aleph_2$-proper) to get PFA ($\aleph_3$-preserving)?

(A poset is δ-preserving if it does not collapse δ as cardinal.)

Answer: Yes!

Main Theorem (Hamkins & J. '07)

If κ is strongly unfoldable and $0^\#$ does not exist, then the PFA lottery preparation of κ forces PFA ($\aleph_2$-preserving) and PFA ($\aleph_3$-preserving) and $\mathrm{PFA}_{\aleph_2}$, with $2^\omega = \kappa = \aleph_2$.

Conclusion:

In order to extract significant strength from PFA, one must collapse $\aleph_3$ to $\aleph_1$!

Combined with the equiconsistency result of Miyamoto '98, we get:

Corollary

The following are equiconsistent *over* ZFC*:*

- *There is a strongly unfoldable cardinal* κ*.*
- $\mathrm{PFA}(\aleph_2\text{-}preserving) + \mathrm{PFA}(\aleph_3\text{-}preserving) + \mathrm{PFA}_{\aleph_2} + 2^\omega = \aleph_2$
- $\mathrm{PFA}_{\aleph_2}$

Question

Do any of the principles PFA ($\aleph_2$-preserving), PFA ($\aleph_3$-preserving), or $\mathrm{PFA}_{\aleph_2}$ imply any of the others? Are the former principles equiconsistent with the latter?

- What happens if $0^\#$ does exist, to the PFA lottery preparation of a strongly unfoldable cardinal?
- Which fragment of PFA can we get from a weakly compact cardinal?

References

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THANK YOU!