The Weak Reflection Principle Versus the Reflection Principle

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ESI Workshop on Large Cardinals and Descriptive Set Theory
Outline

1. Weak Reflection Principle
2. WRP and RP
3. Partial Solutions
4. Solution
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Generalized Stationarity

Notation

For a set $X$, let $P_{\omega_1}(X)$ denote the set $\{a \subseteq X : |a| < \omega_1\}$.

Let $X$ be an uncountable set.

Definition

A set $S \subseteq P_{\omega_1}(X)$ is stationary if for any function $F : [X]^{<\omega} \rightarrow X$, there is a set $b$ in $S$ which is closed under $F$. 
Generalized Stationarity

Definition

A set $C \subseteq P_{\omega_1}(X)$ is **club** if:

- whenever $\langle a_n : n < \omega \rangle$ is an increasing sequence of sets in $C$, then $\bigcup_{n < \omega} a_n \in C$,
- for all $b$ in $P_{\omega_1}(X)$, there is $c \in C$ with $b \subseteq c$.

Proposition

A set $S \subseteq P_{\omega_1}(X)$ is stationary iff $S$ has non-empty intersection with every club $C \subseteq P_{\omega_1}(X)$.

**Notation**

Let $X \subseteq Y$ be uncountable sets, and let $S \subseteq P_{\omega_1}(Y)$ be stationary. We say that $S$ reflects to $X$ if $S \cap P_{\omega_1}(X)$ is stationary in $P_{\omega_1}(X)$. 
Weak Reflection Principle

Definition
Let $\lambda \geq \omega_2$ be a cardinal. The **Weak Reflection Principle for $\lambda$**, or $\text{WRP}(\lambda)$, is the statement: for every stationary set $S \subseteq P_{\omega_1}(\lambda)$, there is a set $X \subseteq \lambda$ of size $\aleph_1$, with $\aleph_1 \subseteq X$, such that $S$ reflects to $X$.

Definition
The **Weak Reflection Principle**, or WRP, is the statement that $\text{WRP}(\lambda)$ holds for all cardinals $\lambda \geq \omega_2$. 
Models of WRP

Theorem

Martin’s Maximum implies the Weak Reflection Principle.

Theorem

Suppose $\kappa$ is a supercompact cardinal. Then the Lévy collapse $\text{Coll}(\omega_1, < \kappa)$ forces the Weak Reflection Principle.
As set-theoretic principles, the Weak Reflection Principle has similar consequences as Martin’s Maximum, and so captures (some of) its large cardinal strength.

**Theorem**

*Martin’s Maximum implies:*

1. $2^\omega = \omega_2$,
2. the non-stationary ideal on $\omega_1$ is saturated,
3. *(Strong) Chang’s Conjecture*
4. *Singular Cardinal Hypothesis*
5. $\neg \square_\kappa$ for all cardinals $\kappa \geq \omega_1$. 
In comparison:

**Theorem**

The Weak Reflection Principle implies:

1. $2^\omega \leq \omega_2$,
2. the non-stationary ideal on $\omega_1$ is presaturated,
3. (Strong) Chang’s Conjecture,
4. Singular Cardinal Hypothesis
5. $\neg \Box_\kappa$ for all cardinals $\kappa \geq \omega_1$. 
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The Reflection Principle is a variation of the Weak Reflection Principle.

**Definition**

Let $\lambda \geq \omega_2$ be a regular cardinal. The Reflection Principle for $\lambda$, or $\text{RP}(\lambda)$, is the statement that for every stationary set $S \subseteq P_{\omega_1}(\lambda)$, there is a set $X \subseteq \lambda$ of size $\aleph_1$, with $\aleph_1 \subseteq X$, and $\text{cf}(\text{ot}(X)) = \omega_1$, such that $S$ reflects to $X$.

**Definition**

The Reflection Principle, or $\text{RP}$, is the statement that $\text{RP}(\lambda)$ holds for all regular $\lambda \geq \omega_2$. 
It is easy to see $\text{RP}(\lambda)$ implies $\text{WRP}(\lambda)$ for any regular $\lambda \geq \omega_2$, and RP implies WRP.

**Theorem**

*Martin’s Maximum implies the Reflection Principle.*

**Theorem**

*If $\kappa$ is a supercompact cardinal, then the Lévy collapse $\text{Coll}(\omega_1, < \kappa)$ forces the Reflection Principle.*
The following questions are obvious and natural, and have been open for some time.

**Question**

*Does the Weak Reflection Principle imply the Reflection Principle?*

**Question**

*For particular values of regular cardinals $\lambda \geq \omega_2$, does $\text{WRP}(\lambda)$ imply $\text{RP}(\lambda)$?*
An Example in Singular Cardinal Combinatorics

Definition

Let $\kappa$ be a cardinal. $\text{ADS}_\kappa$ is the statement that there exists a sequence $\langle A_i : i < \kappa^+ \rangle$ satisfying:

- each $A_i$ is a cofinal subset of $\kappa$ with order type $\text{cf}(\kappa)$,
- for all $\beta < \kappa^+$, there is a function $g : \beta \rightarrow \kappa$ such that $\langle A_i \setminus g(i) : i < \beta \rangle$ is a pairwise disjoint sequence.

If $\kappa$ is regular, then $\text{ADS}_\kappa$ holds.

If $\kappa$ is singular and $\square_\kappa$ holds, then $\text{ADS}_\kappa$ holds.
Theorem (Cummings, Foreman, Magidor)

Let $\lambda$ be a singular cardinal with cofinality $\omega$. Then $RP(\lambda^+)$ implies $\neg ADS_\lambda$.

This theorem gives an elegant proof that RP implies SCH.

By pcf theory, if $\lambda$ is the least cardinal where SCH fails, then $\text{cf}(\lambda) = \omega$ and $ADS_\lambda$ holds.

So RP implies SCH.
By a much more difficult argument Shelah proved:

**Theorem**

*WRP implies SCH.*

Generally, it tends to be easier to prove things from $\text{RP}(\lambda)$ than from $\text{WRP(}\lambda)$. 
Question

For particular values of regular cardinals $\lambda$, does $RP(\lambda)$ imply $WRP(\lambda)$?

The most basic case of this question is when $\lambda$ equals $\omega_2$.

Question

Does $WRP(\omega_2)$ imply $RP(\omega_2)$?
WRP($\omega_2$) versus RP($\omega_2$)

**Lemma**

WRP($\omega_2$) holds iff for any stationary set $S \subseteq P_{\omega_1}(\omega_2)$, there is an uncountable ordinal $\alpha$ in $\omega_2$ such that $S$ reflects to $\alpha$.

**Lemma**

RP($\omega_2$) holds iff for any stationary set $S \subseteq P_{\omega_1}(\omega_2)$, there is an ordinal $\alpha$ in $\omega_2$ with cofinality $\omega_1$ such that $S$ reflects to $\alpha$. 
Consistency of $\text{RP}(\omega_2)$

Theorem

If $\kappa$ is a weakly compact cardinal, then the Lévy collapse $\text{Coll}(\omega_1, < \kappa)$ forces $\text{RP}(\omega_2)$.

Theorem

$\text{WRP}(\omega_2)$ and $\text{RP}(\omega_2)$ are both equiconsistent with a weakly compact cardinal.
Consequences of WRP\((\omega_2)\)

Although relatively weak in large cardinal strength, the Weak Reflection Principle for \(\omega_2\) has some interesting combinatorial consequences.

**Theorem**

\(WRP(\omega_2)\) implies:

1. \(2^\omega \leq \omega_2\),
2. \(\neg \Box \omega_1\),
3. for every stationary set \(S \subseteq \omega_2 \cap \text{cof}(\omega)\), there is an ordinal \(\alpha\) in \(\omega_2 \cap \text{cof}(\omega_1)\) such that \(S \cap \alpha\) is stationary in \(\alpha\),
4. if CH fails, then there does not exist a special Aronszajn tree on \(\omega_2\).
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Question

Does \( \text{WRP}(\omega_2) \) imply \( \text{RP}(\omega_2) \)?

In other words, if every stationary subset of \( P_{\omega_1}(\omega_2) \) reflects to an uncountable ordinal in \( \omega_2 \), does this imply every stationary subset of \( P_{\omega_1}(\omega_2) \) reflects to an ordinal in \( \omega_2 \cap \text{cof}(\omega_1) \)?

A standard argument shows:

Proposition

Suppose \( \diamondsuit_A \) holds for every stationary set \( A \subseteq \omega_2 \cap \text{cof}(\omega) \).

Then every stationary set \( S \subseteq P_{\omega_1}(\omega_2) \) has a stationary subset \( T \) which does not reflect to any ordinal in \( \omega_2 \cap \text{cof}(\omega) \).
Proof.

Let $S \subseteq P_{\omega_1}(\omega_2)$ be stationary.

Let $A = \{\alpha \in \omega_2 \cap \text{cof}(\omega) : S \text{ reflects to } \alpha\}$.

Case 1: $A$ is non-stationary.
Then let $C \subseteq \omega_2$ be club with $A \cap C = \emptyset$.
Let $T = \{a \in S : \text{sup}(a) \in C\}$.

Case 2: $A$ is stationary.
Let $\langle f_\alpha : \alpha \in A \rangle$ be a $\diamondsuit_A$-sequence.
For each $\alpha$ in $A$, $S$ reflects to $\alpha$.
So choose $b_\alpha$ in $S$ with $\text{sup}(b_\alpha) = \alpha$ such that $b_\alpha$ is closed under $f_\alpha$.
Let $T = \{b_\alpha : \alpha \in A\}$. 
Corollary

Suppose $\Diamond_A$ holds for every stationary set $A \subseteq \omega_2 \cap \text{cof}(\omega)$. Then $\text{WRP}(\omega_2)$ implies $\text{RP}(\omega_2)$.

Proof.

Let $S \subseteq P_{\omega_1}(\omega_2)$.

Choose a stationary set $T \subseteq S$ which does not reflect to any ordinal in $\omega_2 \cap \text{cof}(\omega)$.

By $\text{WRP}(\omega_2)$, there is an uncountable $\alpha$ in $\omega_2$ such that $T$ reflects to $\alpha$.

By the choice of $T$, $\text{cf}(\alpha) = \omega_1$. 
Classically, the existence of such diamonds was known to follow from GCH (Gregory 1976).

More recently, Shelah proved they follow from $2^{\omega_1} = \omega_2$.

**Theorem (Shelah)**

Assume $2^{\omega_1} = \omega_2$. Then for every stationary set $A \subseteq \omega_2 \cap \text{cof}(\omega)$, $\lozenge_A$ holds.

**Corollary**

Assume $2^{\omega_1} = \omega_2$. Then $\text{WRP}(\omega_2)$ implies $\text{RP}(\omega_2)$. 
This conclusion was known somewhat earlier than Shelah’s theorem on diamonds, by a different argument.

**Theorem (Koenig-Larson-Yoshinobu, 2007)**

Assume $2^{\omega_1} = \omega_2$. Then $WRP(\omega_2)$ implies $RP(\omega_2)$. 
Consider the possibility $\text{WRP}(\omega_2) \land \neg \text{RP}(\omega_2)$.

(1) By $\neg \text{RP}(\omega_2)$, there exists a stationary set $S \subseteq P_{\omega_1}(\omega_2)$ which does not reflect to any ordinal in $\omega_2$ with cofinality $\omega_1$.

(2) By $\text{WRP}(\omega_2)$, every stationary subset of $S$ reflects to an uncountable ordinal in $\omega_2$.

So by (1) and (2), every stationary subset of $S$ reflects to an uncountable ordinal in $\omega_2$ with cofinality $\omega$. 
Recently Hiroshi Sakai constructed a model satisfying this conclusion.

**Theorem (Sakai, 2008)**

Assume $GCH$ and $\square_{\omega_1}$. Then there is a forcing poset $\mathbb{P}$ which forces that there exists a stationary set $S \subseteq P_{\omega_1}(\omega_2)$ such that every stationary subset of $S$ reflects to an uncountable ordinal in $\omega_2$ with cofinality $\omega$.

Note that this “local reflection” is obtained without assuming any large cardinals.
Let $\langle c_\alpha : \alpha \in \omega_2 \rangle$ be a $\square_{\omega_1}$-sequence.

Define $S = \{ a \in P_{\omega_1}(\omega_2) : \text{ot}(c_{\text{sup}(a)}) \in a \cap \omega_1 \}$.

**Lemma**

The set $S$ is a stationary subset of $P_{\omega_1}(\omega_2)$ which does not reflect to any ordinal in $\omega_2 \cap \text{cof}(\omega_1)$.

**Theorem (Sakai)**

There exists a countably distributive, $\omega_2$-c.c. forcing iteration which destroys the stationarity of every stationary subset of $S$ which does not reflect to any uncountable ordinal in $\omega_2 \cap \text{cof}(\omega)$.
Sakai’s proof depends on the $\square_{\omega_1}$-sequence from which the set $S$ is defined.

However, $\text{WRP}(\omega_2)$ implies $\neg \square_{\omega_1}$.

So Sakai’s Theorem cannot be applied directly to prove the consistency of $\text{WRP}(\omega_2) \land \neg \text{RP}(\omega_2)$. 
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Let us consider the traditional method for obtaining reflection in $P_{\omega_1}(\omega_2)$.

In some model $\mathcal{W}$ of ZFC, we have an elementary embedding

$$j : V \rightarrow M$$

with critical point $\omega_2^V$, where $V$ and $M$ are inner models.
Reflection using an Elementary Embedding

For example, let \( \kappa \) be a measurable cardinal in a model \( V_0 \), with elementary embedding \( j : V_0 \rightarrow M_0 \).

Let \( H \) be a generic filter for \( j(\text{Coll}(\omega_1, < \kappa)) \) over \( V_0 \), and let \( G = H \cap \text{Coll}(\omega_1, < \kappa) \).

Let \( W = V_0[H], \ V = V_0[G], \) and \( M = M_0[H] \).

Since \( j'' G = G \subseteq H \), in \( W \) we can extend \( j \) to \( j : V \rightarrow M \) with critical point \( \kappa = \omega_2^V \).
In $W$ we have $j : V \to M$ with critical point $\omega_2^V$.

Let $T$ be a stationary subset of $P_{\omega_1}(\omega_2)$ in $V$.

For each $a$ in $T$, $a = j(a) \in j(T)$. So $T \subseteq j(T) \cap P_{\omega_1}(\omega_2^V)$.

But $\omega_2^V < j(\omega_2^V) = \omega_2^M$. 
Reflection using an Elementary Embedding

It follows that if $T$ is stationary in $M$, then $j(T)$ reflects to an uncountable ordinal in $\omega_2^M$.

By elementarity, in $V$, $T$ reflects to an uncountable ordinal in $\omega_2$. 
Reflection using an Elementary Embedding

Typically, $T$ is shown to be stationary in $M$ by arguing that $M$ is a generic extension by a proper forcing poset.

For example, consider the case above of $j : V_0[G] \rightarrow M_0[H]$, where $H$ is a generic filter for the Lévy collapse $j(\text{Coll}(\omega_1, < \kappa))$ and $G = H \cap \text{Coll}(\omega_1, < \kappa)$.

If $T$ is stationary in $P_{\omega_1}(\kappa)$ in $V_0[G]$, it is also stationary in $M_0[G]$, and $M = M_0[H] = M_0[G][H']$, where $H'$ is a generic filter for the proper forcing poset $\text{Coll}(\omega_1, [\kappa, j(\kappa)])$.

Since $\text{Coll}(\omega_1, [\kappa, j(\kappa)])$ is proper, $T$ remains stationary in $M$. 
Thus we have two separate methods for obtaining reflection.

1. The traditional method of using an elementary embedding.

2. Sakai’s argument for obtaining local reflection using iterated forcing.
We had the idea to combine parts of Sakai’s argument with the traditional method of obtaining reflection using an elementary embedding, to obtain a model satisfying $\text{WRP}(\omega_2) \land \neg \text{RP}(\omega_2)$.

$\text{WRP}(\omega_2)$ can be factored into two separate “local reflection” statements, by the following easy lemma.

**Lemma**

Suppose $S$ is a stationary subset of $P_{\omega_1}(\omega_2)$. Assume:

- every stationary subset of $S$ reflects to an uncountable ordinal in $\omega_2$ with cofinality $\omega$,
- every stationary subset of $P_{\omega_1}(\omega_2) \setminus S$ reflects to an uncountable ordinal in $\omega_2$ with cofinality $\omega_1$.

Then $\text{WRP}(\omega_2)$ holds.
Combining Both Methods

We would like a stationary set $S \subseteq P_{\omega_1}(\omega_2)$ satisfying:

1. every stationary subset of $S$ reflects to an uncountable ordinal in $\omega_2$ with cofinality $\omega$,
2. every stationary subset of $P_{\omega_1}(\omega_2) \setminus S$ reflects to an uncountable ordinal in $\omega_2$ with cofinality $\omega_1$.

We obtain (1) using a Sakai-style iteration.

We obtain (2) using an elementary embedding.

The set $S$ is added generically, rather than defined using a $\square_{\omega_1}$-sequence.
WRP(\(\omega_2\)) does not imply RP(\(\omega_2\))

**Theorem (K., 2009)**

Assume \(\kappa\) is a \(\kappa^+\)-supercompact cardinal and \(2^\kappa = \kappa^+\). Then there is a forcing poset \(\mathbb{P}\) which collapses \(\kappa\) to become \(\omega_2\), and forces:

\[
WRP(\omega_2) \land \neg RP(\omega_2).
\]
Fix a cardinal $\kappa$ which is $\kappa^+$-supercompact and $2^\kappa = \kappa^+$.

The forcing poset $\mathbb{P}$ in the theorem is of the form:

$$\text{COLL}(\omega_1, < \kappa) \ast \mathbb{P}_\kappa \ast \dot{\mathbb{Q}},$$

where:

(1) $\text{COLL}(\omega_1, < \kappa)$ is the Lévy collapse for collapsing $\kappa$ to become $\omega_2$,

(2) $\mathbb{P}_\kappa$ generically adds a stationary set $S \subseteq P_{\omega_1}(\omega_2)$, using countable conditions, which does not reflect to any ordinal in $\omega_2 \cap \text{cof}(\omega_1)$,

(3) $\mathbb{Q}$ is a Sakai-style iteration with countable support which destroys the stationarity of every subset of $S$ which does not reflect to an ordinal in $\omega_2 \cap \text{cof}(\omega)$. 
A particularly difficult part of the proof is to show that in a generic extension by $\mathbb{P}$:

Every stationary subset of $P_{\omega_1}(\omega_2) \setminus S$ reflects to an ordinal in $\omega_2 \cap \text{cof}(\omega_1)$.

The forcing poset $\mathbb{P}$ is not proper, so in general does not preserve stationary sets.

The proof involves a complicated factoring of $j(\mathbb{P})$, which has a tail forcing which preserves stationary subsets of $P_{\omega_1}(\omega_2)$ which are disjoint from $S$. 
Open Questions

Question
Does $\text{WRP}(\omega_3)$ imply $\text{RP}(\omega_3)$?

Question
Does $\text{WRP}$ imply $\text{RP}(\omega_2)$?

Question
Is the supercompactness used in the proof of the theorem necessary?
For a preprint which includes a proof of the main theorem of the talk, see my website:

http://math.berkeley.edu/~jkrueger