The Weak Reflection Principle Versus the Reflection Principle

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Outline

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Generalized Stationarity

Notation

For a set X, let $P_{\omega_1}(X)$ denote the set $\{a \subseteq X : |a| < \omega_1\}$.

Let X be an uncountable set.

Definition

A set $S \subseteq P_{\omega_1}(X)$ is stationary if for any function $F : [X]^{<\omega} \to X$, there is a set b in S which is closed under F.

Generalized Stationarity

Definition

A set $C \subseteq P_{\omega_1}(X)$ is club if:

- whenever $\langle a_n : n < \omega \rangle$ is an increasing sequence of sets in C, then $\bigcup_{n<\omega} a_n \in C$,
- for all b in $P_{\omega_1}(X)$, there is $c \in C$ with $b \subseteq c$.

Proposition

A set $S \subseteq P_{\omega_1}(X)$ is stationary iff S has non-empty intersection with every club $C \subseteq P_{\omega_1}(X)$.

Weak Reflection Principle

Foreman-Magidor-Shelah (1988) introduced the Weak Reflection Principle, in the context of Martin's Maximum.

Notation

Let $X \subseteq Y$ be uncountable sets, and let $S \subseteq P_{\omega_1}(Y)$ be stationary. We say that S reflects to X if $S \cap P_{\omega_1}(X)$ is stationary in $P_{\omega_1}(X)$.

Weak Reflection Principle

Definition

Let $\lambda \geq \omega_2$ be a cardinal. The Weak Reflection Principle for λ, or $\mathrm{WRP}(\lambda)$, is the statement: for every stationary set $S \subseteq P_{\omega_1}(\lambda)$, there is a set $X \subseteq \lambda$ of size $\aleph_1$, with $\aleph_1 \subseteq X$, such that S reflects to X.

Definition

The Weak Reflection Principle, or WRP, is the statement that $\mathrm{WRP}(\lambda)$ holds for all cardinals $\lambda \geq \omega_2$.

Models of WRP

Theorem

Martin's Maximum implies the Weak Reflection Principle.

Theorem

Suppose κ is a supercompact cardinal. Then the Lévy collapse $\textsc{Coll}(\omega_1, {<}\kappa)$ forces the Weak Reflection Principle.

WRP and MM

As set-theoretic principles, the Weak Reflection Principle has similar consequences as Martin's Maximum, and so captures (some of) its large cardinal strength.

Theorem

Martin's Maximum implies:

1. $2^\omega = \omega_2$,
2. *the non-stationary ideal on* ω_1 *is saturated,*
3. *(Strong) Chang's Conjecture*
4. *Singular Cardinal Hypothesis*
5. $\neg\Box_\kappa$ *for all cardinals* $\kappa \geq \omega_1$.

WRP and MM

In comparison:

Theorem

The Weak Reflection Principle implies:

1. $2^\omega \leq \omega_2$,
2. *the non-stationary ideal on* ω_1 *is presaturated,*
3. *(Strong) Chang's Conjecture,*
4. *Singular Cardinal Hypothesis*
5. $\neg\Box_\kappa$ *for all cardinals* $\kappa \geq \omega_1$.

Outline

The Reflection Principle

The Reflection Principle is a variation of the Weak Reflection Principle.

Definition

Let $\lambda \geq \omega_2$ be a regular cardinal. The Reflection Principle for λ, or $\mathrm{RP}(\lambda)$, is the statement that for every stationary set $S \subseteq P_{\omega_1}(\lambda)$, there is a set $X \subseteq \lambda$ of size $\aleph_1$, with $\aleph_1 \subseteq X$, and $\mathrm{cf}(\mathrm{ot}(X)) = \omega_1$, such that S reflects to X.

Definition

The Reflection Principle, or RP, is the statement that $\mathrm{RP}(\lambda)$ holds for all regular $\lambda \geq \omega_2$.

Models of RP

It is easy to see $\mathrm{RP}(\lambda)$ implies $\mathrm{WRP}(\lambda)$ for any regular $\lambda \geq \omega_2$, and RP implies WRP.

Theorem

Martin's Maximum implies the Reflection Principle.

Theorem

If κ is a supercompact cardinal, then the Lévy collapse $\mathrm{Coll}(\omega_1, {<}\kappa)$ forces the Reflection Principle.

WRP Versus RP

The following questions are obvious and natural, and have been open for some time.

Question

Does the Weak Reflection Principle imply the Reflection Principle?

Question

For particular values of regular cardinals $\lambda \geq \omega_2$*, does* $WRP(\lambda)$ *imply* $RP(\lambda)$*?*

An Example in Singular Cardinal Combinatorics

Definition

Let κ be a cardinal. ADS_κ is the statement that there exists a sequence $\langle A_i : i < \kappa^+ \rangle$ satisfying:

- each A_i is a cofinal subset of κ with order type $\mathrm{cf}(\kappa)$,
- for all $\beta < \kappa^+$, there is a function $g : \beta \to \kappa$ such that $\langle A_i \setminus g(i) : i < \beta \rangle$ is a pairwise disjoint sequence.

If κ is regular, then ADS_κ holds.

If κ is singular and $\square_\kappa$ holds, then ADS_κ holds.

RP and the Singular Cardinal Hypothesis

Theorem (Cummings, Foreman, Magidor)

Let λ be a singular cardinal with cofinality ω. Then $RP(\lambda^+)$ implies $\neg ADS_\lambda$.

This theorem gives an elegant proof that RP implies SCH.

By pcf theory, if λ is the least cardinal where SCH fails, then $\mathrm{cf}(\lambda) = \omega$ and ADS_λ holds.

So RP implies SCH.

WRP and the Singular Cardinal Hypothesis

By a much more difficult argument Shelah proved:

Theorem

WRP implies SCH.

Generally, it tends to be easier to prove things from $\mathrm{RP}(\lambda)$ than from $\mathrm{WRP}(\lambda)$.

$\mathrm{WRP}(\omega_2)$ versus $\mathrm{RP}(\omega_2)$

Question

For particular values of regular cardinals λ, does $RP(\lambda)$ imply $WRP(\lambda)$?

The most basic case of this question is when λ equals ω_2.

Question

Does $WRP(\omega_2)$ imply $RP(\omega_2)$?

$\mathrm{WRP}(\omega_2)$ versus $\mathrm{RP}(\omega_2)$

Lemma

$WRP(\omega_2)$ holds iff for any stationary set $S \subseteq P_{\omega_1}(\omega_2)$, there is an uncountable ordinal α in ω_2 such that S reflects to α.

Lemma

$RP(\omega_2)$ holds iff for any stationary set $S \subseteq P_{\omega_1}(\omega_2)$, there is an ordinal α in ω_2 with cofinality ω_1 such that S reflects to α.

Consistency of $\mathrm{RP}(\omega_2)$

Theorem

If κ is a weakly compact cardinal, then the Lévy collapse $\mathrm{Coll}(\omega_1, <\kappa)$ forces $RP(\omega_2)$.

Theorem

$WRP(\omega_2)$ and $RP(\omega_2)$ are both equiconsistent with a weakly compact cardinal.

Consequences of $\mathrm{WRP}(\omega_2)$

Although relatively weak in large cardinal strength, the Weak Reflection Principle for ω_2 has some interesting combinatorial consequences.

Theorem

$WRP(\omega_2)$ implies:

1. $2^\omega \leq \omega_2$,
2. $\neg\Box_{\omega_1}$,
3. *for every stationary set $S \subseteq \omega_2 \cap \mathrm{cof}(\omega)$, there is an ordinal α in $\omega_2 \cap \mathrm{cof}(\omega_1)$ such that $S \cap \alpha$ is stationary in α,*
4. *if CH fails, then there does not exist a special Aronszajn tree on ω_2.*

Outline

Diamonds

Question

Does $WRP(\omega_2)$ imply $RP(\omega_2)$?

In other words, if every stationary subset of $P_{\omega_1}(\omega_2)$ reflects to an uncountable ordinal in ω_2, does this imply every stationary subset of $P_{\omega_1}(\omega_2)$ reflects to an ordinal in $\omega_2 \cap \mathrm{cof}(\omega_1)$?

A standard argument shows:

Proposition

Suppose $\diamondsuit_A$ holds for every stationary set $A \subseteq \omega_2 \cap \mathrm{cof}(\omega)$.

Then every stationary set $S \subseteq P_{\omega_1}(\omega_2)$ has a stationary subset T which does not reflect to any ordinal in $\omega_2 \cap \mathrm{cof}(\omega)$.

Diamonds

Proof.

Let $S \subseteq P_{\omega_1}(\omega_2)$ be stationary.

Let $A = \{\alpha \in \omega_2 \cap \mathrm{cof}(\omega) : S \text{ reflects to } \alpha\}$.

Case 1: A is non-stationary.
Then let $C \subseteq \omega_2$ be club with $A \cap C = \emptyset$.
Let $T = \{a \in S : \sup(a) \in C\}$.

Case 2: A is stationary.
Let $\langle f_\alpha : \alpha \in A \rangle$ be a $\diamondsuit_A$-sequence.
For each α in A, S reflects to α.
So choose b_α in S with $\sup(b_\alpha) = \alpha$ such that b_α is closed under f_α.
Let $T = \{b_\alpha : \alpha \in A\}$.

Diamonds

Corollary

Suppose $\diamondsuit_A$ holds for every stationary set $A \subseteq \omega_2 \cap \mathrm{cof}(\omega)$. Then $WRP(\omega_2)$ implies $RP(\omega_2)$.

Proof.

Let $S \subseteq P_{\omega_1}(\omega_2)$.

Choose a stationary set $T \subseteq S$ which does not reflect to any ordinal in $\omega_2 \cap \mathrm{cof}(\omega)$.

By $\mathrm{WRP}(\omega_2)$, there is an uncountable α in ω_2 such that T reflects to α.

By the choice of T, $\mathrm{cf}(\alpha) = \omega_1$.

Diamonds

Classically, the existence of such diamonds was known to follow from GCH (Gregory 1976).

More recently, Shelah proved they follow from $2^{\omega_1} = \omega_2$.

Theorem (Shelah)

Assume $2^{\omega_1} = \omega_2$. Then for every stationary set $A \subseteq \omega_2 \cap \mathrm{cof}(\omega)$, $\diamondsuit_A$ holds.

Corollary

Assume $2^{\omega_1} = \omega_2$.
Then $WRP(\omega_2)$ implies $RP(\omega_2)$.

Diamonds

This conclusion was known somewhat earlier than Shelah's theorem on diamonds, by a different argument.

Theorem (Koenig-Larson-Yoshinobu, 2007)

Assume $2^{\omega_1} = \omega_2$. *Then* $WRP(\omega_2)$ *implies* $RP(\omega_2)$.

$\mathrm{WRP}(\omega_2) \wedge \neg\mathrm{RP}(\omega_2)$

Consider the possibility $\mathrm{WRP}(\omega_2) \wedge \neg\mathrm{RP}(\omega_2)$.

(1) By $\neg\mathrm{RP}(\omega_2)$, there exists a stationary set $S \subseteq P_{\omega_1}(\omega_2)$ which does not reflect to any ordinal in ω_2 with cofinality ω_1.

(2) By $\mathrm{WRP}(\omega_2)$, every stationary subset of S reflects to an uncountable ordinal in ω_2.

So by (1) and (2), every stationary subset of S reflects to an uncountable ordinal in ω_2 with cofinality ω.

Sakai's Theorem

Recently Hiroshi Sakai constructed a model satisfying this conclusion.

Theorem (Sakai, 2008)

Assume GCH and $\square_{\omega_1}$. Then there is a forcing poset $\mathbb{P}$ which forces that there exists a stationary set $S \subseteq P_{\omega_1}(\omega_2)$ such that every stationary subset of S reflects to an uncountable ordinal in ω_2 with cofinality ω.

Note that this "local reflection" is obtained without assuming any large cardinals.

Sakai's Theorem

Let $\langle c_\alpha : \alpha \in \omega_2 \rangle$ be a $\square_{\omega_1}$-sequence.

Define $S = \{a \in P_{\omega_1}(\omega_2) : \mathrm{ot}(c_{\sup(a)}) \in a \cap \omega_1\}$.

Lemma

The set S is a stationary subset of $P_{\omega_1}(\omega_2)$ which does not reflect to any ordinal in $\omega_2 \cap \mathrm{cof}(\omega_1)$.

Theorem (Sakai)

There exists a countably distributive, ω_2-c.c. forcing iteration which destroys the stationarity of every stationary subset of S which does not reflect to any uncountable ordinal in $\omega_2 \cap \mathrm{cof}(\omega)$.

But $\mathrm{WRP}(\omega_2) \Rightarrow \neg\Box_{\omega_1}$

Sakai's proof depends on the $\Box_{\omega_1}$-sequence from which the set S is defined.

However, $\mathrm{WRP}(\omega_2)$ implies $\neg\Box_{\omega_1}$.

So Sakai's Theorem cannot be applied directly to prove the consistency of $\mathrm{WRP}(\omega_2) \wedge \neg\mathrm{RP}(\omega_2)$.

Outline

Reflection using an Elementary Embedding

Let us consider the traditional method for obtaining reflection in $P_{\omega_1}(\omega_2)$.

In some model W of ZFC, we have an elementary embedding

$$j : V \longrightarrow M$$

with critical point ω_2^V, where V and M are inner models.

Reflection using an Elementary Embedding

For example, let κ be a measurable cardinal in a model V_0, with elementary embedding $j : V_0 \to M_0$.

Let H be a generic filter for $j(\mathrm{Coll}(\omega_1, <\kappa))$ over V_0, and let $G = H \cap \mathrm{Coll}(\omega_1, <\kappa)$.

Let $W = V_0[H]$, $V = V_0[G]$, and $M = M_0[H]$.

Since $j''G = G \subseteq H$, in W we can extend j to $j : V \to M$ with critical point $\kappa = \omega_2^V$.

Reflection using an Elementary Embedding

In W we have $j : V \to M$ with critical point ω_2^V.

Let T be a stationary subset of $P_{\omega_1}(\omega_2)$ in V.

For each a in T, $a = j(a) \in j(T)$. So $T \subseteq j(T) \cap P_{\omega_1}(\omega_2^V)$.

But $\omega_2^V < j(\omega_2^V) = \omega_2^M$.

Reflection using an Elementary Embedding

It follows that if T is stationary in M, then $j(T)$ reflects to an uncountable ordinal in ω_2^M.

By elementarity, in V, T reflects to an uncountable ordinal in ω_2.

Reflection using an Elementary Embedding

Typically, T is shown to be stationary in M by arguing that M is a generic extension by a proper forcing poset.

For example, consider the case above of $j : V_0[G] \to M_0[H]$, where H is a generic filter for the Lévy collapse $j(\mathrm{Coll}(\omega_1, <\kappa))$ and $G = H \cap \mathrm{Coll}(\omega_1, <\kappa)$.

If T is stationary in $P_{\omega_1}(\kappa)$ in $V_0[G]$, it is also stationary in $M_0[G]$, and $M = M_0[H] = M_0[G][H']$, where H' is a generic filter for the proper forcing poset $\mathrm{Coll}(\omega_1, [\kappa, j(\kappa)))$.

Since $\mathrm{Coll}(\omega_1, [\kappa, j(\kappa)))$ is proper, T remains stationary in M.

Two Methods for Obtaining Reflection

Thus we have two separate methods for obtaining reflection.

(1) The traditional method of using an elementary embedding.

(2) Sakai's argument for obtaining local reflection using iterated forcing.

Combining Both Methods

We had the idea to combine parts of Sakai's argument with the traditional method of obtaining reflection using an elementary embedding, to obtain a model satisfying $\mathrm{WRP}(\omega_2) \wedge \neg\mathrm{RP}(\omega_2)$.

$\mathrm{WRP}(\omega_2)$ can be factored into two separate "local reflection" statements, by the following easy lemma.

Lemma

Suppose S is a stationary subset of $P_{\omega_1}(\omega_2)$. Assume:

- *every stationary subset of S reflects to an uncountable ordinal in ω_2 with cofinality ω,*
- *every stationary subset of $P_{\omega_1}(\omega_2) \setminus S$ reflects to an uncountable ordinal in ω_2 with cofinality ω_1.*

Then $WRP(\omega_2)$ holds.

Combining Both Methods

We would like a stationary set $S \subseteq P_{\omega_1}(\omega_2)$ satisfying:

1. every stationary subset of S reflects to an uncountable ordinal in ω_2 with cofinality ω,
2. every stationary subset of $P_{\omega_1}(\omega_2) \setminus S$ reflects to an uncountable ordinal in ω_2 with cofinality ω_1.

We obtain (1) using a Sakai-style iteration

We obtain (2) using an elementary embedding.

The set S is added generically, rather than defined using a $\square_{\omega_1}$-sequence.

$\mathrm{WRP}(\omega_2)$ does not imply $\mathrm{RP}(\omega_2)$

Theorem (K., 2009)

Assume κ is a κ^+-supercompact cardinal and $2^\kappa = \kappa^+$. Then there is a forcing poset $\mathbb{P}$ which collapses κ to become ω_2, and forces:

$$WRP(\omega_2) \wedge \neg RP(\omega_2).$$

Comments on the Proof

Fix a cardinal κ which is κ^+-supercompact and $2^\kappa = \kappa^+$.

The forcing poset $\mathbb{P}$ in the theorem is of the form:

$$\text{Coll}(\omega_1, {<}\kappa) * \dot{\mathbb{P}}_\kappa * \dot{\mathbb{Q}},$$

where:

(1) $\text{Coll}(\omega_1, {<}\kappa)$ is the Lévy collapse for collapsing κ to become ω_2,

(2) $\mathbb{P}_\kappa$ generically adds a stationary set $S \subseteq P_{\omega_1}(\omega_2)$, using countable conditions, which does not reflect to any ordinal in $\omega_2 \cap \mathrm{cof}(\omega_1)$,

(3) $\mathbb{Q}$ is a Sakai-style iteration with countable support which destroys the stationarity of every subset of S which does not reflect to an ordinal in $\omega_2 \cap \mathrm{cof}(\omega)$.

Comments on the Proof

A particularly difficult part of the proof is to show that in a generic extension by $\mathbb{P}$:

Every stationary subset of $P_{\omega_1}(\omega_2) \setminus S$ reflects to an ordinal in $\omega_2 \cap \mathrm{cof}(\omega_1)$.

The forcing poset $\mathbb{P}$ is not proper, so in general does not preserve stationary sets.

The proof involves a complicated factoring of $j(\mathbb{P})$, which has a tail forcing which preserves stationary subsets of $P_{\omega_1}(\omega_2)$ which are disjoint from S.

Open Questions

Question

Does $WRP(\omega_3)$ imply $RP(\omega_3)$?

Question

Does WRP imply $RP(\omega_2)$?

Question

Is the supercompactness used in the proof of the theorem necessary?

For a preprint which includes a proof of the main theorem of the talk, see my website:

http://math.berkeley.edu/∼jkrueger