Partitions and Indivisibility Properties of Countable Dimensional Vector Spaces

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Finite Dimensional Vector Space over a Finite Field

Theorem (Graham, Leeb and Rothschild, 1972)

For all $d, k, t \geq 0$, there exists $n = GLR^t(d, k)$ with the property that for any n-dimensional vector space V over $\mathbb{F}_q$ and any colouring of all t-dimensional (affine) subspaces into k colours, there exists a d-dimensional (affine) subspace $W \subset V$ such that all its t-dimensional (affine) subspaces have the same colour.

Corollary (t=1 for subspaces)

For all d, k, there exists $n = GLR(d, k)$ such that:

for any n-dimensional vector space V over $\mathbb{F}_q$

and any colouring of the lines of V into k colours,

there exists a d-dimensional subspace $W \subset V$ all of whose lines are the same colour.

Finite Dimensional Vector Space over a Finite Field (t=0)

Corollary (t=0 for affine subspaces)

For all d, k, there exists such that:

for any n-dimensional vector space V over $\mathbb{F}_q$

and any colouring of V into k colours,

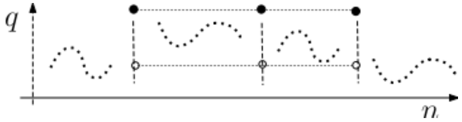
there exists a monochromatic d-dimensional affine subspace $W \subset V$.

Proof.

(Hales-Jewett 63) For $d = 1$:

There exists n such that for any $\chi : q^n \to k$, there exists a monochromatic combinatorial line.

This is a one-dimensional affine subspace (of the form $w + < \vec{1} >$)



What about vector spaces of countable dimension over any field?

Indivisibility Properties

Definition

A relational structure $\mathcal{R}$ is:

1. *indivisible* if for every partition (P_0, P_1) of R,
   there exists an element $\epsilon \in Emb(\mathcal{R})$ and an $i \in 2$ with $\epsilon[R] \subseteq P_i$.
2. A relational structure $\mathcal{R}$ is said to have a *uniform partition* if there is a finite partition $(U_i : i \in n)$ of R such that
   2.1 for all $\epsilon \in Emb(\mathcal{R})$ and all $i \in n$, $age(\epsilon[\mathcal{R}] \downarrow U_i) = age(\mathcal{R})$,
   2.2 for every partition (P_0, P_1) of R and any $i \in n$, there exists an element $\epsilon \in Emb(\mathcal{R})$ and $j \in 2$ with $(\epsilon[R] \cap U_i) \subseteq P_j$.
3. *weakly indivisible* if for every partition (P_0, P_1) of R with $age(\mathcal{R} \downarrow P_0) \neq age(\mathcal{R})$,
   there exists an element $\epsilon \in Emb(\mathcal{R})$ with $\epsilon[R] \subseteq P_1$.
4. *age indivisible* if for every partition (P_0, P_1) of R,
   there is an $i \in 2$ with $age(\mathcal{R} \downarrow P_i) = age(\mathcal{R})$.

Affine Cycle Structure of a Vector Space

Observation

A vector space V can be viewed as a relational structure $\mathcal{V}$ by interpreting every equivalence class of affine cycles of V as a relation on V.
In this setting:

- *$Aut(\mathcal{V}) = IGL(\mathcal{V})$ (Inhomogeneous General Linear group of V).*
  That is all invertible affine transformation from V to V (maps of the form $\alpha(v) = w + \lambda(v)$ for some invertible transformation λ.)
- *$Emb(\mathcal{V})$ = the set of affine embeddings of V into V.*
  That is all affine embeddings from V to V (maps of the form $\alpha(v) = w + \lambda(v)$ for some embedding λ.)

Note that every finite substructure of $\mathcal{V}$ generates affinely a finite dimensional affine subspace of V and hence every element of the age of $\mathcal{V}$ can be affinely embedded into a finite dimensional affine subspace of V.

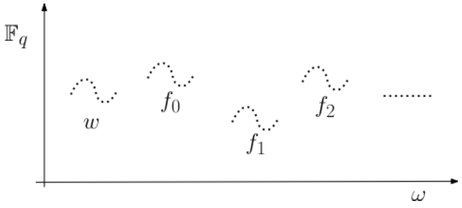
Vector Spaces of Countable Dimension

Fix a vector space V of countable dimension over a field $\mathbb{F}$, we may assume that $V = \mathbb{F}^{[\omega]}$.

Observation

Every infinite dimensional affine subspace $w + W$ of V contains an infinite sequence $(w + f_i : i \in \omega)$ such that

$$w \lll domf_i \lll domf_{i+1} \text{ for all } i \in \omega.$$



Indivisibility

Theorem (Hindman, 1974)

$\mathbb{F}_2^{[\omega]}$ is indivisible.
(i.e. the countable dimensional vector space over $\mathbb{F}_2$ is indivisible).

Proof.

We have $\mathbb{F}_2^{[\omega]}$ = all finite subsets of ω.
Moreover if $f \lll g \in \mathbb{F}_2^{[\omega]}$, then $f + g =$ "$f \cup g$".
So the Theorem follows from (is equivalent to) Hindman's Finite Unions Theorem. □

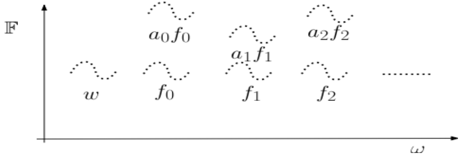
Indivisibility

Theorem (Folklore)

If $\mathbb{F} \neq \mathbb{F}_2$, then $V = \mathbb{F}^{[\omega]}$ is divisible (not indivisible).
(In fact V does not have a uniform and even a canonical partition.)

Proof.

If $dom(f) =< x_i : i \in n >$, then let
$\mathrm{osc}(f) = |\{i \in n-1 : f(x_i) \neq f(x_{i+1})\}|$.
Wlog we may assume $f_i(max\ dom(f_i)) = f_{i+1}(min\ dom(f_{i+1})$.
Choose $a_i \in \mathbb{F} \setminus \{0\}$ with $a_i \neq a_{i+1}$ ($\mathbb{F} \neq \mathbb{F}_2$).



Then

- $\mathrm{osc}(w + f_0 + f_1 + \cdots + f_n) := K$.
- $\mathrm{osc}(w + f_0 + a_1(f_1 + \cdots + f_n)) = K + 1$.
- $\mathrm{osc}(w + f_0 + a_1 f_1 + a_2(f_2 + \cdots + f_n)) = K + 2$.
- $\ldots$

$\square$

Weak Indivisibility – Finite Fields

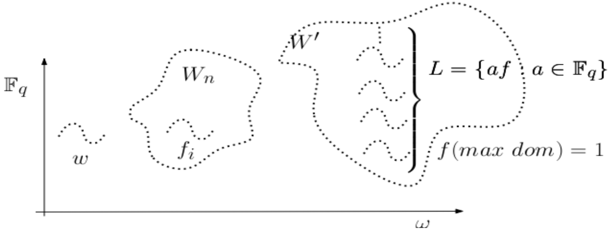
Theorem (LNPS, 2008)

$\mathbb{F}_q^{[\omega]}$ *is weakly indivisible.*

That is if the countable dimensional vector space over $\mathbb{F}_q$ is partitioned into two colours such that the first colour does not contain affine subspaces of unbounded finite dimensions, then there exists an affine subspace of infinite dimension in the second colour.

Proof.

Fix $\Delta : \mathbb{F}_q^{[\omega]} \to \{red,\ blue\}$ with no red affine subspaces of dimension k. We need to construct a blue affine subspace W of infinite dimension.



For each $f_i \in W_n$ define a colouring Γ of every line $L = \langle f \rangle \in W'$ by:

$$\Gamma(L) = \gamma_L : \mathbb{F}_q \to \{red, blue\} \text{ where } \gamma_L(a) = \Delta(w + f_i + af)$$

Use GLR! □

Weak Indivisibility – Infinite Fields

Theorem

Every vector space $\mathcal{V}$ over $\mathbb{Q}$ is not weakly indivisible.

Proof.

Baumgartner (75) proved the following more precise result.

Proposition

A vector space $\mathcal{V}$ over $\mathbb{Q}$ (of any dimension) contains a subset A which meets every infinite arithmetic progression, but does not contain any 3-element arithmetic progression.

$\square$

Age Indivisibility

By a compactness argument:
A relational structure $\mathcal{R}$ is age indivisible if and only if the age of $\mathcal{R}$ is a Ramsey family.

That is, if for every element $\mathcal{A}$ in the age of $\mathcal{R}$ with base A there exists an element $\mathcal{B} \in age(\mathcal{R})$ so that for every partition (P_0, P_1) of the base set of $\mathcal{B}$ there exists an embedding ϵ and an $i \in 2$ with $\epsilon[A] \subseteq P_i$.

By the Hales-Jewett Theorem, the age of any vector space has the Ramsey property.

Summary

- $\mathbb{F}_2^{[\omega]}$ is indivisible.
- $\mathbb{F}_q^{[\omega]}$ is weakly indivisible, and for $q \neq 2$ not indivisible (no uniform partition).
- $\mathbb{Q}^{[\omega]}$ is age indivisible, not weakly indivisible.

Mid Point Structure

Let $\mathcal{V} = \mathbb{Q}^{[\omega]}$, and $\mathbb{M}_{\mathcal{V}} = (V, \mu)$ where

$$\mu(a, b, c) \text{ if and only if } 2b = a + c$$

Then again $Emb(\mathbb{M}_{\mathcal{V}}) =$ affine embeddings of $\mathcal{V}$.

Theorem (LNPS, 2008)

$\mathbb{M}_{\mathcal{V}}$ *is:*

1. *Age indivisible.*
2. *Not weakly indivisible.*

Mid Point Structure

Failure of weak indivisibility.

Same as before (Baumgartner) $\square$

Age Indivisibility.

$\mathbb{M}_{\mathcal{V}}$ and $\mathbb{M}_{\mathbb{N}}$ have the same age, so it suffices to show that $\mathbb{M}_{\mathbb{N}}$ is age indivisible.
Let $A \subset \mathbb{N}$ and $n \in \mathbb{N}$, we need to show that $\mathbb{M}_{\mathbb{N}} \restriction [0, n[$ is embeddable into either

$$\mathbb{M}_{\mathbb{N}} \restriction A \text{ or into } \mathbb{M}_{\mathbb{N}} \restriction (\mathbb{N} \setminus A).$$

But this follows form Van der Waerden's theorem on arithmetic progressions. $\square$

$\mathbb{BS}_{\mathbb{Q}}^{\infty}$

Consider the countable homogeneous metric space $\mathbb{BS}_{\mathbb{Q}}^{\infty}$ (whose class of finite metric subspaces is exactly those with rational values embedding isometrically into the unit sphere of ℓ_2).

Theorem (Nguyen Van Thé and Sauer, 2008)

$\mathbb{BS}_{\mathbb{Q}}^{\infty}$ *is age indivisible, not weakly indivisible.*

Remarks

Examples of age indivisible, not weakly indivisible countable relational structures.

- $\mathbb{Q}^{[\omega]}$ and $\mathbb{BS}_{\mathbb{Q}}^{\infty}$ are homogeneous but have infinitely many relations.
- $\mathbb{M}_{\mathcal{V}}$ for $\mathcal{V} = \mathbb{Q}^{[\omega]}$ has one relation, but is not homogeneous. It can be made homogeneous by adding infinitely many relations.

Question

Can one find a "simple" age indivisible countable homogeneous relational structure not weakly indivisible?

Conclusion

Question

What other infinite generalisations are there to Graham, Leeb and Rothschild's 1972 Theorem?