Eventually different forcing at the second level of the projective hierarchy

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Regularity properties

Regularity Properties: Lebesgue measurability, Baire property, the perfect set property...

Problem. ZFC does not prove that all projective sets are “regular”. For instance, the model \( L \) has a \( \Delta^1_2 \) set that is not Lebesgue measurable and does not have the Baire and perfect set property.

Q. Can we characterize the following statements in set-theoretic terms:

\[
\text{LM}(\Delta^1_2) : \text{“every } \Delta^1_2 \text{ set is Lebesgue measurable”}
\]
\[
\text{LM}(\Sigma^1_2) : \text{“every } \Sigma^1_2 \text{ set is Lebesgue measurable”}
\]
Random reals

**Theorem** (Solovay). $\text{LM}(\Sigma^1_2)$ if and only if for every $x$, the set of random reals over $L[x]$ is a measure one set.

**Solovay-style characterization theorem**

Remember that a real is random over $M$ if and only if it is not a member of any measure zero Borel set with a Borel code in $M$.

$\omega_1$ is inaccessible by reals: “for all $x$, $\omega_1^{L[x]}$ is countable.”

**Proposition.** If $\omega_1$ is inaccessible by reals, then $\text{LM}(\Sigma^1_2)$.

**Proof.** The union of all null sets coded in $L[x]$ is a union of size $\omega_1^{L[x]} < \omega_1$ of null sets, so it has measure zero. But its complement is the set of random reals. q.e.d.
At the $\Delta^1_2$-level

Theorem (Judah-Shelah). $\text{LM}(\Delta^1_2)$ if and only if for every $x$, there is a random real over $L[x]$.

Judah-Shelah-style characterization theorem

Corollary. In the $\omega_1$-iteration of random forcing, $\text{LM}(\Delta^1_2)$ holds.

Corollary. $\text{LM}(\Delta^1_2)$ is strictly weaker than “$\omega_1$ is inaccessible by reals”.
Generalisations (1)

The two characterisation theorems are not just true in the case of random forcing. For instance:

**Theorem.** $\text{BP}(\Sigma^1_2)$ if and only if for every $x$, the set of Cohen reals over $L[x]$ is a comeager set.

**Proposition.** If $\omega_1$ is inaccessible by reals, then $\text{BP}(\Sigma^1_2)$.

**Theorem.** $\text{BP}(\Delta^1_2)$ if and only if for every $x$, there is a Cohen real over $L[x]$.

**Proposition.** $\text{BP}(\Delta^1_2)$ is strictly weaker than “$\omega_1$ is inaccessible by reals”.
Generalisations (2)

Even more generally, a forcing notion $\mathbb{P}$ defines an ideal $\mathcal{I}_\mathbb{P}$, a corresponding notion of measurability, and a notion of genericity. We write $\text{Meas}_\mathbb{P}(\Gamma)$ for “all sets in $\Gamma$ are $\mathbb{P}$-measurable”.

A false hope:

- $\text{Meas}_\mathbb{P}(\Sigma^1_2)$ if and only if for every $x$, the set of $\mathbb{P}$-generics over $L[x]$ is $\text{co-}\mathcal{I}_\mathbb{P}$. ("Solovay Theorem")
- $\text{Meas}_\mathbb{P}(\Delta^1_2)$ if and only if for every $x$, there is a $\mathbb{P}$-generic over $L[x]$. ("Judah-Shelah Theorem")

It will turn out that these are not true in general, and a refinement is necessary.
A concrete example: Hechler forcing

**Hechler forcing** $\mathbb{D}$ consists of pairs $\langle s, f \rangle$ where $s \in \omega^{<\omega}$ and $f \in \omega^\omega$ with

$$\langle s, f \rangle \leq \langle t, g \rangle \text { iff } s \supseteq t \text { and } \forall n \geq \text{lh}(t)(g(n) \leq f(n)) \quad (1)$$

and

$$\forall n \in \text{lh}(s) \setminus \text{lh}(t)(g(n) \leq s(n)) \quad (2)$$

The conditions of Hechler forcing define a topology called the **dominating topology**. We call a set $\mathbb{D}$-measurable if it has the Baire property in the dominating topology and let the ideal $\mathcal{I}_\mathbb{D}$ be the set of all sets meager in the dominating topology.

Again, a real is **Hechler** over $M$ if it is not an element of any Borel set meager in the dominating topology and coded in $M$. 
Theorem (Brendle-L. 1998). The following are equivalent:

- $\text{Meas}_D(\Sigma^1_2)$,
- for every $x$, the set of Hechler reals over $L[x]$ is co-meager in the dominating topology,
- $\omega_1$ is inaccessible by reals.

Solovay-style characterization

Theorem (Brendle-L. 1998). The following are equivalent:

- $\text{Meas}_D(\Delta^1_2)$,
- for every $x$, there is a Hechler real over $L[x]$,
- $\text{BP}(\Sigma^1_2)$.

Judah-Shelah-style characterization
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How regular are the projective sets?

Generic reals and regularity

Ikegami’s characterization theorems

Eventually different forcing:

\[ \Sigma^1_2 \]

\[ \Delta^1_2 \]

A diagram of implications

\[ \Sigma^1_2(R) = \Delta^1_2(R) \]

\[ \Sigma^1_2(C) = \Delta^1_2(D) \]

\[ \Sigma^1_2(L) = \Delta^1_2(L) \]

\[ \Delta^1_2(C) \]

\[ \Sigma^1_2(V) \]

\[ \Sigma^1_2(M) = \Delta^1_2(M) \]

\[ \Delta^1_2(V) \]

\[ \Sigma^1_2(S) = \Delta^1_2(S) \]

\[ \Sigma^1_2(B) = \Delta^1_2(A) \]

\[ \Delta^1_2(B) \]

ev. diff.
Abstract Solovay and Judah-Shelah theorems.

We mentioned a (vain) hope for abstract Solovay and Judah-Shelah theorems:

- $\text{Meas}_P(\Sigma^1_2)$ if and only if for every $x$, the set of $P$-generics over $L[x]$ is co-$\mathcal{I}_P$. *Solovay*
- $\text{Meas}_P(\Delta^1_2)$ if and only if for every $x$, there is a $P$-generic over $L[x]$. *Judah-Shelah*

**Definition** (Brendle-Halbeisen-L.-Ikegami). A real $x$ is $P$-quasigeneric over $M$ if if for all Borel codes $c \in M$ such that $B_c \in \mathcal{I}_P^*$, we have that $r \notin B_c$. Here,

$$\mathcal{I}_P^* := \{ X ; \forall T \in P \exists S \in P(S \leq T \land [S] \cap X \in \mathcal{I}_P) \}.$$

For random, Cohen and Hechler reals, being generic is equivalent to being quasigeneric.
Meas$_{P}(\Sigma^1_2)$ if and only if for every $x$, the set of $P$-generics over $L[x]$ is co-$I_P$. (Solovay)

Meas$_{P}(\Delta^1_2)$ if and only if for every $x$, there is a $P$-generic over $L[x]$. (Judah-Shelah)

A real $x$ is $P$-quasigeneric over $M$ if if for all Borel codes $c \in M$ such that $B_c \in I_P^*$, we have that $r \notin B_c$. Here,

$$I_P^* := \{X ; \forall T \in P \exists S \in P(S \leq T \land [S] \cap X \in I_P)\}.$$ 

Meas$_{P}(\Sigma^1_2)$ if and only if for every $x$, the set of $P$-quasigenerics over $L[x]$ is co-$I_P$. (“Solovay Theorem”)

Meas$_{P}(\Delta^1_2)$ if and only if for every $x$, there is a $P$-quasigeneric over $L[x]$. (“Judah-Shelah Theorem”)

The characterizations of Brendle-L. (1998) for Sacks, Miller, and Laver forcing fit into this template and become Judah-Shelah-style characterizations.
Abstract Judah-Shelah Theorem (Ikegami 2007). If $\mathbb{P}$ is a proper and strongly arboreal forcing notion such that $\{c; c$ is a Borel code and $B_c \in I^*_\mathbb{P}\}$ is $\Sigma^1_2$, then the following are equivalent:

1. $\Sigma^1_3$-$\mathbb{P}$-absoluteness,
2. every $\Delta^1_2$ set is $\mathbb{P}$-measurable, and
3. for every real $x$ and every $T \in \mathbb{P}$, there is a $I^*_\mathbb{P}$-quasigeneric real in $[T]$ over $L[x]$.

Abstract Solovay Theorem (Ikegami 2007). If $\mathbb{P}$ is a proper and strongly arboreal forcing notion such that $\{c; c$ is a Borel code and $B_c \in I^*_\mathbb{P}\}$ is $\Sigma^1_2$ and $I^*_\mathbb{P}$ is Borel generated, then the following are equivalent:

1. every $\Sigma^1_2$ set is $\mathbb{P}$-measurable, and
2. for every real $x$, the set $\{y; y$ is not $I^*_\mathbb{P}$-quasigeneric over $L[x]\}$ belongs to $I^*_\mathbb{P}$. 
Eventually different forcing (1).

Eventually different forcing $\mathbb{E}$ consists of pairs $\langle s, F \rangle$, where $s \in \omega^{<\omega}$ and $F$ is a finite set of reals with

$$\langle s, F \rangle \leq \langle t, G \rangle \quad \text{iff} \quad t \subseteq s, \ G \subseteq F, \ \text{and} \ \forall i \in \text{dom}(s \setminus t) \forall g \in G(s(i) \neq g(i)).$$

Eventually different forcing is a c.c.c. forcing that generates the eventually different topology refining the standard topology on Baire space.

**Proposition** (Łąbędzki 1997). The meager sets in the eventually different topology form an ideal $I_{\mathbb{E}}$ which has a basis of Borel sets.

**Theorem** (Łąbędzki 1997). A real $x$ is $\mathbb{E}$-generic over $M$ if and only if it is $\mathbb{E}$-quasigeneric over $M$. 
Eventually different forcing (2).

Let \( \langle f_\alpha; \alpha < \omega_1 \rangle \) be a family of eventually different functions. Let

\[
E_\alpha := \{ x \in \omega^\omega; \exists^\infty k \in \omega (x(k) = f_\alpha(k)) \}.
\]

These sets are nowhere dense in the eventually different topology.

**Theorem** (Brendle). If \( G \) is meager in the eventually different topology and \( \langle f_\alpha; \alpha < \omega_1 \rangle \) a family of eventually different functions then the set \( \{ \alpha; E_\alpha \subseteq G \} \) is countable.

**Corollary** (Łabędzki). The additivity of \( I_\mathbb{D} \) is \( \aleph_1 \).
A Solovay theorem for $\mathbb{E}$.

Abstract Solovay Theorem (Ikegami 2007). If $P$ is a proper and strongly arboreal forcing notion such that $\{c ; c \text{ is a Borel code and } B_c \in \mathcal{I}_P^*\}$ is $\Sigma^1_2$ and $\mathcal{I}_P$ is Borel generated, then the following are equivalent:

1. every $\Sigma^1_2$ set is $P$-measurable, and
2. for every real $x$, the set $\{y ; y \text{ is not } \mathcal{I}_P^*-\text{quasigeneric over } \mathbb{L}[x]\}$ belongs to $\mathcal{I}_P^*$.

Theorem. The following are equivalent:

1. $\text{Meas}_\mathbb{E}(\Sigma^1_2)$ and
2. for every $x$, the set of $\mathbb{E}$-generics over $\mathbb{L}[x]$ is comeager in the eventually different topology.
3. $\omega_1$ is inaccessible by reals.

\[ E_\alpha := \{x \in \omega^\omega ; \exists^\infty k \in \omega(x(k) = f_\alpha(k))\}. \]

Theorem (Brendle). If $G$ is meager in the eventually different topology and $\langle f_\alpha ; \alpha < \omega_1 \rangle$ a family of eventually different functions then the set $\{\alpha ; E_\alpha \subseteq G\}$ is countable.

Proof. "(ii)\Rightarrow(iii)": Suppose $\omega^\mathbb{L}[x]_1 = \omega_1$. In $\mathbb{L}[x]$, there is a family $\langle f_\alpha ; \alpha < \omega_1 \rangle$ of eventually different functions. All $E_\alpha$ are nowhere dense and coded in $\mathbb{L}[x]$, so no $\mathbb{E}$-generic over $\mathbb{L}[x]$ can lie in one of the $E_\alpha$. So, the complement of the generic reals cannot be meager by Brendle’s theorem. Contradiction! q.e.d.
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How regular are the projective sets?

Generic reals and regularity

Ikegami's characterization theorems

Eventually different forcing: \( \Sigma_2^1 \)

Eventually different forcing: \( \Delta_2^1 \)

The Diagram again

\[
\begin{align*}
\Sigma_2^1(\mathbb{D}) &= \Delta_2^1(\mathbb{B}) = \Delta_2^1(\mathbb{A}) \\
\Sigma_2^1(\mathbb{R}) &= \Delta_2^1(\mathbb{R}) \\
\Sigma_2^1(\mathbb{C}) &= \Delta_2^1(\mathbb{D}) \\
\Sigma_2^1(\mathbb{L}) &= \Delta_2^1(\mathbb{L}) \\
\Sigma_2^1(\mathbb{M}) &= \Delta_2^1(\mathbb{M}) \\
\Sigma_2^1(\mathbb{S}) &= \Delta_2^1(\mathbb{S}) \\
\Sigma_2^1(\mathbb{V}) &= \Delta_2^1(\mathbb{V}) \\
\end{align*}
\]
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Eventually different forcing: $\Sigma^1_2$

Eventually different forcing: $\Delta^1_2$

The Diagram again

The Diagram again

$\Sigma^1_2(R) = \Delta^1_2(R)$

$\Sigma^1_2(L) = \Delta^1_2(L)$

$\Sigma^1_2(M) = \Delta^1_2(M)$

$\Sigma^1_2(S) = \Delta^1_2(S)$

$\Sigma^1_2(\mathbb{E}) = \Sigma^1_2(\mathbb{D})$

$\Sigma^1_2(\mathbb{B}) = \Delta^1_2(\mathbb{A})$

$\Delta^1_2(\mathbb{B})$

$\Sigma^1_2(\mathbb{V})$

$\Delta^1_2(\mathbb{V})$

$\Sigma^1_2(\mathbb{C}) = \Delta^1_2(\mathbb{D})$

$\Delta^1_2(\mathbb{C})$

ev. diff.
A Judah-Shelah theorem for $\mathbb{E}$.

Abstract Judah-Shelah Theorem (Ikegami 2007). If $\mathbb{P}$ is a proper and strongly arboreal forcing notion such that $\{c : c \text{ is a Borel code and } B_c \in \mathcal{I}_\mathbb{P}^*\}$ is $\Sigma^1_2$, then the following are equivalent:

1. $\Sigma^1_3$-$\mathbb{P}$-absoluteness,
2. every $\Delta^1_2$ set is $\mathbb{P}$-measurable, and
3. for every real $x$ and every $T \in \mathbb{P}$, there is a $\mathcal{I}_\mathbb{P}^*$-quasigeneric real in $[T]$ over $L[x]$.

Theorem. The following are equivalent:

1. $\text{Meas}_{\mathbb{E}}(\Delta^1_2)$, and
2. for every $x$, there is an $\mathbb{E}$-generic over $L[x]$. 
Locating $\Delta^1_2(E)$

- The $\omega_1$-iteration of $E$ produces a model of $\text{Meas}_E(\Delta^1_2)$ without dominating or random reals, therefore $\text{LM}(\Delta^1_2)$ and $\text{Meas}_L(\Delta^1_2)$ are false there. In particular, $\text{Meas}_E(\Sigma^1_2)$ and $\text{Meas}_E(\Delta^1_2)$ are not equivalent.
- In the $\omega_1$-iteration of Cohen forcing, we do not have an eventually different real. In particular, $\text{Meas}_E(\Delta^1_2)$ is false.
- Every $E$-generic is also Cohen generic, so $\text{Meas}_E(\Delta^1_2)$ implies $\text{BP}(\Delta^1_2)$.
- Since the $\omega_1$-iteration of random forcing does not add Cohen reals, $\text{Meas}_E(\Delta^1_2)$ is false there.
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Eventually different forcing: $\Sigma^1_2$

Eventually different forcing: $\Delta^1_2$
The final diagram

\[
\begin{align*}
\Sigma_2^1(\mathcal{E}) &= \Sigma_2^1(\mathcal{D}) \\
\Sigma_2^1(\mathcal{B}) &= \Delta_2^1(\mathcal{A}) \\
\Sigma_2^1(\mathcal{R}) &= \Delta_2^1(\mathcal{R}) \\
\Sigma_2^1(\mathcal{C}) &= \Delta_2^1(\mathcal{D}) \\
\Delta_2^1(\mathcal{E}) &= \Delta_2^1(\mathcal{E}) \\
\Delta_2^1(\mathcal{B}) &= \Delta_2^1(\mathcal{B}) \\
\Sigma_2^1(\mathcal{L}) &= \Delta_2^1(\mathcal{L}) \\
\Delta_2^1(\mathcal{C}) &= \Delta_2^1(\mathcal{C}) \\
\Sigma_2^1(\mathcal{V}) &= \Sigma_2^1(\mathcal{V}) \\
\Delta_2^1(\mathcal{V}) &= \Delta_2^1(\mathcal{V}) \\
\Sigma_2^1(\mathcal{M}) &= \Delta_2^1(\mathcal{M}) \\
\Delta_2^1(\mathcal{M}) &= \Delta_2^1(\mathcal{M}) \\
\Sigma_2^1(\mathcal{S}) &= \Delta_2^1(\mathcal{S}) \\
\end{align*}
\]
We still have to give a model of $\text{Meas}_D(\Delta^1_2) \land \neg \text{Meas}_E(\Delta^1_2)$.

**Dichotomy for iterated Hechler forcing.** Let $(P_\alpha, D_\alpha ; \alpha < \gamma)$ be a finite support iteration of Hechler forcing. Let $x$ be a real in the $P_\gamma$-generic extension. Then

1. either $x$ is dominating over $V$
2. or $x$ is not eventually different over $V$.

**Corollary.** In the $\omega_1$-finite support iteration of Hechler forcing, $\text{Meas}_E(\Delta^1_2)$ fails.