Eventually different forcing at the second level of the projective hierarchy

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Regularity properties

Regularity Properties: Lebesgue measurability, Baire property, the perfect set property...

Problem. ZFC does not prove that all projective sets are "regular". For instance, the model $\mathbf{L}$ has a $\mathbf{\Delta}^1_2$ set that is not Lebesgue measurable and does not have the Baire and perfect set property.

Q. Can we characterize the following statements in set-theoretic terms:

$$\mathrm{LM}(\mathbf{\Delta}^1_2) : \text{“every } \mathbf{\Delta}^1_2 \text{ set is Lebesgue measurable”}$$

$$\mathrm{LM}(\mathbf{\Sigma}^1_2) : \text{“every } \mathbf{\Sigma}^1_2 \text{ set is Lebesgue measurable”}$$

Random reals

Theorem (Solovay). $\mathrm{LM}(\boldsymbol{\Sigma}^1_2)$ if and only if for every x, the set of random reals over $\mathbf{L}[x]$ is a measure one set.

Solovay-style characterization theorem

Remember that a real is random over M if and only if it is not a member of any measure zero Borel set with a Borel code in M.

ω_1 is inaccessible by reals: "for all x, $\omega_1^{\mathbf{L}[x]}$ is countable."

Proposition. If ω_1 is inaccessible by reals, then $\mathrm{LM}(\boldsymbol{\Sigma}^1_2)$.

Proof. The union of all null sets coded in $\mathbf{L}[x]$ is a union of size $\omega_1^{\mathbf{L}[x]} < \omega_1$ of null sets, so it has measure zero. But its complement is the set of random reals. q.e.d.

At the $\mathbf{\Delta}^1_2$-level

Theorem (Judah-Shelah). $\mathrm{LM}(\mathbf{\Delta}^1_2)$ if and only if for every x, there is a random real over $\mathbf{L}[x]$.

Judah-Shelah-style characterization theorem

Corollary. In the ω_1-iteration of random forcing, $\mathrm{LM}(\mathbf{\Delta}^1_2)$ holds.

Corollary. $\mathrm{LM}(\mathbf{\Delta}^1_2)$ is strictly weaker than "ω_1 is inaccessible by reals".

Generalisations (1)

The two characterisation theorems are not just true in the case of random forcing. For instance:

Theorem. $\mathrm{BP}(\mathbf{\Sigma}^1_2)$ if and only if for every x, the set of Cohen reals over $\mathbf{L}[x]$ is a comeager set.

Proposition. If ω_1 is inaccessible by reals, then $\mathrm{BP}(\mathbf{\Sigma}^1_2)$.

Theorem. $\mathrm{BP}(\mathbf{\Delta}^1_2)$ if and only if for every x, there is a Cohen real over $\mathbf{L}[x]$.

Proposition. $\mathrm{BP}(\mathbf{\Delta}^1_2)$ is strictly weaker than "ω_1 is inaccessible by reals".

Generalisations (2)

Even more generally, a forcing notion $\mathbb{P}$ defines an ideal $\mathcal{I}_{\mathbb{P}}$, a corresponding notion of measurability, and a notion of genericity. We write $\mathrm{Meas}_{\mathbb{P}}(\Gamma)$ for "all sets in Γ are $\mathbb{P}$-measurable".

A false hope:

- $\mathrm{Meas}_{\mathbb{P}}(\mathbf{\Sigma}^1_2)$ if and only if for every x, the set of $\mathbb{P}$-generics over $\mathbf{L}[x]$ is co-$\mathcal{I}_{\mathbb{P}}$. ("Solovay Theorem")
- $\mathrm{Meas}_{\mathbb{P}}(\mathbf{\Delta}^1_2)$ if and only if for every x, there is a $\mathbb{P}$-generic over $\mathbf{L}[x]$. ("Judah-Shelah Theorem")

It will turn out that these are not true in general, and a refinement is necessary.

A concrete example: Hechler forcing

Hechler forcing $\mathbb{D}$ consists of pairs $\langle s, f\rangle$ where $s \in \omega^{<\omega}$ and $f \in \omega^\omega$ with

$$\langle s, f\rangle \leq \langle t, g\rangle \text{ iff } s \supseteq t \text{ and } \forall n \geq \mathrm{lh}(t)(g(n) \leq f(n)) \quad (1)$$
$$\text{and } \forall n \in \mathrm{lh}(s)\backslash\mathrm{lh}(t)(g(n) \leq s(n)) \quad (2)$$

The conditions of Hechler forcing define a topology called the dominating topology. We call a set $\mathbb{D}$-measurable if it has the Baire property in the dominating topology and let the ideal $\mathcal{I}_{\mathbb{D}}$ be the set of all sets meager in the dominating topology.

Again, a real is Hechler over M if it is not an element of any Borel set meager in the dominating topology and coded in M.

Theorem (Brendle-L. 1998). The following are equivalent:

- $\mathrm{Meas}_{\mathbb{D}}(\mathbf{\Sigma}^1_2)$,
- for every x, the set of Hechler reals over $\mathbf{L}[x]$ is co-meager in the dominating topology,
- ω_1 is inaccessible by reals.

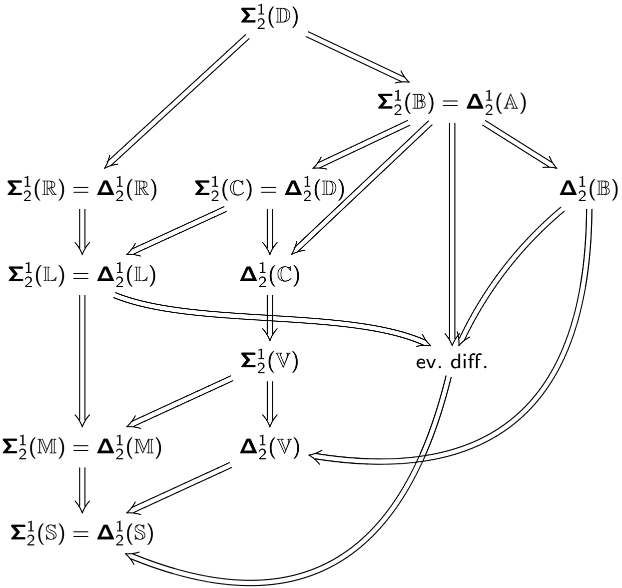
Solovay-style characterization

Theorem (Brendle-L. 1998). The following are equivalent:

- $\mathrm{Meas}_{\mathbb{D}}(\mathbf{\Delta}^1_2)$,
- for every x, there is a Hechler real over $\mathbf{L}[x]$,
- $\mathrm{BP}(\mathbf{\Sigma}^1_2)$.

Judah-Shelah-style characterization

A diagram of implications



Abstract Solovay and Judah-Shelah theorems.

We mentioned a (vain) hope for abstract Solovay and Judah-Shelah theorems:

- $\mathrm{Meas}_{\mathbb{P}}(\mathbf{\Sigma}^1_2)$ if and only if for every x, the set of $\mathbb{P}$-generics over $\mathbf{L}[x]$ is co-$\mathcal{I}_{\mathbb{P}}$. Solovay
- $\mathrm{Meas}_{\mathbb{P}}(\mathbf{\Delta}^1_2)$ if and only if for every x, there is a $\mathbb{P}$-generic over $\mathbf{L}[x]$. Judah-Shelah

Definition (Brendle-Halbeisen-L.-Ikegami). A real x is $\mathbb{P}$-quasigeneric over M if if for all Borel codes $c \in M$ such that $\mathrm{B}_c \in \mathcal{I}^*_{\mathbb{P}}$, we have that $r \notin \mathrm{B}_c$. Here,

$$\mathcal{I}^*_{\mathbb{P}} := \{X\,;\,\forall T \in \mathbb{P} \exists S \in \mathbb{P}(S \leq T \wedge [S] \cap X \in \mathcal{I}_{\mathbb{P}})\}.$$

For random, Cohen and Hechler reals, being generic is equivalent to being quasigeneric.

- $\mathrm{Meas}_{\mathbb{P}}(\mathbf{\Sigma}^1_2)$ if and only if for every x, the set of $\mathbb{P}$-generics over $\mathbf{L}[x]$ is co-$\mathcal{I}_{\mathbb{P}}$. Solovay
- $\mathrm{Meas}_{\mathbb{P}}(\mathbf{\Delta}^1_2)$ if and only if for every x, there is a $\mathbb{P}$-generic over $\mathbf{L}[x]$. Judah-Shelah

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$\mathrm{Meas}_{\mathbb{P}}(\mathbf{\Sigma}^1_2)$ if and only if for every x, the set of $\mathbb{P}$-quasigenerics over $\mathbf{L}[x]$ is co-$\mathcal{I}_{\mathbb{P}}$. ("Solovay Theorem")

$\mathrm{Meas}_{\mathbb{P}}(\mathbf{\Delta}^1_2)$ if and only if for every x, there is a $\mathbb{P}$-quasigeneric over $\mathbf{L}[x]$. ("Judah-Shelah Theorem")

The characterizations of Brendle-L. (1998) for Sacks, Miller, and Laver forcing fit into this template and become Judah-Shelah-style characterizations.

Abstract Judah-Shelah Theorem (Ikegami 2007). If $\mathbb{P}$ is a proper and strongly arboreal forcing notion such that $\{c\,;\,c$ is a Borel code and $\mathrm{B}_c \in \mathcal{I}^*_{\mathbb{P}}\}$ is $\mathbf{\Sigma}^1_2$, then the following are equivalent:

1. $\mathbf{\Sigma}^1_3$-$\mathbb{P}$-absoluteness,
2. every $\mathbf{\Delta}^1_2$ set is $\mathbb{P}$-measurable, and
3. for every real x and every $T \in \mathbb{P}$, there is a $\mathcal{I}^*_{\mathbb{P}}$-quasigeneric real in $[T]$ over $\mathbf{L}[x]$.

Abstract Solovay Theorem (Ikegami 2007). If $\mathbb{P}$ is a proper and strongly arboreal forcing notion such that $\{c\,;\,c$ is a Borel code and $\mathrm{B}_c \in \mathcal{I}^*_{\mathbb{P}}\}$ is $\mathbf{\Sigma}^1_2$ and $\mathcal{I}_{\mathbb{P}}$ is Borel generated, then the following are equivalent:

1. every $\mathbf{\Sigma}^1_2$ set is $\mathbb{P}$-measurable, and
2. for every real x, the set $\{y\,;\,y$ is not $\mathcal{I}^*_{\mathbb{P}}$-quasigeneric over $\mathbf{L}[x]\}$ belongs to $\mathcal{I}^*_{\mathbb{P}}$.

Eventually different forcing (1).

Eventually different forcing $\mathbb{E}$ consists of pairs $\langle s, F\rangle$, where $s \in \omega^{<\omega}$ and F is a finite set of reals with

$$\langle s, F\rangle \leq \langle t, G\rangle \quad \text{iff} \quad t \subseteq s,\ G \subseteq F, \text{ and } \forall i \in \mathrm{dom}(s \backslash t)\, \forall g \in G(s(i) \neq g(i)).$$

Eventually different forcing is a c.c.c. forcing that generates the eventually different topology refining the standard topology on Baire space.

Proposition (Łabędzki 1997). The meager sets in the eventually different topology form an ideal $\mathcal{I}_{\mathbb{E}}$ which has a basis of Borel sets.

Theorem (Łabędzki 1997). A real x is $\mathbb{E}$-generic over M if and only if it is $\mathbb{E}$-quasigeneric over M.

Eventually different forcing (2).

Let $\langle f_\alpha; \alpha < \omega_1 \rangle$ be a family of eventually different functions. Let

$$E_\alpha := \{x \in \omega^\omega; \exists^\infty k \in \omega(x(k) = f_\alpha(k))\}.$$

These sets are nowhere dense in the eventually different topology.

Theorem (Brendle). If G is meager in the eventually different topology and $\langle f_\alpha; \alpha < \omega_1 \rangle$ a family of eventually different functions then the set $\{\alpha; E_\alpha \subseteq G\}$ is countable.

Corollary (Łabędzki). The additivity of $\mathcal{I}_{\mathbb{D}}$ is $\aleph_1$.

A Solovay theorem for $\mathbb{E}$.

Abstract Solovay Theorem (Ikegami 2007). If $\mathbb{P}$ is a proper and strongly arboreal forcing notion such that $\{c\,;\ c$ is a Borel code and $\mathrm{B}_c \in \mathcal{I}^*_{\mathbb{P}}\}$ is $\mathbf{\Sigma}^1_2$ and $\mathcal{I}_{\mathbb{P}}$ is Borel generated, then the following are equivalent:

1. every $\mathbf{\Sigma}^1_2$ set is $\mathbb{P}$-measurable, and
2. for every real x, the set $\{y\,;\ y$ is not $\mathcal{I}^*_{\mathbb{P}}$-quasigeneric over $\mathbf{L}[x]\}$ belongs to $\mathcal{I}^*_{\mathbb{P}}$.

Theorem. The following are equivalent:

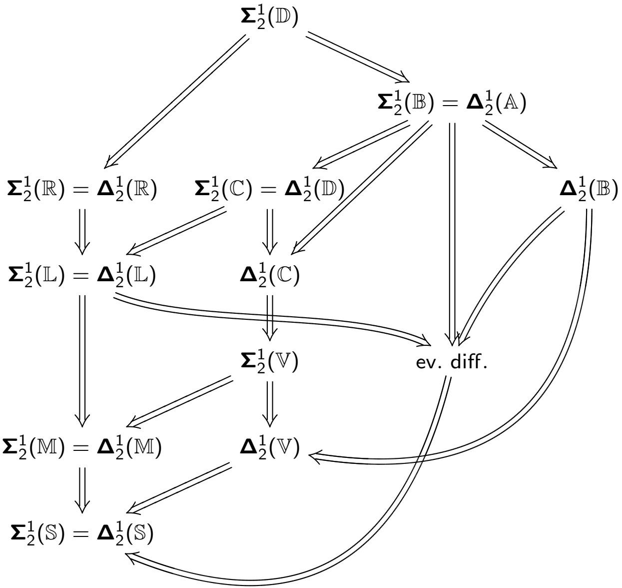
1. $\mathrm{Meas}_{\mathbb{E}}(\mathbf{\Sigma}^1_2)$ and
2. for every x, the set of $\mathbb{E}$-generics over $\mathbf{L}[x]$ is comeager in the eventually different topology.
3. ω_1 is inaccessible by reals.

$$E_\alpha := \{x \in \omega^\omega\,; \exists^\infty k \in \omega (x(k) = f_\alpha(k))\}.$$

Theorem (Brendle). If G is meager in the eventually different topology and $\langle f_\alpha\,; \alpha < \omega_1\rangle$ a family of eventually different functions then the set $\{\alpha; E_\alpha \subseteq G\}$ is countable.

Proof. "(ii)⇒(iii)": Suppose $\omega_1^{\mathbf{L}[x]} = \omega_1$. In $\mathbf{L}[x]$, there is a family $\langle f_\alpha; \alpha < \omega_1\rangle$ of eventually different functions. All E_α are nowhere dense and coded in $\mathbf{L}[x]$, so no $\mathbb{E}$-generic over $\mathbf{L}[x]$ can lie in one of the E_α. So, the complement of the generic reals cannot be meager by Brendle's theorem. Contradiction! q.e.d.

The Diagram again



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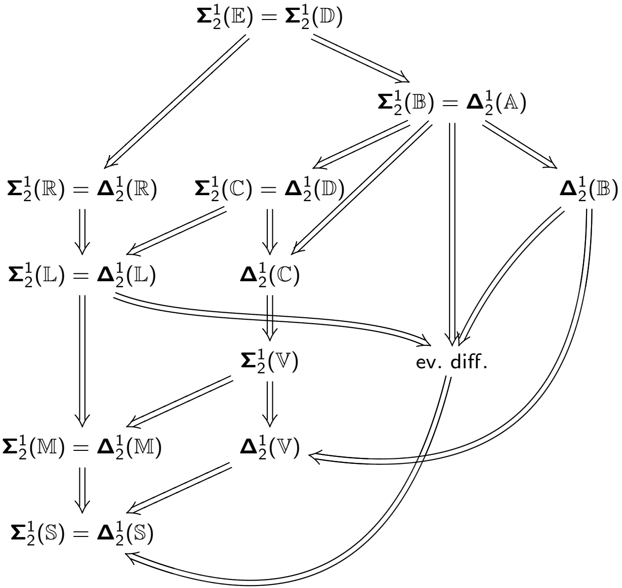

A Judah-Shelah theorem for $\mathbb{E}$.

Abstract Judah-Shelah Theorem (Ikegami 2007). If $\mathbb{P}$ is a proper and strongly arboreal forcing notion such that $\{c\,;\, c$ is a Borel code and $\mathrm{B}_c \in \mathcal{I}^*_{\mathbb{P}}\}$ is $\mathbf{\Sigma}^1_2$, then the following are equivalent:

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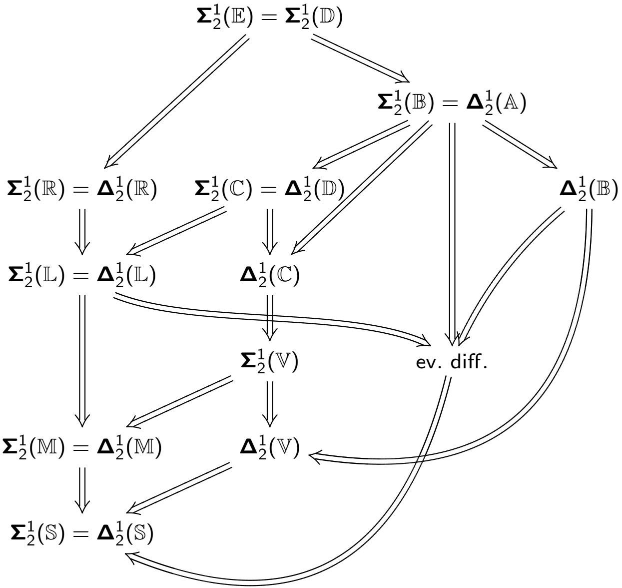
Theorem. The following are equivalent:

1. $\mathrm{Meas}_{\mathbb{E}}(\mathbf{\Delta}^1_2)$, and
2. for every x, there is an $\mathbb{E}$-generic over $\mathbf{L}[x]$.

Locating $\mathbf{\Delta}^1_2(\mathbb{E})$

- The ω_1-iteration of $\mathbb{E}$ produces a model of $\mathrm{Meas}_{\mathbb{E}}(\mathbf{\Delta}^1_2)$ without dominating or random reals, therefore $\mathrm{LM}(\mathbf{\Delta}^1_2)$ and $\mathrm{Meas}_{\mathbb{L}}(\mathbf{\Delta}^1_2)$ are false there. In particular, $\mathrm{Meas}_{\mathbb{E}}(\mathbf{\Sigma}^1_2)$ and $\mathrm{Meas}_{\mathbb{E}}(\mathbf{\Delta}^1_2)$ are not equivalent.
- In the ω_1-iteration of Cohen forcing, we do not have an eventually different real. In particular, $\mathrm{Meas}_{\mathbb{E}}(\mathbf{\Delta}^1_2)$ is false.
- Every $\mathbb{E}$-generic is also Cohen generic, so $\mathrm{Meas}_{\mathbb{E}}(\mathbf{\Delta}^1_2)$ implies $\mathrm{BP}(\mathbf{\Delta}^1_2)$.
- Since the ω_1-iteration of random forcing does not add Cohen reals, $\mathrm{Meas}_{\mathbb{E}}(\mathbf{\Delta}^1_2)$ is false there.

The final diagram



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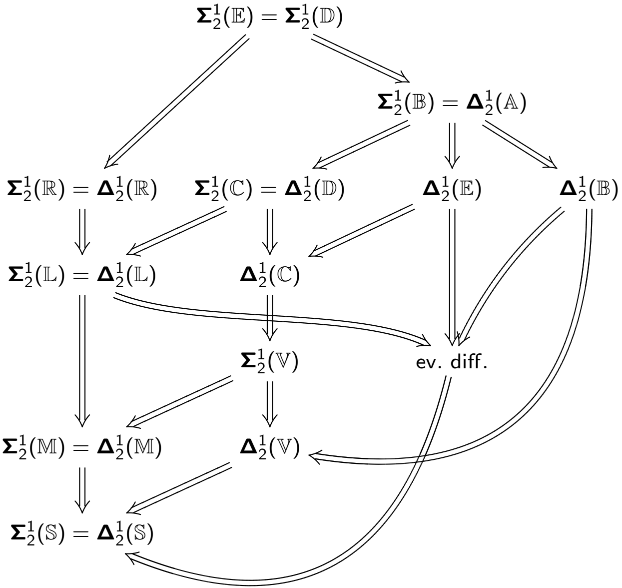

We still have to give a model of $\mathrm{Meas}_{\mathbb{D}}(\mathbf{\Delta}^1_2) \wedge \neg\mathrm{Meas}_{\mathbb{E}}(\mathbf{\Delta}^1_2)$.

Dichotomy for iterated Hechler forcing. Let $(\mathbb{P}_\alpha, \dot{\mathbb{D}}_\alpha\,;\, \alpha < \gamma)$ be a finite support iteration of Hechler forcing. Let x be a real in the $\mathbb{P}_\gamma$-generic extension. Then

1. either x is dominating over V
2. or x is not eventually different over V.

Corollary. In the ω_1-finite support iteration of Hechler forcing, $\mathrm{Meas}_{\mathbb{E}}(\mathbf{\Delta}^1_2)$ fails.