Metric structures and applications to the theory of topological groups

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Polish groups

Definition.

A *Polish group* is a topological group (G, τ) such that τ is separable and completely metrisable.

Example.

Consider the permutation group of the integers, denoted by $\mathcal{S}_\infty$.
If $\sigma, \tau \in \mathcal{S}_\infty$, let

$$d(\sigma, \tau) = \inf\{2^{-n} \colon \sigma_{|n} = \tau_{|n}\} .$$

This is a left invariant separable (ultra)metric.
It is not complete; however the following metric is:

$$\tilde{d}(\sigma, \tau) = d(\sigma, \tau) + d(\sigma^{-1}, \tau^{-1}) .$$

First-order logic and subgroups of $\mathcal{S}_\infty$

Definition.

A first-order (relational) structure $\mathcal{M} = (M, (R_i)_{i \in I})$ is a set M along with a family of finitary relations $R_i \subseteq M^{k_i}$ ($k_i \in \mathbb{N}$).
We always assume that equality is included in our list of relations; we say that $\mathcal{M}$ is *countable* if M is.

Observation.

Let $\mathcal{M}$ be a countable first-order structure; endow its automorphism group $Aut(\mathcal{M})$ with the topology induced by the product topology on M^M. Then $Aut(\mathcal{M})$ is a Polish group and is isomorphic to a closed subgroup of $\mathcal{S}_\infty$.

Ultrahomogeneity

Definition.

Let $\mathcal{M}$ be a countable first-order structure.
Two tuples $\overline{a} = (a_1, \ldots, a_n)$ and $\overline{b} = (b_1, \ldots, b_n)$ in M^n have the same *quantifier-free type* if, whenever $k_i = k \leq n$ and $j_1, \ldots, j_k \in \{1, \ldots, n\}$,

$$R_i(a_{j_1}, \ldots, a_{j_k}) \Leftrightarrow R_i(b_{j_1}, \ldots, b_{j_k}) \ .$$

$\mathcal{M}$ is *ultrahomogeneous* if whenever $\overline{a}$ and $\overline{b}$ have the same quantifier-free type there is some $g \in Aut(\mathcal{M})$ such that $g(\overline{a}) = \overline{b}$.

Theorem (folklore)

Any closed subgroup of $\mathcal{S}_\infty$ is (isomorphic to) the automorphism group of some countable ultrahomogeneous first-order structure.

Metric structures

Definition.

A (relational) metric structure $\mathcal{M}$ is a family $(M, d, (P_i)_{i \in I})$ where :

- (M, d) is a complete metric space of diameter less than 1.
- each P_i is a uniformly continuous function from some M^{k_i}, endowed with the sup-metric, to $[0, 1]$.

$\mathcal{M}$ is *Polish* if (M, d) is a Polish metric space; we extend the notion of quantifier-free type in the obvious way.

Examples.

Countable first-order structures, Polish metric spaces.

Automorphism groups of metric structures

Observation.

$Aut(\mathcal{M})$ is closed in $Iso(M, d)$. Thus $G = Aut(\mathcal{M})$, endowed with the pointwise convergence topology, is a Polish group.
It is also natural to endow G with the uniform metric

$$d_u(g, h) = \sup\{d(g(x), h(x)) \colon x \in M\}$$

This metric is complete, biinvariant, and refines the topology of G. It is in general nonseparable.

Polish groups as automorphism groups

Definition.

A Polish metric structure $\mathcal{M}$ is *approximately ultrahomogeneous* if for any tuples $\overline{a}$, $\overline{b}$ with the same quantifier-free type and any $\varepsilon > 0$ there exists $g \in Aut(\mathcal{M})$ such that

$$d(g(\overline{a}), \overline{b}) \leq \varepsilon \ .$$

Observation.

Any Polish group G is (isomorphic to) the automorphism group of some Polish approximately ultrahomogeneous metric structure.
The associated uniform distance d_u generates the coarsest bi-invariant uniformity which extends the topology of G.

Example: bounded Urysohn spaces

The *Urysohn space of diameter* 1 is the unique Polish metric space of diameter 1 which is both:

- ultrahomogeneous
- universal for all separable metric spaces of diameter 1.

Its isometry group $Iso(\mathbb{U}_1)$ is as above endowed with the pointwise convergence topology, which turns it into a universal Polish group, while the uniform metric is both non separable and non discrete.

Example: automorphisms of a standard probability space

Denote by $Aut(X,\mu)$ the automorphism group of a standard probability space (X,μ).
Recall that the following distance on the measure algebra $MALG_\mu$ is Polish:

$$d(A,B) = \mu(A \Delta B) \ .$$

Then $Aut(X,\mu)$ is the automorphism group of $(MALG_\mu, d, d(.,\emptyset))$.
The associated topology is the usual Polish topology on $Aut(X,\mu)$, while the uniform metric is Lipschitz-equivalent to

$$d_u(S,t) = \mu(\{x \colon S(x) \neq T(x)\}) \ .$$

Again the uniform metric is non separable (any two rotations with different angles are at distance 1) and non discrete.

Example: the unitary group

Denote by $\mathcal{U}(\ell_2)$ the unitary group of the separable Hilbert space ℓ_2.
It is naturally identified with the set of isometries of the unit ball; consider the corresponding action.
The associated Polish topology is the so-called *strong operator topology*, while the uniform metric is just the usual operator norm

$$d(S,T) = ||S-T|| = \sup\{||(S-T)(x)|| \colon ||x|| = 1\} \ .$$

Yet again this group, with the uniform metric, is non separable and non discrete.

Ample generics

Definition.

Let a Polish group G act on itself by conjugacy (denote the action by .), and then let G act on G^n by the diagonal product of this action.
Then G has *ample generics* if for any n there is a comeager orbit in G^n.
The only known examples are subgroups of $\mathcal{S}_\infty$ and their coarsest biinvariant uniformity is discrete.
In the three examples previously discussed, each conjugacy class is meager so the groups don't have ample generics.

Question.

If G has ample generics, is its coarsest biinvariant uniformity necessarily discrete? Does G embed into $\mathcal{S}_\infty$?

Topometric groups

Definition.

A *Polish topometric group* is a triple (G, τ, d) such that

- (G, τ) is a Polish group;
- d is a complete biinvariant distance that refines τ;
- d is τ-lower semicontinuous (i.e $\{(g, h) \colon d(g, h) \leq r\}$ is τ-closed for any r).

Example.

Any Polish group (G, τ) endowed with either the coarsest biinvariant metric d_u which refines τ, or the discrete metric.

Ample metric generics

Definition.
If (G, τ, d) is a topometric group, $U \subseteq G$ and $\varepsilon > 0$, we let

$$(U)_\varepsilon = \{g \in G \colon d(g, U) < \varepsilon\} .$$

We again let G act on G^n by the diagonal product of the conjugacy action.
We say that (G, τ, d) has *ample generics* if for any n there exists some $\overline{g} \in G^n$ such that, for any $\varepsilon > 0$, $(G.\overline{g})_\varepsilon$ is τ-comeager in G.
We say that a Polish group (G, τ) has *ample metric generics* if (G, τ, d_u) has ample generics.

Examples

Consequence of work of Kechris, Rosendal.

$Aut(X, \mu)$, $Iso(\mathbb{U}_1)$ and $\mathcal{U}(\ell_2)$ all have ample metric generics.

Proof.

Based on a lemma that says that these groups are "uniformly approximated" by automorphism groups of countable first order structures with ample generics.

From ample generics to ample metric generics

Lemma.

Assume that $\mathcal{M}$ is a Polish metric structure, $N \subseteq M$ is countable, dense, and there is a first-order structure $\mathcal{N}$ with universe N such that any automorphism of $\mathcal{N}$ extends to an automorphism of $\mathcal{M}$.
Assume also that for any $\varepsilon > 0$, any $n_1, \dots, n_k \in N$ there exists $\delta > 0$ such that :
Whenever $\varphi \in Aut(\mathcal{M})$ satisfies $d(\varphi(n_i), n_i) \leq \delta$ for all i, there exists some $\psi \in Aut(\mathcal{N})$ with

- $\psi(n_i) = n_i$ for all i and
- $\forall m \in M \; d(\psi(m), \varphi(m)) < \varepsilon$.

Finally, assume that $Aut(\mathcal{N})$ has ample generics. Then $Aut(\mathcal{M})$ has ample metric generics.

Are there other examples?

We now have a new (and weaker than the usual one) notion of "generic" element: an element is generic if and only if the uniform closure of its conjugacy class is comeager.
If G satisfies the assumptions of our lemma, then this set of generic elements is dense G_δ.
Also, if G has ample metric generics then any element of G is the product of two generic elements.

Questions.

Is the set of generic elements always Borel ? G_δ? Are there examples of groups with ample generics that cannot be obtained from our lemma?

Applications of ample metric generics: automatic continuity

Theorem.
Let (G, τ, d) be a Polish topometric group with ample generics, H a separable topological group, and $\varphi\colon G \to H$ a morphism.
Assume that $\varphi\colon (G, d) \to H$ is continuous; then $\varphi\colon (G, \tau) \to H$ is continuous.

Corollary (of this and work of Tsankov, Kittrell)
Any automorphism from $Aut(X, \mu)$, with its usual Polish topology, into a separable group is necessarily continuous.

Question.
What about $Iso(\mathbb{U}_1)$ and $\mathcal{U}(\ell_2)$? The same result seems likely to hold (?).

Applications of ample metric generics: bounded orbits

Definition.
Let $\mathcal{M}$ be a Polish metric structure and $G = Aut(\mathcal{M})$. Say that $\mathcal{M}$ is *approximately oligomorphic* if for any n and any $\varepsilon > 0$ there is a finite set A such that $G.A$ is ε-dense in M^n.

Theorem.
Assume that $(Aut(\mathcal{M}), \tau, d_u)$ has ample generics, and that $\mathcal{M}$ is approximately oligomorphic.
Consider now an action by isometries of $G = Aut(\mathcal{M})$ on a metric space X, and suppose that for any x the map $g \mapsto g.x$ is continuous from (G, d_u) to X. Then all G-orbits are bounded.

Back to our examples

Observation.

$Aut(X, \mu)$, $Iso(\mathbb{U}_1)$ and $\mathcal{U}(\ell_2)$ are approximately oligomorphic.

Known results.

B. Miller proved that $Aut(X, \mu)$ has the Bergman property; for $\mathcal{U}(\ell_2)$ this is a theorem of E. Ricard and C. Rosendal.

Question.

Is it true that $Iso(\mathbb{U}_1)$ also has the Bergman property?