Forceless, ineffective, powerless proofs of descriptive set-theoretic dichotomy theorems

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Part I

Introduction

I. Introduction
Historical remarks

Descriptive set theory is concerned with the nature of definable sets, particularly definable subsets of the real numbers.

The study of such sets began with Cantor's observation that closed subsets of the real numbers satisfy the continuum hypothesis.

This was the first of many dichotomy theorems which have played a fundamental role in the subject ever since.

Although some of these results have fairly straightforward proofs using derivatives (in the sense of the Bendixson-Cantor argument), others seem to require sophisticated tools from mathematical logic.

Question

Do all of these theorems have classical proofs?

Question

Is there a single driving force lying beneath them all?

I. Introduction
Desirable properties of a solution

We should avoid using effective descriptive set theory, forcing, reflection, and uncountably many iterates of the power set axiom.

Although our focus will be on Borel sets, the ideas behind our arguments should generalize to other natural classes of definable sets.

Ideally we would like to isolate a common core from which all dichotomy theorems can be easily established.

Observation

There is a natural family of graph-theoretic dichotomy theorems which have classical proofs using nothing more than derivatives and, in some cases, the first separation theorem.

Observation

These theorems can be used to reduce many other dichotomy theorems to topological analogs which can be established using straightforward Baire category arguments.

Part II

The $\mathcal{G}_0$ dichotomy

Notation

Let $[X]^2$ denote the family of 2-element subsets of X.

Definition

A *graph* on X is a set $\mathcal{G} \subseteq [X]^2$.

Definition

A set $B \subseteq X$ is $\mathcal{G}$*-discrete* if $\mathcal{G} \restriction B = \emptyset$.

Definition

A Y*-coloring* of $\mathcal{G}$ is a map $c \colon X \to Y$ with the property that for all $y \in Y$, the set $c^{-1}(\{y\})$ is $\mathcal{G}$-discrete.

Definition (Kechris-Solecki-Todorcevic)

Let $\mathcal{G}_0$ denote the natural example of an acyclic graph on 2^ω with the property that every $\mathcal{G}_0$-discrete set with the Baire property is meager. To be precise, fix sequences $s_n \in 2^n$ which are *dense* in the complete binary tree, in the sense that $\forall s \in 2^{<\omega} \exists n \in \omega\ (s \sqsubseteq s_n)$, and define $\mathcal{G}_0 = \{\{s_n {}^\frown 0 {}^\frown x, s_n {}^\frown 1 {}^\frown x\} \mid n \in \omega \text{ and } x \in 2^\omega\}$.

Theorem (Kechris-Solecki-Todorcevic)

Suppose that X is a Hausdorff space and $\mathcal{G}$ is an analytic graph on X. Then exactly one of the following holds:

1. There is a Borel ω-coloring of $\mathcal{G}$.
2. There is a continuous homomorphism from $\mathcal{G}_0$ to $\mathcal{G}$.

The Kechris-Solecki-Todorcevic argument uses the effective theory.

Observation

There is a classical proof using a derivative and separation.

Observation

The $\mathcal{G}_0$ dichotomy easily implies a number of other theorems which themselves can be proven using derivatives.

These include the Dougherty-Jackson-Kechris characterization of smooth countable equivalence relations, Feng's special case of the open coloring axiom, Souslin's perfect set theorem, and:

Theorem (Lusin-Novikov)

Suppose that X and Y are Hausdorff spaces and $R \subseteq X \times Y$ is an analytic set whose vertical sections are all countable. Then there are Borel sets $R_n \subseteq R$ which are graphs of partial functions and have the property that $R = \bigcup_{n \in \omega} R_n$.

A sketch of the proof (part 1 of 3)

Define $\mathcal{G} = \{\{(x, y), (x', y')\} \in [R]^2 \mid x = x'\}$.

If there is a Borel ω-coloring c of $\mathcal{G}$, then the sets $R_n = c^{-1}(\{n\})$ are graphs of partial functions and $R = \bigcup_{n \in \omega} R_n$.

Suppose, towards a contradiction, that there is no such coloring.

Then there is a continuous homomorphism ϕ from $\mathcal{G}_0$ to $\mathcal{G}$.

A sketch of the proof (part 2 of 3)

Recall that $x E_0 y \iff \forall^\infty n \in \omega \, (x(n) = y(n))$.

Set $\phi_X = \mathrm{proj}_X \circ \phi$.

Clearly ϕ_X is a continuous homomorphism from $\mathcal{G}_0$ to $\Delta(X)$.

It follows that ϕ_X is a continuous homomorphism from E_0 to $\Delta(X)$.

Then ϕ_X is constant. Let x_0 denote its constant value.

A sketch of the proof (part 3 of 3)

Define an equivalence relation E on 2^ω by $xEy \iff \phi(x) = \phi(y)$.

Then every E-class is $\mathcal{G}_0$-discrete, and therefore meager.

The Kuratowski-Ulam theorem implies that E is meager.

Mycielski's theorem ensures that there are perfectly many E-classes.

Then there are perfectly many elements of R_{x_0}, a contradiction. ✓

Observation

Similar arguments yield more difficult dichotomy theorems.

These include Silver's characterization of equivalence relations with perfectly many classes and the Friedman-Harrington-Kechris characterization of quasi-metric spaces with perfect discrete subsets.

There are quite a few dichotomy theorems which do not appear to be straightforward corollaries of the $\mathcal{G}_0$ dichotomy.

Observation

Nevertheless, many are corollaries of natural generalizations.

Part III

The directed $\mathcal{G}_0$ dichotomy

Louveau has noted that the $\mathcal{G}_0$ dichotomy generalizes to digraphs.

The directed $\mathcal{G}_0$ dichotomy implies Louveau's generalization of Silver's theorem to quasi-orders, which gives a two-element basis for the class of uncountable co-analytic quasi-orders.

The directed $\mathcal{G}_0$ dichotomy also yields the Friedman-Shelah characterization of separable linear quasi-orders, which ensures the inexistence of co-analytic Souslin lines.

Part IV

The n-dimensional $\mathcal{G}_0$ dichotomy

IV. The n-dimensional $\mathcal{G}_0$ dichotomy

The theorem and its corollaries

Lecomte has noted that the $\mathcal{G}_0$ dichotomy generalizes to n-dimensional hypergraphs.

The n-dimensional $\mathcal{G}_0$ dichotomy implies generalizations of the van Engelen-Kunen-Miller theorems characterizing vector spaces which are unions of countably many low-dimensional subspaces.

The n-dimensional $\mathcal{G}_0$ dichotomy also yields n-dimensional analogs of Feng's special case of the open coloring axiom.

Part V

The $(<\omega)$-dimensional $\mathcal{G}_0$ dichotomy

The $\mathcal{G}_0$ dichotomy generalizes to $(<\omega)$-dimensional hypergraphs.

The $(<\omega)$-dimensional $\mathcal{G}_0$ dichotomy yields a characterization of vector spaces with perfect linearly independent subsets.

Part VI

The asymptotic $\mathcal{G}_0$ dichotomy

Suppose that $(\mathcal{G}_n)_{n \in \omega}$ is a sequence of graphs on X.

Definition

A set $B \subseteq X$ is *asymptotically $(\mathcal{G}_n)_{n \in \omega}$-discrete* if it is $\mathcal{G}_n$-discrete for all but finitely many $n \in \omega$.

Definition

An *asymptotic Y-coloring* of $(\mathcal{G}_n)_{n \in \omega}$ is a map $c \colon X \to Y$ such that for all $y \in Y$, the set $c^{-1}(\{y\})$ is asymptotically $(\mathcal{G}_n)_{n \in \omega}$-discrete.

Definition

A sequence $p \in \omega^\omega$ is *strongly dominated* by a sequence $q \in \omega^\omega$, or $p \ll q$, if $q(n+1) > p \circ q(n)$ for all $n \in \omega$.

Definition

Suppose that $p \in \omega^\omega$. A *p-homomorphism* from a sequence $(\mathcal{G}_n)_{n \in \omega}$ of graphs on X to a sequence $(\mathcal{H}_n)_{n \in \omega}$ of graphs on Y is a function $\phi \colon X \to Y$ for which there exists $q \in \omega^\omega$ such that $p \ll q$ and ϕ is a homomorphism from $\mathcal{G}_n$ to $\mathcal{H}_{q(n)}$ for all $n \in \omega$.

Definition

Let $(\mathcal{G}_{0,n})_{n\in\omega}$ denote the natural example of an ω-sequence of graphs on 2^ω, whose union is acyclic, with the property that every asymptotically $(\mathcal{G}_{0,n})_{n\in\omega}$-discrete set with the Baire property is meager.

To be precise, fix sequences $s_n \in 2^n$ which are *dense* in the complete binary tree, again in the sense that $\forall s \in 2^{<\omega} \exists n \in \omega \; (s \sqsubseteq s_n)$.

For each $n \in \omega$, define $\mathcal{G}_{0,n} = \{\{s_n{}^\frown 0^\frown x, s_n{}^\frown 1^\frown x\} \mid x \in 2^\omega\}$.

Theorem

Suppose that X is a Hausdorff space and $(\mathcal{G}_n)_{n \in \omega}$ is a sequence of analytic graphs on X. Then exactly one of the following holds:

1. There is a Borel asymptotic ω-coloring of $(\mathcal{G}_n)_{n \in \omega}$.
2. For every sequence $p \in \omega^\omega$, there is a continuous p-homomorphism from $(\mathcal{G}_{0,n})_{n \in \omega}$ to $(\mathcal{G}_n)_{n \in \omega}$.

The asymptotic $\mathcal{G}_0$ dichotomy implies natural strengthenings of the locally countable special cases of many dichotomy theorems.

The asymptotic $\mathcal{G}_0$ dichotomy also yields a characterization of real-valued functions $f(x, y)$ of the form $u(x) + v(y)$.

The asymptotic $\mathcal{G}_0$-dichotomy gives a characterization of real-valued cocycles which admit invariant σ-finite measures of a given type.

Notation

Let $E_{\mathcal{G}}$ denote the equivalence relation generated by $\mathcal{G}$.

Definition

A *transversal* of an equivalence relation is a set which includes exactly one point from every equivalence class.

Theorem (Hjorth)

Suppose that X is a Polish space and $\mathcal{T}$ is an acyclic Borel graph on X. Then exactly one of the following holds:

1. There is a Borel transversal of $E_{\mathcal{T}}$.
2. There is a continuous embedding of E_0 into $E_{\mathcal{T}}$.

Theorem

Suppose that X is a Hausdorff space and $\mathcal{T}$ is an acyclic analytic graph on X. Then at least one of the following holds:

1. There is a co-analytic transversal of $E_{\mathcal{T}}$.
2. There is a continuous embedding of E_0 into $E_{\mathcal{T}}$.

Part VII

The $(\mathcal{G}_0, \mathcal{H}_0)$ dichotomy

VII. The $(\mathcal{G}_0, \mathcal{H}_0)$ dichotomy
The canonical object

Definition

Let $(\mathcal{G}_0^{\text{even}}, \mathcal{H}_0^{\text{odd}})$ denote the natural example of a pair of disjoint graphs on 2^ω, whose union is acyclic, with the property that $\mathcal{G}_0^{\text{even}}$ intersects every non-meager square with the Baire property and $\mathcal{H}_0^{\text{odd}}$ intersects every non-meager rectangle with the Baire property.

To be precise, fix sequences $s_{2n} \in 2^{2n}$ which are dense in the complete binary tree, as well as pairs $s_{2n+1} \in 2^{2n+1} \times 2^{2n+1}$ which are *dense* in the square of the complete binary tree, in the sense that $\forall s \in 2^{<\omega} \times 2^{<\omega} \exists n \in \omega \forall i \in 2\ (s(i) \sqsubseteq s_{2n+1}(i))$.

Define $\mathcal{G}_0^{\text{even}} = \{\{s_{2n} \frown 0 \frown x, s_{2n} \frown 1 \frown x\} \mid n \in \omega \text{ and } x \in 2^\omega\}$ and $\mathcal{H}_0^{\text{odd}} = \{\{s_{2n+1}(0) \frown 0 \frown x, s_{2n+1}(1) \frown 1 \frown x\} \mid n \in \omega \text{ and } x \in 2^\omega\}$.

Theorem

Suppose that X is a Hausdorff space and $\mathcal{G}$ and $\mathcal{H}$ are analytic graphs on X. Then exactly one of the following holds:

1. There exist a Borel homomorphism ϕ from $\mathcal{H}$ to $\Delta(2^\omega)$ and a Borel function $\psi \colon X \to \omega$ such that $\phi \times \psi$ is a coloring of $\mathcal{G}$.
2. There is a continuous homomorphism $\pi \colon 2^\omega \to X$ from the pair $(\mathcal{G}_0^{\text{even}}, \mathcal{H}_0^{\text{odd}})$ to the pair $(\mathcal{G}, \mathrm{TrCl}(\mathcal{H}))$.

Definition

An equivalence relation E is *smooth* if it is Borel reducible to $\Delta(2^\omega)$.

Theorem (Harrington-Kechris-Louveau)

Suppose that X is a Hausdorff space and E is a bi-analytic equivalence relation on X. Then exactly one of the following holds:

1. The equivalence relation E is smooth.
2. There is a continuous embedding of E_0 into E.

By altering our choice of $\mathcal{G}$ and $\mathcal{H}$, we obtain related theorems.

Theorem

Suppose that X is a Hausdorff space, E is a co-analytic equivalence relation on X, and F is an analytic subequivalence relation of E. Then exactly one of the following holds:

1. There is a smooth equivalence relation between F and E.
2. There is a continuous embedding of (E_0, E_0) into (F, E).

Definition

Let F_0 denote the index 2 subequivalence relation of E_0 given by $xF_0y \iff \forall^\infty n \in \omega \left(\sum_{i \in n} x(i) \equiv \sum_{i \in n} y(i) \pmod 2\right)$.

Theorem

Suppose that X is a Hausdorff space, E is a co-analytic equivalence relation on X, F is a co-analytic subequivalence relation of E, and $[E : F] = 2$. Then exactly one of the following holds:

1. There is an F-invariant Borel set $B \subseteq X$ such that B/F is a transversal of E/F.
2. There is a continuous embedding of (F_0, E_0) into (F, E).

Part VIII

The directed $(\mathcal{G}_0, \mathcal{H}_0)$ dichotomy

The $(\mathcal{G}_0, \mathcal{H}_0)$ dichotomy has a generalization to pairs of digraphs.

The directed $(\mathcal{G}_0, \mathcal{H}_0)$ dichotomy can be used to establish the Kanovei-Louveau characterization of linearizable quasi-orders.

The directed $(\mathcal{G}_0, \mathcal{H}_0)$ dichotomy also yields the Harrington-Marker-Shelah result on the cofinality of the set of lexicographical orderings.

The directed $(\mathcal{G}_0, \mathcal{H}_0)$ dichotomy implies the Harrington-Marker-Shelah Borel Dilworth theorem.

Part IX

Broader notions of definability

A simplification of the classical proof of the $\mathcal{G}_0$ dichotomy gives:

Theorem

Suppose that X is a Hausdorff space and $\mathcal{G}$ is a κ-Souslin graph on X. Then at least one of the following holds:

1. There is a κ-coloring of $\mathcal{G}$.
2. There is a continuous homomorphism from $\mathcal{G}_0$ to $\mathcal{G}$.

Definition

A set $B \subseteq X$ is *ω-universally Baire* if for every continuous function $\phi\colon 2^\omega \to X$, the set $\phi^{-1}(B)$ has the Baire property.

Theorem

Suppose that X is a Hausdorff space and E is a co-κ-Souslin equivalence relation on X which is ω-universally Baire. Then at least one of the following holds:

1. The equivalence relation E has at most κ-many classes.
2. The equivalence relation E has at least perfectly many classes.

The other graph-theoretic dichotomy theorems have similar generalizations to the κ-Souslin case.

Theorem

Suppose that X is a Hausdorff space and E is a bi-κ-Souslin equivalence relation on X which is ω-universally Baire. Then at least one of the following holds:

1. There is a reduction of E to $\Delta(2^\kappa)$.
2. There is a continuous embedding of E_0 into E.

It would be desirable to obtain the analogous result in which the former condition is strengthened so that the reduction is κ^+-Borel.

This sort of generalization appears to be a consequence of analogous graph-theoretic dichotomies, such as the following:

Theorem (Hjorth)

Suppose that X is a Hausdorff space and $\mathcal{G}$ is a κ-Souslin graph on X. Then at least one of the following holds:

1. There is a κ^+-Borel κ-coloring of $\mathcal{G}$.
2. There is a continuous homomorphism from $\mathcal{G}_0$ to $\mathcal{G}$.

Part X

Open questions

X. Open questions
Filling in the gaps

Definition

We say that a set is *weakly κ-Souslin* if it is a continuous image of a κ^+-Borel subset of κ^ω.

Question

Does the first separation theorem hold for weakly κ-Souslin sets?

Question

Let $\mathtt{GC}$ abbreviate the statement that for every graph $\mathcal{G}$ on the real numbers, at least one of the following holds:

1. There is an ordinal-valued coloring of $\mathcal{G}$.
2. There is a continuous homomorphism from $\mathcal{G}_0$ to $\mathcal{G}$.

Does $\mathtt{ZF} + \mathtt{BP} + \mathtt{GC}$ have strength?

Lecomte has established a somewhat subtle analog of the $\mathcal{G}_0$ dichotomy for ω-dimensional hypergraphs.

Question

What is the analogous result for perfect-dimensional hypergraphs?