Towers in Boolean algebras
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Definitions

A *tower* in a Boolean algebra A is a strictly increasing sequence $\langle a_\alpha : \alpha < \kappa \rangle$ of elements of A with sum 1, where κ is an infinite regular cardinal. Clearly towers exist in any atomless BA. On the other hand, the BA of finite and cofinite subsets of ω_1 does not have a tower.

For any BA A we define

- $\mathfrak{t}_{\mathrm{spect}}(A) = \{\kappa\colon$ there is a tower of length κ in $A\}$.
- $\mathfrak{t}(A) = \min(\mathfrak{t}_{\mathrm{spect}}(A))$.

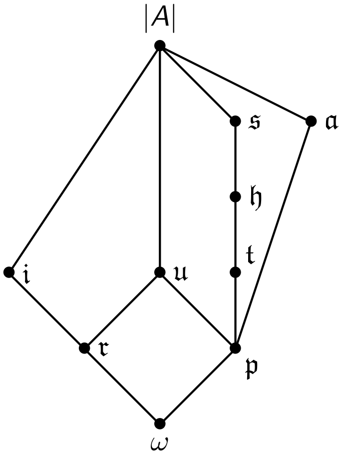
Survey of results

Towers have been studied a lot for the BA $\mathfrak{F} \stackrel{\text{def}}{=} \mathscr{P}(\omega)/\mathrm{fin}$. Consistently, $\mathfrak{t}(\mathfrak{F}) < 2^\omega$, for example. There are consistency results concerning the relationships of $\mathfrak{t}$ to other cardinal invariants on this BA. For example, for any BA B let

- $\mathfrak{a}(B) = \min\{|X| : X \text{ is an infinite partition of unity of } B\}$;
- $\mathfrak{s}(B) = \min\{|X| : X \text{ is splitting in } B\}$; (This means that for every nonzero $b \in B$ there is an $x \in X$ such that $b \cdot x \neq 0 \neq b \cdot -x$.)
- $\mathfrak{p}(B) = \min\{|X| : \sum X = 1$ but $\sum F \neq 1$ for every finite subset F of X.

Then it is consistent to have $\mathfrak{t}(\mathfrak{F}) < \mathfrak{a}(\mathfrak{F}), \mathfrak{s}(\mathfrak{F})$. On the other hand it is still an open problem, as far as I know, whether it is consistent to have $\mathfrak{p}(\mathfrak{F}) < \mathfrak{t}(\mathfrak{F})$.

Small cardinal functions on Boolean algebras



The functions for atomless Boolean algebras

For BAs in general one can easily find examples with $\mathfrak{p}(B) < \mathfrak{t}(B)$. Let κ be an uncountable cardinal, and let A be atomless and κ^+-saturated, and let B be the subalgebra of ${}^\kappa A$ consisting of functions which are 0 except at finitely many places, or 1 except at finitely many places. The relationship of $\mathfrak{t}$ with various other invariants for BAs in general is explored in Monk [01] (JSL, vol. 66); some of the problems mentioned in that paper have been solved. One of those problems is to find a BA A in ZFC such that $\mathfrak{s}(A) < \mathfrak{a}(A)$. This was solved in McKenzie, Monk [04] (J. Symb. Logic vol. 69). The theorem involves towers, and is as follows:

Theorem 1 *Let κ and λ be regular cardinals with $\aleph_0 < \kappa < \lambda$. Then there is a BA A such that $\mathfrak{t}(A) = \mathfrak{s}(A) = \kappa$ and $\mathfrak{a}(A) = \lambda$.*

SKETCH OF PROOF

- A set $U \subseteq {}^{\kappa}2$ is $(<\kappa)$-*defined* iff $\exists D \in [\kappa]^{<\kappa} \forall f, g \in {}^{\kappa}2[f \in U \wedge f \restriction D = g \restriction D \to g \in U]$. Let B_0 be the set of all $(<\kappa)$-defined subsets of ${}^{\kappa}2$; it is a field of sets. Let $S = \{s_\alpha : \alpha < \kappa\}$ with $s_\alpha = \{f \in {}^{\kappa}2 : f(\alpha) = 1\}$.
- $S \subseteq B$ has the *co-κ-splitting property in B* iff $\forall b \in B \backslash \{0\} \exists T \in [\kappa]^{<\kappa} \forall \beta \in \kappa \backslash T[b \cdot s_\beta \neq 0 \neq b \cdot -s_\beta]$. Clearly S has the co-κ-splitting property in B_0.
- If S has the co-κ-splitting property in B and X is a partition of unity (resp. a tower of size less than κ) in B, then there is a BA $B' \geq B$ such that X does not have sum 1 in B', and S has the co-κ-splitting property in B'.

- If S has the co-κ-splitting property in $B \geq B_0$, then there is an extension $B' \geq B$ such that S has the co-κ-splitting property in B' and B' does not have any partition of unity of infinite size less than λ, and does not have any tower size less than κ.
- The final algebra is $B' \oplus D$ with D the BA of finite and cofinite subsets of λ, and $\oplus$ is the free product operation.

Another of the problems from Monk [01] is to find an interval algebra A such that $\mathfrak{a}(A) < \mathfrak{t}(A)$. Given a linearly ordered set L, the *interval algebra* on L is the field of subsets of L generated by the half-open intervals $[a, \infty)$. This problem was solved in Monk [02], using a dense linear order L without endpoints in which every element has character (ω, ω_1) or (ω_1, ω) and every gap has character (ω_1, ω_1). Such linear orders exist by Hausdorff [1908] (Math. Ann. vol. 65.) Then A is the interval algebra on $M + (1 + M) \times \mathbb{Z} \times L + M$, where $|L| = \aleph_\alpha$ and M is an $\eta_{\alpha+1}$ set.

The following theorem from Monk [07] (Alg. Univ., vol. 56) is surpisingly easy to prove. An interested reader could supply the details in the following sketch of the proof.

Theorem 2. *If M is a nonempty set of regular cardinals, then there is a BA A such that $\mathfrak{t}_{\mathrm{spect}}(A) = M$.*

Lemma 3. *For any BA A there is a BA B with $A \leq B$ such that no tower in A remains a tower in B.*

Proof. Let B be the power set of the set of all ultrafilters on A and use the Stone isomorphism of A into B. $\square$

Lemma 4. *If A is an atomless BA and κ is a regular cardinal, then $A \leq B$ for some atomless BA B such that $\mathfrak{t}_{\mathrm{spect}}(B) = \{\kappa\}$.*

Proof. We define $\langle C_\alpha : \alpha \leq \kappa \rangle$ by recursion. Let $C_0 = A$, $C_\lambda = \bigcup_{\alpha<\lambda} C_\alpha$ for λ limit, and let $C_{\alpha+1}$ be obtained from C_α by applying Lemma 3 and then extending to an atomless BA $C_{\alpha+1}$. It is easy to check that $B = C_\kappa$ is as desired. $\square$

Proof of the theorem. Let f be a surjection from an uncountable set I onto M, and for each $i \in I$ let B_i be such that $\mathfrak{t}_{\mathrm{spect}}(B_i) = \{f(i)\}$. Let C be the subalgebra of $\prod_{i\in I} B_i$ consising of elements which are either 0 at all but finitely many places, or 1 at all but finitely many places. It is easy to check that C is as desired. $\square$

The existence of towers

As mentioned on slide 1, towers do not always exist. A natural but vague open problem is to characterize those BAs for which towers exist. Failing to solve this, we can look at special classes of BAs and ask the same question. We have already observed that towers exist in any atomless BA. Our simple BA without a tower given on slide 1 is superatomic (has only atomic subalgebras and homomorphic images). Of course an interval algebra on an infinite ordinal, while superatomic, has a tower. So a more feasible looking form of our natural question is to characterize those superatomic BAs which have towers. We have no results on this. We now turn to interval algebras and pseudo-tree algebras.

Existence of towers in interval algebras

The following theorem, whose proof is not very difficult, characterizes existence of towers in interval algebras.

Theorem 5. *Suppose that L is an infinite linear order with first element 0, and κ is a regular cardinal. Then the following conditions are equivalent.*
(i) $\mathrm{Intalg}(L)$ *has a tower of order type κ.*
(ii) One of the following holds:
(a) There is a $c \in L$ and a strictly decreasing sequence $\langle a_\alpha : \alpha < \kappa \rangle$ of elements of (c, ∞) coinitial with c.
(b) There is a $c \in L \cup \{\infty\}$ and a strictly increasing sequence $\langle a_\alpha : \alpha < \kappa \rangle$ of elements of $[0, c)$ cofinal in c.
(c) There exist a strictly increasing sequence $\langle b_\alpha : \alpha < \kappa \rangle$ of elements of L and a strictly decreasing sequence $\langle c_\alpha : \alpha < \kappa \rangle$ of elements of L such that $b_\alpha < c_\beta$ for all $\alpha, \beta < \kappa$, and there is no element $d \in L$ such that $b_\alpha < d < c_\beta$ for all $\alpha, \beta < \kappa$.

This theorem can be used to give an example of an interval algebra without towers. We start out with a dense linear order M such that every point of M has character (ω_1, ω_1), the gaps of M have characters (ω, ω_1) and (ω_1, ω), M has no first or last element, and M has cofinality and coinitiality ω_1. The existence of M follows from a theorem in Hausdorff [1908] (Math. Ann. vol 65). We replace each element of M by $\omega^* + \omega$, put ω to the left of the result, and ω^* to the right.

Existence of towers in pseudo-tree algebras

A *pseudo-tree* is a partially ordered set T such that for each $t \in T$ the set $\{s \in T : s < t\}$ is linearly ordered; in case this set is well-ordered, we call T a tree. For any pseudo-tree T, the *tree algebra* $\mathrm{Treealg}(T)$ of T is the subalgebra of $\mathscr{P}(T)$ generated by the sets $T \uparrow t \stackrel{\mathrm{def}}{=} \{s \in T : t \leq s\}$. Pseudo-trees generalize both trees and linear orders. A relatively deep theorem is that pseudo-tree algebras coincide with those BAs isomorphically embeddable in interval algebras. (Purisch, Nikiel in 1994 proved this topologically, and then Heindorf in 1997 gave a fairly simple algebraic proof (Proc. AMS vol. 125).)

We have a characterization of those pseudo-trees T such that $\mathrm{Treealg}(T)$ does not have a tower. (To appear in **Order**.) The characterization follows from the following theorem.

Theorem 6. *Let* κ *be a regular cardinal and let* T *be a pseudo-tree with a minimum element. Then the following conditions are equivalent.*

(i) $\mathrm{Treealg}(T)$ *has a tower* $\langle a_\alpha : \alpha < \kappa \rangle$.

(ii) There is a sequence $\langle x_\alpha : \alpha < \kappa \rangle$ *of elements of* T *such that* $x_\alpha \leq x_\beta$ *whenever* $\alpha < \beta < \kappa$, *either the sequence is strictly increasing or has a constant value, and one of the following conditions holds:*

(a) $\langle x_\alpha : \alpha < \kappa \rangle$ *is strictly increasing, and there is a finite set* F *of incomparable elements of* T *such that* $x_\alpha < v$ *for every* $v \in F$, *and* $\forall w \in T[\forall \alpha < \kappa(x_\alpha < w) \Rightarrow \exists v \in F(v \leq w)]$.

(b) There exist countable sets Y, Z *and for each* $y \in Y$ *a strictly decreasing sequence* $\langle t_{y\alpha} : \alpha < \kappa \rangle$ *of elements of* T, *such that* $Z \subseteq T$, *and the following conditions hold:*

(I) $x_\beta < t_{y\alpha}$ *for each* $y \in Y$ *and all* $\alpha, \beta < \kappa$.

(II) $x_\beta < z$ *for each* $z \in Z$ *and each* $\beta < \kappa$.

(III) The members of Z *are pairwise incomparable.*

(IV) If y and z are distinct members of Y and $\alpha, \beta < \kappa$, then $t_{y\alpha}$ and $t_{z\beta}$ are incomparable.

(V) If $y \in Y$, $\alpha < \kappa$, and $z \in Z$, then $t_{y\alpha}$ and z are incomparable.

(VI) If κ is uncountable, then Y and Z are finite, and Y is nonempty.

(VII) If κ is uncountable, then there is a $y \in Y$ such that $\neg\exists v \in T\forall\delta, \beta < \kappa[x_\beta < v < t_{y\alpha}]$.

(VIII) If κ is uncountable and $x_\beta < v$ for all $\beta < \kappa$, then one of the following holds:

(A) There is a $z \in Z$ such that $z \leq v$.

(B) There is a $y \in Y$ such that $v < t_{y\alpha}$ for all $\alpha < \kappa$.

(C) There exist $y \in Y$ and $\alpha < \kappa$ such that $t_{y\alpha} \leq v$.

(IX) If κ is uncountable, $x_\beta < v$ for all $\beta < \kappa$, and there is a $y \in Y$ such that $v < t_{y\alpha}$ for all $\alpha < \kappa$, then there is a $y \in Y$ such that $v < t_{y\alpha}$ for all $\alpha < \kappa$, while there is no w such that $v < w < t_{y\alpha}$ for all $\alpha < \kappa$.

(X) If $\kappa = \omega$ and $Y = \emptyset$, then Z is infinite.

(XI) If $\kappa = \omega$ and F is a finite set of elements of T each greater than each x_β for $\beta < \kappa$, then one of the following conditions holds:

(A) There is an $s \in Z$ such that $\forall x \in F(x \not\leq s)$.

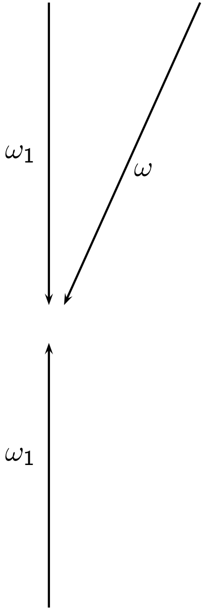
(B) There exist $y \in Y$ and $l \in \omega$ such that $\forall x \in F(x \not\leq t_{yl})$.

(XII) If $\kappa = \omega$, $x_\beta < w$ for each $\beta < \kappa$, and F is a finite subset of $(T \uparrow w)\backslash\{w\}$, then one of the following conditions holds:

(A) There is an $s \in Z$ such that w and s are comparable and $\forall x \in F(x \not\leq s)$.

(B) There exist $y \in Y$ and $l \in \omega$ such that w and t_{yl} are comparable and $\forall x \in F(x \not\leq t_{yl})$.

A simple application of Theorem 6 is illustrated in the following diagram.



(no towers)

The proof of Theorem 6 is long. We give most details for the following special case.

Theorem 7. *Let T be an infinite tree with only one root. Then the following conditions are equivalent:*
(i) $\mathrm{Treealg}(T)$ *has a tower.*
(ii) One of the following conditions holds:
(a) T has an element with exactly ω immediate successors.
(b) T has a chain of countable limit length with at most ω immediate successors.
(c) For some uncountable regular cardinal κ, T has an chain of order type κ with only finitely many immediate successors.

(ii)(a)$\Rightarrow$(i): Say t has the immediate successors $\langle s_i : i \in \omega \rangle$. For each $i \in \omega$ let

$$x_i = [T \backslash (T \uparrow t)] \cup \bigcup_{j<i} (T \uparrow s_j).$$

(ii)(b)$\Rightarrow$(i): Suppose that $\langle t_\xi : \xi < \alpha \rangle$ is a chain, α countable limit, and the set of immediate successors of this chain is a finite set F. We may assume that $\alpha = \omega$. For each $i \in \omega$ let

$$x_i = [T \backslash (T \uparrow t_i)] \cup \bigcup_{s \in F} (T \uparrow s).$$

The case of infinitely many immediate successors uses the idea of (ii)(a)$\Rightarrow$(i).

(ii)(c)⇒(i): Suppose that $\langle t_\alpha : \alpha < \kappa \rangle$ is a chain with the finite set F of immediate successors. For each $\alpha < \kappa$ let

$$x_\alpha = [T \backslash (T \uparrow t_\alpha)] \cup \bigcup_{s \in F} (T \uparrow s).$$

The direction (i)⇒(ii) is much harder. We only sketch a special case, in which T has a tower of order type an uncountable regular cardinal κ; then we want to show that (ii)(c) holds.

- $\mathrm{Treealg}(T)$ is isomorphic to $\mathrm{Treealg}(T')$, where T' has a single root r. This is easy to prove.

- Let $\langle a_\alpha : \alpha < \kappa \rangle$ be a tower. Then we can write each a_α in the following form:

$$a_\alpha = \bigcup_{t \in M_\alpha} e_{\alpha t}; \text{ here } e_{\alpha t} = (T \uparrow t) \setminus \bigcup_{s \in N_{\alpha t}} (T \uparrow s),$$

  where M_α is a finite subset of T, $N_{\alpha t}$ is a finite set of pairwise incomparable elements of $(T \uparrow t) \setminus \{t\}$, $e_{\alpha t} \cap e_{\alpha u} = \emptyset$ for $t \neq u$, and $t \notin N_{\alpha u}$ for $t \neq u$.
- There is an $\alpha < \kappa$ such that $r \in a_\alpha$. This is not hard to prove. So, we may assume that $r \in a_\alpha$ for all $\alpha < \kappa$.
- If $\alpha < \beta$ and $s \in N_{\beta r}$, then there is a unique $t \in N_{\alpha r}$ such that $t \leq s$. This is clear. Hence
- If $\alpha < \beta$, then $e_{\alpha r} \subseteq e_{\beta r}$.

- If $\alpha < \kappa$ and $t \in N_{\alpha r}$, then there is a $\beta \in (\alpha, \kappa)$ such that $t \in a_\beta$.
- Now let $T' = \{(s, \alpha) : \alpha < \kappa \text{ and } s \in N_{\alpha r}\}$. We define $(s, \alpha) < (t, \beta)$ iff $\alpha < \beta$ and $s \leq t$. This makes T' into a tree of height κ. Then there is a function $f \in P_{\alpha<\kappa} N_{\alpha r}$ such that $f(\alpha) < f(\beta)$ whenever $\alpha < \beta$.
- There do not exist $\alpha < \kappa$ and $s \in T$ such that $f(\beta) = (s, \beta)$ for all $\beta \geq \alpha$.
- There are strictly increasing sequences $\langle \alpha_\xi : \xi < \kappa \rangle$ of ordinals and $\langle s_\xi : \xi < \kappa \rangle$ of elements of T such that $s_\xi \in N_{\alpha_\xi}$ for all $\xi < \kappa$. Let $\beta = \sup_{\xi<\kappa} \alpha_\xi$. So β is a limit ordinal of cofinality κ.

- Now assume that (ii)(c) fails. Then $\langle s_\xi : \xi < \kappa \rangle$ has infinitely many immediate successors. For each $\xi < \kappa$ let $P(\xi)$ be the set of all immediate successors u of s_ξ such that $u \leq v$ for some $v \in M_{\alpha_\xi}$.
- If $\xi < \eta < \kappa$, then $P(\xi) \subseteq P(\eta)$.
- There is a $\xi < \kappa$ such that $P(\xi) = P(\eta)$ for all $\eta \in (\xi, k)$. Let t be an immediate successor of $\langle s_\xi : \xi < \kappa \rangle$ which is not in $P(\xi)$. A contradiction is now easily obtained.

Draft versions of the references used above can be obtained at
http://euclid.colorado.edu/˜monkd