A reflection principle together with the continuum arbitrarily large (2)

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In this talk we will show how to force some strong failures of Weak Club Guessing together with the continuum larger than $\omega_2$.

(●) Recall that a sequence $\{A_\delta : \delta \in Lim\}$ is said to be a ladder system iff for every $\delta$, $A_\delta$ is a cofinal subset of $\delta$ of order type $\omega$.

(●) Weak Club Guessing (WCG) There is a ladder system $\{A_\delta : \delta \in Lim\}$ such that for every club $C \subseteq \omega_1$ there is $\delta \in C$ such that $A_\delta \cap C$ is infinite.

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But we are interested in weakenings and not in strengthenings of WCG.
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One weak form of WCG considered in the literature is the statement we may call *Very Weak Club Guessing* (VWCG). It says that there is a collection \( \{ A_\delta : \delta \in \omega_1 \} \) of subsets of \( \omega_1 \) of order type \( \omega \) such that every club of \( \omega_1 \) has infinite intersection with some \( A_\delta \).

Given a cardinal \( \kappa \) (possibly finite), WCG\(^\kappa\) says that there exist a system \( \{ A_\delta^\alpha : \alpha \in \kappa, \delta \in Lim \} \) such that for every \( \alpha \) and \( \delta \), \( A_\delta^\alpha \) is a cofinal subset of \( \delta \) of order type \( \omega \) and such that every club subset of \( \omega_1 \) has an infinite intersection with one of them.

Note that WCG and VWCG are respectively equal to the parameterized principles WCG\(^1\) and WCG\(^\omega_1\).
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Given a cardinal \( \kappa \) (possibly finite), WCG\(^\kappa\) says that there exist a system \( \{ A_\alpha^\delta : \alpha \in \kappa, \delta \in \text{Lim} \} \) such that for every \( \alpha \) and \( \delta \), \( A_\alpha^\delta \) is a cofinal subset of \( \delta \) of order type \( \omega \) and such that every club subset of \( \omega_1 \) has an infinite intersection with one of them.

Note that WCG and VWCG are respectively equal to the parameterized principles WCG\(^1\) and WCG\(^{\aleph_1}\).
Proposition

WCG is equivalent to $\text{WCG}^\omega$.

Proof.

We will see that $\text{WCG}^\omega$ implies WCG. So, let

\[
\{A^n_\delta : n \in \omega, \delta \in \text{Lim}\}
\]

be a system witnessing $\text{WCG}^\omega$. We define a ladder system \(\{B_\delta : \delta \in \text{Lim}\}\) as follows. First, for each \(\delta \in \text{Lim}\) fix an increasing cofinal sequence \(\{\delta_n : n \in \omega\} \subseteq \delta\) of order type \(\omega\). Now define \(B_\delta = \bigcup\{B^n_\delta : n \in \omega\}\), where \(B^n_\delta\) is equal to \(A^n_\delta \setminus \delta_n\). It is easy to check that \(\{B_\delta : \delta \in \text{Lim}\}\) witnesses WCG.

Certainly, VWCG follows from CH, but \(\neg\text{WCG}\) is compatible with CH (See Shelah’s [656]).
Proposition

WCG is equivalent to $WCG^{\aleph_0}$.

Proof.

We will see that $WCG^{\aleph_0}$ implies WCG. So, let

$\{A^n_\delta : n \in \omega, \delta \in Lim\}$ be a system witnessing $WCG^{\aleph_0}$. We define a ladder system $\{B_\delta : \delta \in Lim\}$ as follows. First, for each $\delta \in Lim$ fix an increasing cofinal sequence $\{\delta_n : n \in \omega\} \subseteq \delta$ of order type $\omega$. Now define $B_\delta = \bigcup \{B^n_\delta : n \in \omega\}$, where $B^n_\delta$ is equal to $A^n_\delta \setminus \delta_n$. It is easy to check that $\{B_\delta : \delta \in Lim\}$ witnesses WCG.

Certainly, VWCG follows from CH, but $\neg WCG$ is compatible with CH (See Shelah’s [656]).
Let us introduce another parameter. Given $\tau$ an indecomposable ordinal (i.e., of the form $\omega^\beta$, $\beta \neq 0$), VWCG$_\tau$ states that there exist a sequence $\{A_\delta : \delta \in \omega_1\}$ such that for every $\delta$, $A_\delta$ is a closed subset of order type an indecomposable ordinal less than or equal than $\tau$ and such that every club $C$ of $\omega_1$ has infinite intersection with one of them.

We will assume that a sequence $\{A_\delta : \delta \in \omega_1\}$ always consists of sets of order type an indecomposable ordinal.

**Remark**

VWCG $\rightarrow$ VWCG$_\tau$. 
Proposition
\[ \neg \text{VWCG} \rightarrow \neg \text{VWCG}_\tau. \]

The proof is by induction. Let \( \mathcal{A} = \{ A_\delta : \delta \in \text{Lim} \} \) be a sequence of sets such that every \( A_\delta \) has order type less or equal than \( \tau \). For each limit ordinal \( \delta \) consider an increasing sequence \( \{ \delta_n : n \in \omega \} \subseteq \text{sup}(A_\delta) \) of accumulation points of the set \( A_\delta \). Further, we can choose this sequence in such a way that

\[ A(\delta, n) := A_{\delta_{n+1}} \setminus A_{\delta_n} \]

has order type an indecomposable ordinal strictly less than \( \tau \).

Fix an increasing and cofinal sequence \( \{ \tau_m : m \in \omega \} \subseteq \tau \) of indecomposable ordinals. Now, consider the system

\[ \mathcal{A} = \{ A(\delta, n) : \delta \in \text{Lim}, n \in \omega \} \]
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has order type an indecomposable ordinal strictly less than \( \tau \).

Fix an increasing and cofinal sequence \( \{ \tau_m : m \in \omega \} \subseteq \tau \) of indecomposable ordinals. Now, consider the system

\[
\mathcal{A} = \{ A(\delta, n) : \delta \in \text{Lim}, n \in \omega \}
\]
and note that for each \( m \in \omega \) there exists a club \( C_m \) such that for every \( \delta \) and for every \( n \) if \( A(\delta, n) \) has order type less or equal than \( \tau_m \), then \( A(\delta, n) \cap C_m \) is finite. Let \( C \) be the intersection of all the \( C_m \). Now define the set \( B_\delta \) as follows:

\[
B_\delta = \{ \delta_n : \delta \in \omega \} \cup \bigcup \{ A(\delta, n) \cap C : n \in \omega \}
\]

Note that this set has order type \( \omega \). Finally find a club \( D \subseteq C \) witnessing that the system \( \mathcal{B} = \{ B_\delta : \delta \in \text{Lim} \} \) does not guess in the very weak sense. It is easy to check that \( D \) also witnesses that \( \mathcal{A} \) does not guess in the \( \text{VWCG}_\tau \)–sense.

**Corollary**

\( \text{VWCG} \leftrightarrow \text{VWCG}_\tau \).
Corollary
The following are equivalent:

a) VWCG

b) If $\mathcal{A}$ is a family of subsets of $\omega_1$ (of order type an indecomposable ordinal) such that $|\mathcal{A}| = \aleph_1$ and such that for every $\gamma < \omega_1$ and every $B \in \mathcal{A}$ the order type of $B \cap \gamma$ is strictly less than $\gamma$, then there exists a club $E$ such that $E$ has finite intersection with all the elements of this family.
Proof.
The idea of the proof is very close to that of the above proposition. Namely, we should start by fragmenting each $B \in A$ into $\omega$ pieces $\{B_n : n \in \omega\}$ in such a way that the order type of each of them is an indecomposable ordinal. Now, if $E$ is the diagonal intersection of a suitable sequence of clubs, we can assume that $E$ has finite intersection with $B_n$ for every $B \in A$ and every natural number $n$. The rest is standard. □
Cohen and Random Reals in the context of Club Guessing (FOLKLORE?)

Proposition
Let $P$ be Cohen forcing. Then, $V^P \models WCG$.

Lemma
Let $P$ be a $\omega_\omega$ bounding notion of forcing and let $\langle A_\delta : \delta \in \omega_1 \rangle \in V^P$ be a sequence of sets of order type $\omega$. Then, for every condition $p$ and for every ordinal $\delta \in \omega_1$, there exists a condition $q$ extending $p$ and a countable set $C_q \in V$ such that $q$ forces that $A_\delta$ is included in $C_q$ and $\sup(A_\delta) = \sup(C_q)$. 
So, if $P$ has in addition the countable chain condition, then there are $\omega$ possibilities for this set $C_q$. But now if we use a diagonalization similar to that of the proof of the equivalence between WCG and $\text{WCG}^{\aleph_0}$, then we get the following:

**Corollary**

Let $P$ be a $\omega \omega$ bounding notion of forcing satisfying the countable chain condition and let $\langle A_\delta : \delta \in \omega_1 \rangle \in V^P$ be a sequence of sets of order type $\omega$. If $\langle A_\delta : \delta \in \omega_1 \rangle \in V^P$ is a sequence of sets which guesses all the clubs of $V$ (in the very weak sense), then there is a sequence of sets in the ground model doing the same.

In particular, random forcing preserves the negation of VWCG.
So, if $P$ has in addition the countable chain condition, then there are $\omega$ possibilities for this set $C_q$. But now if we use a diagonalization similar to that of the proof of the equivalence between WCG and $\text{WCG}^{\aleph_0}$, then we get the following:

**Corollary**

Let $P$ be a $\omega_1$-bounding notion of forcing satisfying the countable chain condition and let $\langle A_\delta : \delta \in \omega_1 \rangle \in V^P$ be a sequence of sets of order type $\omega$. If $\langle A_\delta : \delta \in \omega_1 \rangle \in V^P$ is a sequence of sets which guesses all the clubs of $V$ (in the very weak sense), then there is a sequence of sets in the ground model doing the same.

In particular, random forcing preserves the negation of VWCG.
Now, we introduce another strengthening of $\neg WCG$ for which the above technique of diagonalization does not seem to apply.

**Definition (Miyamoto)**

$\text{Code(}\text{even–odd})$ states that for every ladder system $\langle A_\delta : \delta \in \text{Lim} \rangle$ and for every $B \subseteq \omega_1$, there exist two clubs $C$ and $D$ of $\omega_1$ such that for every $\delta \in C$ which is a limit point

1) If $\delta \in B$, then $|A_\delta \cap D| < \aleph_0$ is odd.

2) If $\delta \notin B$, then $|A_\delta \cap D| < \aleph_0$ is even.

**Proposition (Miyamoto)**

$\text{BPFA} \rightarrow \text{Code(}\text{even–odd}) \rightarrow (2^{\aleph_0} = 2^{\aleph_1} + \neg WCG)$. 
We prove that each instance of Code\((\text{even–odd})\) follows from the forcing axiom defined in the previous talk. This will show the consistency of Code\((\text{even–odd})\) together \(c > \aleph_2\).

So, let \(\langle A_\delta : \delta \in \text{Lim}\rangle\) be a ladder system and \(B\) a subset of \(\omega_1\). Now, consider the notion of forcing \(P\) defined as follows: Its elements are pairs \((f, \langle b_\delta : \delta \in D\rangle)\) such that:
(a) There exists a normal function \( F : \omega_1 \rightarrow \omega_1 \) such that \( f \) is a finite subset of \( F \).

(b) Let \( C = \text{range}(f) \). If we denote by \( LIND \) the set of all those ordinals which are a limit of indecomposables, then \( D \) is included in the set of all ordinals in \( C \cap LIND \) which are fixed points of \( f \).

(c) For each \( \delta \in D \), \( C \cap A_\delta = b_\delta \). Further, if \( b_\delta \) is odd (even), then \( \delta \in B (\delta \notin B) \).

(d) For every \( \delta' \in D \) and every \( \delta \in C \) with \( \delta < \delta' \) there exists a finite subset \( b_{\delta, \delta'} \subseteq (\delta + 1) \setminus b_{\delta'} \) such that \( q|_\alpha \) forces that the union of \( b_{\delta, \delta'} \) and \( b_{\delta'} \) is equal to the initial segment \( A_{\delta'} \cap (\delta + 1) \) of \( A_{\delta'} \).

(e) For every \( \delta' \in D \) and every \( \delta \in C \) with \( \delta < \delta' \), the function \( f \) omits all points of \( b_{\delta, \delta'} \). That is, if \( \gamma \in b_{\delta, \delta'} \), then there exist \( \pi, \beta \) and \( \beta' \) such that \( \beta < \gamma < \beta' \) and \( (\pi, \beta), (\pi + 1, \beta') \in f \).
(a) There exists a normal function $F : \omega_1 \rightarrow \omega_1$ such that $f$ is a finite subset of $F$.

(b) Let $C = \text{range}(f)$. If we denote by $\text{LIND}$ the set of all those ordinals which are a limit of indecomposables, then $D$ is included in the set of all ordinals in $C \cap \text{LIND}$ which are fixed points of $f$.

(c) For each $\delta \in D$, $C \cap A_\delta = b_\delta$. Further, if $b_\delta$ is odd (even), then $\delta \in B$ ($\delta \notin B$).

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(e) For every $\delta' \in D$ and every $\delta \in C$ with $\delta < \delta'$, the function $f$ omits all points of $b_{\delta, \delta'}$. That is, if $\gamma \in b_{\delta, \delta'}$, then there exist $\pi$, $\beta$ and $\beta'$ such that $\beta < \gamma < \beta'$ and $(\pi, \beta), (\pi + 1, \beta') \in f$. 
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The following result is a first step for showing that if there exists a generic which intersects $\aleph_1$ dense sets, then this forcing adds an instance of $\text{Code}(\text{even–odd})$.

**Lemma**

For every countable ordinal $\beta$, and every condition $q = (f, \langle b_\delta : \delta \in D \rangle)$ there exists a condition $q'$ extending $q$ and such that $\beta \in \text{Dom}(f')$. 
The following result is a first step for showing that if there exists a generic which intersects $\aleph_1$ dense sets, then this forcing adds an instance of $Code(\text{even–odd})$.

**Lemma**

*For every countable ordinal $\beta$, and every condition $q = (f, \langle b_\delta : \delta \in D \rangle)$ there exists a condition $q'$ extending $q$ and such that $\beta \in \text{Dom}(f')$.***
(Sketch) Fix a normal function $F : \omega_1 \rightarrow \omega_1$ such that $f \subseteq F$. Let us assume that there exists $\delta \in D$ such that $\delta > \beta$. Let $\delta_\beta$ be the minimum of the set $D \setminus \beta$. The most difficult case is when $\text{Dom}(f) \cap [\beta, \delta_\beta) = \emptyset$ (the other cases are easier since by condition (e) we are asking to omit all the bad points). Let $\delta'_\beta < \delta_\beta$ be the first indecomposable ordinal which is above both $\beta$ and $\mu = \max(\mathcal{C} \cap \delta_\beta)$. Let $\eta$ be the maximum of the set

$$\{\beta\} \cup \bigcup\{A_{\delta, i}^\alpha \cap \delta'_\beta : \delta \in D \setminus \delta'_\beta\}.$$ 

Let $\tau$ be such that $f(\tau) = \mu$, and let $\varepsilon$ be the unique ordinal such that $\tau + 1 + \varepsilon = \beta$. Finally, let $f' = f \cup \{(\tau + 1, \eta + 1), (\beta, \eta + 1 + \varepsilon)\}$. It is clear that the result of replacing $f$ with $f'$ in $q$ is a condition $q'$ as required.
SOME ENIGMATIC REMARKS

a) There is a variety of strengthenings of $\neg WCG$ similar to $\text{Code}(\text{even–odd})$ which can be forced with $c > \aleph_2$.

b) Doing some minor variations in the definition of this forcing $P$ we can also argue that $FA(\Gamma_\kappa)$ implies $\neg VWCG$.

c) If we restrict the class $\Gamma_\kappa$ to this type of posets, then the proof of the consistency of this forcing axiom together with $2^{\aleph_0} > \aleph_2$ becomes considerably simpler (in that case the side conditions are elementary substructures $N$ of $H(\omega_2)$ and making a promise means to put $\delta_N$ as a fixed point). This is because in the case of $\text{Code}(\text{even–odd})$ we try to omit a final segment of a ladder system (which can be seen as a small or null set), while in the case of $\neg \mathcal{U}$ for example, we are trying to omit a larger set.
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