A reflection principle together with the continuum arbitrarily large (2)

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In this talk we will show how to force some strong failures of Weak Club Guessing together with the continuum larger than ω_2.

- (•) Recall that a sequence $\{A_\delta : \delta \in Lim\}$ is said to be a ladder system iff for every δ, A_δ is a cofinal subset of δ of order type ω.

- (•) *Weak Club Guessing* (WCG) There is a ladder system $\{A_\delta : \delta \in Lim\}$ such that for every club $C \subseteq \omega_1$ there is $\delta \in C$ such that $A_\delta \cap C$ is infinite.

If we say that this intersection must include a final segment of A_δ, then we get *Club Guessing* (CG)
But we are interested in weakenings and not in strengthenings of WCG

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One weak form of WCG considered in the literature is the statement we may call *Very Weak Club Guessing* (VWCG). It says that there is a collection $\{A_\delta : \delta \in \omega_1\}$ of subsets of ω_1 of order type ω such that every club of ω_1 has infinite intersection with some A_δ.

Given a cardinal κ (possibly finite), WCG^κ says that there exist a system $\{A^\alpha_\delta : \alpha \in \kappa, \delta \in \mathit{Lim}\}$ such that for every α and δ, A^α_δ is a cofinal subset of δ of order type ω and such that every club subset of ω_1 has an infinite intersection with one of them.

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Proposition

WCG *is equivalent to* $\mathrm{WCG}^{\aleph_0}$*.*

Proof.

We will see that $\mathrm{WCG}^{\aleph_0}$ implies WCG. So, let $\{A^n_\delta : n \in \omega, \delta \in \mathit{Lim}\}$ be a system witnessing $\mathrm{WCG}^{\aleph_0}$. We define a ladder system $\{B_\delta : \delta \in \mathit{Lim}\}$ as follows. First, for each $\delta \in \mathit{Lim}$ fix an increasing cofinal sequence $\{\delta_n : n \in \omega\} \subseteq \delta$ of order type ω. Now define $B_\delta = \bigcup\{B^n_\delta : n \in \omega\}$, where B^n_δ is equal to $A^n_\delta \setminus \delta_n$. It is easy to check that $\{B_\delta : \delta \in \mathit{Lim}\}$ witnesses WCG. $\square$

Certainly, VWCG follows from CH, but $\neg$WCG is compatible with CH (See Shelah's [656]).

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Let us introduce another parameter. Given τ an indecomposable ordinal (i.e., of the form ω^β, $\beta \neq 0$), VWCG_τ states that there exist a sequence $\{A_\delta : \delta \in \omega_1\}$ such that for every δ, A_δ is a closed subset of order type an indecomposable ordinal less than or equal than τ and such that every club C of ω_1 has infinite intersection with one of them

We will assume that a sequence $\{A_\delta : \delta \in \omega_1\}$ always consists of sets of order type an indecomposable ordinal.

Remark

$\mathsf{VWCG} \to \mathsf{VWCG}_\tau$.

Proposition

$\neg$VWCG $\rightarrow \neg$VWCG$_\tau$.

The proof is by induction. Let $\mathcal{A} = \{A_\delta : \delta \in Lim\}$ be a sequence of sets such that every A_δ has order type less or equal than τ. For each limit ordinal δ consider an increasing sequence $\{\delta_n : n \in \omega\} \subseteq sup(A_\delta)$ of accumulation points of the set A_δ. Further, we can choose this sequence in such a way that

$$A(\delta, n) := A_{\delta_{n+1}} \setminus A_{\delta_n}$$

has order type an indecomposable ordinal strictly less than τ. Fix an increasing and cofinal sequence $\{\tau_m : m \in \omega\} \subseteq \tau$ of indecomposable ordinals. Now, consider the system

$$\mathcal{A} = \{A(\delta, n) : \delta \in Lim, n \in \omega\}$$

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and note that for each $m \in \omega$ there exists a club C_m such that for every δ and for every n if $A(\delta, n)$ has order type less or equal than τ_m, then $A(\delta, n) \cap C_m$ is finite. Let C be the intersection of all the C_m. Now define the set B_δ as follows:

$$B_\delta = \{\delta_n : \delta \in \omega\} \cup \bigcup \{A(\delta, n) \cap C : n \in \omega\}$$

Note that this set has order type ω. Finally find a club $D \subseteq C$ witnessing that the system $\mathcal{B} = \{B_\delta : \delta \in Lim\}$ does not guess in the very weak sense. It is easy to check that D also witnesses that $\mathcal{A}$ does not guess in the VWCG_τ–sense.

Corollary

$\mathrm{VWCG} \leftrightarrow \mathrm{VWCG}_\tau$.

Corollary

The following are equivalent:

*a)*VWCG

b) If $\mathcal{A}$ is a family of subsets of ω_1 (of order type an indecomposable ordinal) such that $|\mathcal{A}| = \aleph_1$ and such that for every $\gamma < \omega_1$ and every $B \in \mathcal{A}$ the order type of $B \cap \gamma$ is strictly less than γ, then there exists a club E such that E has finite intersection with all the elements of this family.

Proof.

The idea of the proof is very close to that of the above proposition. Namely, we should start by fragmenting each $B \in \mathcal{A}$ into ω pieces $\{B_n : n \in \omega\}$ in such a way that the order type of each of them is an indecomposable ordinal. Now, if E is the diagonal intersection of a suitable sequence of clubs, we can assume that E has finite intersection with B_n for every $B \in \mathcal{A}$ and every natural number n. The rest is standard. $\square$

Cohen and Random Reals in the context of Club Guessing (FOLKLORE?)

Proposition

Let P be Cohen forcing. Then, $V^P \models WCG$.

Lemma

Let P be a ${}^\omega\omega$ bounding notion of forcing and let $\langle A_\delta : \delta \in \omega_1 \rangle \in V^P$ be a sequence of sets of order type ω. Then, for every condition p and for every ordinal $\delta \in \omega_1$, there exists a condition q extending p and a countable set $C_q \in V$ such that q forces that A_δ is included in C_q and $sup(A_\delta) = sup(C_q)$.

So, if P has in addition the countable chain condition, then there are ω possibilities for this set C_q. But now if we use a diagonalization similar to that of the proof of the equivalence between WCG and $\mathrm{WCG}^{\aleph_0}$, then we get the following:

Corollary

Let P be a ${}^{\omega}\omega$ bounding notion of forcing satisfying the countable chain condition and let $\langle A_\delta : \delta \in \omega_1 \rangle \in V^P$ be a sequence of sets of order type ω. If $\langle A_\delta : \delta \in \omega_1 \rangle \in V^P$ is a sequence of sets which guesses all the clubs of V (in the very weak sense), then there is a sequence of sets in the ground model doing the same.

In particular, random forcing preserves the negation of VWCG.

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In particular, random forcing preserves the negation of VWCG.

Now, we introduce another strengthening of $\neg$WCG for which the above technique of diagonalization does not seem to apply.

Definition (Miyamoto)

Code(even–odd) states that for every ladder system $\langle A_\delta : \delta \in Lim\rangle$ and for every $B \subseteq \omega_1$, there exist two clubs C and D of ω_1 such that for every $\delta \in C$ which is a limit point
1) If $\delta \in B$, then $|A_\delta \cap D| < \aleph_0$ is odd.
2) If $\delta \notin B$, then $|A_\delta \cap D| < \aleph_0$ is even.

Proposition (Miyamoto)

BPFA $\rightarrow$ *Code*(*even–odd*) $\rightarrow$ $(2^{\aleph_0} = 2^{\aleph_1} + \neg\mathrm{WCG})$.

We prove that each instance of *Code*(even–odd) follows from the forcing axiom defined in the previous talk. This will show the consistency of *Code*(even–odd) together $\mathfrak{c} > \aleph_2$.

So, let $\langle A_\delta : \delta \in Lim \rangle$ be a ladder system and B a subset of ω_1
Now, consider the notion of forcing P defined as follows:
Its elements are pairs $(f, \langle b_\delta : \delta \in D \rangle)$ such that:

(a) There exists a normal function $F : \omega_1 \longrightarrow \omega_1$ such that f is a finite subset of F.

(b) Let $C = range(f)$. If we denote by $LIND$ the set of all those ordinals which are a limit of indecomposables, then D is included in the set of all ordinals in $C \cap LIND$ which are fixed points of f.

(c) For each $\delta \in D$, $C \cap A_\delta = b_\delta$. Further, if b_δ is odd (even), then $\delta \in B$ ($\delta \notin B$).

(d) For every $\delta' \in D$ and every $\delta \in C$ with $\delta < \delta'$ there exists a finite subset $b_{\delta, \delta'} \subseteq (\delta + 1) \setminus b_{\delta'}$ such that $q|_\alpha$ forces that the union of $b_{\delta,\delta'}$ and $b_{\delta'}$ is equal to the initial segment $A_{\delta'} \cap (\delta + 1)$ of $A_{\delta'}$.

(e) For every $\delta' \in D$ and every $\delta \in C$ with $\delta < \delta'$, the function f omits all points of $b_{\delta, \delta'}$. That is, if $\gamma \in b_{\delta, \delta'}$, then there exist π, β and β' such that $\beta < \gamma < \beta'$ and (π, β), $(\pi + 1, \beta') \in f$.

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The following result is a first step for showing that if there exists a generic which intersects $\aleph_1$ dense sets, then this forcing adds an instance of *Code*(even–odd).

Lemma

For every countable ordinal β, and every condition $q = (f, \langle b_\delta : \delta \in D \rangle)$ there exists a condition q' extending q and such that $\beta \in Dom(f')$.

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For every countable ordinal β*, and every condition* $q = (f, \langle b_\delta : \delta \in D \rangle)$ *there exists a condition* q' *extending* q *and such that* $\beta \in Dom(f')$.

(Sketch) Fix a normal function $F : \omega_1 \longrightarrow \omega_1$ such that $f \subseteq F$. Let us assume that there exists $\delta \in D$ such that $\delta > \beta$. Let δ_β be the minimum of the set $D \setminus \beta$. The most difficult case is when $Dom(f) \cap [\beta, \delta_\beta) = \emptyset$ (the other cases are easier since by condition (e) we are asking to omit all the bad points)
Let $\delta'_\beta < \delta_\beta$ be the first indecomposable ordinal which is above both β and $\mu = max(C \cap \delta_\beta)$. Let η be the maximum of the set

$$\{\beta\} \cup \bigcup \{A_\delta^{\alpha, i} \cap \delta'_\beta : \delta \in D \setminus \delta'_\beta\}.$$

Let τ be such that $f(\tau) = \mu$, and let ε be the unique ordinal such that $\tau + 1 + \varepsilon = \beta$. Finally, let $f' = f \cup \{(\tau + 1, \eta + 1), (\beta, \eta + 1 + \varepsilon)\}$. It is clear that the result of replacing f with f' in q is a condition q' as required.

SOME ENIGMATIC REMARKS

a) There is a variety of strengthenings of $\neg$WCG similar to *Code*$(\mathrm{even\text{–}odd})$ which can be forced with $\mathfrak{c} > \aleph_2$.

b) Doing some minor variations in the definition of this forcing P we can also argue that $FA(\Gamma_\kappa)$ implies $\neg$VWCG.

c) If we restrict the class Γ_κ to this type of posets, then the proof of the consistency of this forcing axiom together with $2^{\aleph_0} > \aleph_2$ becomes considerably simpler (in that case the side conditions are elementary substructures N of $H(\omega_2)$ and making a promise means to put δ_N as a fixed point). This is because in the case of *Code*$(\mathrm{even\text{–}odd})$ we try to omit a final segment of a ladder system (which can be seen as a small or null set), while in the case of $\neg\mho$ for example, we are trying to omit a larger set.

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