A universality property for analytic equivalence relations and quasi-orders

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Analytic and Borel equivalence relations

A pair (X, E) is said to be an analytic equivalence relation (resp. Borel equivalence relation) if X is a Polish space (or even just a standard Borel space) and E is an equivalence relation on X which is analytic (resp. Borel) as a subset of $X \times X$.

Any two uncountable standard Borel spaces are Borel-isomorphic, hence we can restrict ourselves to the case $X = {}^{\omega}2$.

A motivation for analyzing analytic equivalence relations and their relationships is given by the various classification problems arising in many areas of mathematics.

Borel reducibility

To compare the complexity of two analytic equivalence relations (X, E) e (Y, F) we use Borel-reducibility:

- $(X, E) \leq_B (Y, F)$ iff there is a Borel function $f : X \to Y$ such that $\forall x_1, x_2 \in X (x_1 \, E \, x_2 \iff f(x_1) \, F \, f(x_2))$;
- (X, E) and (Y, F) are Borel-equivalent, $(X, E) \sim_B (Y, F)$ in symbols, iff $(X, E) \leq_B (Y, F)$ and $(Y, F) \leq_B (X, E)$.
- (X, E) is said to be complete if for every analytic equivalence relation (Y, F) one has $(Y, F) \leq_B (X, E)$.

Intuitively: "$(X, E) \leq_B (Y, F)$" = "(X, E) is not more complicated than (Y, F)".

The structure of analytic equivalence relation under $\leq_B$ is quite complicated!

Example: Polish group actions

A Polish group $G = (G, e, \cdot)$ is a group equipped with a Polish topology τ such that the map $G \times G \to G \colon (x, y) \mapsto x \cdot y^{-1}$ is continuous.

If X is a Polish space, a continuous function $a \colon G \times X \to X$ is said to be action of G on X if $a(e, x) = x$ and $a(g, a(h, x)) = a(g \cdot h, x)$ for every $x \in X$ and $g, h \in G$.

The orbit equivalence relation E_a induced by a on X,
where $x \, E_a \, y \iff \exists g \in G (a(g, x) = y)$,
is an analytic equivalence relation.

We can also just require X to be standard Borel and a to be a Borel function: in this case X is said to be a Borel G-space.

Example: isomorphism relation

Let $\mathcal{L}$ be a countable language. We can assume that each countable $\mathcal{L}$-structure has domain ω, so that we can identify such a structure with an element of the Cantor space ${}^{\omega}2$.

Example: $\mathcal{L} = \{R\}$, R binary relational symbol. Every $x \in {}^{\omega}2$ codes the structure $\mathcal{A}_x = (\omega, R^{\mathcal{A}_x})$, where $n \, R^{\mathcal{A}_x} \, m$ iff $x(\langle n, m \rangle) = 1$.

Every isomorphism between countable $\mathcal{L}$-structures is simply a permutation of ω.

The isomorphism relation on a Borel set of countable $\mathcal{L}$-structures is an analytic equivalence relation.

Remark: The isomorphism relation among countable $\mathcal{L}$-structures coincides with the orbit equivalence relation induced by the canonical action ("logic action") $j_{\mathcal{L}}$ of the group S_∞ on ${}^{\omega}2$.

Example: isomorphism relation

$Mod_{\mathcal{L}}$ = (codes for) countable $\mathcal{L}$-structures

S_∞ = Polish group of permutations on ω.

Definition

A set $X \subseteq Mod_{\mathcal{L}}$ is said to be invariant if it is closed under isomorphism (i.e. closed with respect to the logic action $j_{\mathcal{L}}$ of S_∞ on $Mod_{\mathcal{L}}$).

Given an $\mathcal{L}_{\omega_1\omega}$-sentence φ, let Mod_φ be the collection of those $x \in Mod_{\mathcal{L}}$ which model φ.

Theorem (Lopez-Escobar)

$X \subseteq Mod_{\mathcal{L}}$ is Borel and invariant iff there is an $\mathcal{L}_{\omega_1\omega}$-sentence φ such that $X = Mod_\varphi$.

Example: isomorphism relation

Isomorphism relations are a very special subclass of the analytic equivalence relations:

- H. Friedman-Stanley: isomorphism on countable trees, on linear orders, and so on are S_∞-complete, i.e. $F \leq_B E$ for every isomorphism relation F;
- there are analytic equivalence relations which are not Borel-reducible to an isomorphism relation;
- in particular, no isomorphism relation can be complete!

Analytic quasi-orders

All the notions and definitions about analytic equivalence relations can be rephrased in the context of quasi-orders (i.e. reflexive and transitive relations):

- a quasi-order R is analytic if its domain X is standard Borel and R is an analytic subset of $X \times X$;
- $R \leq_B S$ iff there is a Borel function $f \colon X \to Y$ (where X and Y are the domains of R and S) such that $x \, R \, y \iff f(x) \, S \, f(y)$;
- R is complete if $S \leq R$ for every analytic quasi-order S, and so on.

Every analytic quasi-order R canonically induce the analytic equivalence relation $E_R = R \cap R^{-1}$, and if R is complete then E_R is complete as well.

Example: embeddability relation

Let $\mathcal{L}$, $Mod_{\mathcal{L}}$, φ and Mod_φ be defined as before.

Definition

Given two $\mathcal{L}$-structures $\mathcal{A}$ and $\mathcal{B}$, we say that $\mathcal{A}$ embeds into $\mathcal{B}$ ($\mathcal{A} \sqsubseteq \mathcal{B}$ in symbols) if there is an injection $f : A \to B$ which is an isomorphism on its image (considered as a substructure of $\mathcal{B}$).

Remark: $\sqsubseteq$ restricted to Mod_φ is an analytic quasi-order, and canonically induces the analytic equivalence relation $\equiv$ of bi-embeddability.

Some examples

Example 1: On well-founded linear orders of length $\leq \alpha$, α a fixed countable ordinal, isomorphism and bi-embeddability coincide.

Example 2: The isomorphism relation on linear orders is S_∞-complete (H. Friedman-Stanley), whereas bi-embeddability on linear orders has $\aleph_1$-many equivalence classes but doesn't Borel-reduces equality on ${}^\omega 2$ (Laver: $\sqsubseteq_{LO}$ is a bqo).

Exmple 3: Embeddability on graphs (hence also bi-embeddability) is complete (Louveau-Rosendal), whereas isomorphism on graphs is S_∞-complete.

Combinatorial trees

A combinatorial tree is a connected acyclic graph. CT denotes the collection of all countable combinatorial trees.

Theorem (Louveau-Rosendal)

$\sqsubseteq_{CT}$ *is a complete analytic quasi-order.*

The proof uses the fact that, given an arbitrary analytic quasi-order R on X, one can code informations about $x \in X$ w.r.t. R into a corresponding combinatorial tree G_x using the distance and the valence function, together with the fact that any embedding between graphs must preserve distances and (at most) increase valences.

"Universality" of embeddability

Theorem (S. D. Friedman-M.)

For every analytic quasi-order R there is an $\mathcal{L}_{\omega_1\omega}$-sentence φ such that R is Borel-equivalent to $\sqsubseteq$ on Mod_φ.

Corollary (S. D. Friedman-M.)

E is an analytic equivalence relation iff there is an $\mathcal{L}_{\omega_1\omega}$-sentence φ such that E is Borel-equivalent to $\equiv$ on Mod_φ.

One can also replace "embeddability between countable $\mathcal{L}$-structures" with homomorphism or weak-homomorphism between countable $\mathcal{L}$-structures.

Sketch of the proof of the main result

Let R be an arbitrary quasi-order on X, and $x \mapsto G_x$ be the Borel map constructed by Louveau and Rosendal (in particular, such map reduces R to $\sqsubseteq$).

Now modify the definition of G_x to obtain a $\hat{G}_x$ such that:

- $x \neq y \Rightarrow \hat{G}_x \ncong \hat{G}_y$;
- each $\hat{G}_x$ is rigid (no nontrivial automorphisms);
- $x \mapsto \hat{G}_x$ is still a Borel reduction of R into $\sqsubseteq$.

The structure $\hat{G}_x$ can be obtained in two different ways:

1. add some new vertices to G_x to code some additional informations (in particular in order to have a rigid structure);
2. add a new binary relational symbol to the language and interpret it as a (special) well-founded linear order on the original G_x (in this case $\hat{G}_x$ will be a so-called ordered combinatorial tree).

Sketch of the proof of the main result

Let R' be the analytic quasi-order on $X \times S_\infty$ defined by $(x,p) \, R' \, (y,q) \iff x \, R \, y$. Clearly $R \sim_B R'$, so it is enough to construct Mod_φ such that $R' \sim_B \sqsubseteq \restriction Mod_\varphi$.

Consider the Borel map f which sends (x,p) to $j_{\mathcal{L}}(p, \hat{G}_x)$. f reduces R' to $\sqsubseteq$ because

$$\begin{aligned}(x,p) \, R' \, (y,q) &\iff x \, R \, y \\ &\iff \hat{G}_x \sqsubseteq \hat{G}_y \\ &\iff f(x,p) \sqsubseteq f(y,q).\end{aligned}$$

Sketch of the proof of the main result

Check that f is injective. Consider $(x, p), (y, q) \in X \times S_\infty$ and assume $f(x, p) = f(y, q)$: since this implies $\hat{G}_x \cong \hat{G}_y$, we have $x = y$; but then $q^{-1} \circ p$ is an automorphism of $\hat{G}_x$, whence $p = q$.

Notice that the $\mathrm{range}(f)$ is invariant by definition of f.

Since f is an injective Borel map defined on a Borel set we get:

- $\mathrm{range}(f)$ is Borel: being Borel and invariant, it coincides with Mod_φ for some $\mathcal{L}_{\omega_1\omega}$-sentence φ;
- f^{-1} is a Borel function, hence a Borel reduction of $\sqsubseteq \restriction Mod_\varphi$ to R'.

This concludes the proof.

Homomorphism and weak-homomorphism

Given two $\mathcal{L}$-structures $\mathcal{A}$ and $\mathcal{B}$, an homomorphism between $\mathcal{A}$ and $\mathcal{B}$ is a function $f \colon A \to B$ which preserves relations and functions in both directions (that is, if e.g. $P \in \mathcal{L}$ is a binary relational symbol then $n\, P^{\mathcal{A}}\, m$ iff $f(n)\, P^{\mathcal{B}}\, f(m)$).

f is said to be a weak-homomorphism if relations and functions are preserved just in one direction (if $n\, P^{\mathcal{A}}\, m$ then $f(n)\, P^{\mathcal{B}}\, f(m)$).

Theorem (S. D. Friedman-M.)

If R is an analytic quasi-order then there is an $\mathcal{L}_{\omega_1\omega}$-sentence φ such that R is Borel-equivalent to the homomorphism (resp. weak-homomorphism) relation on Mod_φ.

Sketch of the proof

Construct $\hat{G}_x$ by adjoining to G_x a suitable strict well-founded linear order $<_x$. Then check that on ordered combinatorial trees of this form weak-homomorphism, homomorphism and embedding coincide.

Let f be a weak-homomorphism between $\hat{G}_x$ and $\hat{G}_y$. First notice that f is injective because $<_y$ is strict and $<_x$ is linear.

Now assume $f(z_0) <_y f(z_1)$: if $z_0 \not<_x z_1$ then $z_1 <_x z_0$ and hence $f(z_1) <_y f(z_0)$, a contradiction!

Finally, let $f(z_0)$ and $f(z_1)$ be two linked vertices of G_y: if z_0 and z_1 are not linked in G_x, then the (unique) proper chain between z_0 and z_1 should be mapped to a proper chain between $f(z_0)$ and $f(z_1)$, a contradiction!

Relationship between isomorphism and bi-embeddability

Let $\mathcal{C}$ be an $\mathcal{L}_{\omega_1\omega}$-elementary class (i.e. $\mathcal{C} = Mod_\varphi$ for some $\mathcal{L}_{\omega_1\omega}$-sentence φ), and let $\cong_\mathcal{C}$, $\sqsubseteq_\mathcal{C}$ and $\equiv_\mathcal{C}$ denote, respectively, isomorphism, embeddability and bi-embeddability on $\mathcal{C}$.

Example

- *if $\mathcal{C}$ is the class of linear orders then $\cong_\mathcal{C}$ is S_∞-complete, whereas $\equiv_\mathcal{C}$ has $\aleph_1$-many classes ($\sqsubseteq_\mathcal{C}$ is a bqo);*
- *if $\mathcal{C}$ is the class of combinatorial trees then $\equiv_\mathcal{C}$ and $\sqsubseteq_\mathcal{C}$ are both complete, whereas $\cong_\mathcal{C}$ is S_∞-complete.*

Three questions

Louveau and Rosendal asked if we can increase the previous gaps between $\cong_{\mathcal{C}}$ and $\equiv_{\mathcal{C}}$.

Question

- *Is there an $\mathcal{L}_{\omega_1\omega}$-elementary class $\mathcal{C}$ such that $\cong_{\mathcal{C}}$ is S_∞-complete but $\equiv_{\mathcal{C}}$ has just countably many classes?*
- *Is there an $\mathcal{L}_{\omega_1\omega}$-elementary class $\mathcal{C}$ such that $\equiv_{\mathcal{C}}$ is complete but $\cong_{\mathcal{C}}$ is not S_∞-complete?*

A similar question, which is related to the Louveau-Rosendal's method for proving completeness of analytic equivalence relations is the following:

Question

Is there an $\mathcal{L}_{\omega_1\omega}$-elementary class $\mathcal{C}$ such that $\equiv_{\mathcal{C}}$ is complete but $\sqsubseteq_{\mathcal{C}}$ is not complete?

$\cong_{\mathcal{C}}$ S_∞-complete, $\equiv_{\mathcal{C}}$ with ω-many classes

Consider the set-theoretical trees defined by Friedman-Stanley in the proof that $\cong$ on trees is S_∞-complete: each of them consists of the tree T_{seq} of all finite sequences of natural numbers plus some new terminal node.

The class of all set-theoretical trees of this form is an $\mathcal{L}_{\omega_1\omega}$-elementary class $\mathcal{C}$, and since any set-theoretical tree can be embedded into T_{seq} we get that $\equiv_{\mathcal{C}}$ has just one equivalence class (whereas $\cong_{\mathcal{C}}$ remains S_∞-complete).

$\equiv_{\mathcal{C}}$ complete, $\cong_{\mathcal{C}}$ not S_∞-complete

Recall that we constructed the combinatorial trees $\hat{G}_x$ in such a way that the map $x \mapsto \hat{G}_x$ reduces equality to $\cong$.

This means that, given an arbitrary analytic quasi-order R, the $\mathcal{L}_{\omega_1\omega}$-elementary class $\mathcal{C} = Mod_\varphi$ given by our main theorem is such that $R \sim_B \sqsubseteq_{\mathcal{C}}$ and simultaneously $= \sim_B \cong_{\mathcal{C}}$ (both equivalences being witnessed by the same functions).

Therefore, if R is complete then the relations $\sqsubseteq_{\mathcal{C}}$ and $\equiv_{\mathcal{C}}$ resulting from the application of the main theorem will be complete, but $\cong_{\mathcal{C}}$ will be Borel-equivalent to equality on ${}^{\omega}2$.

Remark: Such result is best possible: if $= \not\leq_B \cong_{\mathcal{C}}$ then $\equiv_{\mathcal{C}}$ cannot be complete since $= \not\leq_B \equiv_{\mathcal{C}}$.

$\equiv_{\mathcal{C}}$ complete, $\sqsubseteq_{\mathcal{C}}$ not complete

Let E be a complete analytic equivalence relation. Since E is, in particular, a quasi-order, we can apply our main theorem to such E: the resulting $\mathcal{C}$ is such that $E \sim_B \sqsubseteq_{\mathcal{C}} = \equiv_{\mathcal{C}}$, hence $\equiv_{\mathcal{C}}$ is complete while $\sqsubseteq_{\mathcal{C}}$ cannot be complete because it is an equivalence relation (a downward closed notion).

More generally, given two analytic equivalence relations E, F and an analytic quasi-order R, our techniques can be used to produce, an $\mathcal{L}_{\omega_1\omega}$ elementary class $\mathcal{C}$ such that E, F, R are Borel-equivalent to, respectively, $\cong_{\mathcal{C}}, \equiv_{\mathcal{C}}, \sqsubseteq_{\mathcal{C}}$ *as long as* $E \sim_B =$ *and* $F \sim_B E_R$.

For example, we can produce an $\mathcal{L}_{\omega_1\omega}$-elementary class $\mathcal{C}$ such that $\cong_{\mathcal{C}}$ and $\equiv_{\mathcal{C}}$ are distinct but still $\cong_{\mathcal{C}} \sim_B \equiv_{\mathcal{C}}$.

Remark: Recently the condition $E \sim_B =$ has been removed... but this is another story!

Other complete analytic quasi-orders

After Louveau-Rosendal's paper, many other natural complete analytic quasi-orders have been discovered: colour-preserving and colour-decreasing embeddability on linear orders, colour-preserving *dop* embeddability on colourings of $\mathbb{Q}$, weak-epimorphism on countable graphs, and so on.

Question

Is it possible to extend the main result to all these quasi-orders?

More precisely, we are asking if for any quasi-order S above is it true that given an arbitrary analytic quasi-order R there is a Borel class $\mathcal{C}$ closed under the natural "isomorphism" relation E associated to S such that $R \sim_B S \restriction \mathcal{C}$. If the answer is positive we say that (S, E) (or just S) is invariantly universal.

The general technique

Let Y be a Polish group acting in a Borel way on X and $G(Y)$ be the standard Borel space of the closed subgroups of Y.

Lemma

Let $X \to G(Y) \colon x \mapsto H_x$ be a Borel map. Then there is a Borel set $Z \subseteq X \times Y$ such that $Z_x = \pi_2(Z \cap (\{x\} \times Y))$ is a Borel transversal for E_x, the equivalence relation on Y whose classes are the (left) cosets of H_x.

Theorem

Suppose (S, E) is a pair of analytic relations on a standard Borel space W such that S is a quasi-order and $E \subseteq E_S$ is a Borel Y-space. Suppose f is a Borel reduction of $\sqsubseteq$ between countable graphs to S which simultaneously reduces isomorphism on $\mathcal{G}$ (the set of all possible $\hat{G}_x$ constructed as above) to E, and assume that the function $\mathcal{G} \to G(Y) \colon \hat{G}_x \mapsto Stab_Y(f(\hat{G}_x))$ is Borel. Then (S, E) is invariantly universal.

Sketch of the proof

Given an arbitrary analytic quasi-order R, apply the lemma to the map $x \mapsto Stab_Y(f(\hat{G}_x))$ and let $Z \subseteq X \times Y$ be the resulting Borel set. Define g on Z by $(x, y) \mapsto a(y, f(\hat{G}_x))$: g reduces R to S because $E \subseteq E_S$.

- g is injective. $g(x_0, y_0) = g(x_1, y_1) \iff f(\hat{G}_{x_0}) \, E \, f(\hat{G}_{x_1}) \iff \hat{G}_{x_0} \cong \hat{G}_{x_1} \iff x_0 = x_1$. Then $y_1^{-1} \circ y_0 \in Stab_Y(f(\hat{G}_{x_0}))$, hence $y_0 = y_1$ because $(x_0, y_0), (x_1, y_1) \in Z$.
- $\mathrm{range}(g)$ is E-saturated. For $y \in Y$ we have that $a(y, f(\hat{G}_x)) = a(\bar{y}, f(\hat{G}_x)) = g(x, \bar{y})$, where $\bar{y}$ is the unique point such that $y \, E_x \, \bar{y}$ and $(x, \bar{y}) \in Z$.

The proof can now be concluded as before.

Weak-epimorphism on graphs

Definition

We call weak-epimorphism a surjective weak-homomorphism.

If φ is an $\mathcal{L}_{\omega_1\omega}$-sentence, the relation $x \preceq_{epi} y$ if and only if x is the weak-epimorphic image of y defined on Mod_φ is an analytic equivalence relation.

Theorem (Camerlo)

$\preceq_{epi}$ on graphs is a complete analytic quasi-order.

Theorem (Camerlo-M.)

$\preceq_{epi}$ is invariantly universal.

Coloured linear orders

A coloured linear order is an element of $LO \times {}^{\omega}C$. Given a quasi-order P on C and coloured linear orders $(L, \varphi), (L', \psi)$, we put $(L, \varphi) \preceq_Q (L', \psi)$ iff there is an embedding f from L to L' such that $\varphi(n) Q \psi(f(n))$.

Theorem

The following are complete analytic quasi-orders:

- *(Marcone-Rosendal)* $\preceq_=$;
- *(Camerlo)* $\preceq_\geq$;
- *(Camerlo)* $\preceq_=$ *restricted to colourings on* $\mathbb{Q}$ *or on any fixed non-scattered linear order.*

Theorem (Camerlo-M.)

All previous quasi-orders are invariantly universal.

Colour preserving *dop* embeddings

A function $f \colon \mathbb{Q} \to \mathbb{Q}$ is *dop* (dense order preserving) if for every $q_0, q_1, r_0, r_1 \in \mathbb{Q}$ such that $f(q_0) < r_0 < r_1 < f(q_1)$ there is a $q \in \mathbb{Q}$ such that $r_0 < f(q) < r_1$. Given two colourings $\varphi, \psi \in {}^C\mathbb{Q}$ and a quasi-order P on C, we put $\varphi \leq^P_{dop} \psi$ iff there is a *dop* function f such that $\varphi(q) P \psi(f(q))$.

Theorem (Camerlo)

The following are complete analytic quasi-orders: $\leq^{=}_{dop}$, $\leq^{\geq}_{dop}$, $\leq^{=_2}_{dop}$.

Theorem (Camerlo-M.)

All previous quasi-orders are invariantly universal.

Open problems

1. Is the relation of "being epimorphic image" (where epimorphism is surjective homomorphism) a complete analytic quasi-order? Is it invariantly universal?
2. Is the relation of elementary embeddability between countable structures a complete analytic quasi-order? Is it invariantly universal?
3. What about complete analytic quasi-orders arising in analysis, such as isometric embeddability between ultrametric Polish spaces, continuous embeddability between compacta (or even just dendrites), and so on? Are they invariantly universal?
4. Is there a complete analytic quasi-order which is not invariantly universal (with respect to some natural analytic equivalence relation)?

Thank you
for your attention!