Structural Ramsey theory and topological dynamics I

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Part I

Outline

Outline and goals

Describe an interaction established by Alekos Kechris, Vladimir Pestov and Stevo Todorcevic between:

- Topological dynamics
  - Extreme amenability.
  - Universal minimal flows.
  - Oscillation stability.
- Combinatorics
  - Finite Ramsey theory on Fraïssé classes.
  - Infinite Ramsey theory on countable ultrahomogeneous structures.

First lecture: fundamentals of Fraïssé theory

- Extreme amenability for topological groups.
- Closed subgroups of S_∞.
- Fundamentals of Fraïssé theory.
- Examples of Fraïssé classes and Fraïssé limits.

Part II

Extreme amenability

Continuous actions and amenable groups

Definition

Let G be a topological group, X a topological space.
A continuous action of G on X is a continuous map $G \times X \longrightarrow X$.

Remark

Such an action is also called a G-flow.
Notation: $G \curvearrowright X$.

Definition

Let G be a topological group.
G is amenable when every continuous action of G on a compact space X has a fixed point, provided X convex subset of a Hausdorff locally convex topological vector space, and the action is affine:

$$\exists x \in X \; \forall g \in G \quad g \cdot x = x.$$

Extremely amenable groups

Definition

Let G be a topological group.
G is extremely amenable when every continuous action of G on a compact space X has a fixed point.

Question (Mitchell, 66)

Is there a non trivial extremely amenable group at all?

Theorem (Herrer-Christensen, 75)

There is a Polish Abelian extremely amenable group.

Theorem (Veech, 77)

Let G be non-trivial and locally compact.
Then G is not extremely amenable.

Extremely amenable groups: examples everywhere!

Examples

1. *$O(\ell_2)$, pointwise convergence topology (Gromov-Milman, 84).*
2. *Measurable maps $[0,1] \to \mathbb{S}^1$ (Furstenberg-Weiss, unpub-Glasner, 98)*

$$d(f,g) = \int_0^1 d(f(x), g(x)) d\mu.$$

3. *$\mathrm{Aut}(\mathbb{Q}, <)$, product topology induced by $\mathbb{Q}^{\mathbb{Q}}$ (Pestov, 98).*
4. *$Homeo_+([0,1])$, $Homeo_+(\mathbb{R})$, pointwise convergence topology (Pestov, 98).*
5. *$\mathrm{iso}(\mathbb{U})$, pointwise convergence topology, $\mathbb{U}$ the Urysohn metric space (Pestov, 02).*

Remark

Examples 3, 4, and 5 by Pestov use some Ramsey theoretic results.

The work of Kechris, Pestov and Todorcevic, I

Definition

S_∞: the group of permutations of $\mathbb{N}$.
Basic open sets: $f \in S_\infty$, $F \subset \mathbb{N}$ finite.

$$U_{f,F} = \{g \in S_\infty : g \restriction F = f \restriction F\}.$$

This topology is Polish (separable, metrizable with a complete metric).

Theorem (Kechris - Pestov - Todorcevic, 05)

There is a link between extreme amenability and Ramsey theory when G is a closed subgroup of S_∞.

Part III

Closed subgroups of S_∞

Ultrahomogeneous structures

Definition

Let $L = \{R_i : i \in I\} \cup \{f_j : j \in J\}$ *be a first order language.*
An L-structure $\mathbb{F}$ *is* ***ultrahomogeneous*** *when every isomorphism between finite substructures of* $\mathbb{F}$ *extends to an automorphism of* $\mathbb{F}$*.*

Example

$L = \{<\}$, $<$ *binary relation symbol.*
$\mathbb{F} = (\mathbb{Q}, <)$.

More examples later.

Closed subgroups of S_∞ and countable ultrahomogeneous structures

Proposition

- *If* $\mathbb{F}$ *is countable (WLOG,* $\mathbb{F} = (\mathbb{N}, \ldots)$*), then* $\mathrm{Aut}(\mathbb{F})$ *is a closed subgroup of* S_∞*.*
- *If* G *closed subgroup of* S_∞*, then there is*
  - L *countable language,*
  - $\mathbb{F}_G = (\mathbb{N}, \ldots)$ *countable ultrahomogeneous* L*-structure*

  such that

  $$G = \mathrm{Aut}(\mathbb{F}_G).$$

  Relations of arity n*: orbits of* $G \curvearrowright \mathbb{N}^n$*.*

Corollary

The closed subgroups of S_∞ *are exactly the automorphism groups of countable ultrahomogeneous structures.*

Part IV

Fraïssé theory

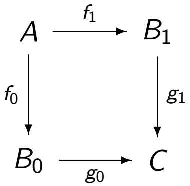
Combinatorial properties of classes of finite structures

L a countable first order language, $\mathcal{K}$ a class of finite L-structures.

Definition

$\mathcal{K}$ satisfies:

1. *hereditarity when it is is closed under substructures.*
2. *amalgamation when for all $A, B_i \in \mathcal{K}$ $(i = 0, 1)$, embeddings $f_i : A \longrightarrow B_i$, there is C and embeddings $g_i : B_i \longrightarrow C$ such that $g_0 \circ f_0 = g_1 \circ f_1$.*



3. *joint embedding property: for all $A, B \in \mathcal{K}$, there is $C \in \mathcal{K}$ such that A, B embed in C.*

Fraïssé classes

Definition

$\mathcal{K}$ is a Fraïssé class when it is countable and satisfies properties 1, 2 and 3.

Examples

- $\mathcal{LO}$ *finite linear orders,* $L = \{<\}$.
- $\mathcal{G}$ *finite graphs,* $L = \{E\}$ *adjacency relation symbol.*
- $\mathcal{M}_{\mathbb{Q}\cap[0,1]}$ *finite metric spaces with rational distances,* $L = \{d_q : q \in \mathbb{Q}\}$ *binary relational language,* $d_q^X(x,y)$ *when* $d^X(x,y) < q$.

Fraïssé's theorem

Proposition

Let $\mathbb{F}$ *be a countable ultrahomogeneous L-structure.*
Age($\mathbb{F}$) *the class of all finite substructures of* $\mathbb{F}$.
Then Age($\mathbb{F}$) *is a Fraïssé class.*

Theorem (Fraïssé, 54)

Let $\mathcal{K}$ *be a Fraïssé class in some language countable L.*
Then up to isomorphism, there is a unique countable ultrahomogeneous L-structure $\mathbb{F}$ *for which*

$$Age(\mathbb{F}) = \mathcal{K}.$$

Notation: $\mathbb{F} = Flim(\mathcal{K})$.

Part V

Examples of Fraïssé classes and Fraïssé limits

Graphs

Fraïssé classes of graphs classified by Lachlan-Woodrow, 80.

Examples

- $\mathcal{CG}$ *finite complete graphs:* $\mathit{Flim}(\mathcal{CG}) = K_\omega$.
  The countable infinite complete graph.
- $\mathcal{G}$ *finite graphs:* $\mathit{Flim}(\mathcal{G}) = \mathcal{R}$.
  The Rado graph, universal for countable graphs.
- $\mathcal{G}_n$ K_n*-free finite graphs:* $\mathit{Flim}(\mathcal{G}_n) = H_n$.
  Henson graphs, universal for countable K_n*-free graphs.*

Oriented graphs

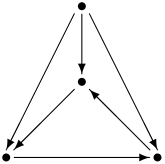
Fraïssé classes of oriented graphs classified by Cherlin, 98.

Examples

- $\mathcal{LO}$ *finite linear orders:* *Flim*$(\mathcal{LO}) = (\mathbb{Q}, <)$.
- $\mathcal{PO}$ *finite partial orders:* *Flim*$(\mathcal{PO}) = \mathbb{P}$.
  The countable ultrahomogeneous poset, universal for all countable posets.

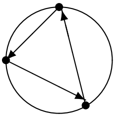
Oriented graphs, cont'd

- $\mathcal{C}$ finite local orders:
  Finite tournaments not embedding



$$\color{red}{Flim(\mathcal{C}) = S(2)}.$$

Vertices: Rational points of $\mathbb{S}^1$ (no antipodal pair).
Arcs: $x \rightarrow y$ iff (counterclockwise angle from x to y) $< \pi$.



Metric spaces

Fraïssé classes of finite metric spaces still not classified.

Examples

- *$\mathcal{M}_S$ finite metric spaces with distances in S (conditions on S needed, see Delhommé-Laflamme-Pouzet-Sauer):*

$$Flim(\mathcal{M}_S) = \mathbb{U}_S.$$

The countable Urysohn space with distances in S, universal for countable metric spaces with distances in S.

  - *Interesting cases: finite, $\mathbb{Q}$, $\mathbb{N}$.*

- *$\mathcal{U}$ finite ultrametric spaces with distances in $\{1/2^n : n \in \mathbb{N}\}$:*

$$\forall x, y, z \quad d(x, z) \leq \max(d(x, y), d(y, z)).$$

$$Flim(\mathcal{U}) = \mathbb{U}^{ult}.$$

Dense subspace of the Baire space $\mathbb{N}^{\mathbb{N}}$ (eventually 0 sequences).

Euclidean metric spaces

Examples

- $\mathcal{E}_{\mathbb{Q}}$ *finite affinely independent Euclidean metric spaces, distances in* $\mathbb{Q}$*:*

  $$\mathit{Flim}(\mathcal{E}_{\mathbb{Q}}) = \ell_2^{\mathbb{Q}}.$$

  Countable dense metric subspace of ℓ_2*.*
- $\mathcal{S}_{\mathbb{Q}}$ *finite affinely independent Euclidean metric spaces, distances in* $\mathbb{Q}$*, circumradius* < 1*:*

  $$\mathit{Flim}(\mathcal{S}_{\mathbb{Q}}) = \mathbb{S}^{\infty}_{\mathbb{Q}}.$$

  Countable dense metric subspace of the unit sphere $\mathbb{S}^{\infty}$ *of* ℓ_2*.*
- *Finite metric subspaces of* $(M^{\omega}, \| \cdot \|_2)$*,* $M \subset \mathbb{R}$ *countable and closed under sufficiently many operations (Jasinski, Hamilton-Loo, 08).*

Structures with operations

Examples

- $\mathcal{BA}$ *finite Boolean algebras,* $L = \{0, 1, -, \wedge, \vee\}$*:*
$$Flim(\mathcal{BA}) = B_\infty.$$
The countable atomless Boolean algebra, universal for countable Boolean algebras.
- $\mathcal{V}_F$ *finite vector spaces,* F *finite field,* $L = \{+\} \cup \{f_\alpha : \alpha \in F\}$*:*
$$Flim(\mathcal{V}_F) = F^{<\omega}.$$
The countable infinite dimensional vector space over F*.*

Summary

- Some Polish, non locally compact groups G are extremely amenable: Every continuous action of G on a compact space has a fixed point.
- When G closed subgroup of S_∞, extreme amenability has a combinatorial characterization. Namely:
- $G = \mathrm{Aut}(\mathbb{F})$, $\mathbb{F}$ a countable ultrahomogeneous first order structure.
- $\mathbb{F}$ is the Fraïssé limit of a class $\mathcal{K}$ of finite structures.
- G is extremely amenable iff some combinatorial phenomenon takes place at the level of $\mathcal{K}$ (Ramsey type properties).